Combinatorics of flag varieties lect3 Hausdorff Institute Winter School 10-14 Jan. 2011 ①

Simple T-modules and simple B-modules

G is generated by $x_\alpha(c)$, $x_{-\alpha}(c)$, $h_{\lambda^\vee}(d)$, $\quad \alpha \in R^+$, $c \in \mathbb{F}$

$\cup\!\mid$

B is generated by $x_\alpha(c)$, $\qquad h_{\lambda^\vee}(d)$, $\quad \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}$, $d \in \mathbb{F}^\times$.

$\cup\!\mid$

T is generated by $\qquad h_{\lambda^\vee}(d)$

comes from

W_0 a $\mathbb{Z}$-reflection group acting on

$\mathfrak{h}_{\mathbb{Z}}$, a $\mathbb{Z}$-lattice

R^+ an index set for reflections in W_0

Define

$$\mathfrak{h}_{\mathbb{Z}}^* = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{h}_{\mathbb{Z}}, \mathbb{Z}) = \{\mu : \mathfrak{h}_{\mathbb{Z}} \to \mathbb{Z} \mid \mu \text{ is } \mathbb{Z}\text{-linear}\}.$$

The simple B modules are $\mathbb{C}_\mu = \mathrm{span}\{v_\mu\}$

corresponding to

$$X^\mu : T \longrightarrow \mathbb{C}^\times \qquad \text{i.e.} \qquad b v_\mu = X^\mu(b) v_\mu$$

$$h_{\lambda^\vee}(d) \longmapsto d^{\langle \mu, \lambda^\vee \rangle} \qquad h_{\lambda^\vee}(d) v_\mu = d^{\langle \lambda^\vee, \mu \rangle} v_\mu$$

$$x_\alpha(c) \longmapsto 1 \qquad x_\alpha(c) v_\mu = v_\mu$$

where $\langle \mu, \lambda^\vee \rangle = \mu(\lambda^\vee)$. So

$$R(T) = K_T(pt) = R(B) = K_T(B)$$

$$= \mathrm{span}\{X^\mu \mid \mu \in \mathfrak{h}_{\mathbb{Z}}^*\} \text{ with } X^\mu X^\nu = X^{\mu+\nu}$$

Example SL_2

$G = SL_2(\mathbb{C})$ is generated by

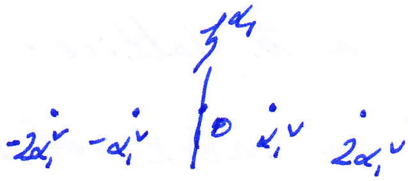
$$x_{\alpha_1}(c) = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, \quad x_{-\alpha_1}(c) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \quad h_{\ell\alpha_1^\vee}(d) = \begin{pmatrix} d^{\ell} & 0 \\ 0 & d^{-\ell} \end{pmatrix}$$

and $G \supseteq B \supseteq T$ is

$$SL_2(\mathbb{C}) \supseteq \left\{ \begin{pmatrix} d & c \\ 0 & d^{-1} \end{pmatrix} \right\} \supseteq \left\{ \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \right\}$$

Here

$W_0 =$ 

and $\mathfrak{h}_{\mathbb{Z}} = \mathbb{Z}\text{-span}\{\alpha_1^\vee\}$ 

and $\mathfrak{h}_{\mathbb{Z}}^* = \text{span}\{\omega_1\}$ where $\langle \omega_1, \alpha_1^\vee \rangle = 1$.

