

c -FUNCTIONS AND KOORNWINDER POLYNOMIALS

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In memory of Ian G. Macdonald.

ABSTRACT

This paper develops the theory of Macdonald–Koornwinder polynomials in parallel analogy with the work done by us in 2024 for the GL_n case [c -functions and Macdonald polynomials, *J. Algebra* 655 (2024), 163–222]. In the context of the type $(C_n^\vee, C_n)$ affine root system, the Macdonald polynomials of other root systems of classical type are specializations of the Koornwinder polynomials. We derive c -function formulas for symmetrizers and use them to give E -expansions, principal specializations and norm formulas for bosonic, mesonic and fermionic Koornwinder polynomials. Finally, we explain the proof of the norm conjectures and constant term conjectures for the Koornwinder case.

1. INTRODUCTION

The Koornwinder polynomials are multivariate generalizations of the classical orthogonal polynomials that appear in the Askey scheme [9]. At the top of this hierarchy, we find the Askey–Wilson polynomials, and the other families of orthogonal polynomials in the Askey scheme are obtained from the Askey–Wilson polynomials by specializing parameters.

Macdonald’s 1987 paper [12] provides a very general framework for associating orthogonal polynomials P_λ to any affine root system. It turns out that the Koornwinder polynomials are the Macdonald polynomials for the affine root system of type $(C_n^\vee, C_n)$, and the Askey–Wilson polynomials are the Macdonald polynomials for the affine root system of type $(C_1^\vee, C_1)$. One of the key features of Macdonald’s picture is that the norms $\langle P_\lambda, P_\lambda \rangle_+$ are generalizations of Macdonald’s ‘constant terms’ $\langle P_0, P_0 \rangle_+$. In this way, Macdonald stated conjectures for the values of $\langle P_\lambda, P_\lambda \rangle_+$, which vastly expanded his earlier constant-term conjectures.

Cherednik introduced the double affine Hecke algebra (DAHA) as a tool for extending Opdam’s ideas to prove the norm conjectures [3]. This perspective pointed to a larger family of orthogonal polynomials $E_{\mu, \nu}$ from which the P_λ are obtained by a process of symmetrization. All of these tools, including the proof of the norm conjectures, were wonderfully expounded in the full generality of a possibly non-reduced

Received 9 October 2024; Revised 20 March 2025

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affine root system in Macdonald's book [13]. For a wonderful history of the exciting trajectory of these amazing developments, see [6].

Particularly in the type GL_n case, the Macdonald polynomials have been of interest to the combinatorial community because of the wealth of wonderful q, t -generalizations of classical combinatorial formulas in symmetric function theory. It is also stimulating that there are many fascinating connections to adjacent fields (representations of p -adic groups and affine Lie algebras, geometry of Hilbert schemes and affine Springer fibres, torus knot invariants, vertex models in statistical mechanics, particle processes in probability, etc.). For this reason, it is desirable to provide expositions of the tools that bridge the language gaps between the generality of affine root systems and the standard conventions in classical symmetric function theory. In [4], we explained how, in the type GL_n case, many of the combinatorial formulas can be understood from the theory of c -functions, which are the analogues of the Harish-Chandra c -functions surveyed by Helgason [7] and that appear everywhere in Macdonald's monograph [13].

In the type GL_n case, the Macdonald polynomials depend on two parameters, q and t . In the Koornwinder case (type $(C_n^\vee, C_n)$), the polynomials depend on six parameters, $q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}$ and $u_n^{\frac{1}{2}}$. Because of this plethora of parameters, sometimes it is not so easy to see how the combinatorial formulas familiar in the type GL_n case generalize to the Koornwinder case. This paper follows the same pattern as our earlier paper [4], generalizing from the type GL_n case to the Koornwinder (type $(C_n^\vee, C_n)$) case.

There is a constantly increasing literature on Koornwinder polynomials. There are interesting technical advances and also fascinating connections to other fields (see, for example, [2, 5, 16, 17, 26, 27]). The foundational work in [14, 15, 18, 20, 24, 25], among others, continues to be extremely useful for clarifying the role and position of the DAHA as a tool for the Koornwinder case. There are also important and very useful surveys of the theory of Koornwinder polynomials (see, for example, [21, 23]). There is a significant intersection between the content of this paper and the content presented in these surveys. We hope that our combinatorial and c -function point of view can be useful in continuing research on Koornwinder polynomials.

The plan of this paper is as follows:

At the end of this section, we include some remarks on the different sets of parameters used in the literature and establish the ones we will use for the paper. Since there are six different parameters to keep track of and lots of literature to navigate, perhaps this dictionary will be useful to readers (as it was for us). Following these remarks, we present a diagram of the affine root systems of classical types together with the specializations of the parameters for obtaining the Macdonald polynomials of the corresponding type from Koornwinder polynomials, which are the Macdonald polynomials for the affine root system of type $(C_n^\vee, C_n)$. A thorough study can be found in [27].

Section 2 is dedicated to the affine Weyl group and root system for type $(C_n^\vee, C_n)$, and we include the affine coroots and affine roots. Our new contribution in section 1 is the diagram giving the relative positions of the affine root systems of classical type and the specializations that give the Macdonald polynomials of the other classical type from Koornwinder polynomials. Although these relationships are, in principle, well known (from Macdonald [11] and Bruhat-Tits [1]), we have not seen this way of presenting this information, which we find very useful, and not broadly known.

In section 3, we introduce the main tools for working with Koornwinder polynomials, including the c -functions and the DAHA. Our contribution here is to provide a framework for the DAHA in terms of c -functions, which makes the, sometimes daunting, formulas for the operators on the polynomial representation seem obvious and natural.

In the second half of section 3, we introduce four families of symmetrizers together with the relations between them, their c -function formulas and the case when the stabilizers are non-trivial. Our contribution here is to treat the four types of symmetrizers in tandem to clarify their role in the theory (and the symmetry between them).

