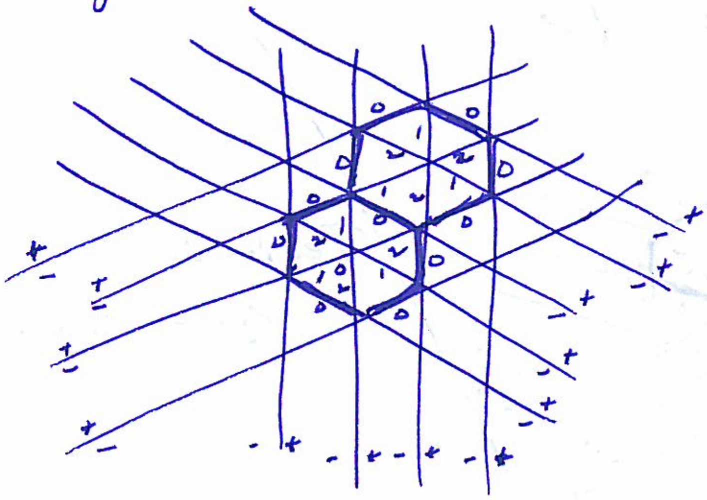
Colloquium: University of Stuttgart

Random Walks, Spherical Functions and Representations

The alcove walk algebra

July 4, 2005

Anfang data: D

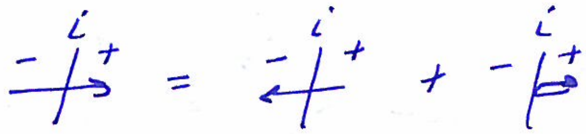


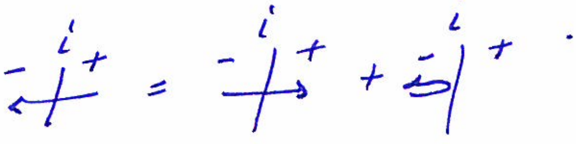
Periodic orientation

The alcove walk algebra has generators

positive i-crossing, negative i-crossing, negative i-fold, positive i-fold

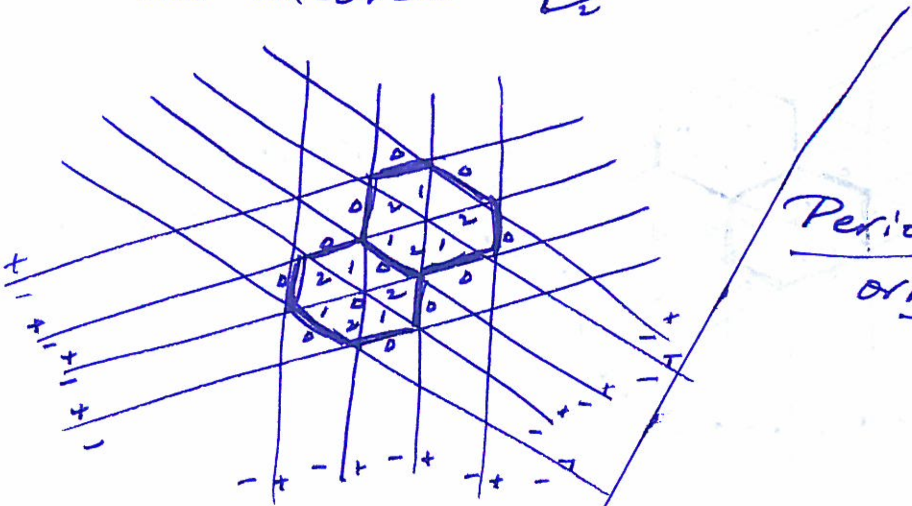
and relations





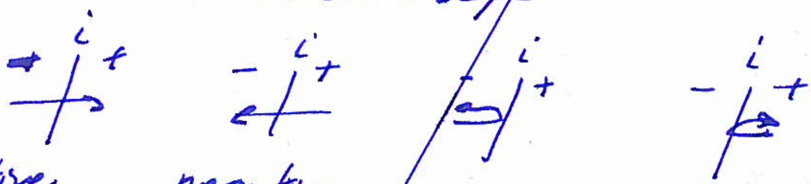
The alcove walk algebra

Fundamental alcove: 



Periodic orientation

Alcove walks have steps

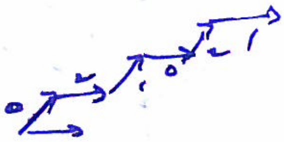


positive i-fold | negative i-crossing | negative i-fold | positive i-fold.

