Introduction to
Buildings and Combinatorial Representation Theory
American Institute of Mathematics (AIM)
March 26, 2007

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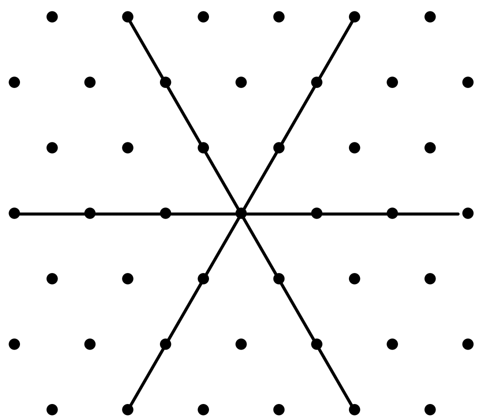
1 Weyl characters

Your favourite group $G^\vee$ (probably $SL_3(\mathbb{C})$) corresponds to

$W = \{\text{chambers}\}$

and

$P^\vee = \{\text{dots}\}$



The irreducible $G^\vee$-modules $L(\lambda^\vee)$ are indexed by $\lambda^\vee \in (P^\vee)^+$ and

$$\mathrm{char}(L(\lambda^\vee)) = \sum_{\mu^\vee \in P^\vee} \mathrm{Card}(B(\lambda^\vee)_{\mu^\vee} x^{\mu^\vee},$$

where

$$B(\lambda^\vee)_{\mu^\vee} = \{\text{Littelmann paths of type } \lambda^\vee \text{ and end } \mu^\vee\}.$$

If

$$G = G(\mathbb{C}((t))), \qquad K = G(\mathbb{C}[[t]]), \quad \text{and} \quad U^- = \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \right\}.$$

then G/K is the *loop Grassmanian* and

$$G = \bigsqcup_{\lambda^\vee \in (P^\vee)^+} K t_{\lambda^\vee} K \qquad \text{and} \qquad G = \bigsqcup_{\mu^\vee \in P^\vee} U^- t_{\mu^\vee} K.$$

The *MV cycles of type* $\lambda^\vee$ *and weight* $\mu^\vee$ are the elements of

$$MV(\lambda^\vee)_{\mu^\vee} = \{\text{irreducible components of } \overline{Kt_{\lambda^\vee}K \cap U^-t_{\mu^\vee}K}\ \},$$

and

$$\mathrm{char}(L(\lambda^\vee)) = \sum_{\mu^\vee} \mathrm{Card}(MV(\lambda^\vee)_{\mu^\vee})x^{\mu^\vee}.$$

2 Hecke algebras

The *spherical* and *affine Hecke algebras* are

$$\tilde{H}_{\mathrm{sph}} = C(K\backslash G/K) \qquad \text{and} \qquad \tilde{H} = C(I\backslash G/I),$$

where

$$\begin{array}{ccccccc} G & = & G(\mathbb{C}((t))) & & & & \\ \cup\!\shortmid & & \cup\!\shortmid & & & & \\ K & = & G(\mathbb{C}[[t]]) & \stackrel{\Phi}{\longrightarrow} & G(\mathbb{C}) & \text{where} & B = \left\{\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\}. \\ \cup\!\shortmid & & \cup\!\shortmid & & \cup\!\shortmid & & \\ I & = & \Phi^{-1}(B) & \longrightarrow & B\,, & & \end{array}$$

The *Satake map* is

$$\begin{array}{rcll} \mathbb{C}[X]^W = Z(\tilde{H}) & \stackrel{\sim}{\longrightarrow} & Z(\tilde{H})\mathbf{1}_0 = \mathbf{1}_0\tilde{H}\mathbf{1}_0 = \tilde{H}_{\mathrm{sph}} & \\ f & \longmapsto & f\mathbf{1}_0 & \\ P_{\lambda^\vee} & \leftarrow & \mathbf{1}_0X^{\lambda^\vee}\mathbf{1}_0 = \chi_{Kt_{\lambda^\vee}K} & \text{``“obvious'' basis} \end{array}$$

