Reflection groups W_0

A <u>reflection</u> is $s \in GL_n(\mathbb{C})$ such that all except one eigenvalue is equal to 1

$$s \text{ is conjugate to } \begin{pmatrix} \xi & & 0 \\ & 1 & \\ 0 & & \ddots & 1 \end{pmatrix}$$

$\left.\begin{matrix} \mathfrak{h}_{\mathbb{Z}} \\ \mathfrak{h}_{\mathbb{Z}}^* \end{matrix}\right\}$ dual $\mathbb{Z}$-vector spaces, $\langle\,,\,\rangle : \mathfrak{h}_{\mathbb{Z}}^* \times \mathfrak{h}_{\mathbb{Z}} \longrightarrow \frac{1}{e}\mathbb{Z}$

where $e \in \mathbb{Z}_{>0}$, and

W_0 is a finite subgroup of $GL(\mathfrak{h}_{\mathbb{Z}})$

generated by

$$R^+ = \{ s \in W_0 \mid s \text{ is a reflection} \}$$

W_0 acts on $\mathfrak{h}_{\mathbb{Z}}^*$ by

$$\langle w\mu, \lambda^\vee \rangle = \langle \mu, w^{-1}\lambda^\vee \rangle, \quad \text{for } \mu \in \mathfrak{h}_{\mathbb{Z}}^*, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}, w \in W_0$$

For each $s \in R^+$, fix $\alpha \in \mathfrak{h}_{\mathbb{Z}}^*$, $\alpha^\vee \in \mathfrak{h}_{\mathbb{Z}}$ such that

$$s\mu = \mu - \langle \mu, \alpha^\vee \rangle \alpha \quad \text{and} \quad s^{-1}\lambda^\vee = \lambda^\vee - \langle \lambda^\vee, \alpha \rangle \alpha^\vee$$

for $\mu \in \mathfrak{h}_{\mathbb{Z}}^*$, $\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}$.

The double affine Weyl group $\tilde{W}$

$$\tilde{W} = \{q^{k/e} X^{\mu} w Y^{\lambda^\vee} \mid k \in \mathbb{Z}, \mu \in \mathfrak{h}^*_{\mathbb{Z}}, w \in W_0, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}\}$$

with

$$q^{\frac{1}{e}} \in Z(\tilde{W}), \quad X^{\mu} X^{\nu} = X^{\mu+\nu}, \quad Y^{\lambda^\vee} Y^{\sigma^\vee} = Y^{\lambda^\vee + \sigma^\vee}$$

$$X^{\mu} Y^{\lambda^\vee} = q^{\langle \mu, \lambda^\vee \rangle} Y^{\lambda^\vee} X^{\mu}$$

$$w X^{\mu} = X^{w\mu} w \quad \text{and} \quad w Y^{\lambda^\vee} = Y^{w\lambda^\vee} w.$$

The affine Weyl group

$$W^\vee = \{X^{\mu} w \mid \mu \in \mathfrak{h}^*_{\mathbb{Z}}, w \in W_0\}$$

acts on $\mathfrak{h}^*_{\mathbb{R}}$,

$$\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{h}^*_{\mathbb{Z}} \quad \text{by} \quad X^{\mu} \cdot \nu = \mu + \nu.$$

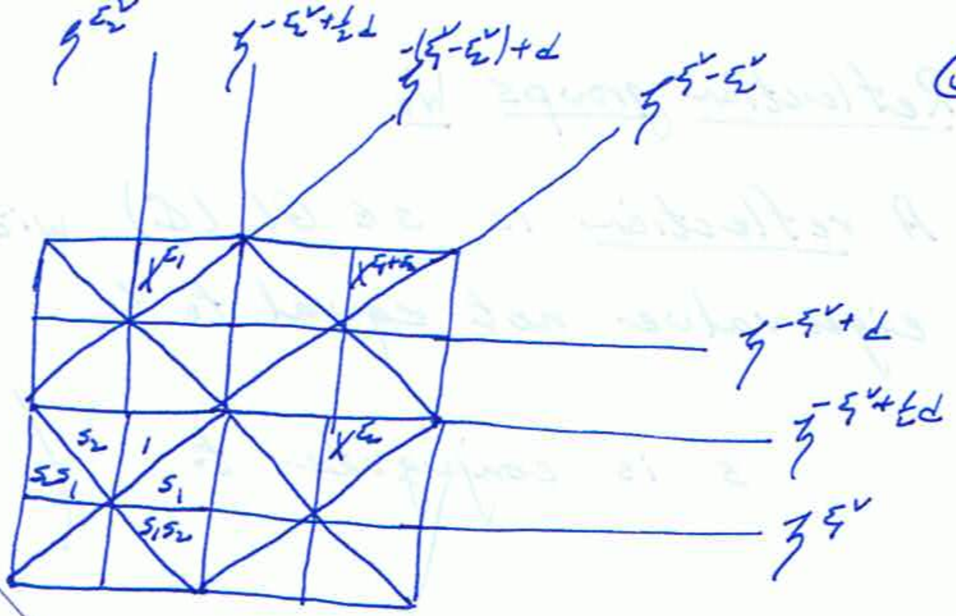
The alcoves are the connected components of