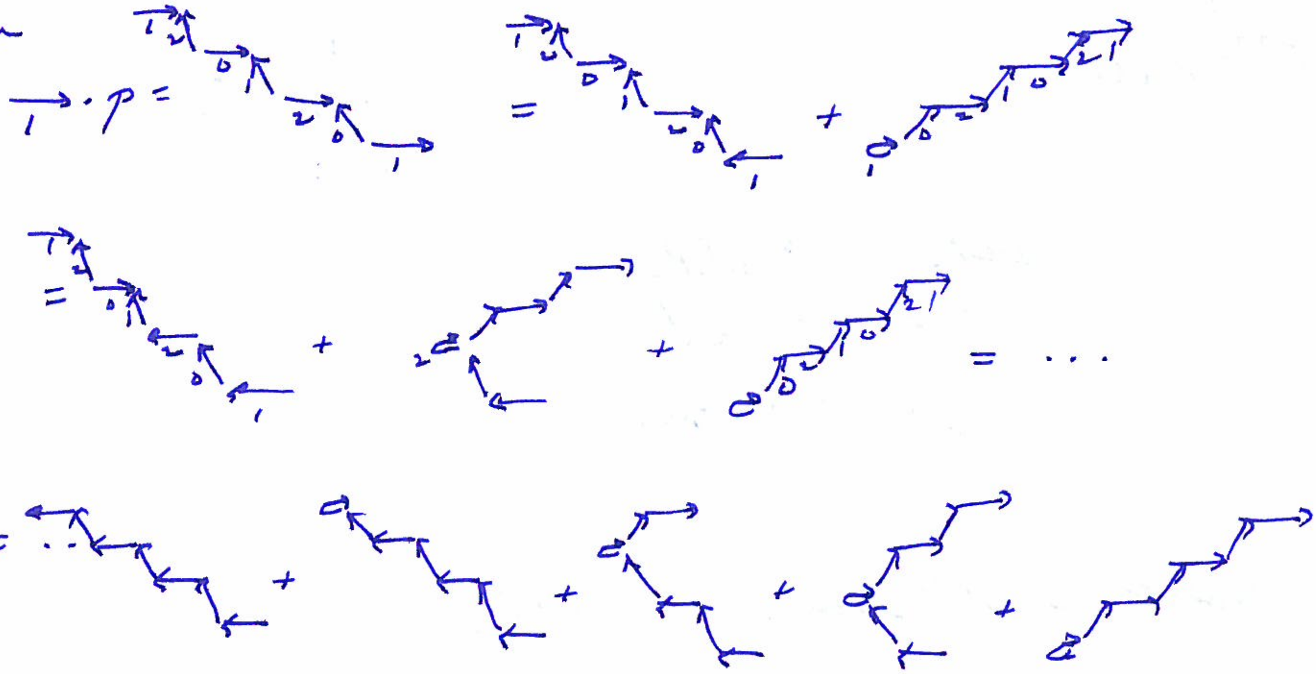
and relations

Straightening: an example

If

$p =$ 

then

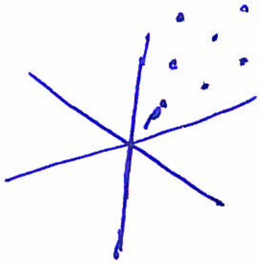


Theorem The alcove walk algebra has basis

$\{$alcove walks$\}$.

