Chevalley groups

Linear algebra theorems 1 and 2

(1) GL_n is generated by elementary matrices

$$x_{\varepsilon_i - \varepsilon_j}(f) = \begin{pmatrix} 1 & & f \\ & \ddots & \\ & & 1 \end{pmatrix}, \quad h_{\varepsilon_i^\vee}(g) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & g & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

(2) $GL_n = \bigsqcup_{w \in W_0} B_0^+ w B_0^+$, where

$$B_0^+ = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \quad \text{and} \quad W_0 = S_n = \left\{ \begin{matrix} \text{permutation} \\ \text{matrices} \end{matrix} \right\}$$

A Chevalley group G is given by generators

$x_\alpha(f)$, $\alpha \in R$, $f \in \mathbb{F}$,

$h_{\lambda^\vee}(g)$, $\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}$, $g \in \mathbb{F}^\times$

with

$$T_0 = \langle h_{\lambda^\vee}(g) \mid \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}, g \in \mathbb{F}^\times \rangle \simeq \mathbb{F}^\times \times \cdots \times \mathbb{F}^\times$$

$$\langle x_\alpha(f), x_{-\alpha}(f) \mid f \in \mathbb{F} \rangle \simeq SL_2$$

$W_0 = N/T$ where N is the normalizer of T_0

is a finite group generated by reflections.

$\mathbb{F} = \mathbb{C}((t))$ is the field of fractions of

$$\mathbb{C}[[t]] = \{a_0 + a_1 t + a_2 t^2 + \cdots \mid a_i \in \mathbb{C}\} \quad \text{and}$$

$$\mathbb{F} = \bigcup_{k \in \mathbb{Z}} t^k \mathbb{C}[[t]] = \{a_k t^k + a_{k+1} t^{k+1} + \cdots \mid a_i \in \mathbb{C}, k \in \mathbb{Z}\}$$

$$\mathbb{C}[[t]]^\times = \{a_0 + a_1 t + a_2 t^2 + \cdots \mid a_i \in \mathbb{C}, a_0 \in \mathbb{C}^\times\}$$

Define

$$x_{\alpha+k\delta}(c) = x_\alpha(ct^k), \quad n_{\alpha+k\delta}(g) = x_{\alpha+k\delta}(g) x_{-\alpha-k\delta}(-g^{-1}) x_{\alpha+k\delta}(g)$$

so that N is generated by T_0 and the $n_{\alpha+k\delta}(g)$

$$W = W_0 \ltimes \mathfrak{h}_{\mathbb{Z}} = \{w y^{\lambda^\vee} \mid w \in W_0, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}\} \quad \text{with}$$

$$y^{\lambda^\vee} y^{\sigma^\vee} = y^{\lambda^\vee + \sigma^\vee} \quad \text{and} \quad w y^{\lambda^\vee} = y^{w\lambda^\vee} w$$

T is the kernel of
$$\begin{aligned} N &\longrightarrow W \\ h_{\lambda^\vee}(t^{-1}) &\longmapsto y^{\lambda^\vee} \\ n_\alpha(1) &\longmapsto s_\alpha \end{aligned}$$

Define

$$U_{\alpha, \geq k} = \{x_\alpha(f) \mid f \in t^k \mathbb{C}[[t]]\}$$

$$U_\alpha = U_{\alpha, \geq -\infty} = \{x_\alpha(f) \mid f \in \mathbb{C}((t))\}$$

and

$$U_0^- = \langle x_{-\alpha}(f) \mid \alpha \in R^+, f \in \mathbb{F} \rangle = \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \right\}.$$

Flag varieties

Define subgroups

$$B_0^+ = \langle T, U_\alpha \mid \alpha \in R^+ \rangle$$

$$I = \left\langle T, \begin{array}{l} U_{\alpha, \geq k}, \ \alpha \in R^+, \ k \in \mathbb{Z}_{\geq 0} \\ U_{-\alpha, \geq k}, \ \alpha \in R^+, \ k \in \mathbb{Z}_{> 0} \end{array} \right\rangle$$

$$K = \left\langle T, \begin{array}{l} U_{\alpha, \geq k} \quad \alpha \in R^+, \ k \in \mathbb{Z}_{\geq 0} \\ U_{-\alpha, \geq k} \quad \alpha \in R^+, \ k \in \mathbb{Z}_{\geq 0} \end{array} \right\rangle$$

