Combinatorics of flag varieties Lect. 2 Hausdorff Institute Winter School, 10-14 Jan. 2011 ①

Re-view We had an

Equivalence of categories

$$\left\{\begin{matrix}\mathbb{Z}\text{-reflection}\\ \text{groups } W_0, \mathfrak{h}_{\mathbb{Z}}, R^+\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Chevalley}\\ \text{groups } G\end{matrix}\right\}$$

$G$ is generated by $x_\alpha(c)$, $x_\alpha(c)$, $h_{\lambda^\vee}(d)$

$\cup|$

$B$ is generated by $x_\alpha(c)$ $h_{\lambda^\vee}(d)$

$\cup|$

$T$ is generated by $h_{\lambda^\vee}(d)$

and our favorite example is

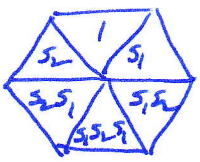
$$GL_n(\mathbb{F}) \supseteq \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} \supseteq \left\{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right\}.$$

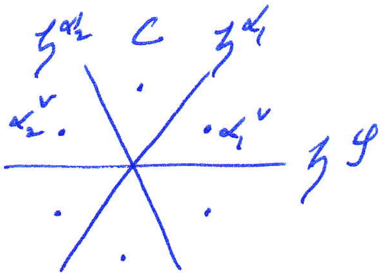
We chose $C$ so that

$W_0$ is presented by $s_1, \dots, s_n$ with

$$s_i^2 = 1 \quad \text{and} \quad \underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}}.$$

Example

$W_0 =$ 


$\mathfrak{h}_{\mathbb{Z}} =$ 


$R^+ = \{\alpha_1, \alpha_2, \varphi\}$.

$\downarrow = s_2 s_1$ is an element of $W_0$

$\overset{5}{\underset{7}{\downarrow}} = x_{\alpha_1}(5) n_{\alpha_1}^{-1} x_{\alpha_2}(7) n_{\alpha_2}^{-1} B$ is a point in $G/B$.

<u>$G/B$ is the flag variety</u>

$$G = \bigsqcup_{w \in W_0} B n_w B$$

and if $n_{\alpha_i}^{-1} = x_{\alpha_i}(-1)\, x_{-\alpha_i}(1)\, x_{\alpha_i}(-1)$ and $w = s_{i_1} \cdots s_{i_\ell}$ is a minimal length path to $w$ then

$$\{ x_{\alpha_{i_1}}(c_1) n_{\alpha_{i_1}}^{-1} \cdots x_{\alpha_{i_\ell}}(c_\ell) n_{\alpha_{i_\ell}}^{-1} B \mid c_1, \ldots, c_\ell \in \mathbb{F} \}$$

are the points of $B n_w B \simeq \mathbb{F}^\ell$ in $G/B$.

$B n_w B$ are the <u>Schubert cells</u>

$\overline{B n_w B}$ are the <u>Schubert varieties</u>

$$n_w B = x_{\alpha_{i_1}}(0) n_{\alpha_{i_1}}^{-1} \cdots x_{\alpha_{i_\ell}}(0) n_{\alpha_{i_\ell}}^{-1} B$$

is the unique <u>T-fixed point</u> on $B n_w B$ (the "center of the cell").

$$\overline{B n_w B} = \bigsqcup_{v \leq w} B n_v B \qquad \text{(Bruhat order).}$$

A <u>generalised cohomology theory</u> is a family of functors

$$E_T : \{T\text{-spaces}\} \longrightarrow \mathcal{C}_T,$$

indexed by groups $T$, which satisfy

<u>Axiom</u> (Künneth = products) something like

$$E_{G\times H}(X\times Y) \to E_G(X) \otimes_E E_H(Y)$$

<u>Axiom</u> (Change of groups) something like

$$E_G(G\times_H X) \xrightarrow{\sim} E_H(X)$$

<u>Axiom</u> (Suspension = periodicity) something like

$$E_G(S^V \wedge X) \xrightarrow{\sim} E_G(S^V) \otimes E_G(X).$$

## K-theory = cohomology

Let $T$ be a group and $X$ a $T$-space

$K_T(X) =$ Grothendieck group of $T$-equiv vector bundles on $X$ $\quad \begin{matrix} V \\ \downarrow \\ X. \end{matrix}$

i.e. generators $[V]$ with relations

$$[V] = [W] \quad \text{if} \quad V \simeq W,$$

$$[V] + [W] = [V \oplus W], \qquad [V][W] = [V \otimes W]$$

$$[U] - [V] + [W] = 0 \quad \text{if} \quad 0 \to U \to V \to W \to 0 \text{ is exact.}$$

**Thm**

(a) $K_T(pt) = R(T)$, the Grothendieck ring of $T$-modules.