The irreducible B-modules are $\mathbb{C}_k = \text{span}\{v_k\}$ corresponding to

$$X^k = X^{k\omega_1} : B \longrightarrow \mathbb{C}^\times$$

$$\begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \longmapsto d^k \qquad \text{i.e.} \qquad \begin{pmatrix} d & c^{+1} \\ 0 & d^{-1} \end{pmatrix} v_k = d^k v_k.$$

$$h_{\ell\alpha_1^\vee}(d) \longmapsto d^{k\ell} = d^{\langle k\omega_1, \ell\alpha_1^\vee \rangle}$$

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \longmapsto 1$$

for $k \in \mathbb{Z} \cong \mathfrak{h}_{\mathbb{Z}}^*$. So

$$R(T) = \text{span}\{X^k \mid k \in \mathbb{Z}\} \text{ with } X^k X^\ell = X^{k+\ell}$$

$$= \mathbb{C}[X, X^{-1}]$$

Hermann Weyl's theorem

The irreducible integrable G-modules $L(\mu)$ are indexed by

$\mu \in \mathfrak{h}_{\mathbb{Z}}^*$ such that $\langle \mu, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for $\alpha \in R^+$

$L(\mu)$ has a unique B-submodule $\mathbb{C}_\mu$,

and is characterized by this B-submodule

i.e. there is a unique $v_\mu \in L(\mu)$ such that

$$x_\alpha(c) v_\mu = 1 \quad \text{and} \quad h_{\lambda^\vee}(d) v_\mu = d^{\langle \mu, \lambda^\vee \rangle} v_\mu .$$

Example SL_2:

The $\mu \in \mathfrak{h}_{\mathbb{Z}}^*$ with $\langle \mu, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$ are

$$\mu = k\omega_1, \quad \text{for } k \in \mathbb{Z}_{\geq 0}$$

So the irreducible SL_2-modules are

$$L(k), \quad \text{for } k \in \mathbb{Z}_{\geq 0}$$

and $L(k)$ contains v_k with

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} v_k = v_k \quad \text{and} \quad \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} v_k = d^k v_k .$$

The G-modules $H^0(G/B, \mathcal{L}_\mu)$.

The line bundle $\mathcal{L}_\mu$ is

$$\begin{array}{ccc} G\times_B \mathbb{C}_\mu & \xrightarrow{p} & G/B \\ (g, cv) & \longmapsto & gB, \end{array} \qquad \text{where } G\times_B \mathbb{C}_\mu = \frac{G\times\mathbb{C}}{\langle (gb, cv) = (g, cbv)\rangle}$$

The vector space of global sections of $\mathcal{L}_\mu$ is

$$H^0(G/B, \mathcal{L}_\mu) = \{ G\times_B \mathbb{C}_\mu \xleftarrow{s} G/B \mid p\circ s = id_{G/B} \}$$

Identify

$$H^0(G/B, \mathcal{L}_\mu) \longleftrightarrow \left\{ f: G\to\mathbb{C} \;\middle|\; \begin{array}{l} f(gb) = f(g)X^\mu(b^{-1}) \\ \text{for } g\in G,\ b\in B \end{array} \right\}$$

