Categories

A category $\mathcal{C}$ is a collection of
objects and morphisms
with composition maps

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Z)$$
$$(f\ ,\ g) \longmapsto g \circ f$$

for which associativity holds and identities exist (if X is an object of $\mathcal{C}$ then there exists an identity morphism $id_X : X \to X$ such that ...)

<u>Examples</u>

Objects	Morphisms
Sets	functions
Groups	Group homomorphisms
Rings	Ring homomorphisms
Vector spaces	linear transformations
A-modules	A-module homomorphisms
Abelian groups	$\mathbb{Z}$-module homomorphisms
Topological spaces	continuous functions
manifolds	smooth maps
complex manifolds	holomorphic maps

Objects	Morphisms
Algebras	homomorphism of algebras
Lie algebras	Lie algebra homomorphisms
varieties	morphisms of varieties
affine varieties	~~morphisms~~ regular functions
schemes	morphisms of schemes
affine schemes	morphisms of affine schemes
sheaves	morphisms of sheaves
vector bundles	morphisms of vector bundles
principal bundles	morphism of principal bundles
categories	functors
functors	natural transformations
complexes	chain maps
homotopy category	chain maps
derived category	morphisms

The category of categories

The category of categories has

Objects	and	Morphisms
categories		functors

A functor $F: \mathcal{A} \to \mathcal{B}$ maps objects to objects and morphisms to morphisms

$$F: \mathcal{A} \to \mathcal{B} \qquad \text{and} \qquad F: \mathrm{Hom}_{\mathcal{A}}(M,N) \to \mathrm{Hom}_{\mathcal{B}}(F(M),F(N))$$
$$M \mapsto F(M) \qquad\qquad f \longmapsto F(f)$$

such that

$$F(\mathrm{id}_M) = \mathrm{id}_{F(M)} \quad \text{and} \quad F(f_1 \circ f_2) = F(f_1) \circ F(f_2).$$

Example Let A and B be algebras with $A \subseteq B$ (e.g. $A = \mathbb{C}S_3$ and $B = \mathbb{C}S_4$). Let

$\mathcal{A}$ be the category of A-modules and

$\mathcal{B}$ the category of B-modules.

Then induction is a functor

$$\mathrm{Ind}_A^B: \mathcal{A} \to \mathcal{B} \qquad \text{and} \qquad \mathrm{Ind}_A^B(f): B \otimes_A M \to B \otimes_A N$$
$$M \mapsto B \otimes_A M \qquad\qquad b \otimes m \longmapsto b \otimes f(m)$$

if $f: M \to N$ is an A-module homomorphism.

The category of functors

Let $\mathcal{A}$ and $\mathcal{B}$ be categories.

The category of functors from $\mathcal{A}$ to $\mathcal{B}$ has

Objects	and	Morphisms
functors $F: \mathcal{A} \to \mathcal{B}$		natural transformations.

A natural transformation $\varphi: F \to G$ is a collection of morphisms

$$\{\varphi_M : F(M) \to G(M) \mid M \in \mathcal{A}\}$$

such that if $f: M \to N$ then

$$\begin{tikzcd} F(M) \arrow[r, "F(f)"] \arrow[d, "\varphi_M"'] & F(N) \arrow[d, "\varphi_N"] \\ G(M) \arrow[r, "G(f)"] & G(N) \end{tikzcd}$$

Example An additive category is a category $\mathcal{A}$ such that

$Hom_{\mathcal{A}}(M,N)$ is an abelian group

and there is a 0 object in $\mathcal{A}$ and direct sums $M \oplus N$ exist in $\mathcal{A}$.

A 2-category is a category $\mathcal{A}$ such that

$Hom_{\mathcal{A}}(M,N)$ is a category

and ...

The category of categories is an example of a 2-category.

Example of a category of functors

Let X be a topological space with topology $\mathcal{T}$

$\mathcal{T}$ is a category with

Objects	and	Morphisms
open sets U		inclusions $U_1 \hookrightarrow U_2$

Let $\mathcal{C}$ be the category of commutative rings with 1.