(b) $K_T(X)$ is a $K_T(pt)$ module.

(c) $K_T(G/B)$ has $R(T)$ basis $\{[\mathcal{O}_{X_v}] \mid v \in W_0\}$

where $\mathcal{O}_{X_v}$ is the structure sheaf of $X_v = \overline{Bn_vB}$.

(d) The Chern character is an isomorphism

$$K_T(\cancel{G/B}\ X) \xrightarrow{\sim} H_T(\cancel{G/B}\ X)^\wedge$$

$$\mathcal{L} \longmapsto 1 + c_1(\mathcal{L}) + \frac{1}{2} c_1(\mathcal{L})^2 + \frac{1}{3!} c_1(\mathcal{L})^3 + \cdots$$

for line bundles $\mathcal{L}$ and $c_1(\mathcal{L}) =$ Chern class of $\mathcal{L}$.

## Remarks

(a) a $T$-equiv vector bundle $\begin{matrix} V \\ \downarrow \\ pt \end{matrix}$ is a $T$-module.

$T$-modules $V$ have composition series.

The simple $T$-modules are $\mathbb{C}_\mu$ corresp to

$$X^\mu : T \longrightarrow \mathbb{C}^\times$$
$$h_{\lambda^\vee}(d) \longmapsto d^{\langle \mu, \lambda^\vee \rangle} \qquad \text{for } \mu \in \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{h}_{\mathbb{Z}}, \mathbb{Z})$$

where $\langle \mu, \lambda^\vee \rangle$ means $\mu(\lambda^\vee)$. So

$$R(T) = K_T(pt) = \mathrm{span}\{X^\mu \mid \mu \in \mathfrak{h}_{\mathbb{Z}}^*\} \quad \text{with } X^\mu X^\nu = X^{\mu+\nu}.$$

(b) By Künneth $pt \times X \cong X$ gives
$$K_T(pt) \otimes K_T(X) \longrightarrow K_T(X).$$

(c) See Grothendieck 1958 Prop 7:

$\mathcal{O}_{X_w}$ comes from $X_w \longrightarrow pt$ and $X_w \hookrightarrow G/B$

and $G = \bigsqcup_{w \in W} B n_w B$ (affine paving)

causes $\{[\mathcal{O}_{X_v}] \mid v \in W_0\}$ to be a basis of $K_T(G/B)$

(d) What is cohomology.

# The nil affine Hecke algebra

Let $\mathfrak{h}_{\mathbb{Z}}^* = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{h}_{\mathbb{Z}}, \mathbb{Z})$. The affine Weyl group is

$$W = \{X^\mu t_w \mid \mu \in \mathfrak{h}_{\mathbb{Z}}^*, w \in W_0\} \quad \text{with}$$

$$X^\mu X^\nu = X^{\mu+\nu}, \quad t_u t_v = t_{uv}, \quad t_w X^\mu = X^{w\mu} t_w.$$

Let

$$R(T) = \mathrm{span}\{X^\mu \mid \mu \in \mathfrak{h}_{\mathbb{Z}}^*\},$$

$$Q(T) = \text{field of fractions of } R(T),$$

$$\mathbb{K} = \mathrm{span}\{X^\mu t_w \mid \mu \in \mathfrak{h}_{\mathbb{Z}}^*, w \in W_0\}$$

$$= R(T)\text{-}\mathrm{span}\{t_w \mid w \in W_0\}$$

$$\mathbb{K}^\wedge = Q(T)\text{-}\mathrm{span}\{t_w \mid w \in W_0\}$$

~~where~~ so that $\mathbb{K}$ is the group algebra of $W$.

If

$$\Delta_i = \frac{1}{1-X^{-\alpha_i}}(1-t_{s_i}) \quad \text{and} \quad \tilde{\Delta}_i = \frac{1}{1-X^{\alpha_i}}(X^{\alpha_i}-t_{s_i})$$

then

$$\Delta_i^2 = \Delta_i \qquad \tilde{\Delta}_i^2 = -\tilde{\Delta}_i$$

$$\underbrace{\Delta_i\Delta_j\Delta_i\cdots}_{m_{ij}} = \underbrace{\Delta_j\Delta_i\Delta_j\cdots}_{m_{ij}} \qquad \underbrace{\tilde{\Delta}_i\tilde{\Delta}_j\tilde{\Delta}_i\cdots}_{m_{ij}} = \underbrace{\tilde{\Delta}_j\tilde{\Delta}_i\tilde{\Delta}_j\cdots}_{m_{ij}}$$

Because of these relations $\mathbb{K}$ or $\mathbb{K}^\wedge$ is often called the nil affine Hecke algebra.