and $P_{\lambda^\vee}$ are the *Hall-Littlewood polynomials.*

$$P_{\lambda^\vee} = \sum_{\mu^\vee \in P^\vee} \mathrm{Card}_q(\mathcal{P}(\lambda^\vee)_{\mu^\vee})x^{\mu^\vee},$$

where

$$\mathcal{P}(\lambda^\vee)_{\mu^\vee} = \{\text{Hecke paths of type } \lambda^\vee \text{ and end } \mu^\vee\} \longleftrightarrow \{\text{slices of } G/K \text{ in } Kt_{\lambda^\vee}K \cap U^-t_{\mu^\vee}K\}$$

and

$$\mathrm{Card}_q(\mathcal{P}(\lambda^\vee)_{\mu^\vee}) = \sum_{p \in \mathcal{P}(\lambda^\vee)_{\mu^\vee}} (\#\text{ of } \mathbb{F}_q \text{ points in slice } p).$$

After normalization,

$$P_{\lambda^\vee}\big|_{q^{-1}=0} = \mathrm{char}(L(\lambda^\vee)).$$

3 Buildings

The group B is a Borel subgroup of $G = G(\mathbb{C})$ and

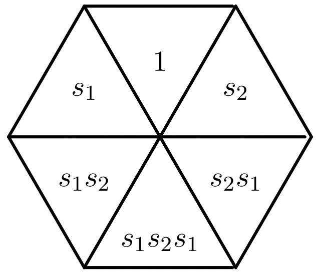
$$G/B = \text{flag variety} = \text{building}.$$

The cell decomposition of G/B is

$$G = \bigsqcup_{w \in W} BwB.$$

Idea: The points of W are regions, or chambers.

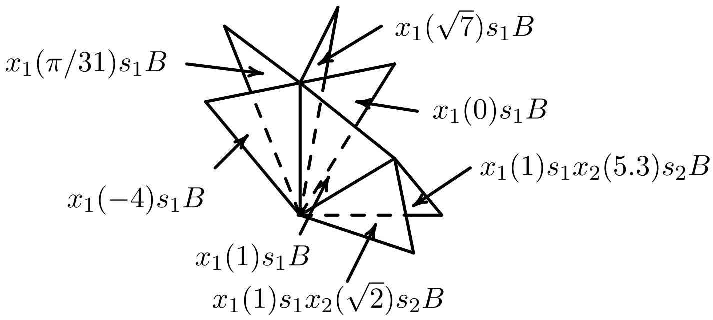
$$W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$$



If $w = s_{i_1} \cdots s_{i_\ell}$ is a minimal length path to w then

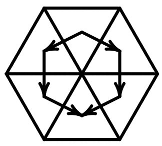
$$BwB = \{x_{i_1}(c_1)s_{i_1} \cdots x_{i_\ell}(c_\ell)s_{i_\ell}B \mid c_1, \ldots, c_\ell \in \mathbb{C}\}, \qquad \text{where} \quad x_i(c) = 1 + cE_{i,i+1},$$

with $E_{i,i+1}$ the matrix with a 1 in the $(i, i+1)$ entry and all other entries 0.
IDEA: The points of G/B are regions, or chambers.



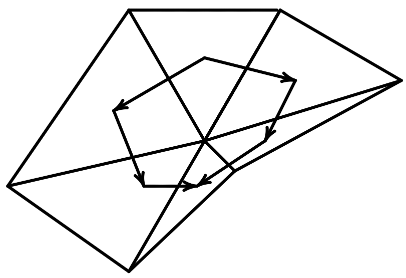
Just as the building of W, the *Coxeter complex*, has relations

$$s_1 s_2 s_1 = s_2 s_1 s_2$$



the building of G/B also has relations

$$x_1(c_1)s_1x_2(c_2)s_2x_1(c_3)s_1 = x_2(c_3)s_2x_1(c_1c_3 - c_2)s_1x_2(c_3)s_2$$



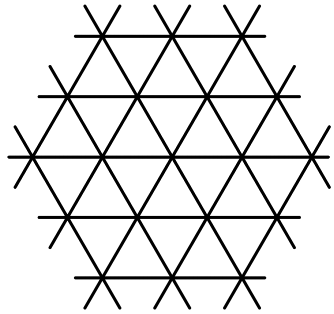
An *apartment* is a subbuilding of G/B that looks like W.
The Borel subgroup of $G = G(\mathbb{C}((t)))$ is I and