Sections 4 and 5 examine the main objects of study:

- (a) electronic Macdonald–Koornwinder polynomials (section 4);
- (b) bosonic Macdonald–Koornwinder polynomials;
- (c) fermionic Macdonald–Koornwinder polynomials; and
- (d) mesonic Macdonald–Koornwinder polynomials.

We introduce the electronic Macdonald–Koornwinder polynomials E_μ as eigenfunctions of Cherednik–Dunkl operators and then give a recursive formula and a creation formula for the E_μ . For the other variants, our study includes

- (a) definition of the Weyl denominators;
- (b) study of the bosonic, fermionic and mesonic spaces;
- (c) formulas for the Poincaré polynomial;
- (d) expansions in terms of E_μ ; and
- (e) principal specializations.

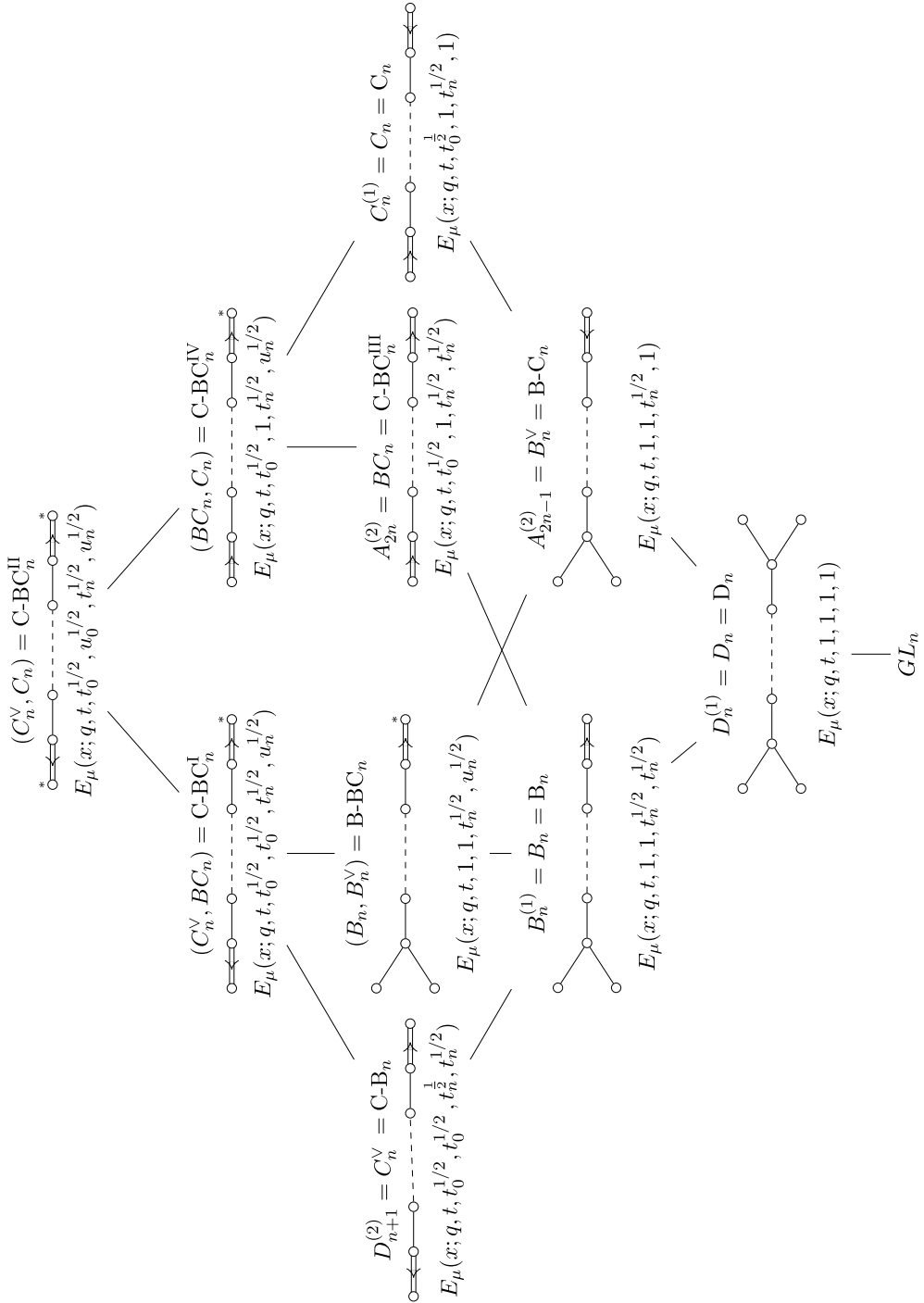
Our contribution here is to put the focus on the fermionic and mesonic Koornwinder polynomials so that the 4-fold structure is clearly visible. This 4-fold structure eventually leads to powerful recursions for computing norms.

Finally, section 6 is dedicated to studying the Macdonald–Koornwinder polynomials as a family of orthogonal polynomials. In particular, we

- (a) define the Macdonald–Koornwinder inner product via multiplication by a kernel and taking the constant term and characterize the electronic and bosonic Macdonald–Koornwinder polynomials in terms of the inner product;
- (b) compute adjoints of the operators from the DAHA;
- (c) prove the going up a level and Weyl character formulas to provide recursions for norms; and
- (d) use the recursions for norms to compute the norms $\langle P_\lambda, P_\lambda \rangle_+$ and the Macdonald constant term for type $(C_n^\vee, C_n)$.

