Heisenberg group

$$\left.\begin{matrix}\mathfrak{h}_{\mathbb{Z}} \\ \mathfrak{h}_{\mathbb{Z}}^*\end{matrix}\right\}$$ dual $\mathbb{Z}$-vectorspaces, $\langle\, ,\rangle : \mathfrak{h}_{\mathbb{Z}}^* \times \mathfrak{h}_{\mathbb{Z}} \to \frac{1}{e}\mathbb{Z}$

The group algebras are

$$K_T(pt) = \mathrm{span}\{X^\mu \mid \mu \in \mathfrak{h}_{\mathbb{Z}}\}, \quad K_{T^\vee}(pt) = \mathrm{span}\{Y^{\lambda^\vee} \mid \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}\}$$

with

$$X^\mu X^\nu = X^{\mu+\nu} \quad \text{and} \quad Y^{\lambda^\vee} Y^{\sigma^\vee} = Y^{\lambda^\vee+\sigma^\vee}.$$

If $\varepsilon_1, \dots, \varepsilon_n$ is a basis of $\mathfrak{h}_{\mathbb{Z}}^*$

$\varepsilon_1^\vee, \dots, \varepsilon_n^\vee$ a basis of $\mathfrak{h}_{\mathbb{Z}}$ then

$$K_T(pt) = \mathbb{C}[X_1^{\pm1}, \dots, X_n^{\pm1}] \quad \text{and} \quad K_{T^\vee}(pt) = \mathbb{C}[Y_1^{\pm1}, \dots, Y_n^{\pm1}]$$

with

$$X^\mu = X_1^{\mu_1} \cdots X_n^{\mu_n}, \quad \text{if } \mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n$$

$$Y^{\lambda^\vee} = Y_1^{\lambda_1^\vee} \cdots Y_n^{\lambda_n^\vee}, \quad \text{if } \lambda^\vee = \lambda_1\varepsilon_1^\vee + \cdots + \lambda_n\varepsilon_n^\vee$$

Let $q^{1/e}$ be a parameter.

The *Heisenberg group* is

$$\{q^{k/e} X^\mu Y^{\lambda^\vee} \mid k \in \mathbb{Z}, \mu \in \mathfrak{h}_{\mathbb{Z}}^*, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}\}$$

with (*) and

$$X^\mu Y^{\lambda^\vee} = q^{\langle\mu, \lambda^\vee\rangle} Y^{\lambda^\vee} X^\mu.$$

Weyl algebras

$$\left.\begin{matrix}\mathfrak{h}_{\mathbb{C}} \\ \mathfrak{h}_{\mathbb{C}}^*\end{matrix}\right\} \text{ dual vector spaces.} \quad \langle\,,\,\rangle : \mathfrak{h}_{\mathbb{C}}^* \times \mathfrak{h}_{\mathbb{C}} \to \mathbb{C}.$$

The symmetric algebras are

$$H_T(pt) = S(\mathfrak{h}_{\mathbb{C}}^*) = \mathbb{C}[x_1, \ldots, x_n]$$

with

$$H_{T^\vee}(pt) = S(\mathfrak{h}_{\mathbb{C}}) = \mathbb{C}[D_1, \ldots, D_n]$$

$x_\mu = \mu_1 x_1 + \cdots + \mu_n x_n$, if $\mu = \mu_1 \varepsilon_1 + \cdots + \mu_n \varepsilon_n$

$D_{\lambda^\vee} = \lambda_1 D_1 + \cdots + \lambda_n D_n$, if $\lambda^\vee = \lambda_1 \varepsilon_1^\vee + \cdots + \lambda_n \varepsilon_n^\vee$.

Let

κ be a parameter.

The Weyl algebra $\mathcal{D}$ is generated by

$\mathbb{C}[D_1, \ldots, D_n]$ and $\mathbb{C}[x_1, \ldots, x_n]$

with

$$D_{\lambda^\vee} x_\mu = x_\mu D_{\lambda^\vee} + \kappa \langle \mu, \lambda^\vee \rangle.$$

$\mathcal{D}$ acts on polynomials: If $\langle \varepsilon_i, \varepsilon_j^\vee \rangle = \delta_{ij}$, $\kappa = 1$

$D_j = \frac{\partial}{\partial x_j}$ then $\left[\frac{\partial}{\partial x_j}, x_i\right] = \frac{\partial}{\partial x_j} x_i - x_i \frac{\partial}{\partial x_j} = \delta_{ij}$.

In physics, sometimes $\kappa = i\hbar$.