by

$$s(gB) = (g, f(g)v_\mu)$$

Note: $(g, f(g)v) = s(gB) = s(gbB) = (gb, f(gb)v_\mu)$

$$= (g, f(gb)\cancel{b}v_\mu) = (g, f(gb)X^\mu(b)v_\mu).$$

The group G acts on $H^0(G/B, \mathcal{L}_\mu)$ by

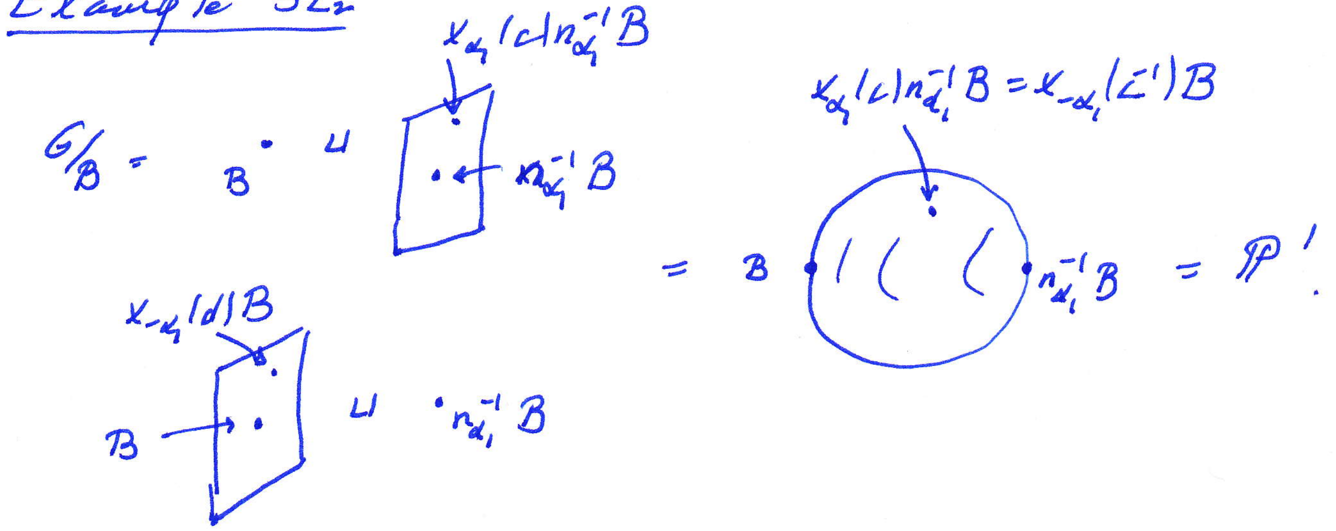
$$(gf)(h) = f(g^{-1}h)$$

Note: $(gf)(hb) = f(g^{-1}hb) = f(g^{-1}h)X^\mu(b^{-1}) = (gf)(h)X^\mu(b^{-1})$

Theorem (Borel-Weil-Bott). As G-modules

$$H^0(G/B, \mathcal{L}_\mu) \cong \begin{cases} L(+w_0\mu), & \text{if } \langle +w_0\mu, \alpha^\vee\rangle \in \mathbb{Z}_{\geq 0} \text{ for } \alpha\in R^+, \\ 0, & \text{otherwise.} \end{cases}$$

Example SL_2



since

$$x_{\alpha_1}(c)n_{\alpha_1}^{-1}B = \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix} B \quad \text{and} \quad \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c^{-1} & 1 \\ 0 & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c^{-1} & 1 \end{pmatrix}$$

The functions $f : SL_2(\mathbb{C}) \to \mathbb{C}$ such that

$$f(gb) = f(g)\,X^{\mu}(b^{-1}), \qquad \text{for } g \in G,\ b \in B$$

are determined by their values on coset reps for G/B

Let

$$g_1(c) = f\begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad g_2(d) = f\begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix}$$

Then $g_1 \in \mathbb{C}[c]$ and $g_2 \in \mathbb{C}[d]$ and

$$g_2(c^{-1}) = g_1(c)\,X^{k}\begin{pmatrix} c & 1 \\ 0 & c^{-1} \end{pmatrix} = g_1(c)\,c^{k}.$$

Example: $k = -7$, $g_1 = c^5$ and $g_2 = d^2$. Then

$$g_2(c^{-1}) = c^{-2} = c^5 c^{-7} = g_1(c)\,c^k.$$

We find

$$H^0(G/B, \mathcal{L}_k) \cong \begin{cases} \operatorname{span}\{1, c, c^2, \ldots, c^{-k}\}, & \text{if } k \in \mathbb{Z}_{\le 0} \\ 0, & \text{if } k \in \mathbb{Z}_{>0} \end{cases}$$