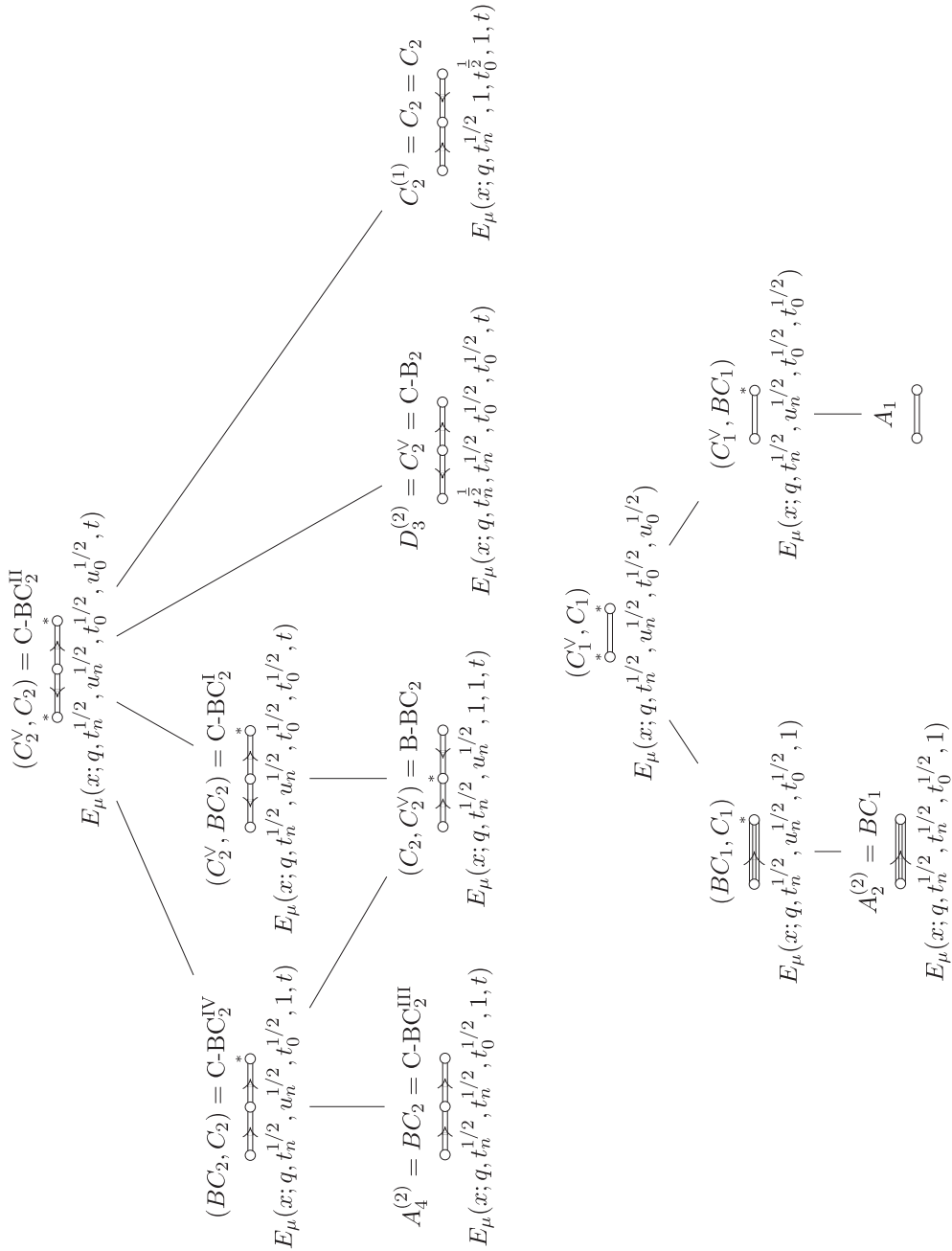
This section follows the same trajectory as that taken in [13, Ch. 5]. Our contribution here is to use the fermionic and mesonic framework to organize the recursions for norms and make the proof of norm conjectures easy and natural.

1.0.1. The poset of affine root systems of classical type



See section 2.3 for the explanation of this diagram.

1.1. The affine root systems for $n = 2$ and $n = 1$



1.2. Remarks on parameters

Depending on the reference, the notation for the parameters varies. In this article, we follow [14] and [18] and mostly use the parameters

$$q, t^{\frac{1}{2}}, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}.$$

REMARK 1.1. In an attempt to relate the parameter notations in [13, §4.7], [18, §3] and [3, Def. 2.1], let

$$\tau'_0 = u_0^{\frac{1}{2}}, \quad \tau'_n = u_n^{\frac{1}{2}}, \quad \tau_0 = t_0^{\frac{1}{2}}, \quad \tau_n = t_n^{\frac{1}{2}} \quad \text{and} \quad \tau_i = \tau'_i = t^{\frac{1}{2}} \quad \text{for } i \in \{1, \dots, n-1\}.$$

The summary of (1.5.1), (4.4.1), (4.4.2), (4.4.3) and (5.1.4) in [13] is that, for an affine root a ,

$$\tau_a = (t_a t_{2a})^{\frac{1}{2}} = q^{\frac{1}{2}\kappa_a} = q^{\frac{1}{2}(k(a)+k(2a))}, \quad \text{and} \quad \tau'_a = t_a^{\frac{1}{2}} = q^{\frac{1}{2}\kappa'_a} = q^{\frac{1}{2}(k(a)-k(2a))}.$$

In our situation,

$$\begin{aligned} t_n^{\frac{1}{2}} &= \tau_n = q^{\frac{1}{2}\kappa_n} = t_{\varepsilon_n}^{\frac{1}{2}} t_{2\varepsilon_n}^{\frac{1}{2}} = q^{\frac{1}{2}k(\varepsilon_n) + \frac{1}{2}k(2\varepsilon_n)} = q^{\frac{1}{2}k_1 + \frac{1}{2}k_2}, \\ u_n^{\frac{1}{2}} &= \tau'_n = q^{\frac{1}{2}\kappa'_n} = t_{\varepsilon_n}^{\frac{1}{2}} = q^{\frac{1}{2}k(\varepsilon_n) - \frac{1}{2}k(2\varepsilon_n)} = q^{\frac{1}{2}k_1 - \frac{1}{2}k_2}, \\ t_0^{\frac{1}{2}} &= \tau_0 = q^{\frac{1}{2}\kappa_0} = t_{-\varepsilon_1 + \frac{1}{2}\delta}^{\frac{1}{2}} t_{-2\varepsilon_1 + \delta}^{\frac{1}{2}} = q^{\frac{1}{2}k(-\varepsilon_1 + \frac{1}{2}\delta) + \frac{1}{2}k(-2\varepsilon_1 + \delta)} = q^{\frac{1}{2}k_3 + \frac{1}{2}k_4}, \\ u_0^{\frac{1}{2}} &= \tau'_0 = q^{\frac{1}{2}\kappa'_0} = t_{-\varepsilon_1 + \frac{1}{2}\delta}^{\frac{1}{2}} = q^{\frac{1}{2}k(-\varepsilon_1 + \frac{1}{2}\delta) - \frac{1}{2}k(-2\varepsilon_1 + \delta)} = q^{\frac{1}{2}k_3 - \frac{1}{2}k_4} \quad \text{and} \\ t^{\frac{1}{2}} &= \tau_i = q^{\frac{1}{2}\kappa} = t_{\varepsilon_i - \varepsilon_{i+1}}^{\frac{1}{2}} = q^{\frac{1}{2}k_5}, \quad \text{for } i \in \{1, \dots, n-1\}, \end{aligned}$$

