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Example: The Weyl character formula

Compute: $\chi : K_T(G/B) \longrightarrow K_T(pt)$

$$\mathcal{L}_\lambda \longmapsto s_\lambda$$

where $\mathcal{L}_\lambda = G \times_B \mathbb{C} v_\lambda^+$.

$$G/B = \bigsqcup_{w \in W} BwB \quad \text{and} \quad (G/B)^T = \{wB \mid w \in W\}.$$

In $K_T(pt)$,

$$[\mathcal{L}_\lambda|_{wB}] = e^{w\lambda}.$$

Since $T_{wB}(G/B) \simeq \mathfrak{g}/w\mathfrak{b}w^{-1}$, the normal bundle to $(G/B)^T$ in G/B is

$$T_{(G/B)^T}(G/B) \simeq w\mathfrak{n}w^{-1} \text{ where}$$

$$\mathfrak{g} = \left(\bigoplus_{\alpha \in R^+} \mathfrak{g}_{-\alpha}\right) \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha,$$

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha, \text{ and}$$

$$\mathfrak{n} = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha.$$

If t is a regular element of T, then

$$[\lambda_T|_{wB}] = \prod_{\alpha\in R^+}(1-e^{w\alpha})(t),$$

so that localisation gives

$$\chi(\mathcal{L}_\lambda) = \sum_i (-1)^i \operatorname{Tr}(t, H^i(G/B, \mathcal{L}_\lambda))$$

$$= \sum_i (-1)^i \operatorname{Tr}(t, H^i((G/B)^T, (\lambda_T^{-1}\otimes\mathcal{L}_\lambda)|_{(G/B)^T}))$$

$$= \operatorname{Tr}(t; H^0((G/B)^T, (\lambda_T^{-1}\otimes\mathcal{L}_\lambda)|_{(G/B)^T}))$$

$$= \sum_{w\in W} \frac{e^{w\lambda}}{\prod_{\alpha\in R^+}(1-e^{w\alpha})}(t)$$

$$= \sum_{w\in W} \det(w)\, w\left(\frac{e^{\lambda+\rho}}{\prod_{\alpha\in R^+}(e^{\alpha/2}-e^{-\alpha/2})}\right),$$

where $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$.

Polytopes and Erhart function

Let $\mathbb{R}^N = \mathbb{R}\text{-span}\{\varepsilon_1, \ldots, \varepsilon_N\}$ and let

$$\begin{aligned} A : \mathbb{R}^N &\longrightarrow \mathfrak{h}^*_{\mathbb{R}} \\ \varepsilon_i &\longmapsto \alpha_i, \end{aligned} \qquad \text{an } r \times N \text{ matrix}$$

For $\mu \in \mathfrak{h}^*_{\mathbb{R}}$ define a convex polytope

$$P(\Phi, \mu) = \{ \gamma \in \mathbb{R}^N_{\geq 0} \mid A\gamma = \mu \}.$$

Let

$$v(\Phi, \mu) = \text{volume of } P(\Phi, \mu),$$

$$k(\Phi, \mu) = \text{Card}(P(\Phi, \mu) \cap \mathbb{Z}^N).$$

The Erhart function is $k(\Phi, \mu)$ and if Φ is a rootsystem then $k(\Phi, \mu)$ is the Kostant partition function.

(Discrete) Laplace transforms

The cone and dual cone of Δ^+ are

$$C(\Delta^+) = \mathbb{R}_{\geq 0}\text{-span}\{\alpha_1, \ldots, \alpha_N\} \text{ and}$$

$$C(\Delta^+)^\vee = \{\lambda^\vee \in \mathfrak{h}_\mathbb{R} \mid \langle \lambda^\vee, \alpha_i \rangle \in \mathbb{R}_{\geq 0} \text{ for } i = 1, \ldots, N\},$$

where $\mathfrak{h}_\mathbb{R} = \mathrm{Hom}(\mathfrak{h}^*_\mathbb{R}, \mathbb{R})$ is the dual of $\mathfrak{h}^*_\mathbb{R}$.

Lemma (see BV Lemma 10) For $\lambda^\vee \in (C(\Delta^+)^\vee)^\circ$,

$$\sum_{\mu \in C(\Delta^+) \cap \mathfrak{h}_\mathbb{Z}} k(\Phi, \mu) e^{-\langle \mu, \lambda^\vee \rangle} = \prod_{\alpha \in \Phi} \frac{1}{1 - e^{-\langle \alpha, \lambda^\vee \rangle}}, \quad \text{and}$$

$$\int_{C(\Delta^+)} v(\Phi, \mu) e^{-\langle \mu, \lambda^\vee \rangle} d\mu = \frac{1}{\prod_{\alpha \in \Phi} \langle \alpha, \lambda^\vee \rangle},$$

where $\mathfrak{h}_\mathbb{Z} = \mathbb{Z}\text{-span}\{\alpha_1, \ldots, \alpha_N\}$.