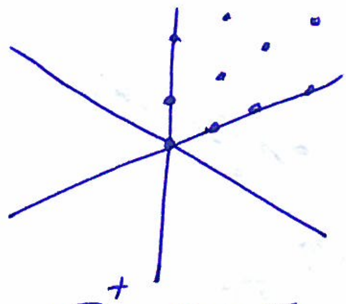
Miracle! $\mathcal{A}$ sees the Littelmann path model for crystal bases of quantum group representations.

Classical stories



$P^{++} = P \cap C$

$$\begin{array}{ccc} P^+ & \xrightarrow{\sim} & P^{++} \\ \lambda & \longmapsto & \lambda+\rho \end{array}$$



$P^{+} = P \cap \bar{C}$

$$\begin{array}{ccc} \mathbb{C}[P]^W & \xrightarrow{\sim} & \mathbb{C}[P]^{\det} \\ f & \longmapsto & a_\rho f \\ s_\lambda & \longleftarrow & a_{\lambda+\rho} \end{array}$$

where $\mathbb{C}[P] = \mathrm{span}\{X^\lambda \mid \lambda \in P\}$ with

$$X^\lambda X^\mu = X^\mu X^\lambda = X^{\lambda+\mu} \quad \text{and} \quad wX^\lambda = X^{w\lambda}.$$

$$\mathbb{C}[P]^W = \{f \in \mathbb{C}[P] \mid wf = f, \text{ for all } w \in W\}$$

$$\mathbb{C}[P]^{\det} = \{f \in \mathbb{C}[P] \mid wf = \det(w) f, \text{ for all } w \in W\}.$$

and

$$a_{\lambda+\rho} = \sum_{w \in W} \det(w)\, w X^{\lambda+\rho}$$

<u>Example</u> If $W = S_n$ and $\mathbb{C}[P] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ then

$$a_\rho = \sum_{w \in S_n} \det(w)\, w(x_1^{n-1} x_2^{n-2} \cdots x_{n-1})$$

$$= \det \begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \end{pmatrix} = \text{Vandermonde}.$$

<u>Hermann Weyl</u>

$G = G(\mathbb{C})$ is the complex reductive group corresponding to the Anfang data.

T = maximal torus

$$P \overset{1-1}{\longleftrightarrow} \{\text{simple } T\text{-modules } X^\mu\}$$

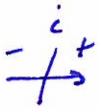
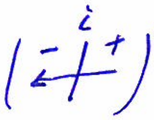
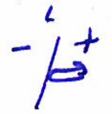
$$P^+ \overset{1-1}{\longleftrightarrow} \{\text{simple } G\text{-modules } L(\lambda)\}$$

Then

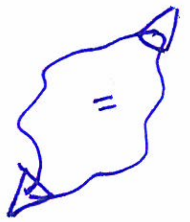
$$\operatorname{Res}^G_T(L(\lambda)) = \sum_{\mu \in P} K_{\lambda\mu} X^\mu = s_\lambda .$$

Modern stories

The *affine Hecke algebra* $\tilde{H}$ is the algebra $\mathcal{A}$ with additional relations

 $= \Big($  $\Big)^{-1}$,  $= -(q-q^{-1})$,  $= (q-q^{-1})$

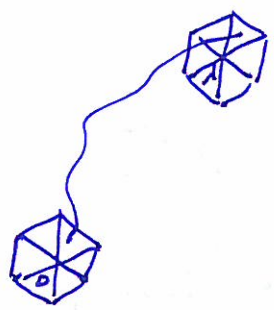
and

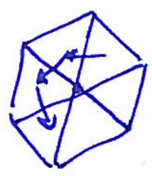
 for nonfolded walks

$$\tilde{W} = \{\text{alcoves}\} = \{wt_\lambda \mid w \in W, \lambda \in P\}.$$

$$W = \left\{ \begin{matrix}\text{alcoves in} \\ 0 \text{ hexagon}\end{matrix} \right\} \qquad P = \left\{ \begin{matrix}\text{centers of} \\ \text{hexagons}\end{matrix} \right\}$$

Let

$X^\lambda =$  a minimal length walk

$T_w^{-1} =$  a minimal length walk.