# "Combinatorial" realization of $K_T(G/B)$

Define $\psi^v \in \mathrm{Fun}(W_0, R(T))$ by

$$t_w = \sum_{v \in W_0} \psi^v(w)\, \tilde{\Delta}_w$$

and set

$$\Psi = R(T)\text{-span}\{\psi^v \mid v \in W_0\}.$$

Theorem (a) GKM condition:

$$\Psi = \left\{ \psi \in \mathrm{Fun}(W_0, Q(T)) \;\middle|\; \begin{array}{l} \psi(s_\alpha w) - \psi(w) \in (1 - X^\alpha) R(T) \\ \text{for } \alpha \in R^+,\ w \in W_0 \end{array} \right\}$$

where $s_\alpha$ is the reflection in $W_0$ corresp. to $\alpha \in R^+$

(b) With pointwise product on $\mathrm{Fun}(W_0, Q(T))$

$$\begin{array}{rcl} K_T(G/B) & \longrightarrow & \Psi \\ [\mathcal{O}_{X_v}] & \longmapsto & \psi^v \end{array}$$ is an $R(T)$-algebra isomorphism.

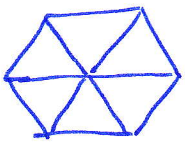
(c) With $\begin{array}{rcl} i_w : pt & \longrightarrow & G/B \\ \cdot & \longmapsto & n_w B, \end{array}$ so $i_w^* : K_T(G/B) \longrightarrow K_T(pt)$

then

$$\psi^v(w) = i_w^*([\mathcal{O}_{X_v}]).$$

# Example $SL_3(\mathbb{C}) = G$

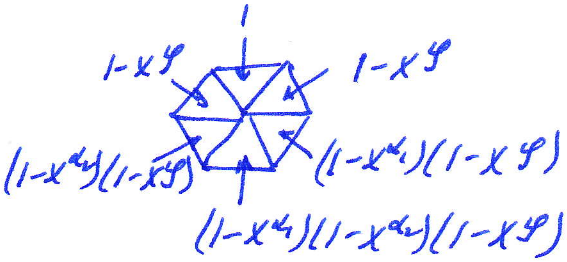


Since $W_0 =$  , $\varphi \in Fun(W_0, R(T))$

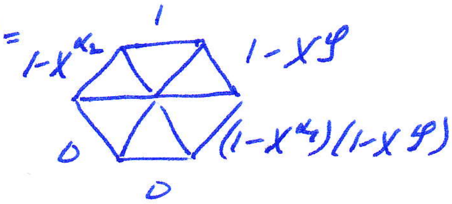
is a hexagon with chamber $w$ labeled $\varphi(w)$

Then $K_T(G/B)$ has basis

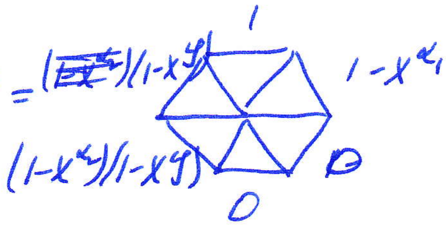
$[\mathcal{O}_{X_1}] =$



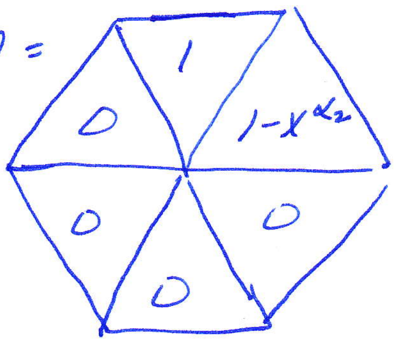

$[\mathcal{O}_{X_{s_1}}] =$



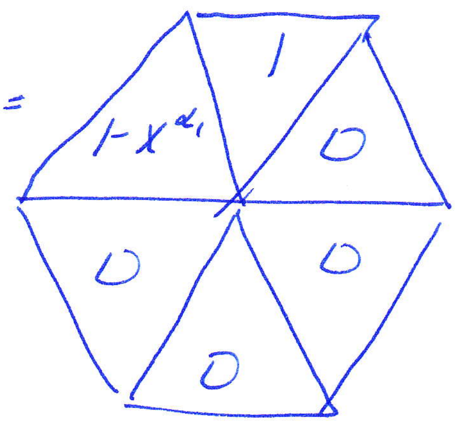

$[\mathcal{O}_{X_{s_2}}] =$




$[\mathcal{O}_{X_{s_1 s_2}}] =$




$[\mathcal{O}_{X_{s_2 s_1}}] =$




$[\mathcal{O}_{X_{s_1 s_2 s_1}}] =$