G/B_0^+ is the <u>flag variety</u>

G/I is the <u>affine flag variety</u>

G/K is the <u>loop Grassmanian</u>

$$G = \bigsqcup_{w \in W_0} B_0^+ w B_0^+ \qquad \text{(Bruhat decomposition)}$$

$$G = \bigsqcup_{w \in W} I w I \quad \text{and} \quad G = \bigsqcup_{v \in W} U_0^- v I$$

$$G = \bigsqcup_{\lambda^\vee \in \overline{C_0^\vee} \cap \mathfrak{h}_{\mathbb{Z}}} K h_{\lambda^\vee}(t^{-1}) K \quad \text{and} \quad G = \bigsqcup_{\mu^\vee \in \mathfrak{h}_{\mathbb{Z}}} K h_{\mu^\vee}(t^{-1}) K$$

(Cartan decomposition) (Iwasawa decomposition).

Hecke algebras = Double coset algebras

Let $\mathfrak{h}^{\alpha_0}, \ldots, \mathfrak{h}^{\alpha_n}$ be the walls of $C^\vee$ and

$$x_i(c) = x_{\alpha_i}(c) \quad \text{and} \quad n_i = n_{\alpha_i}(1).$$

<u>Theorem</u> Let $\vec{w} = s_{i_1} \cdots s_{i_\ell}$ be a reduced word for w.

$$IwI = \{ x_{i_1}(c_1) n_{i_1}^{-1} \cdots x_{i_\ell}(c_\ell) n_{i_\ell}^{-1} I \mid c_1, \ldots, c_\ell \in \mathbb{C} \}$$

so that

$$IwI \overset{1-1}{\longleftrightarrow} \{ \text{labeled paths of type } \vec{w} \}$$

To <u>prove</u> this by induction compute

$$IwI \cdot Is_jI = \{ x_{i_1}(c_1) n_{i_1}^{-1} \cdots x_{i_\ell}(c_\ell) n_{i_\ell}^{-1} I \cdot x_j(c) n_j^{-1} I \}.$$

<u>Case 1</u> If $ws_j > w$ then

$$x_{i_1}(c_1) n_{i_1}^{-1} \cdots x_{i_\ell}(c_\ell) n_{i_\ell}^{-1} x_j(c) n_j^{-1} I \in Iws_jI.$$

<u>Case 2</u> If $ws_j < w$ and $c = 0$ then

$$x_{i_1}(c_1) n_{i_1}^{-1} \cdots x_{i_{\ell-1}}(c_{\ell-1}) n_{i_{\ell-1}}^{-1} x_j(c_\ell) n_j^{-1} x_j(0) n_j^{-1} I$$

$$= x_{i_1}(c_1) n_{i_1}^{-1} \cdots x_{i_{\ell-1}}(c_{\ell-1}) n_{i_{\ell-1}}^{-1} I \in Iws_jI$$

<u>Case 3</u> $ws_j < w$ and $c \neq 0$. Then

$$x_{i_1}(c_1)n_{i_1}^{-1} \cdots x_{i_{\ell-1}}(c_{\ell-1})n_{i_{\ell-1}}^{-1}\, x_j(c_\ell)n_j^{-1}\, x_j(c)n_j^{-1} I$$

$$= x_{i_1}(c_1)n_{i_1}^{-1} \cdots x_{i_{\ell-1}}(c_{\ell-1})n_{i_{\ell-1}}^{-1}\, x_j(c_\ell + c^{-1})n_j^{-1} I \in IwI.$$

If $\mathbb{C}$ is replaced by $\mathbb{F}_q$, then

$$IwI \cdot Is_jI = \begin{cases} Iws_jI, & \text{if } ws_j > w, \\ (q-1)IwI + q\,Is_jI, & \text{if } ws_j < w. \end{cases}$$

