Combinatorics of flag varieties: Minicourse at Hausdorff Institute for Mathematics Winter School

10-14 January 2011

The flag variety is G/B

G is a Chevalley group
$\cup|$
B is a Borel subgroup

The data for G is

W_0, a finite $\mathbb{Z}$-reflection group acting on
$\mathfrak{h}_{\mathbb{Z}}$, a $\mathbb{Z}$ lattice.
$\mathbb{F}$, a field or commutative ring.

Reflections in W_0 are indexed by R^+

G is given by generators

$x_\alpha(c)$, $x_{-\alpha}(c)$, $h_{\lambda^\vee}(d)$, for $\alpha \in R^+$, $c \in \mathbb{F}$
$\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}$, $d \in \mathbb{F}^\times$

with relations

$\sim\sim\sim$

B is the subgroup of G generated by

$x_\alpha(c)$ and $h_{\lambda^\vee}(d)$, for $\alpha \in R^+$, $c \in \mathbb{F}$
$\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}$, $d \in \mathbb{F}^\times$

* a reflection is a matrix with exactly one eigenvalue $\neq 1$, i.e. $\begin{pmatrix} \zeta & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$ is an example.

Examples

GL_n: $W_0 = S_n$ acts on

$\mathfrak{h}_{\mathbb{Z}} = \mathbb{Z}\text{-span}\{\varepsilon_1^\vee, \ldots, \varepsilon_n^\vee\}$ by permuting $\varepsilon_1^\vee, \ldots, \varepsilon_n^\vee$

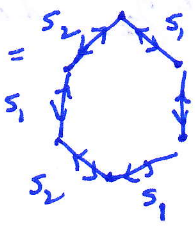
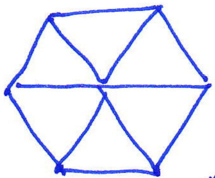
$R^+ = \{(ij) \mid 1 \le i < j \le n\} = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n\}$.

$GL_n(\mathbb{F})$ is generated by

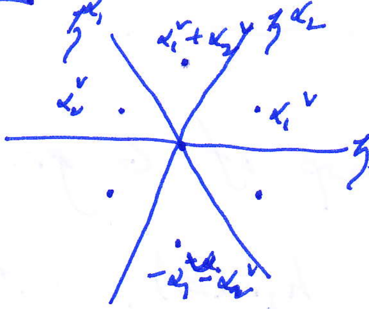
$$x_{ij}(c) = \begin{array}{c c} & \quad\quad j \\ i & \begin{pmatrix} 1 & & c \\ & \ddots & \\ & & 1 \end{pmatrix} \end{array} \qquad x_{ji}(c) = \begin{array}{c c} & i \\ & \\ j & \begin{pmatrix} 1 & & \\ & \ddots & \\ c & & 1 \end{pmatrix} \end{array}$$

$$h_{\lambda^\vee}(d) = \begin{pmatrix} d^{\lambda_1} & & \\ & \ddots & \\ & & d^{\lambda_n} \end{pmatrix} \quad \text{for } \lambda = \lambda_1\varepsilon_1^\vee + \cdots + \lambda_n\varepsilon_n^\vee.$$

SL_3

$W_0 =$  $=$  $=$ generators: s_1, s_2

relations: $s_1 s_2 s_1 = s_2 s_1 s_2$, $s_i^2 = 1$

$\mathfrak{h}_{\mathbb{Z}} = \mathbb{Z}\text{-span}\{\alpha_1^\vee, \alpha_2^\vee\} =$  , $R^+ = \{\alpha_1, \alpha_2, \varphi\}$

$SL_3(\mathbb{F})$ is generated by

$$x_{\alpha_1}(c) = \begin{pmatrix} 1 & c & \\ & 1 & \\ & & 1 \end{pmatrix} \quad x_{\alpha_2}(c) = \begin{pmatrix} 1 & & \\ & 1 & c \\ & & 1 \end{pmatrix} \quad x_{\varphi}(c) = \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$x_{-\alpha_1}(c) = \begin{pmatrix} 1 & & \\ c & 1 & \\ & & 1 \end{pmatrix} \quad x_{-\alpha_2}(c) = \begin{pmatrix} 1 & & \\ & 1 & \\ & c & 1 \end{pmatrix} \quad x_{-\varphi}(c) = \begin{pmatrix} 1 & & \\ & 1 & \\ c & & 1 \end{pmatrix}$$

points of G/B = coset representatives

Let C be a fundamental chamber for

W_0 acting on $\mathfrak{h}_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{h}_{\mathbb{Z}}$

$s_1, s_2, \ldots, s_n$ reflections in $\mathfrak{h}^{\alpha_1}, \ldots, \mathfrak{h}^{\alpha_n}$

the walls of C.

