Symmetric functions

$\left.\begin{array}{l}\mathfrak{h}_{\mathbb{Z}}\\ \mathfrak{h}_{\mathbb{Z}}^*\end{array}\right\}$ dual $\mathbb{Z}$-vector spaces, $\langle\ \rangle : \mathfrak{h}_{\mathbb{Z}}^* \times \mathfrak{h}_{\mathbb{Z}} \to \frac{1}{c}\mathbb{Z}$

W_0 a finite subgroup of $GL(\mathfrak{h}_{\mathbb{Z}})$ generated by reflections. Then W_0 acts on the group algebra

$$K_{T^\vee}(pt) = \mathrm{span}\{ y^{\lambda^\vee} \mid \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}} \}$$

with $y^{\lambda^\vee} y^{\sigma^\vee} = y^{\lambda^\vee + \sigma^\vee}$ by $w y^{\lambda^\vee} = y^{w\lambda^\vee}$.

The <u>algebra of symmetric functions</u> is

$$K_{T^\vee}(pt)^{W_0} = \{ f \in K_{T^\vee}(pt) \mid wf = f, \text{ for all } w \in W_0 \}$$

Then

$$K_{T^\vee}(pt)^{\det} = \{ f \in K_{T^\vee}(pt) \mid wf = \det(w) f, \text{ for all } w \in W_0 \}$$

is a free $K_{T^\vee}(pt)^{W_0}$-module of rank 1.

$K_{T^\vee}(pt)^{W_0}$ and $K_{T^\vee}(pt)^{\det}$ have bases

$$m_{\lambda^\vee} = \pi_0 y^{\lambda^\vee} \quad \text{and} \quad a_{\lambda^\vee+\rho^\vee} = \varepsilon_0 y^{\lambda^\vee}, \qquad \lambda^\vee \in P^+ = \mathfrak{h}_{\mathbb{Z}}/W_0$$

where

$$\pi_0 = \sum_{w \in W_0} w \quad \text{and} \quad \varepsilon_0 = \sum_{w \in W_0} \det(w^{-1}) w$$

Weyl character formula

$$
\begin{array}{rcl}
K_{T^\vee}(pt)^{W_0} & \longrightarrow & K_{T^\vee}(pt)^{\det} \\
f & \longmapsto & a_\rho f
\end{array}
$$

naive basis $\quad m_{\lambda^\vee}$

$s_{\lambda^\vee} \longleftarrow\!\shortmid\ a_{\lambda^\vee+\rho^\vee}$ naive basis

$m_{\lambda^\vee}$ are the monomial symmetric functions

$s_{\lambda^\vee}$ are the Weyl characters or Schur functions

Note:

$$K_{T^\vee}(pt)^{W_0} = K_{G^\vee}(pt) = K(G^\vee\text{-modules})$$

and $s_{\lambda^\vee}$ are the classes of the simple modules.

Double affine Hecke algebra $\tilde{H}$

$$\tilde{H} = \mathrm{span}\{q^{k/e} X^{\mu} T_w Y^{\lambda^\vee} \mid k \in \mathbb{Z}, \mu \in \mathfrak{h}_{\mathbb{Z}}^*, w \in W_0, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}\}$$

$$\cup\!\mathsf{I}$$

$$H^\vee = \mathrm{span}\{X^{\mu} T_w \mid \mu \in \mathfrak{h}_{\mathbb{Z}}^*, w \in W_0\}$$

$$\cup\!\mathsf{I}$$

$$H_0 = \mathrm{span}\{T_w \mid w \in W_0\}$$

with

$$q^{1/e} \in Z(\tilde{H}), \quad X^{\mu} X^{\nu} = X^{\mu+\nu}, \quad Y^{\lambda^\vee} Y^{\sigma^\vee} = Y^{\lambda^\vee + \sigma^\vee}$$

H_0 is generated by $T_1, \ldots, T_n$ with

$$T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) T_i + 1 \quad \text{and} \quad \underbrace{T_i T_j T_i \cdots}_{m_{ij}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij}}$$

where $\frac{\pi}{m_{ij}} = \mathfrak{h}^{\alpha_i} \measuredangle \mathfrak{h}^{\alpha_j}$.

