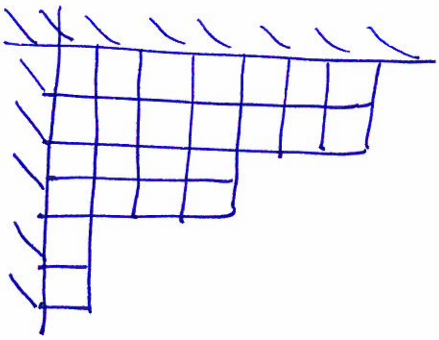
Combinatorial Representation Theory I – Towers and Centralizers

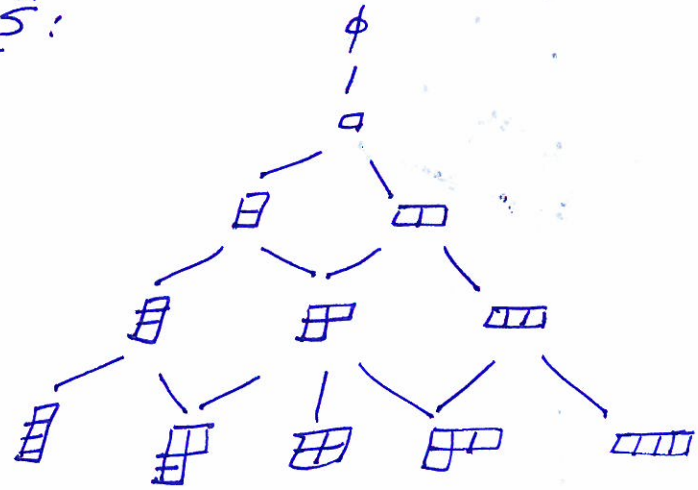
Towers

University of Zaragoza, Feb. 22, 2005

A partition is a collection of boxes in a corner.



The tower $\hat{S}$:

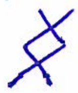


The group algebra of the symmetric group

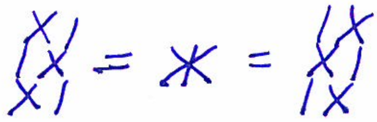
Generators: $s_1, s_2, \ldots, s_{k-1}$

$s_i =$ 

Relations:

$s_i^2 = 1$, 

$s_i s_j = s_j s_i$, 

$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$. 

Representation Theory

Let A be an algebra

Problem 1: Classify/Index the simple modules

Problem 2: Construct the simple A-modules.

Combinatorial Representation Theory

Answer 1 Let $\hat{A} = \{$set of combinatorial objects$\}$

Then

$$\begin{array}{ccc} \hat{A} & \longleftrightarrow & \{\text{simple } \hat{A}\text{-modules}\} \\ \lambda & \longmapsto & A^\lambda \end{array}$$

is a bijection.

Answer 2: Let $\lambda \in \hat{A}$ and let

$$\hat{A}^\lambda = \{\text{set combinatorial objects depending on } \lambda\}$$

Let $A^\lambda = \text{span}\{v_T \mid T \in \hat{A}^\lambda\}$

so that the v_T are a basis of A^λ

Define an A-action on A^λ by

$$a v_T = \sum_S a_{ST} v_S$$

where

$$a_{ST} = \text{nice formula}.$$

For towers:

$$\hat{S}_n = \{\lambda \text{ on level } n \text{ of the tower}\}.$$

$$\hat{S}_n^\lambda = \{\text{paths } T = (\phi \to T^{(1)} \to \cdots \to T^{(n)} = \lambda) \text{ in } \hat{S}\}.$$

Let

$$S_n^\lambda = \text{span}\{v_T \mid T \in \hat{S}_n^\lambda\}$$

and define

$$s_i v_T = \frac{1}{c(T(i)) - c(T(i+1))} v_T + \left(1 + \frac{1}{c(T(i)) - c(T(i+1))}\right) v_{s_i T}$$

where $T(i)$ is the box added $T^{(i+1)} \to T^{(i)}$

$c(T(i))$ is the diagonal number of $T(i)$

$s_i T$ is the same as T except $T^{(i)}$ and $T^{(i+1)}$ are switched.