Then

$$H = \text{span}\{T_w^{-1} \mid w \in W\} \qquad \text{and}$$

$$\mathbb{C}[P] = \text{span}\{X^\lambda \mid \lambda \in P\}$$

are subalgebras of $\tilde{H}$.

$$
\begin{array}{ccccc}
\mathbb{C}[P]^W = Z(\tilde{H}) & \longrightarrow & \mathbb{1}_0 \tilde{H} \mathbb{1}_0 & \longrightarrow & \varepsilon_0 \tilde{H} \mathbb{1}_0 \\
f & \longmapsto & f\mathbb{1}_0 & & \\
 & & h & \longmapsto & A_\rho h \\
s_\lambda & \longleftarrow\!\shortmid & C'_{n_\lambda} & \longleftarrow\!\shortmid & A_{\lambda+\rho} = \varepsilon_0 X^{\lambda+\rho} \mathbb{1}_0 \\
P_\lambda & \longleftarrow\!\shortmid & \mathbb{1}_0 X^\lambda \mathbb{1}_0 & &
\end{array}
$$

where $\mathbb{1}_0, \varepsilon_0 \in H$ are given by

$$\mathbb{1}_0^2 = \mathbb{1}_0 \quad \text{and} \quad T_w \mathbb{1}_0 = q^{-\ell(w)} \mathbb{1}_0$$

$$\varepsilon_0^2 = \varepsilon_0 \quad \text{and} \quad \varepsilon_0 T_w^{-1} = \det(w)\, q^{\ell(w)} \mathbb{1}_0 .$$

P_λ is <u>Macdonald's</u> <u>spherical</u> <u>function</u>

C'_{n_λ} is a <u>Kazhdan-Lusztig basis</u> <u>element</u> corresp. to n_λ, the maximal length element in $W t_\lambda W$.

<u>Function spaces</u>

$G(\mathbb{C})$ is the complex reductive group corresp. to the Anfang data.

$\mathbb{F} = \mathbb{Q}_p$, $\mathfrak{o} = \mathbb{Z}_p$, and $k = \mathbb{F}_p$.

$$\begin{array}{llll} & G = G(\mathbb{F}) & & \\ & \cup\!\vert & & \\ \text{max'l compact} & K = G(\mathfrak{o}) & \longrightarrow G(k) & = \text{finite Chevalley group} \\ & \cup\!\vert & \cup\!\vert & \\ \text{Iwahori subgroup} & I & \longrightarrow B = \left\{\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\} & = \text{Borel subgroup} \end{array}$$

Then

$$\tilde{H} \cong C_c(I\backslash G/I) \quad \text{and} \quad \mathbb{1}_0 \tilde{H} \mathbb{1}_0 \cong C_c(K\backslash G/K)$$

where

$$C_c(K\backslash G/K) = \left\{ f : G \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ is continuous, compact support} \\ f(k_1 g k_2) = f(g) \text{ for } k_1, k_2 \in K \end{array} \right\}$$

P_λ corresponds to $\chi_{Kt_\lambda K}$.

<u>Changing fields</u>

(1) If $\mathbb{F} = \mathbb{R}$ and $G = GL_n(\mathbb{R})$ then $K = O_n(\mathbb{R})$

and $\mathbb{1}_0 \tilde{H} \mathbb{1}_0$ is spherical harmonics.

(2) If $\mathbb{F} = \mathbb{C}((t))$ then $\mathfrak{o} = \mathbb{C}[[t]]$ and $k = \mathbb{C}$

and this is <u>Geometric</u> <u>Langlands</u>.

$\mathbb{1}_0 \tilde{H} \mathbb{1}_0$ is the convolution algebra of perverse sheaves on the Loop Grassmannian

$\mathbb{C}[P]^W$ is the representation ring of the Langlands dual group.