Total Residue Polytopes

Following Baldoni-Vergne, Transf. Gps $\underline{\underline{13}}$ no. 3-4, 2008

Let $\mathfrak{h}_{\mathbb{R}}$ be a real vector space.

$\mathfrak{h}_{\mathbb{R}}^* = Hom(\mathfrak{h}_{\mathbb{R}}, \mathbb{R})$ the dual space

Let $\alpha_1, \ldots, \alpha_N \in \mathfrak{h}_{\mathbb{R}}^*$ s.t. $\mathbb{R}\text{span}\{\alpha_1, \ldots, \alpha_N\} = \mathfrak{h}_{\mathbb{R}}^*$

Φ = multiset $(\alpha_1, \ldots, \alpha_N)$

Δ^* = set $\{\alpha_1, \ldots, \alpha_N\}$

$$S(\mathfrak{h}_{\mathbb{R}}^*) = \mathbb{C}[\varepsilon_1, \ldots, \varepsilon_r]$$

$$R_\Delta = \mathbb{C}[x_1, \ldots, x_r, \alpha_1^{-1}, \ldots, \alpha_N^{-1}]$$

$$S_\Delta = \left\{ \frac{1}{\prod_{\alpha_i \in S} \alpha_i} \;\middle|\; S \subseteq \{\alpha_1, \ldots, \alpha_N\} \text{ and } S \text{ is a basis of } \mathfrak{h}_{\mathbb{R}}^* \right\}$$

Then

$$R_\Delta = D_{>0} R_\Delta \oplus S_\Delta$$

where $D_{>0} = \mathbb{R}\text{span}\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_r}\right\}$

The <u>total</u> <u>residue</u> is the projection

$$Tres_\Delta : R_\Delta \longrightarrow S_\Delta$$

~~Define, for $\mu \in \mathfrak{h}_{\mathbb{R}}^*$~~

$$\cancel{J_\Phi(\mu) = Tres_\Delta\left(\frac{e^\mu}{\alpha_1 \cdots \alpha_N}\right) \text{ and } K_\Phi(\mu) = Tres\left(\frac{e^\mu}{(1-e^{-\alpha_1})\cdots(1-e^{-\alpha_N})}\right)}$$

The Jeffrey-Kirwan and Dahmen-Micchelli formulas

A chamber of $C(\Delta^+)$ is a connected component of

$$C(\Delta^+)_{reg} = \Big(\bigcup_{\substack{v^+ \subseteq \Delta^+ \\ Card(v^+) < r}} \mathbb{R}_{\geq 0}\text{-span}(v^+) \Big)^c$$

For each chamber c of $C(\Delta^+)$ define a linear map

$$\langle\langle c, \cdot \rangle\rangle : S_\Delta \longrightarrow \mathbb{R}$$
$$f \longmapsto \langle\langle c, f \rangle\rangle$$

by setting

$$\Big\langle\Big\langle c, \frac{1}{\prod_{\alpha_i \in S} \alpha_i} \Big\rangle\Big\rangle = \begin{cases} \dfrac{1}{vol\big(\sum_{\alpha_i \in S} [0,1]\alpha_i\big)}, & \text{if } c \subseteq \mathbb{R}_{\geq 0}\text{-span}(S), \\ 0, & \text{if } c \cap \mathbb{R}_{\geq 0}\text{-span}(S) = \phi \end{cases}$$

<u>Theorem</u> (a) If c is a chamber of $C(\Delta^+)$, $\mu \in c$

$$v(\Phi, \mu) = \Big\langle\Big\langle c, \mathrm{Tres}\Big(\frac{e^\mu}{\alpha_1 \cdots \alpha_N} \Big) \Big\rangle\Big\rangle$$

(b) If Φ is unimodular and $Z(\Phi) = \sum_{\alpha_i \in \Phi} [0,1]\alpha_i$.
Let c be a chamber of $C(\Delta^+)$ and $\mu \in c - Z(\Phi)$. Then

$$k(\Phi, \mu) = \Big\langle\Big\langle c, \mathrm{Tres}\Big(\frac{e^\mu}{(1-e^{\alpha_1}) \cdots (1-e^{\alpha_N})} \Big) \Big\rangle\Big\rangle$$

The multiset Φ is <u>unimodular</u> if

$$\mathrm{vol}\left(\sum_{\alpha_i \in S} [0,1]\alpha_i\right) = 1, \text{ for all bases } S \subseteq \Delta.$$

The <u>zonotopes</u> generated by Φ is

$$Z(\Phi) = \sum_{\alpha_i \in \Phi} [0,1]\alpha_i .$$

<u>Copy here the quote from Brion - Vergne</u>

<u>From Atiyah-Bott</u>

Let T be a torus acting on a symplectic manifold M with moment map

$$\mu : M \longrightarrow \mathfrak{g}^* = \mathbb{R}^d$$

In the case when the vertices of

$P(\Phi, \cancel{\gamma})$ are integral

there is ~~a~~ projective algebraic variety $M(\cancel{\gamma})$ with a T-action and a line bundle L such that

(a) $P(\Phi, \cancel{\gamma})$ is the image of $M(\cancel{\gamma})$ under the moment map.