and the formulas in [13, (1.5.1)] correspond to interchanging κ_0 and κ'_n . The parameter notations τ_i , τ'_i , k_1 , $\dots$, k_5 will not be used in the remainder of this paper. $\square$

REMARK 1.2. **Askey–Wilson parameters.** In type $(C_1^\vee, C_1)$, the bosonic Macdonald polynomials $P_\lambda(q, t_1, u_1, t_0, u_0)$ are also known as the Askey–Wilson polynomials. Following [14, §3], the correspondence to the original Askey–Wilson parameters is given by

$$q = q, \quad a = q^{\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{\frac{1}{2}}, \quad b = -q^{\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{-\frac{1}{2}}, \quad c = t_n^{\frac{1}{2}} u_n^{\frac{1}{2}}, \quad d = -t_n^{\frac{1}{2}} u_n^{-\frac{1}{2}}. \quad (1)$$

These conversions are equivalent to

$$t_0 = -q^{-1}ab, \quad t_n = -cd, \quad u_0 = -ab^{-1}, \quad u_n = -cd^{-1},$$

and it is useful to note that

$$a + b = q^{\frac{1}{2}} t_0^{\frac{1}{2}} \left(u_0^{\frac{1}{2}} - u_0^{-\frac{1}{2}} \right) \quad \text{and} \quad c + d = t_n^{\frac{1}{2}} \left(u_n^{\frac{1}{2}} - u_n^{-\frac{1}{2}} \right).$$

Up to permutations of a, b, c, d , these parameters are used in [19, (1)], in [2, (17)], in [5, Def. 2.2] and, with different notation, in [13, (5.1.14)]. $\square$

2. THE AFFINE WEYL GROUP AND ROOT SYSTEM FOR TYPE $(C_n^\vee, C_n)$

The affine root system of type $(C_n^\vee, C_n)$ is the structure that holds the combinatorics of Koornwinder polynomials in place as they are the Macdonald polynomials for this affine root system. The affine Weyl group W plays the role of the group of symmetries of the affine root system. In this section, we introduce the definitions and notations for working with the affine Weyl group W and the affine root system of type $(C_n^\vee, C_n)$.

The coroots $S^\vee$ and the roots S for the affine root system of type $(C_n^\vee, C_n)$ play just slightly different roles in the theory, especially in the computations involving Koornwinder polynomials. One of the challenges in this work is to keep these two mirror worlds in proper focus. For this purpose, in section 2.2,

we carefully lay out two versions of the affine Weyl group W , one denoted W_X , which acts on the coroots (with five orbits), and one denoted W_Y , which acts on the roots (with five orbits). While the groups W_X , W_Y and W are all isomorphic, being pedantic about the notation at this early stage prevents future headaches.

To conclude this section, we present a brief explanation of the reasoning for how the Macdonald polynomials of other classical types (such as B_n , C_n , BC_n , $A_{2n-1}^{(2)}$, $D_{n+1}^{(2)}$, etc.) are derived from the Koornwinder polynomials by specialization. This specialization process is summarized in the diagram included in section 1.0.1.

2.1. The affine Weyl group W and the finite Weyl group W_{fin}

Use a graphical notation for relations so that

$$\begin{array}{lcl} \begin{array}{c} g_i \\ \circ \end{array} & \begin{array}{c} g_j \\ \circ \end{array} & \text{means } g_i g_j = g_j g_i, \\ \begin{array}{c} g_i \\ \circ \end{array} \text{---} \begin{array}{c} g_j \\ \circ \end{array} & & \text{means } g_i g_j g_i = g_j g_i g_j, \text{ and} \\ \begin{array}{c} g_i \\ \circ \end{array} \text{=}= \begin{array}{c} g_j \\ \circ \end{array} & & \text{means } g_i g_j g_i g_j = g_j g_i g_j g_i. \end{array}$$

The *affine Weyl group* is the group W presented by generators $s_0, s_1, \dots, s_n$ and relations

$$s_i^2 = 1 \quad \text{and} \quad \begin{array}{c} s_0 \text{---} s_1 \text{---} s_2 \text{---} \cdots \text{---} s_{n-2} \text{---} s_{n-1} \text{---} s_n \\ \circ \text{---} \circ \text{---} \circ \text{---} \cdots \text{---} \circ \text{---} \circ \text{---} \circ \end{array}$$

The *finite Weyl group* is the subgroup W_{fin} generated by $s_1, \dots, s_n$.

Let $w \in W$. The *length* of w , $\ell(w)$, is the minimal $\ell \in \mathbb{Z}_{\geq 0}$ such that

$$w = s_{i_1} \dots s_{i_\ell} \quad \text{with} \quad i_1, \dots, i_\ell \in \{0, 1, \dots, n-1, n\}.$$

The expression $w = s_{i_1} \dots s_{i_\ell}$ is a *reduced word* for w , and any other expression of the form $w = s_{j_1} \dots s_{j_k}$, with $j_1, \dots, j_k \in \{0, 1, \dots, n-1, n\}$, has $k \geq \ell(w)$.