Rational Cherednik algebras

W_0 is a finite subgroup of $GL(\mathfrak{h}_\mathbb{C})$

generated by

$$R^+ = \{ s \in W_0 \mid s \text{ is a reflection} \}$$

The group algebra is

$$\mathbb{C}W_0 = \text{span}\{ t_w \mid w \in W_0 \} \quad \text{with} \quad t_{w_1} t_{w_2} = t_{w_1 w_2}.$$

(W_0 acts on $\mathfrak{h}_\mathbb{C}^*$ by $\langle w\mu, \lambda^\vee \rangle = \langle \mu, w^{-1}\lambda^\vee \rangle$.)

For $s \in R^+$ fix $\alpha_s \in \mathfrak{h}_\mathbb{C}^*$ and $\alpha_s^\vee \in \mathfrak{h}_\mathbb{C}$ so that

$$s\mu = \mu - \langle \mu, \alpha_s^\vee \rangle \alpha_s \quad \text{and} \quad s^{-1}\lambda^\vee = \lambda^\vee - \langle \lambda^\vee, \alpha_s \rangle \alpha_s^\vee,$$

$$\alpha_{wsw^{-1}} = w\alpha_s \quad \text{and} \quad \alpha^\vee_{wsw^{-1}} = \alpha_s^\vee \quad \text{for } w \in W_0.$$

Fix parameters

$$c_s, \ s \in R^+, \qquad \text{with } c_s = c_{wsw^{-1}} \text{ for } w \in W_0.$$

The <u>rational</u> <u>Cherednik</u> <u>algebra</u> $\mathbb{H}$ is gen. by

$$\mathbb{C}[D_1, \ldots, D_n], \quad \mathbb{C}[x_1, \ldots, x_n] \quad \text{and } \mathbb{C}W_0$$

with

$$t_w x_\mu = x_{w\mu} t_w, \qquad t_w D_{\lambda^\vee} = D_{w\lambda^\vee} t_w$$

$$D_{\lambda^\vee} x_\mu = x_\mu D_{\lambda^\vee} + \kappa \langle \mu, \lambda^\vee \rangle - \sum_{s \in R^+} c_s \langle \lambda^\vee, \alpha_s \rangle \langle \mu, \alpha_s^\vee \rangle t_s.$$

Dunkl operators

For $p \in \mathbb{C}[x_1, \dots, x_n]$,

$$D_{\lambda^\vee} p = p D_{\lambda^\vee} + \kappa (d_{\lambda^\vee} p) - \sum_{s \in R^+} c_s \langle \lambda^\vee, \alpha_s \rangle (\Delta_s p) t_s$$

where $d_{\lambda^\vee} : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}[x_1, \dots, x_n]$ is given by

$$d_{\lambda^\vee}(x_\mu) = \langle \mu, \lambda^\vee \rangle, \qquad d_{\lambda^\vee}(p_1 p_2) = p_1 d_{\lambda^\vee}(p_2) + d_{\lambda^\vee}(p_1) p_2,$$

and $\Delta_s : H_T(pt) \to H_T(pt)$ is

$$\Delta_s p = \frac{p - sp}{x_{\alpha_s}}, \quad \text{the } \underline{\text{BGG-operator}}.$$

The subalgebra

$\mathbb{H}$ generated by $\mathbb{C}W_0$ and $\mathbb{C}[D_1, \dots, D_n]$

has a 1-dim'l module $\mathrm{span}\{\mathbb{1}\}$ given by

$$t_w \mathbb{1} = \mathbb{1} \quad \text{and} \quad D_{\lambda^\vee} \mathbb{1} = 0.$$

The <u>polynomial</u> <u>representation</u> of $\tilde{\mathbb{H}}$ is

$$\mathrm{Ind}_{\mathbb{H}}^{\tilde{\mathbb{H}}}(\mathbb{1}) = \tilde{\mathbb{H}} \cdot \mathbb{1} = \mathbb{C}[x_1, \dots, x_n] \cdot \mathbb{1}.$$

$D_{\lambda^\vee}$ acts on $\tilde{\mathbb{H}} \cdot \mathbb{1}$ by the <u>Dunkl operator</u>

$$D_{\lambda^\vee} = \kappa d_{\lambda^\vee} - \sum_{s \in R^+} c_s \langle \lambda^\vee, \alpha_s \rangle \frac{1}{x_{\alpha_s}} (1 - s)$$