$$\mathfrak{h}^*_{\mathbb{R}} \setminus \bigcup_{\substack{\alpha^\vee \in R^+ \\ k \in \mathbb{Z}}} \mathfrak{h}^{\alpha^\vee + \frac{kd}{e}} \quad \text{where}$$

$$\mathfrak{h}^{\alpha^\vee + \frac{kd}{e}} = \{\nu \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \nu, \alpha^\vee \rangle = -k/e\}$$

Fix an alcove C with $0 \in \bar{C}$.

The affine Weyl group $W^\vee$



$W_0 \overset{1\text{-}1}{\longleftrightarrow} \{\text{alcoves in the } 0\text{-octagon}\}$ $\qquad \mathfrak{h}^*_{\mathbb{Z}} = \{\text{octagons}\}$

$\tilde{W} \overset{1\text{-}1}{\longleftrightarrow} \{\text{alcoves in } \Omega^\vee \times \mathfrak{h}^*_{\mathbb{R}}\}$

Fix an alcove C with $0 \in \bar{C}$ (closure of C).

$\mathfrak{h}^{\alpha_0^\vee}, \dots, \mathfrak{h}^{\alpha_n^\vee}$ are the walls of C

$s_0^\vee, \dots, s_n^\vee$ the corresponding reflections

$\Omega^\vee = \{ g^\vee \in W^\vee \mid g^\vee C = C \}$.

$W^\vee$ is presented by generators

$s_0^\vee, \dots, s_n^\vee$ and $\Omega^\vee$ with

$\underbrace{s_i^\vee s_j^\vee \cdots}_{m_{ij}^\vee} = \underbrace{s_j^\vee s_i^\vee \cdots}_{m_{ij}^\vee}$, where $\pi/m_{ij}^\vee = \angle(\mathfrak{h}^{\alpha_i^\vee}, \mathfrak{h}^{\alpha_j^\vee})$

$(s_i^\vee)^2 = 1$ and $g^\vee s_i^\vee (g^\vee)^{-1} = s_{g(i)}^\vee$ if $g^\vee \mathfrak{h}^{\alpha_i^\vee} = \mathfrak{h}^{\alpha_{g(i)}^\vee}$.

The affine braid group $\mathcal{B}^\vee$

$\mathcal{B}^\vee$ has generators $T_0^\vee, \ldots, T_n^\vee$ and $\Omega^\vee$ with

$$\underbrace{T_i^\vee T_j^\vee \cdots}_{m_{ij}^\vee} = \underbrace{T_j^\vee T_i^\vee \cdots}_{m_{ij}^\vee} \quad \text{and} \quad g^\vee T_i^\vee (g^\vee)^{-1} = T_{g(i)}^\vee$$

The periodic orientation has

(a) 1 on the positive side of $\mathfrak{h}^{\alpha^\vee}$, $\alpha^\vee \in R_+^\vee$,

(b) $\mathfrak{h}^{\alpha^\vee + kd}$ and $\mathfrak{h}^{\alpha^\vee}$ have parallel orientations.