2.1.1. Translation presentation of W

Define $h_{\varepsilon_1}, \dots, h_{\varepsilon_n} \in W$ by

$$h_{\varepsilon_1} = s_0 s_1 s_2 \cdots s_{n-1} s_n s_{n-1} \cdots s_2 s_1, \quad \text{and} \quad h_{\varepsilon_j} = s_j h_{\varepsilon_{j-1}} s_j, \quad \text{for } j \in \{2, \dots, n\}.$$

For $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$, define the *translation* h_μ by

$$h_\mu = (h_{\varepsilon_1})^{\mu_1} \cdots (h_{\varepsilon_n})^{\mu_n} \tag{2}$$

and define $u_\mu \in W$ and $v_\mu \in W_{\text{fin}}$ by the equation

$$h_\mu = u_\mu v_\mu, \quad \text{where } v_\mu \in W_{\text{fin}} \text{ and } u_\mu \text{ is minimal length in the coset } h_\mu W_{\text{fin}}. \tag{3}$$

Define an action of W_{fin} on $\mathbb{Z}^n$ by

$$\begin{aligned} s_i(\mu_1, \dots, \mu_n) &= (\mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \mu_i, \mu_{i+2}, \dots, \mu_n), \quad \text{for } i \in \{1, \dots, n-1\}, \\ s_n(\mu_1, \dots, \mu_n) &= (\mu_1, \dots, \mu_{n-1}, -\mu_n). \end{aligned} \tag{4}$$

Then

$$W = \mathbb{Z}^n \rtimes W_{\text{fin}}.$$

In other words, if $\mu, \nu \in \mathbb{Z}^n$ and $\nu \in W_{\text{fin}}$, then

$$\nu h_\mu = h_{\nu\mu} \nu, \quad h_\mu h_\nu = h_{\mu+\nu} \quad \text{and} \quad W = \{h_\mu \nu \mid \mu \in \mathbb{Z}^n, \nu \in W_{\text{fin}}\}.$$

REMARK 2.1. In [4], we use t_μ to denote the translations in the type A case. We use the notation h_μ for type $(C_n^\vee, C_n)$ to avoid conflict with the set of parameters for Koornwinder polynomials (specified in Section 3.1).

2.2. Affine coroots, affine roots and the groups W_X and W_Y

In this section, we set up the notation for the affine Weyl groups W_X and W_Y . Both groups W_X and W_Y are isomorphic to W , but they serve slightly different roles and it is necessary to set up the notation to distinguish them.

2.2.1. The dual lattices $\mathfrak{a}_{\mathbb{Z}}$ and $\mathfrak{a}_{\mathbb{Z}}^*$

Let $\varepsilon_1, \dots, \varepsilon_n$ and $\varepsilon_1^\vee, \dots, \varepsilon_n^\vee$ be symbols and define dual lattices (that is, dual free $\mathbb{Z}$ -modules)

$$\mathfrak{a}_{\mathbb{Z}}^* = \{\gamma_1 \varepsilon_1 + \dots + \gamma_n \varepsilon_n \mid \gamma_1, \dots, \gamma_n \in \mathbb{Z}\} \quad \text{and} \quad \mathfrak{a}_{\mathbb{Z}} = \{\mu_1 \varepsilon_1^\vee + \dots + \mu_n \varepsilon_n^\vee \mid \mu_1, \dots, \mu_n \in \mathbb{Z}\}$$

with $\mathbb{Z}$ -bilinear pairing

$$\langle \cdot, \cdot \rangle : \mathfrak{a}_{\mathbb{Z}}^* \times \mathfrak{a}_{\mathbb{Z}} \rightarrow \mathbb{Z} \quad \text{given by} \quad \langle \varepsilon_i, \varepsilon_j^\vee \rangle = \delta_{ij}.$$

Both $\mathfrak{a}_{\mathbb{Z}}$ and $\mathfrak{a}_{\mathbb{Z}}^*$ are isomorphic to $\mathbb{Z}^n$.

2.2.2. The affine coroots for type $(C_n^\vee, C_n)$

Let

$$Q^\vee = \mathbb{Z}\text{-span}\{\varepsilon_1^\vee, \dots, \varepsilon_n^\vee, \frac{1}{2}K\}$$

be the $\mathbb{Z}$ -vector space spanned by symbols $\varepsilon_1^\vee, \dots, \varepsilon_n^\vee$ and $\frac{1}{2}K$. The affine Weyl group W_X is the group of $\mathbb{Z}$ -linear transformations of $Q^\vee$ generated by the transformations $s_0^\vee, s_1^\vee, \dots, s_n^\vee$ given as follows: If $\lambda^\vee = \lambda_1 \varepsilon_1^\vee + \dots + \lambda_n \varepsilon_n^\vee + \frac{k}{2}K$, then

$$\begin{aligned} s_0^\vee \lambda^\vee &= -\lambda_1 \varepsilon_1^\vee + \lambda_2 \varepsilon_2^\vee + \dots + \lambda_n \varepsilon_n^\vee + \left(\frac{k}{2} + \lambda_1\right) K, \\ s_n^\vee \lambda^\vee &= \lambda_1 \varepsilon_1^\vee + \dots + \lambda_{n-1} \varepsilon_{n-1}^\vee - \lambda_n \varepsilon_n^\vee + \frac{k}{2} K, \quad \text{and} \\ s_i^\vee \lambda^\vee &= \lambda_1 \varepsilon_1^\vee + \dots + \lambda_{i-1} \varepsilon_{i-1}^\vee + \lambda_{i+1} \varepsilon_i^\vee + \lambda_i \varepsilon_{i+1}^\vee + \lambda_{i+2} \varepsilon_{i+2}^\vee + \dots + \lambda_n \varepsilon_n^\vee + \frac{k}{2} K, \end{aligned} \quad (5)$$