The trigonometric Cherednik algebra $\tilde{H}_{gr}$

W_0 is a finite subgroup of $GL(\mathfrak{h}_{\mathbb{R}})$ generated by R^+. Then W_0 has a presentation by generators $s_1, \ldots, s_n$ and relations

$$s_i^2 = 1 \quad \text{and} \quad \underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}}$$

where $\pi/m_{ij} = \mathfrak{h}^{\alpha_i} \measuredangle \mathfrak{h}^{\alpha_j}$. Let $\mathbb{C}[y_1, \ldots, y_n] = S(\mathfrak{h}_{\mathbb{C}})$ with

$$y_{\lambda^\vee} = \lambda_1 y_1 + \cdots + \lambda_n y_n \quad \text{if} \quad \lambda^\vee = \lambda_1 \varepsilon_1^\vee + \cdots + \lambda_n \varepsilon_n^\vee.$$

The trigonometric Hecke algebra $\tilde{H}_{gr}$ is gen. by

$$\mathbb{C}[y_1, \ldots, y_n], \quad \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \quad \text{and} \quad \mathbb{C}W_0$$

with

$$t_w X^\mu = X^{w\mu} t_w,$$

$$t_{s_i} y_{\lambda^\vee} = y_{s_i \lambda^\vee} t_{s_i} + c_{s_i} \langle \lambda^\vee, \alpha_i \rangle, \quad \text{for } i = 1, \ldots, n$$

$$y_{\lambda^\vee} X^\mu = X^\mu y_{\lambda^\vee} + \kappa \langle \mu, \lambda^\vee \rangle X^\mu - \sum_{s \in R^+} c_s \langle \lambda^\vee, \alpha_s \rangle \frac{X^\mu - X^{s\mu}}{1 - X^{\alpha_s}} t_s$$

Dunkl-Cherednik operators

Note: $K_T(pt) = \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ and $\pi_s : K_T(pt) \to K_T(pt)$,

$$\pi_s X^\mu = \frac{X^\mu - X^{s\mu}}{1 - X^\alpha}$$ is a Demazure operator.

The subalgebra H_{gr}

H_{gr} generated by $\mathbb{C}W_0$ and $\mathbb{C}[y_1, \ldots, y_n]$

has a 1-dim'l module $\mathrm{span}\{\mathbb{1}\}$ given by

$$t_w \mathbb{1} = \mathbb{1} \quad \text{and} \quad y_{\lambda^\vee} \mathbb{1} = \langle \rho_c, \lambda^\vee \rangle \mathbb{1}$$

where $\langle \rho_c, \alpha_i^\vee \rangle = c_{s_i}$ for $i = 1, \ldots, n$.

The polynomial representation of $\tilde{H}_{gr}$ is

$$\mathrm{Ind}_{H_{gr}}^{\tilde{H}_{gr}}(\mathbb{1}) = \tilde{H}_{gr} \mathbb{1} = \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \mathbb{1}.$$

The Dunkl-Cherednik operator is

$$y_{\lambda^\vee} = \langle \rho_c, \lambda^\vee \rangle + \kappa \partial_{\lambda^\vee} - \sum_{s \in R^+} c_s \langle \lambda^\vee, \alpha_s \rangle \frac{1}{1 - X^\alpha} (1 - s)$$

where $\partial_{\lambda^\vee} : \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \to \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ is given by

$$\partial_{\lambda^\vee}(X^\mu) = \langle \mu, \lambda^\vee \rangle X^\mu.$$

Quantisation

$$H_T(pt) = \mathbb{C}[x_1, \dots, x_n] \xrightarrow{ch} K_T(pt) = \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$$

$$e^{x_\mu} \longleftarrow X^\mu$$

Let

$X^\mu = e^{h x_\mu}$. Then $\partial_{\lambda^\vee} = \frac{1}{h} d_{\lambda^\vee}$

and

$$\begin{array}{ccc}
\tilde{\mathbb{H}} & \xrightarrow{\quad ch \quad} & \tilde{H}_{qr} \\
e^{h x_\mu} & \longleftarrow & X^\mu \\
\frac{1}{h} d_\lambda & \longleftarrow & \partial_{\lambda^\vee} \\
t_w & \longleftarrow & t_w
\end{array}$$

is an "isomorphism" with

$$y_{\lambda^\vee} = \langle \rho_c, \lambda^\vee \rangle + \frac{1}{h} D_{\lambda^\vee} + \sum_{s \in R^+} c_s \langle \lambda^\vee, \alpha_s \rangle \frac{1}{h x_{\alpha_s}} \left(1 - \frac{h x_{\alpha_s}}{1 - e^{h x_{\alpha_s}}} \right) (1 - t_s)$$