A sheaf (of rings) on X is a contravariant functor

$$\mathcal{F} : \mathcal{T} \to \mathcal{C}$$

A morphism of sheaves is a morphism of functors $\mathcal{F}$ to $\mathcal{G}$.

The category of sheaves is the category of functors $\mathcal{T} \to \mathcal{C}$.

The category of complexes

Let $\mathcal{A}$ be a category (e.g. $\mathcal{A}$ is the category of A-modules)

The category of complexes over $\mathcal{A}$ $\mathrm{Kom}(\mathcal{A})$ has

Objects		Morphisms
complexes over $\mathcal{A}$	and	chain maps.

A complex M over $\mathcal{A}$ is a sequence of morphisms

$$\cdots \longrightarrow M^i \xrightarrow{d^i} M^{i+1} \xrightarrow{d^{i+1}} \cdots \quad \text{with } d_{i+1} \circ d_i = 0$$

i.e. a $\mathbb{Z}$-graded A-module M with a map $d : M \to M$ with $\deg d = 1$ and $d^2 = 0$.

A chain map $f : M \to N$ is a collection of morphisms

$$\{ f^i : M^i \to N^i \mid i \in \mathbb{Z} \}$$

such that

$$\begin{tikzcd} M^i \arrow[r, "d^i"] \arrow[d, "f^i"'] & M^{i+1} \arrow[d, "f^{i+1}"] \\ N^i \arrow[r, "d^{i+1}"'] & N^{i+1} \end{tikzcd} \quad \text{commutes.}$$

Totalization

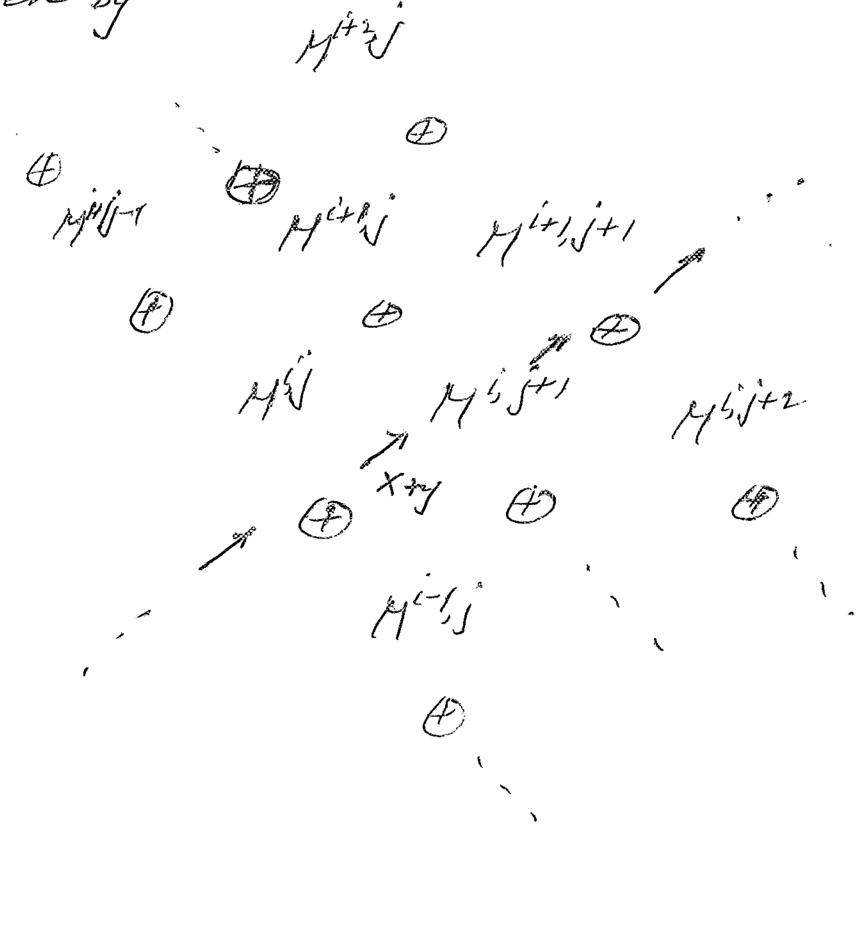
Elements of the category $Kom(Kom(\mathcal{A}))$ look like

$$
\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r] & M^{i,j+1} \arrow[r, "x^{i,j+1}"] \arrow[u] & M^{i+1,j+1} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & M^{i,j} \arrow[r, "x^{i,j}"'] \arrow[u, "y^{i,j}"] & M^{i+1,j} \arrow[r] \arrow[u, "y^{i+1,j}"'] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}
$$