for $i \in \{1, \dots, n-1\}$. Let

$$s_{\varepsilon_1}^\vee = s_1^\vee \dots s_n^\vee \dots s_1^\vee, \quad h_{\varepsilon_1} = s_0^\vee s_{\varepsilon_1}^\vee = s_0^\vee s_1^\vee \dots s_n^\vee \dots s_1^\vee \quad \text{and} \quad h_{\varepsilon_{i+1}} = s_i^\vee h_{\varepsilon_i} s_i^\vee,$$

for $i \in \{1, \dots, n-1\}$. Then $h_{\varepsilon_1} \lambda^\vee = s_0^\vee s_{\varepsilon_1}^\vee \lambda^\vee = \lambda_1 \varepsilon_1^\vee + \dots + \lambda_n \varepsilon_n^\vee + \left(\frac{k}{2} - \lambda_1\right) K$ and

$$h_{\varepsilon_i} \lambda^\vee = \lambda_1 \varepsilon_1^\vee + \dots + \lambda_n \varepsilon_n^\vee + \left(\frac{k}{2} - \lambda_i\right) K, \quad \text{for } i \in \{1, \dots, n\}.$$

If $\gamma = \gamma_1 \varepsilon_1 + \cdots \gamma_n \varepsilon_n$ and $\lambda^\vee = \lambda_1 \varepsilon_1^\vee + \cdots + \lambda_n \varepsilon_n^\vee + \frac{k}{2} K$, then

$$h_\gamma \lambda^\vee = h_{\varepsilon_1}^{\gamma_1} \cdots h_{\varepsilon_n}^{\gamma_n} \lambda^\vee = \lambda^\vee + \left(\frac{k}{2} - (\lambda_1 \gamma_1 + \cdots + \lambda_n \gamma_n) \right) K = \lambda^\vee + \left(\frac{k}{2} - \langle \gamma, \lambda^\vee \rangle \right) K,$$

and special cases of this last formula are

$$h_\gamma \varepsilon_i^\vee = \varepsilon_i^\vee - \gamma_i K \quad \text{and} \quad h_\gamma K = K. \quad (6)$$

The set of coroots $S^\vee$ for type $(C_n^\vee, C_n)$ is the union of the five W_X -orbits given by

$$\begin{aligned} O_1^\vee &= W_X \cdot \alpha_n^\vee = W_X \cdot \varepsilon_n^\vee = \{\pm \varepsilon_i^\vee + rK \mid i \in \{1, \dots, n\}, r \in \mathbb{Z}\}, \\ 2O_1^\vee &= W_X \cdot 2\alpha_n^\vee = W_X \cdot 2\varepsilon_n^\vee = \{\pm 2\varepsilon_i^\vee + 2rK \mid i \in \{1, \dots, n\}, r \in \mathbb{Z}\}, \\ O_3^\vee &= W_X \cdot \alpha_0^\vee = W_X \cdot \left(-\varepsilon_1^\vee + \frac{1}{2}K \right) = \left\{ \pm \left(\varepsilon_i^\vee + \frac{1}{2}(2r+1)K \right) \mid i \in \{1, \dots, n\}, r \in \mathbb{Z} \right\}, \\ 2O_3^\vee &= W_X \cdot 2\alpha_0^\vee = W_X \cdot (-2\varepsilon_1^\vee + K) = \{\pm 2\varepsilon_i^\vee + (2r+1)K \mid i \in \{1, \dots, n\}, r \in \mathbb{Z}\}, \\ O_5^\vee &= W_X \cdot \alpha_1^\vee = W_X \cdot (\varepsilon_1^\vee - \varepsilon_2^\vee) = \left\{ \begin{array}{l} \pm(\varepsilon_i^\vee + \varepsilon_j^\vee) + rK, \\ \pm(\varepsilon_i^\vee - \varepsilon_j^\vee) + rK \end{array} \mid i, j \in \{1, \dots, n\}, i < j, r \in \mathbb{Z} \right\}, \end{aligned}$$

where

$$\begin{array}{ccccc} \alpha_0^\vee = -\varepsilon_1^\vee + \frac{1}{2}K & & \alpha_i^\vee = \varepsilon_i^\vee - \varepsilon_{i+1}^\vee & & \alpha_n^\vee = \varepsilon_n^\vee \\ 2\alpha_0^\vee = -2\varepsilon_1^\vee + K & \longleftrightarrow & & \longleftrightarrow & 2\alpha_n^\vee = 2\varepsilon_n^\vee \end{array} \quad (7)$$

REMARK 2.2. Throughout this paper, we present several diagrams imitating the Dynkin diagram with the labelling related to the coroots (like the one above), roots, or parameters. These are merely intended for conceptual association rather than to specify relations between the objects.

REMARK 2.3. In (7) and (9), we deviate from a common identification of the coroot space and the coroot space via a formula $\alpha^\vee = (2\alpha)/\langle \alpha, \alpha \rangle$. We avoid this formula in this paper as it conflicts with an ‘XY-symmetry’ that is special to the type $(C_n^\vee, C_n)$ case. Hopefully, the discomfort of deviating from the ‘standard’ check notation in root system literature will be alleviated by the advantages of avoiding some of the complications for the type $(C_n^\vee, C_n)$ case, which are dealt with in a case-by-case analysis in, for example, [13, (1.4.3), (1.5.1), (5.1.14), (5.1.16)] and other proofs in Chapter 5 of [13].