__Theorem__ W_0 is presented by $s_1, \ldots, s_n$ with

$$s_i^2 = 1 \quad \text{and} \quad \underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ factors}}$$

where $\pi/m_{ij} = \mathfrak{h}^{\alpha_i} \measuredangle \mathfrak{h}^{\alpha_j}$.

Let

$$n_{\alpha_i}^{-1} = x_{\alpha_i}(-1)\, x_{\alpha_i}(1)\, x_{\alpha_i}(-1).$$

__Theorem__ (see Steinberg Yale lecture notes Ch. 8).

$$G = \bigsqcup_{w \in W_0} BwB, \quad \text{and}$$

if $w = s_{i_1} \cdots s_{i_\ell}$ is a minimal length path to w then

$$\{ x_{\alpha_{i_1}}(c_1) n_{\alpha_{i_1}}^{-1} \cdots x_{\alpha_{i_\ell}}(c_\ell) n_{\alpha_{i_\ell}}^{-1} \mid c_1, \ldots, c_\ell \in \mathbb{F} \}$$

is a set of representatives of the right B-cosets in BwB,

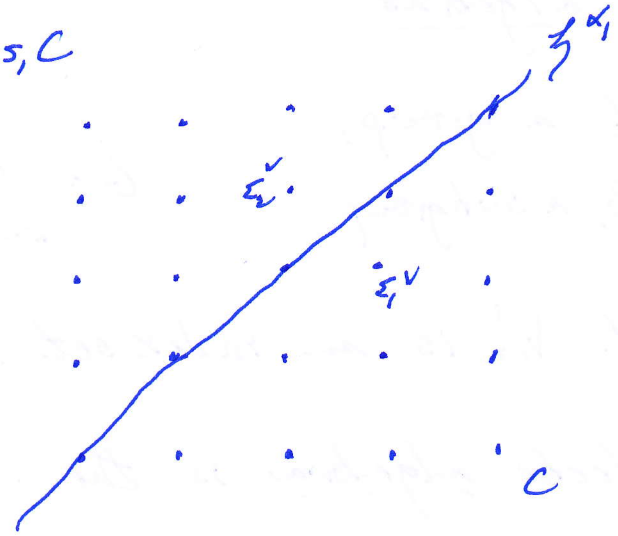
i.e. the points of G/B in BwB.

Example $GL_2(\mathbb{C})$

$$W_0 = \bullet\!\overset{s_1}{\longleftrightarrow}\!\bullet = \{1, s_1\}$$

$$\mathfrak{h}_{\mathbb{Z}} = \mathbb{Z}\text{-span}\{\varepsilon_1^\vee, \varepsilon_2^\vee\}$$

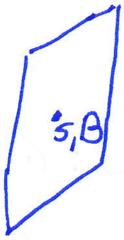
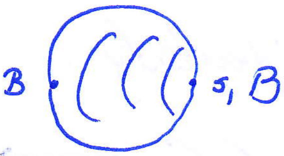
$$R^+ = \{\alpha_1\}$$



$$G = B \sqcup Bs_1B, \quad \text{and}$$

$$Bs_1B = \{x_{\alpha_1}(c)n_{\alpha_1}^{-1}B \mid c \in \mathbb{C}\} = \left\{\begin{pmatrix} c & -1 \\ 1 & 0\end{pmatrix} B \;\middle|\; c \in \mathbb{C}\right\}$$

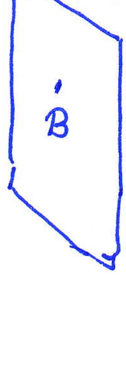
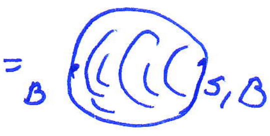
so that

$G/B = pt \sqcup \mathbb{C} = \dot{B} \sqcup$  $=$  $= \mathbb{P}^1$

especially since $\begin{pmatrix} c & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} c^{-1} & 1 \\ 0 & c\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c^{-1} & 1\end{pmatrix}$ so

$$\begin{pmatrix} c & -1 \\ 1 & 0\end{pmatrix} B = \begin{pmatrix} 1 & 0 \\ c^{-1} & 1\end{pmatrix} B, \quad \text{if } c \neq 0$$

and

$G/B = \left\{\begin{pmatrix} 1 & 0 \\ d & 1\end{pmatrix} B \;\middle|\; d \in \mathbb{C}\right\} \sqcup s_1B =$  $\sqcup\, \dot{s_1B} =$ 