Let $\mathcal{A}$ be an abelian category (i.e. $Hom_{\mathcal{A}}(X,Y)$ are abelian groups, there exists a 0 object and direct sums $X \oplus Y$).

The <u>totalization</u> <u>functor</u> $tot: Kom(Kom(\mathcal{A})) \longrightarrow Kom(\mathcal{A})$ is given by

$tot(M) =$



Cohomology

An *abelian category* is a category $\mathcal{A}$ such that $\mathrm{Hom}_{\mathcal{A}}(M,N)$ are abelian groups, there exists a 0 object and direct sums $M \oplus N$ and kernels and cokernels exist.

Let $\mathcal{A}$ be an abelian category and let $\mathrm{Kom}(\mathcal{A})$ be the category of complexes over $\mathcal{A}$

$$M = (\cdots \to M^i \xrightarrow{d^i} M^{i+1} \to \cdots) \quad \text{with } d_{i+1} d_i = 0$$

The *cohomology* of a complex (M,d) is

$$H(M) = \frac{Z(M)}{B(M)}, \text{ where } Z(M) = \ker d \text{ and } B(M) = \mathrm{im}\, d.$$

and

$$\begin{aligned} H(f) : H(M) &\to H(N) \\ [c] &\mapsto [f(c)], \end{aligned}$$ if $f : M \to N$ is a morphism in $\mathrm{Kom}(\mathcal{A})$.

The derived category $D(\mathcal{A})$

Let $\mathcal{A}$ be an abelian category and let $Kom(\mathcal{A})$ be the category of complexes over $\mathcal{A}$ and let $H(M)$ be the cohomology of a complex M.

A <u>quasiisomorphism</u> is a morphism $f: M \to N$ in $Kom(\mathcal{A})$ such that $H(f): H(M) \to H(N)$ is an isomorphism.

The <u>derived category</u> of $\mathcal{A}$ is the category $D(\mathcal{A})$ with a functor $Q: Kom(\mathcal{A}) \to D(\mathcal{A})$ such that

(a) if f is a quasiisomorphism then $Q(f)$ is an isomorphism,

(b) If $F: Kom(\mathcal{A}) \to \mathcal{C}$ is a functor that takes quasiisomorphisms to isomorphisms then there exists a unique functor $\hat{F}: D(\mathcal{A}) \to \mathcal{C}$ such that

$$\begin{tikzcd} Kom(\mathcal{A}) \arrow[r, "Q"] \arrow[dr, "F"'] & D(\mathcal{A}) \arrow[d, "\hat{F}"] \\ & \mathcal{C} \end{tikzcd}$$

The homotopy category

Let $\mathcal{A}$ be an abelian category and $\mathrm{Kom}(\mathcal{A})$ the category of complexes over $\mathcal{A}$.

Let M and N be objects of $\mathrm{Kom}(\mathcal{A})$.

A <u>homotopy</u> between morphisms $f: M \to N$ and $g: M \to N$ is a collection of morphisms

$$\{h^i : M^i \to N^{i+1} \mid i \in \mathbb{Z}\} \quad \text{such that}$$

$$f^i - g^i = h^{i+1} \circ d^i + d^{i-1} \circ h^i$$

If f and g are homotopic then $H(f) = H(g)$.

The <u>homotopy category</u> $\mathrm{Ho}(\mathcal{A})$ has

Objects		Morphisms
complexes over $\mathcal{A}$	and	chain maps modulo homotopy equivalence.