2.2.3. The affine roots for type $(C_n^\vee, C_n)$

Let

$$Q = \mathbb{Z}\text{-span} \left\{ \varepsilon_1, \dots, \varepsilon_n, \frac{1}{2}\delta \right\}$$

be the $\mathbb{Z}$ -vector space spanned by symbols $\varepsilon_1, \dots, \varepsilon_n$ and $\frac{1}{2}\delta$. The affine Weyl group W_Y is the group of $\mathbb{Z}$ -linear transformations of Q generated by the transformations $s_0, s_1, \dots, s_n$ given as follows: If $\mu =$

$\mu_1 \varepsilon_1 + \cdots + \mu_n \varepsilon_n + \frac{k}{2} \delta$, then

$$\begin{aligned} s_0 \mu &= -\mu_1 \varepsilon_1 + \mu_2 \varepsilon_2 + \cdots + \mu_n \varepsilon_n + \left(\frac{k}{2} + \lambda_1 \right) \delta, \\ s_n \mu &= \mu_1 \varepsilon_1 + \cdots + \mu_{n-1} \varepsilon_{n-1} - \mu_n \varepsilon_n + \frac{k}{2} \delta, \quad \text{and} \\ s_i \mu &= \mu \varepsilon_1 + \cdots + \mu_{i-1} \varepsilon_{i-1} + \mu_{i+1} \varepsilon_i + \mu_i \varepsilon_{i+1} + \mu_{i+2} \varepsilon_{i+2} + \cdots + \mu_n \varepsilon_n + \frac{k}{2} \delta, \end{aligned} \quad (8)$$

for $i \in \{1, \dots, n-1\}$. Let

$$s_{\varepsilon_1} = s_1 \cdots s_n \cdots s_1, \quad h_{\varepsilon_1^\vee} = s_0 s_{\varepsilon_1} = s_0 s_1 \cdots s_n \cdots s_1 \quad \text{and} \quad h_{\varepsilon_{i+1}^\vee} = s_i t_{\varepsilon_i^\vee} s_i,$$

for $i \in \{1, \dots, n-1\}$. Then $h_{\varepsilon_1^\vee} \mu = s_0 s_{\varepsilon_1} \mu = \mu_1 \varepsilon_1 + \cdots + \mu_n \varepsilon_n + \left(\frac{k}{2} - \mu_1 \right) \delta$ and

$$h_{\varepsilon_i^\vee} \mu = \mu_1 \varepsilon_1 + \cdots + \mu_n \varepsilon_n + \left(\frac{k}{2} - \mu_i \right) \delta, \quad \text{for } i \in \{1, \dots, n\}.$$

If $v^\vee = v_1 \varepsilon_1^\vee + \cdots + v_n \varepsilon_n^\vee$ and $\mu = \mu_1 \varepsilon_1 + \cdots + \mu_n \varepsilon_n + \frac{k}{2} \delta$, then

$$h_{v^\vee} \mu = h_{\varepsilon_1^\vee}^{v_1} \cdots h_{\varepsilon_n^\vee}^{v_n} \mu = \mu - \left(\frac{k}{2} + (\mu_1 v_1 + \cdots + \mu_n v_n) \right) \delta = \mu + \left(\frac{k}{2} - \langle \mu, v^\vee \rangle \right) \delta,$$

and special cases of this last formula are

$$h_{v^\vee} \varepsilon_i = \varepsilon_i - v_i \delta \quad \text{and} \quad h_{v^\vee} \delta = \delta.$$

The set of roots S for type $(C_n^\vee, C_n)$ is the union of the five W_Y -orbits given by

$$\begin{aligned} O_1 &= W_Y \cdot \alpha_n = W_Y \cdot \varepsilon_n = \{ \pm \varepsilon_i + r \delta \mid i \in \{1, \dots, n\}, r \in \mathbb{Z} \}, \\ 2O_1 &= O_2 = W_Y \cdot 2\alpha_n = W_Y \cdot 2\varepsilon_n = \{ \pm 2\varepsilon_i + 2r \delta \mid i \in \{1, \dots, n\}, r \in \mathbb{Z} \}, \\ O_3 &= W_Y \cdot \alpha_n = W_Y \cdot \left(-\varepsilon_1 + \frac{1}{2} \delta \right) = \left\{ \pm \left(\varepsilon_i + \frac{1}{2} (2r+1) \delta \right) \mid i \in \{1, \dots, n\}, r \in \mathbb{Z} \right\}, \\ 2O_3 &= O_4 = W_Y \cdot 2\alpha_n = W_Y \cdot (-2\varepsilon_1 + \delta) = \{ \pm 2\varepsilon_i + (2r+1) \delta \mid i \in \{1, \dots, n\}, r \in \mathbb{Z} \}, \\ O_5 &= W_Y \cdot \alpha_1 = W_Y \cdot (\varepsilon_1 - \varepsilon_2) = \left\{ \begin{array}{l} \pm(\varepsilon_i + \varepsilon_j) + r \delta \\ \pm(\varepsilon_i - \varepsilon_j) + r \delta \end{array} \mid i, j \in \{1, \dots, n\}, i < j, r \in \mathbb{Z} \right\}, \end{aligned}$$

where

$$\begin{array}{ccccccc} \alpha_0 = -\varepsilon_1 + \frac{1}{2} \delta & \circ \leftarrow & \alpha_i = \varepsilon_i - \varepsilon_{i+1} & \circ \cdots \cdots \circ & \alpha_n = \varepsilon_n & \circ \rightarrow & 2\alpha_n = 2\varepsilon_n \\ 2\alpha_0 = -2\varepsilon_1 + \delta & & & & & & \end{array} \quad (9)$$

2.3. Other classical types

With the notations as in section 2.2.3, each affine root system of classical type is a subset of S . The irreducible affine root systems of classical type (and the appropriate specializations for obtaining the Macdonald polynomials of each type from the Macdonald polynomials of type $(C_n^\vee, C_n)$) are given by the diagram in section 1.0.1. The middle notation for each root system is the notation in Macdonald [13, §1.3], the right notation is that of Bruhat-Tits [1] and the left notation is that of Kac [8, Ch. 6].

For instance, when both O_1 and $2O_1$ are present, then the factor corresponding to the root ε_n is

(The notation $\frac{1}{\Delta_{\varepsilon_n} \Delta_{2\varepsilon_n}}$ for this factor is as in [13, (5.1.3)]; the notation that we use for this factor in section 6.1 is $\kappa_{\varepsilon_n}^X$.) If only the orbit O_1 is present, then the factor is

If only the orbit $2O_1$ is present, then the factor is

In this way, the parameter specializations in the diagram in section 1.0.1 are determined by which orbits of roots are present in the root system. Using the parameters t_a defined in (13), the specialization is given by setting $t_a = 1$ when a is not in the W_Y -orbit of the smaller root system.