The proof

For SL_2: Suppose $f_\ell \in H_0(G/B, \mathcal{L}_k)$ is determined by

$$f_\ell\begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix} = c^\ell.$$

Then

$$\begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} f_\ell = d^{-k-2\ell} f_\ell \quad \text{and} \quad \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} f_\ell = \sum_{j=0}^{\ell} \binom{\ell}{j} (-a)^{\ell-j} f_j$$

so that $\mathbb{C} f_0$ is the unique B-submodule in $H^0(G/B, \mathcal{L}_k)$ with

$$\begin{pmatrix} d & a \\ 0 & d^{-1} \end{pmatrix} f_0 = d^{-k} f_0 . \quad \text{and so}$$

$$H^0(G/B, \mathcal{L}_k) \cong \begin{cases} L(-k), & \text{if } k \in \mathbb{Z}_{\le 0} \\ 0, & \text{otherwise} \end{cases}$$

In general: Let $w_0 = s_{i_1} \cdots s_{i_N}$ be a reduced word for the longest element of W_0. Suppose $f_\gamma \in H^0(G/B, \mathcal{L}_\mu)$ is determined by

$$f_\gamma(x_{\alpha_{i_1}}(c_1) n_{\alpha_{i_1}}^{-1} \cdots n_{\alpha_{i_N}}^{-1}) = c_1^{\gamma_1} c_2^{\gamma_2} \cdots c_N^{\gamma_N}.$$

Then

$$(h_{\lambda^\vee}(d) f_\gamma)(x_{\alpha_{i_1}}(c_1) n_{\alpha_{i_1}}^{-1} \cdots n_{\alpha_{i_N}}^{-1}) = f_\gamma(h_{\lambda^\vee}(d^{-1}) x_{\alpha_{i_1}}(c_1) \cdots n_{\alpha_{i_N}}^{-1})$$

$$= f_\gamma(x_{\alpha_{i_1}}(d^{-\langle \lambda^\vee, \alpha_{i_1} \rangle} c_1) \cdots n_{\alpha_{i_N}}^{-1} h_{w_0\lambda^\vee}(d^{-1})) = \cdots$$

and $\mathbb{C} f_0$ is the unique B-submodule in $H^0(G/B, \mathcal{L}_\mu)$ with

$$h_{\lambda^\vee}(d) f_0 = X^{w_0\mu}(h_{\lambda^\vee}(d)) f_0 .$$

The computation

$$\left(\begin{pmatrix} d & 0\\ 0 & d^{-1}\end{pmatrix} f_\ell\right)\begin{pmatrix} c & -1\\ 1 & 0\end{pmatrix} = f_\ell\left(\begin{pmatrix} d^{-1} & 0\\ 0 & d\end{pmatrix}\begin{pmatrix} c & -1\\ 1 & 0\end{pmatrix}\right)$$

$$= f_\ell\left(\begin{pmatrix} cd^{-2} & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix} d & 0\\ 0 & d^{-1}\end{pmatrix}\right) = (cd^{-2})^{\ell} d^{-k} = d^{-k-2\ell} f_\ell\begin{pmatrix} c & -1\\ 1 & 0\end{pmatrix}$$

so that

$$\begin{pmatrix} d & 0\\ 0 & d^{-1}\end{pmatrix} f_\ell = d^{-k-2\ell} f_\ell .$$

and

$$\begin{pmatrix} 1 & a\\ 0 & 1\end{pmatrix} f_\ell \begin{pmatrix} c & -1\\ 1 & 0\end{pmatrix} = f_\ell\left(\begin{pmatrix} 1 & -a\\ 0 & 1\end{pmatrix}\begin{pmatrix} c & -1\\ 1 & 0\end{pmatrix}\right) = f_\ell\begin{pmatrix} c-a & -1\\ 1 & 0\end{pmatrix}$$

$$= (c-a)^{\ell} = \sum_{j=0}^{\ell} \binom{\ell}{j}(-a)^{\ell-j} c^{j}$$

$$= \left(\sum_{j=0}^{\ell} \binom{\ell}{j}(-a)^{\ell-j} f_j\right)\begin{pmatrix} c & -1\\ 1 & 0\end{pmatrix}$$