(b) The vertices of $P(\Phi, \cancel{\gamma})$ are the images of the T-fixed pts $M(\gamma)^T$

(c) $\sum_\gamma P(\Phi, \gamma) e^\gamma = [H^0(M(\gamma), L)] \in K_T(pt)$

and $H^i(M(\gamma), L) = 0$ for $i > 0$.

Moment maps

A symplectic manifold is a manifold M with a 2-form $\omega \in \Omega^2(M)$ such that $d\omega = 0$. the symplectic volume $\frac{\omega^n}{n!}$ is nowhere 0 on M. (nondegenerate).

Hamiltonian vector fields are the image of

$$\mathcal{O}_M \longrightarrow \text{vectorfields on } M$$
$$f \longmapsto \omega(df, \cdot)$$

A Hamiltonian action of T on M is an action $T \times M \to M$ such that

if $x \in \mathfrak{h}$ then the corresponding vector field on M is Hamiltonian,

where $\mathfrak{h} = Lie(T)$.

The moment map is

$$\mu : M \longrightarrow \mathfrak{h}^* \quad \text{where} \quad \mu_m : \mathfrak{h} \to \mathbb{R}$$
$$m \longmapsto \mu_m(x) = f_x(m) \qquad\qquad x \longmapsto f_x(m)$$

where f_x is the function on M corresponding to the vector field x.

(a) The support of $\mu_*\left(\frac{\omega^n}{n!}\right)$ is a convex polytope

(b) (Duistermaat-Heckman) The measure $\mu_*\left(\frac{\omega^n}{n!}\right)$ is piecewise polynomial.

(c) Atiyah and Bott explain: how to think of

~~$\int_M \frac{\omega^n}{n!} e^{-\mu} = \pi_*(e^{\omega-\mu})$ where~~

$$\int_M \frac{\omega^n}{n!} e^{-\mu} \quad \text{as} \quad \pi_*(e^{\omega^\#}) \text{ where}$$

$$\pi_* : H_T^*(M) \to H_T^*(pt).$$

Let G be a compact Lie group acting on a symplectic manifold M. by a Hamiltonian action:

$$\mu : M \to \mathfrak{g}^* \quad \text{the moment map.}$$

(d) Im μ is a union of coadjoint orbits.

~~Theorem (Borel-Bott-Weil) The simple G-modules are~~

~~$V^\lambda = H^0(G/B, \mathcal{L}_\lambda)$~~

Representations of G

Let T be a maximal torus of G and
$\mathfrak{h} = Lie(T)$.

Let V be a ~~G~~-module with a ~~G~~ invariant inner product. Then the moment map on $\mathbb{P}(V)$ is given by

$$\mu : \mathbb{P}(V) \longrightarrow \mathfrak{h}^*$$
$$[v] \longmapsto \mu_v, \quad \text{where } \mu_v(X) = \frac{1}{i}\frac{\langle X v, v\rangle}{\langle v, v\rangle}$$

Let V be a G-module. Then there exists a line bundle $\mathcal{L}_\lambda = G \times_B \mathbb{C} v_\lambda^+$ on G/B such that

$$V = H^0(G/B, \mathcal{L}_\lambda)$$

If λ is regular ($\mathcal{L}_\lambda$ is ample) then

$$G/B \hookrightarrow \mathbb{P}(V) \xrightarrow{\mu} \mathfrak{h}^*$$
$$gB \longmapsto [g v_\lambda^+] \longmapsto \mu_{g v_\lambda^+} \longmapsto \mu_{g v_\lambda^+}$$

is moment map for a T-action on G/B

<u>Coadjoint orbits</u> $\mathcal{O}_\lambda = G\cdot\lambda$ for $\lambda \in \mathfrak{g}^*$

[CG, 1.1.5] $\mathcal{O}_\lambda$ has a symplectic structure. (Kirillov-Kostant Souriau).
These provide a moment map:

$$\mathcal{O}_\lambda \xrightarrow{\mu} \mathfrak{g}^* \longrightarrow i\mathfrak{h}^*_{\mathbb{R}}.$$

and $\operatorname{im}\mu = \operatorname{Conv}(W\cdot\lambda)$.

We know <u>lots</u> about the image of the G-invariant measure on $\mathcal{O}_\lambda$.

<u>The cotangent bundle</u> $T^*(G/\mathfrak{p})$

[CG Ex 1.1.3] $T^*(M)$ has a symplectic structure.

and a moment map

$$T^*(G/\mathfrak{p}) \xrightarrow{\mu} \mathfrak{g}^* \qquad [CG\ 1.4.10]$$