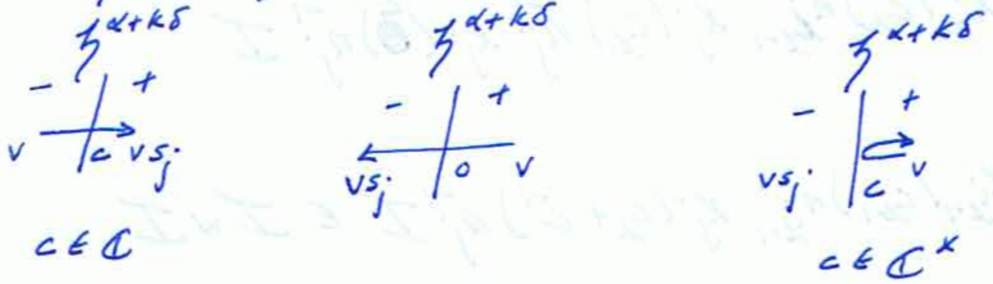
Letting $T_j = Is_jI$ the double coset algebra has

$$T_j^2 = (q-1)T_j + q, \quad \text{and}$$

$$\underbrace{T_iT_jT_i\cdots}_{m_{ij}} = \underbrace{T_jT_iT_j\cdots}_{m_{ij}} \quad \text{where } \frac{\pi}{m_{ij}} = \sphericalangle\, \mathfrak{z}^{\alpha_i} \,\mathfrak{z}^{\alpha_j}.$$

Labeled positively folded alcove walks

A step of type j is



<u>Theorem</u> Let $\vec{w} = s_{i_1} \cdots s_{i_\ell}$ be a reduced word.

$$U_0^- v I \cap I w I \overset{1-1}{\longleftrightarrow} \left\{ \begin{matrix} \text{labeled positively folded} \\ \text{paths of type } i_1, \dots, i_\ell \\ \text{ending in } v \end{matrix} \right\}$$

To <u>prove</u> this by induction compute

$$(U_0^- v I \cap I w I) \cdot I s_j I.$$

Using induction assume

$$x_{i_1}(c_1) n_{i_1}^{-1} \cdots x_{i_\ell}(c_\ell) n_{i_\ell}^{-1} = \underbrace{x_{\gamma_1}(c_1')}_{\in U_0^-} \cdots \underbrace{x_{\gamma_\ell}(c_\ell')}_{\in U_0^-} n_v \underbrace{b}_{\in I}$$

Let $\tilde{c} \in \mathbb{C}$ and $b' \in I$ such that

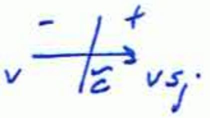
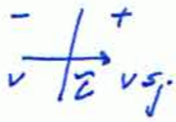
$$x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') n_v b x_j(c) n_j^{-1} = x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') n_v x_j(\tilde{c}) n_j^{-1} b'.$$

<u>Case 1</u> $x_{v\alpha_j}(1) \in U_0^-$. Then

$$x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') n_v x_j(\check{c}) n_j^{-1} b'$$

$$= x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') x_{v\alpha_j}(\pm\check{c}) n_{vs_j} \cdot b'$$

gives

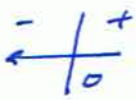
 becomes 

<u>Case 2</u> $x_{v\alpha_j}(1) \notin U_0^-$ but $\check{c} = 0$. Then

$$x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') n_v x_j(0) n_j^{-1} b'$$

$$= x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') x_{-v\alpha_j}(0) n_{vs_j} \cdot b'$$

gives

 becomes 

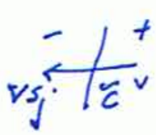
<u>Case 3</u> $x_{v\alpha_j}(1) \notin U_0^-$ and $\check{c} \neq 0$. Then

$$x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') n_v x_j(\check{c}) n_j^{-1} b'$$

$$= x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') x_{-v\alpha_j}(\pm\check{c}^{-1}) n_v b'',$$

where $b'' = x_{\alpha_j}(-\check{c}) h_{\alpha_j^\vee}(\check{c}) b' \in I$.

Hence

 becomes