$H^\vee$ has a unique 1-dim'l module

$$\mathrm{span}\{\mathbb{1}\} \quad \text{with} \quad T_i \mathbb{1} = t^{\frac{1}{2}} \mathbb{1} \text{ for } i = 1, \ldots, n.$$

The <u>polynomial</u> <u>representation</u> of $\tilde{H}$ is

$$\mathrm{Ind}_{H^\vee}^{\tilde{H}}(\mathbb{1}) = \tilde{H}\mathbb{1} = \mathrm{span}\{q^{k/e} Y^{\lambda^\vee} \mathbb{1} \mid k \in \mathbb{Z}, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}\}$$

$$= K_{T^\vee}(pt)\mathbb{1}.$$

Macdonald polynomials

Let $\mathbb{1}_0, \varepsilon_0 \in H_0$ be such that

$$\mathbb{1}_0 T_i = t^{\frac{1}{2}} \mathbb{1}_0 \quad \text{and} \quad \varepsilon_0 T_i = (-t^{-\frac{1}{2}}) \varepsilon_0$$

for $i = 1, \ldots, n$. At $t=1$, $\mathbb{1}_0 = \pi_0$ and $\varepsilon_0 = \varepsilon_0$.

Then

$$K_{T^\vee}(pt)\mathbb{1} = \tilde{H}\mathbb{1} \quad \supseteq \quad \mathbb{1}_0 \tilde{H} \mathbb{1} = K_{T^\vee}(pt)^{W_0}\mathbb{1}$$

The nonsymmetric Macdonald polynomial $E_{\lambda^\vee} = E_{\lambda^\vee}(q,t)$ in $K_{T^\vee}(pt)$ is given by

(a) $E_{\lambda^\vee}\mathbb{1}$ is an eigenvector of all X^μ (acting on $\tilde{H}\mathbb{1}$)

(b) $E_{\lambda^\vee} = y^{\lambda^\vee} +$ lower stuff

The symmetric Macdonald polynomial $P_{\lambda^\vee} = P_{\lambda^\vee}(q,t)$ in $K_{T^\vee}(pt)^{W_0}$ is given by

$$P_{\lambda^\vee}\mathbb{1} = \mathbb{1}_0 E_{\lambda^\vee}\mathbb{1}$$

Define $A_{\lambda^\vee+\rho^\vee} = A_{\lambda^\vee+\rho^\vee}(q,t)$ in $K_{T^\vee}(pt)$ by

$$A_{\lambda^\vee+\rho^\vee}\mathbb{1} = \varepsilon_0 E_{\lambda^\vee+\rho^\vee}\mathbb{1}.$$

Big picture

$$K_{T^\vee}(pt)^{W_0}\mathbf{1} = \mathbf{1}_0 \tilde{H}\mathbf{1} \longrightarrow \varepsilon_0 \tilde{H}\mathbf{1}$$

$$f\mathbf{1} \longmapsto A_{\rho^\vee}(q,t) f\mathbf{1}$$

$$\mathbf{1}_0 E_{\lambda^\vee}\mathbf{1} = P_{\lambda^\vee}(q,t)\mathbf{1}$$

$$P_{\lambda^\vee}(q,qt)\mathbf{1} \longleftarrow A_{\lambda^\vee+\rho^\vee}(q,t)\mathbf{1} = \varepsilon_0 E_{\lambda^\vee+\rho^\vee}\mathbf{1}$$

At $q=0$ This picture becomes

$$K_{T^\vee}(pt)^{W_0}\mathbf{1} = \mathbf{1}_0 H \mathbf{1}_0 \longrightarrow \varepsilon_0 H \mathbf{1}_0$$

$$f\mathbf{1}_0 \longmapsto A_{\rho^\vee}(0,t) f\mathbf{1}_0$$

$$\mathbf{1}_0 Y^{\lambda^\vee}\mathbf{1}_0 = P_{\lambda^\vee}(0,t)\mathbf{1}_0$$

$$s_{\lambda^\vee}\mathbf{1}_0 = P_{\lambda^\vee}(0,0)\mathbf{1}_0 \longleftarrow A_{\lambda^\vee+\rho^\vee}(0,t)\mathbf{1}_0 = \varepsilon_0 Y^{\lambda^\vee+\rho^\vee}\mathbf{1}_0$$

where $H = \mathrm{span}\{T_w Y^{\lambda^\vee} \mid w\in W_0,\ \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}\}$

At $q=0,\ t=1$ This becomes

$$K_{T^\vee}(pt)^{W_0} \longrightarrow K_{T^\vee}(pt)^{\det}$$

$$f \longmapsto a_{\rho^\vee} f$$

$$\pi_0 Y^{\lambda^\vee} = m_{\lambda^\vee}$$

$$s_{\lambda^\vee} \longleftarrow a_{\lambda^\vee+\rho^\vee} = \varepsilon_0 Y^{\lambda^\vee+\rho^\vee}$$