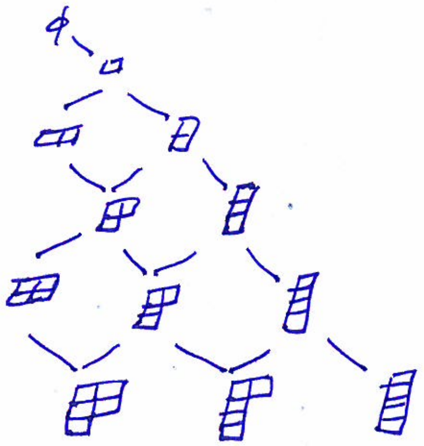
$v_{s_i T} = 0$ if $s_i T \notin S_n^\lambda$

Theorem

$$\begin{aligned} \hat{S}_n &\longleftrightarrow \{\text{irreducible } S_n\text{-modules}\} \\ \lambda &\longmapsto S_n^\lambda \end{aligned}$$

Examples of towers

$\hat{T}$



Temperley-Lieb algebras

Generators: $E_1, \ldots, E_{k-1}$

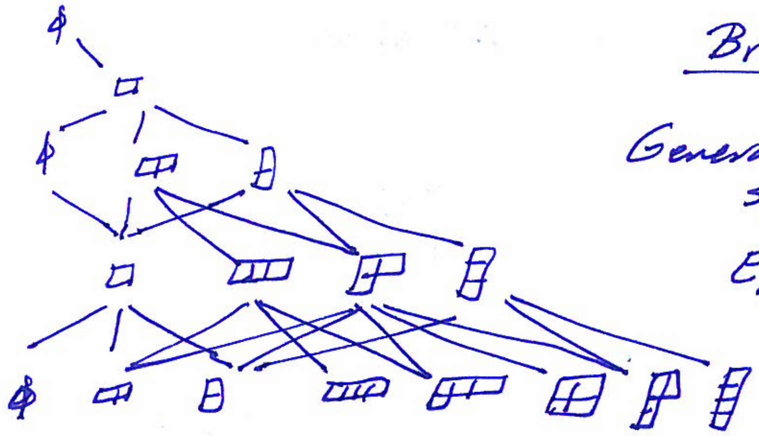
$$\begin{array}{c} 1 \;\cdots\; i \;\; i{+}1 \;\cdots\; n \\ E_i = |\,|\,|\,|\; \substack{\smile\\\frown}\; |\,|\,|\,| \end{array}$$

Relations: $E_i^2 = nE_i$

$$E_i E_{i\pm1} E_i = E_i$$

$$E_i E_j = E_j E_i, \quad \text{if } |i-j|>1.$$

$\hat{B}$



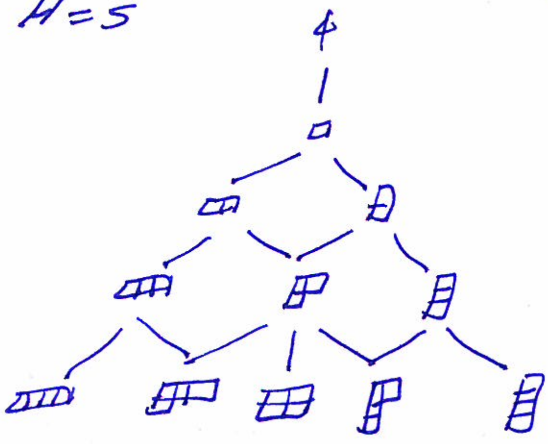
Brauer algebras

Generators:

$$s_1, s_2, \ldots, s_{k-1}$$

$$E_1, E_2, \ldots, E_{k-1}$$

$\hat{H} = \hat{S}$



Iwahori-Hecke algebras

Generators: $T_1, \ldots, T_{k-1}$

$$T_i = |\,|\,|\,|\times|\,|\,|\,|$$

Relations:

$$T_i^2 = (q-q^{-1})T_i + 1$$

$$T_i T_j = T_j T_i$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}.$$

Main Tool for study: Centralizers

Idea: Study A by studying where A "comes from". Then transfer information.