Hecke algebras

Let

G be a group,

B a subgroup,

$$G = \bigsqcup_{w \in W_0} B n_w B.$$

W_0 an index set for B-double cosets in G

The <u>Hecke algebra</u> H is the subalgebra of

$$\mathbb{C}G = \mathrm{span}\{g \in G\} \qquad \begin{pmatrix}\text{product} \\ \text{from } G\end{pmatrix}$$

spanned by

$$\chi_{BwB} = \frac{1}{|B|} \sum_{x \in BwB} x, \qquad H = \mathrm{span}\{\chi_w \mid w \in W_0\}.$$

Often we identify

$$\mathbb{C}G \longrightarrow \{f : G \to \mathbb{C}\}$$

$$\sum_{g \in G} f(g) g \longleftarrow f$$

so that

$$H = \{f : G \to \mathbb{C} \mid f(b_1 g b_2) = f(g) \text{ for } b_1, b_2 \in B, g \in G\}$$

with product "convolution" or "correspondences"

G/B = the flag variety - Some computations in H

$\boxed{\text{Think } GL_2(\mathbb{F}_q)}$

The identity in H:

$$\chi_B = \frac{1}{|B|}\sum_{b\in B} b, \quad \text{so} \quad b'\chi_B = \chi_B \text{ for } b'\in B,$$
$$\chi_B^2 = \chi_B$$

A double coset:

$$\chi_{Bs_1B} = \frac{1}{|B|}\sum_{x\in Bs_1B} x = \sum_{c\in\mathbb{F}_q} x_{\alpha_1}(c)n_{\alpha_1}^{-1}\chi_B.$$

A "point in G/B":

$$\sum_{y\in x_{\alpha_1}(d)n_{\alpha_1}^{-1}B} y = x_{\alpha_1}(d)n_{\alpha_1}^{-1}\chi_B$$

A "point in G/B" multiplied with a double coset

$$x_{\alpha_1}(d)n_{\alpha_1}^{-1}\chi_B\cdot\chi_{Bs_1B} = x_{\alpha_1}(d)n_{\alpha_1}^{-1}\chi_{Bs_1B}$$

$$= x_{\alpha_1}(d)n_{\alpha_1}^{-1}\sum_{c\in\mathbb{F}_q} x_{\alpha_1}(c)n_{\alpha}^{-1}\chi_B$$

$$= x_{\alpha_1}(d)n_{\alpha_1}^{-1}x_{\alpha_1}(0)n_{\alpha_1}^{-1}\chi_B + \sum_{c\in\mathbb{F}_q^\times} x_{\alpha_1}(d+c^{-1})x_{\alpha_1}(-c^{-1})x_{-\alpha_1}(c)x_{\alpha_1}(-c^{-1})\chi_B$$

$$= \chi_B + \sum_{c\in\mathbb{F}_q^\times} x_{\alpha_1}(d+c^{-1})n_{\alpha_1}^{-1}\chi_B.$$

A double coset multiplied with a double coset:

$$\chi_{Bs_1B}^2 = \sum_{d\in\mathbb{F}_q} x_{\alpha_1}(d)\, n_{\alpha_1}^{-1} \chi_B \cdot \chi_{Bs_1B}$$

$$= \sum_{d\in\mathbb{F}_q} \Big(\chi_B + \sum_{c\in\mathbb{F}_q^\times} x_{\alpha_1}(c^{-1}+d)\, n_{\alpha_1}^{-1} \chi_B \Big)$$

$$= q\chi_B + (q-1)\chi_{Bs_1B}$$

If $T_1 = q^{-\frac{1}{2}} \chi_{Bs_1B}$ then

$$T_1^2 = 1 + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})T_1 .$$

<u>Theorem</u> (a) W_0 is presented by generators $s_1, \ldots, s_n$ with

$$s_i^2 = 1 \quad \text{and} \quad \underbrace{s_i s_j s_i \cdots}_{m_{ij}\text{ factors}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}\text{ factors}}$$

(b) H is presented by generators $T_1, \ldots, T_n$ and relations

$$T_i^2 = 1 + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})T_i \quad \text{and} \quad \underbrace{T_i T_j T_i \cdots}_{m_{ij}\text{ factors}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij}\text{ factors}}$$