Remarks

(1) At $q \neq 0$, $Z(\tilde{H})$ is trivial $\left(Z(\tilde{H}) = \mathbb{C}[q^{\pm 1/e}] \right)$

At $q=0$, $Z(\tilde{H})$ is big, and contains

$$K_{T^\vee}(pt)^{W_0} = Z(H) \qquad \left(\begin{array}{l}\text{theorem of}\\ \text{Bernstein}\end{array}\right).$$

(2) The <u>Satake</u> <u>isomorphism</u> is

$$K_{T^\vee}(pt)^{W_0} \mathbb{1}_0 \longleftarrow \mathbb{1}_0 H \mathbb{1}_0$$

$$P_{\lambda^\vee}(0,t) \mathbb{1}_0 \longleftarrow\!\shortmid\; \mathbb{1}_0 y^{\lambda^\vee} \mathbb{1}_0 \qquad \text{and}$$

$P_{\lambda^\vee}(0,t)$ is the <u>Macdonald</u> <u>spherical</u> <u>function</u>, or <u>Hall-Littlewood</u> <u>polynomial</u>.

(3) $H \cong$ Grothendieck ring (product is convolution) of I equiv. perverse sheaves on G/I

$\mathbb{1}_0 H \mathbb{1}_0 =$ Groth. ring of K-equiv. perverse sheaves on G/K.

$G/I =$ affine flag variety $\quad G/K =$ loop Grassmanian

$\{ s_{\lambda^\vee} \mathbb{1}_0 \}$ is the Kazhdan-Lusztig basis of $\mathbb{1}_0 H \mathbb{1}_0$

i.e. $s_{\lambda^\vee} \mathbb{1}$ is the image of $IC(K h_{\lambda^\vee}(\varepsilon^{\prime})K, \bar{\mathbb{Q}}_\ell)$.

(4) $\left.\begin{array}{l} K_{T^\vee}(pt)^{det} \\ \varepsilon_0 \tilde{H}\mathbb{1} \\ \varepsilon_0 H \mathbb{1}_0 \end{array}\right\}$ are "Fock spaces"

and $\mathbb{1}_0 \tilde{H} \mathbb{1} \xrightarrow{\sim} \varepsilon_0 \tilde{H}\mathbb{1}$ are "boson-Fermion correspondences". The big picture at $q=0$ is a 1981 paper of Lusztig which kicked off "Geometric Langlands".

(5) In $\tilde{H}$

$$T_i X^\mu = X^{s_i\mu} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{X^\mu - X^{s_i\mu}}{1 - X^{\alpha_i}} \qquad \left(\begin{array}{l}\text{Bernstein} \\ \text{-Lusztig relation}\end{array}\right)$$

is equivalent to

$$\tau_i X^\mu = X^{s_i\mu} \tau_i, \text{ where}$$

$$\tau_i = T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X^{\alpha_i}} = T_i^{-1} + \frac{(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) X^{\alpha_i}}{1 - X^{\alpha_i}} \qquad \text{(intertwiner)}$$

If $y^{\lambda^\vee} = s_{i_1} \cdots s_{i_\ell}$ is a minimal length walk to $y^{\lambda^\vee}$ in W, then in $\tilde{H}$,

$$y^{\lambda^\vee} = T_{i_1}^{\epsilon_1} \cdots T_{i_\ell}^{\epsilon_\ell} \quad \text{where}$$

$$\epsilon_k = \begin{cases} +1, & \text{if the } k^{\text{th}} \text{ step is } \xrightarrow{-\,|\,+} \\ -1, & \text{if the } k^{\text{th}} \text{ step is } \xleftarrow{-\,|\,+} \end{cases}$$

and

$$E_{\lambda^\vee} = \tau_{i_1} \cdots \tau_{i_\ell} \mathbb{1}$$

Using folded alcove walks this can be expanded to give a formula

$$E_{\lambda^\vee} = \sum_{\substack{\text{folded alcove} \\ \text{paths } p}} \begin{pmatrix} \text{explicit} \\ \text{coeffs} \end{pmatrix} y^{\mathrm{end}(p)}$$

which has similar coefficients to the Haglund-Haiman-Loehr formula for $E_{\lambda^\vee}$ in type Gl_n, and generalizes

$$s_{\lambda^\vee} = \sum_{\substack{\text{column strict} \\ \text{tableaux } p}} y^{\mathrm{wt}(p)} = \sum_{\substack{\text{Littelmann} \\ \text{paths } p}} y^{\mathrm{end}(p)}$$

and the ~~labeled~~ positively folded walks labeling points in MV intersections $IwI \cap U^- vI$