Case 1: Let G be a group, V a G-module.

$$A_k = \mathrm{End}_G(V^{\otimes k}).$$

Case 2: Let G be a group, B a subgroup.

$$A = \mathrm{End}_G(\mathrm{Ind}_B^G(\mathrm{triv})).$$

New versions:

Case 1: Let $\mathcal{U}$ be a Hopf algebra, V a $\mathcal{U}$-module

$$A_k = \mathrm{End}_{\mathcal{U}}(\mathrm{Res}_{\mathcal{U}}^{\mathcal{U}\otimes\cdots\otimes\mathcal{U}}(V^{\otimes k}))$$

Case 2: Let $\iota: B \hookrightarrow R$ be an inclusion of rings. Let V be a simple B-module.

$$A = \mathrm{End}_R(\mathrm{Ind}_B^R(V))$$

Case 3: Let $\mu: M \to N$ be a morphism of G-varieties. Let

$$\mu_*: \left\{ \begin{matrix}\text{sheaves}\\ \text{on } M\end{matrix} \right\} \longrightarrow \left\{ \begin{matrix}\text{sheaves}\\ \text{on } N\end{matrix} \right\}$$ the induced map.

Let $\mathcal{C}_M$ be a perverse sheaf on M.

$$A = \mathrm{End}_{D(N)}(\mu_*\mathcal{C}_M) = \bigoplus_i \mathrm{Ext}^i_{D(N)}(\mu_*\mathcal{C}_M, \mu_*\mathcal{C}_M).$$

Examples: Case 1

1.1) $G = GL_n(\mathbb{C})$, $V = \mathbb{C}^n$. Then

$$\mathbb{C}S_k \twoheadrightarrow \mathrm{End}_G(V^{\otimes k}).$$

1.2) $G = O_n(\mathbb{C})$ or $Sp_n(\mathbb{C})$, $V = \mathbb{C}^n$

$$\text{Brauer algebra} \twoheadrightarrow \mathrm{End}_G(V^{\otimes k})$$

1.3) $G = S_n$, $V = \mathbb{C}^n$

$$\text{Partition algebra} \twoheadrightarrow \mathrm{End}_G(V^{\otimes k}).$$

1.4) $U = U_q\mathfrak{sl}_2$, $V = \mathbb{C}^2$

$$\text{Temperley-Lieb algebra} \twoheadrightarrow \mathrm{End}_U(V^{\otimes k})$$

1.5) $U = U_q\mathfrak{gl}_n$ or $U = U_q\mathfrak{sl}_n$, $V = \mathbb{C}^n$

$$\text{Iwahori-Hecke algebra} \twoheadrightarrow \mathrm{End}_U(V^{\otimes k})$$

1.6) $U = U_q\mathfrak{so}_n$ or $U_q\mathfrak{sp}_n$, $V = \mathbb{C}^n$

$$\text{BMW algebra} \twoheadrightarrow \mathrm{End}_U(V^{\otimes k})$$

1.7) $U = U_q\mathfrak{gl}_n$ or $U_q\mathfrak{sl}_n$, $V = \mathbb{C}^n$

$$\text{Affine Hecke algebra} \twoheadrightarrow \mathrm{End}_U(M \otimes V^{\otimes k})$$

for any U-module M.

2.1) $G = GL_n(\mathbb{F}_q)$, $B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$, $V = \mathbb{C}$

Iwahori-Hecke algebra $\twoheadrightarrow \mathrm{End}_G(\mathrm{Ind}_B^G(\mathbb{C}))$.

2.2) $G = G(\mathbb{F}_q)$ a finite Chevalley group

B = Borel subgroup

Iwahori-Hecke algebra $\twoheadrightarrow \mathrm{End}_G(\mathrm{Ind}_B^G(\mathbb{C}))$

2.3) $G = G(\mathbb{Q}_p)$ a p-adic group.

I = Iwahori subgroup

Affine Hecke algebra $\twoheadrightarrow \mathrm{End}_G(\mathrm{Ind}_I^G(\mathbb{C}))$

3.1) $G = GL_n(\mathbb{C})$ acts, by conjugation, on

$\mathcal{N}$ = {nilpotent matrices}

Let

$\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ be a desingularization

Then

Affine Hecke algebra $\twoheadrightarrow \mathrm{End}_{D(\mathcal{N})}(\mu_* \mathcal{C}_{\tilde{\mathcal{N}}})$

where $\mathcal{C}_{\tilde{\mathcal{N}}}$ is the constant perverse sheaf on $\tilde{\mathcal{N}}$.