If $X^\mu = g^\vee s_{i_1}^\vee \cdots s_{i_\ell}^\vee$ is a minimal length walk to X^μ (i.e. a reduced word) define

$$X^\mu = g^\vee (T_{i_1}^\vee)^{\epsilon_1^\vee} \cdots (T_{i_\ell}^\vee)^{\epsilon_\ell^\vee} \quad \text{where}$$

$$\epsilon_k^\vee = \begin{cases} -1, & \text{if the } k^{\text{th}} \text{ step is } \overset{-}{\longrightarrow}\!\!|\overset{+}{} \\ +1, & \text{if the } k^{\text{th}} \text{ step is } \overset{-}{}|\!\!\overset{+}{\longleftarrow} \end{cases}$$

For example,

$$X^{\varepsilon_1} = (T_0^\vee)^{-1} (T_1^\vee)^{-1} (T_2^\vee)^{-1} (T_1^\vee)^{-1}$$

$$X^{\varepsilon_2} = T_1^\vee (T_0^\vee)^{-1} (T_1^\vee)^{-1} (T_2^\vee)^{-1}.$$

Then

$X = \{X^\mu \mid \mu \in \mathfrak{h}_{\mathbb{Z}}^*\}$ is an abelian subgroup of $\mathcal{B}^\vee$.

The double affine braid group $\tilde{\mathcal{B}}$.

$W^\vee = \{ X^\mu w \mid \mu \in \mathfrak{h}^*_{\mathbb{Z}}, w \in W_0 \}$ acts on

$\tilde{Y} = \{ q^{k/2} y^{\lambda^\vee} \mid \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}, k \in \mathbb{Z} \}$ by conjugation.

Write

$$v y^{\lambda^\vee} v^{-1} = y^{v\lambda^\vee}, \quad \text{for } v \in W^\vee, \ \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}.$$

The double affine braid group is generated by $\mathcal{B}^\vee$ and $\tilde{Y}$ with

$$q^{\frac{1}{2}} \in Z(\tilde{\mathcal{B}}), \qquad g^\vee y^{\lambda^\vee} (g^\vee)^{-1} = y^{g^\vee \lambda^\vee}, \quad \text{and}$$

$$(T_i^\vee)^{-1} y^{\lambda^\vee} = \begin{cases} y^{s_i\lambda^\vee} (T_i^\vee)^{-1}, & \text{if } \langle \lambda^\vee, \alpha_i \rangle = 0, \\ y^{s_i \lambda^\vee} T_i^\vee, & \text{if } \langle \lambda^\vee, \alpha_i \rangle = 1. \end{cases}$$

Introduce parameters

$$t^{\frac{1}{2}}_{\alpha+k\delta} = t^{\frac{1}{2}}_{u(\alpha+k\delta)} \quad \text{for } u \in W^\vee$$

and let

$$t_i^{\frac{1}{2}} = t_{\alpha_i}^{\frac{1}{2}} \quad \text{for} \quad i = 0, \ldots, n.$$

The double affine Hecke algebra $\tilde{H}$

$\tilde{H}$ is the quotient of $\mathbb{C}\tilde{\mathcal{B}}$ by

$$(T_i - t_i^{\frac{1}{2}})(T_i + t_i^{-\frac{1}{2}}) = 0, \text{ for } i = 0, \ldots, n.$$

Then $\tilde{H}$ has basis

$$\{q^{k/e} X^{\mu} T_w Y^{\lambda^\vee} \mid k \in \mathbb{Z}, \mu \in \mathfrak{h}^*_{\mathbb{Z}}, w \in W_0, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}\}$$

with

$$T_w = T_{i_1} \cdots T_{i_\ell} \quad \text{if} \quad w = s_{i_1} \cdots s_{i_\ell} \text{ is reduced.}$$

The trivial representation of the affine Hecke algebra

$$H = \text{span}\{T_w Y^{\lambda^\vee} \mid w \in W_0, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}\}$$

is $\text{span}\{\mathbb{1}\}$ with

$$T_i \mathbb{1} = t_i^{\frac{1}{2}} \mathbb{1} \text{ and } g\mathbb{1} = \mathbb{1}, \text{ for } g \in \Omega$$

and $i = 0, \ldots, n$.

The polynomial representation of $\tilde{H}$ is

$$\text{Ind}_H^{\tilde{H}}(\mathbb{1}) = \text{span}\{q^{k/e} X^{\mu} \mathbb{1} \mid k \in \mathbb{Z}, \mu \in \mathfrak{h}_{\mathbb{Z}}\} = \mathbb{C}[X]\mathbb{1}.$$