G/I is the *affine flag variety*

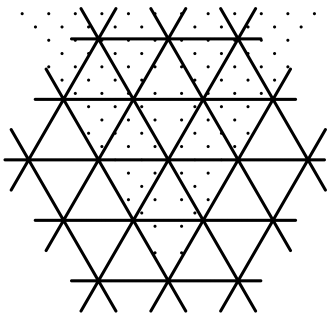
with

$$G = \bigsqcup_{w \in \tilde{W}} IwI, \qquad \text{where} \quad \tilde{W} = W \ltimes P^\vee$$

is the *affine Weyl group*



The *affine building* G/I has *sectors*



since $\quad G = \bigsqcup_{v \in \tilde{W}} U^- vI.$

4 MV polytopes

Let

$$T = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \qquad \text{and} \quad \text{let } V \text{ be a } T\text{-module}$$

with T-invariant inner product $\langle , \rangle$ (such that $\langle v, v \rangle = 0 \Leftrightarrow v = 0$). Let

$$\mathfrak{h} = Lie(T) \qquad\qquad \text{and} \qquad\qquad \mathbb{P}V = \{[v] \mid v \in V, v \neq 0\},$$

where $[v] = \text{span}\{v\}$. The *moment map* on $\mathbb{P}V$ is

$$\begin{array}{rccc} \mu\colon & \mathbb{P}V & \to & \mathfrak{h}^* \\ & [v] & \longmapsto & \mu_v \end{array} \qquad \text{where} \qquad \mu_v(h) = \frac{\langle hv, v \rangle}{\langle v, v \rangle}.$$

Now let $V = L(\gamma)$ be a simple G-module ($G = G(\mathbb{C})$) with highest weight vector v^+. Then

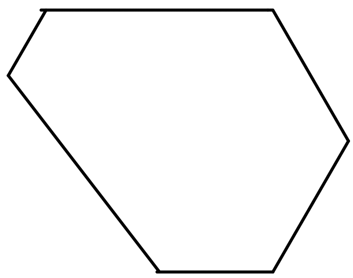
$$B[v^+] = [v^+] \qquad \text{and} \qquad G[v^+] \subseteq \mathbb{P}V$$

is the image of G/B in $\mathbb{P}V$. The *moment map* on G/B (associated to γ) is

$$\begin{array}{rccccc} \mu\colon & G/B & \to & \mathbb{P}V & \to & \mathfrak{h}^* \\ & gB & \mapsto & g[v^+] & \mapsto & \mu_{gv^+} \end{array}$$

Joel(Kamnitzer)'s favourite case is G/K with $\gamma = \omega_0$ (the fundamental weight corresponding to the added node on the extended Dynkin diagram) and

$$\mu(\text{MV cycle of type } \lambda^\vee \text{ and weight } \mu^\vee) = (\textit{MV polytope of type } \lambda^\vee \textit{ and weight } \mu^\vee)$$



5 Tropicalization

Let $G = G(\mathbb{C}((t)))$.

$$\mathbb{C}((t)) = \{a_\ell t^\ell + a_{\ell+1}t^{\ell+1} + \cdots \mid \ell \in \mathbb{Z}, a_i \in \mathbb{C}\}.$$

Points of G/I are

$$gI, \qquad \text{where} \quad g = (g_{ij}), \quad g_{ij} \in \mathbb{C}((t)).$$

The *valuation* on $\mathbb{C}((t))$

$$v(a_\ell t^\ell + a_{\ell+1}t^{\ell+1} + \cdots) = \ell,$$

is like log

$$v(f_1 f_2) = v(f_1) + v(f_2) \qquad \text{and} \qquad v(f_1 + f_2) = \min(v(f_1), v(f_2)).$$

Then $v(gI)$ is a *tropical point* of $v(G/I)$, the *tropical flag variety.* An *amoeba*, or *tropical subvariety*, is the image, under v, of a subvariety of G/I.