This section collects the tools for working with Koornwinder polynomials as polynomials in $x_1^{\pm 1}, \dots, x_n^{\pm 1}$ that depend on six parameters. The root system of type $(C_n^\vee, C_n)$ provides the structure for organizing the many symmetries between the variables and the various parameters, and this section specifies carefully the links to the root system. The c -functions, introduced in section 3.3, are a core structure to providing explicit formulas for Koornwinder expansions, specializations and norm formulas. With the notation for the c -functions in hand, section 3.4 briefly describes the relations of the DAHA. These relations provide a convenient summary of the calculus of the operators on polynomials, which are used in the rest of the paper.

Let

and let

$$\mathbb{K} = \mathbb{C}(q, t^{\frac{1}{2}}, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}) \quad \text{be the field of fractions of } \mathbb{C}[q, t^{\frac{1}{2}}, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}].$$

The field $\mathbb{K}$ will be the base field for most algebras in this paper.

Recalling the simple coroots and simple roots from (7) and (9), set

$$\begin{array}{ccccccc}
t_{\alpha_0^\vee} = u_n & \text{---} & t_{\alpha_i^\vee} = u_{\alpha_i^\vee} = t & \text{---} & t_{\alpha_n^\vee} = t_n \\
u_{\alpha_0^\vee} = u_0 & \text{---} & & \text{---} & & \text{---} & u_{\alpha_n^\vee} = t_0
\end{array} \quad (11)$$

and

$$\begin{array}{ccccc} t_{\alpha_0} = t_0 & & t_{\alpha_i} = u_{\alpha_i} = t & & t_{\alpha_n} = t_n \\ u_{\alpha_0} = u_0 & \circ \rightrightarrows \circ & \cdots & \circ \rightrightarrows \circ & u_{\alpha_n} = u_n \end{array} \quad (12)$$

Let $S^\vee$ be the set of affine coroots, and let S be the set of affine roots. Define $t_{\alpha^\vee}$, $u_{\alpha^\vee}$ and t_α , u_α for arbitrary coroots $\alpha^\vee$ and arbitrary roots α by requiring

$$t_{w\alpha} = t_\alpha \quad \text{and} \quad t_{w\alpha^\vee} = t_{\alpha^\vee}, \quad \text{for } w \in W, \quad \alpha^\vee \in S^\vee \quad \text{and} \quad \alpha \in S. \quad (13)$$

The difference between the parameters for the coroots in (11) and the parameters for the roots in (12) corresponds exactly to introduction of the ‘dual labels’ in [13, (1.5.1)].

3.2. The polynomial rings $\mathbb{K}[X]$ and $\mathbb{K}[Y]$

Let $\mathbb{K}[X] = \mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ be the Laurent polynomial ring in the variables $X_1, \dots, X_n$. Identify $\mathbb{K}[X]$ with the group algebra of $Q = \mathbb{Z}\text{-span}\{\varepsilon_1, \dots, \varepsilon_n, \frac{1}{2}\delta\}$ via the notations

$$q^{\frac{1}{2}} = X^{\frac{1}{2}\delta} \quad \text{and} \quad X_i = X^{\varepsilon_i}, \quad \text{and} \quad q^{\frac{k}{2}} X_1^{\mu_1} \cdots X_n^{\mu_n} = X^{\frac{k}{2}\delta + \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n} = X^\mu,$$

for $i \in \{1, \dots, n\}$ and $\mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n + \frac{k}{2}\delta \in Q$. The image of the simple roots in $\mathbb{K}[X]$ is given by

$$\begin{array}{ccccc} X^{\alpha_0} = q^{\frac{1}{2}} X_1^{-1} & & X^{\alpha_i} = X_i X_{i+1}^{-1} & & X^{\alpha_n} = X_n \\ \circ \rightrightarrows \circ & \cdots & \circ \rightrightarrows \circ & \cdots & \circ \rightrightarrows \circ \end{array} \quad (14)$$

Let $\mathbb{K}[Y] = \mathbb{K}[Y_1^{\pm 1}, \dots, Y_n^{\pm 1}]$ be the Laurent polynomial ring in the variables $Y_1, \dots, Y_n$. Identify $\mathbb{K}[Y]$ with the group algebra of $Q^\vee = \mathbb{Z}\text{-span}\{\varepsilon_1^\vee, \dots, \varepsilon_n^\vee, \frac{1}{2}K\}$ via the notations

$$q^{\frac{1}{2}} = Y^{-\frac{1}{2}K} \quad \text{and} \quad Y_i = Y^{\varepsilon_i^\vee}, \quad \text{and} \quad q^{-\frac{k}{2}} Y_1^{\lambda_1} \cdots Y_n^{\lambda_n} = Y^{\frac{k}{2}K + \lambda_1\varepsilon_1^\vee + \cdots + \lambda_n\varepsilon_n^\vee} = Y^{\lambda^\vee},$$

for $i \in \{1, \dots, n\}$ and $\lambda^\vee = \lambda_1\varepsilon_1^\vee + \cdots + \lambda_n\varepsilon_n^\vee + \frac{k}{2}K \in Q^\vee$. The image of the simple coroots in $\mathbb{K}[Y]$ is given by

$$\begin{array}{ccccc} Y^{\alpha_0^\vee} = q^{-\frac{1}{2}} Y_1^{-1} & & Y^{\alpha_i^\vee} = Y_i Y_{i+1}^{-1} & & Y^{\alpha_n^\vee} = Y_n \\ \circ \rightrightarrows \circ & \cdots & \circ \rightrightarrows \circ & \cdots & \circ \rightrightarrows \circ \end{array} \quad (15)$$

3.3. c -functions

Let $\mathbb{K}(Y)$ be the field of fractions of the Laurent polynomial ring $\mathbb{K}[Y]$. For a coroot $\alpha^\vee$, let (see [13, (4.2.2) and (4.3.9)])

$$c_{\alpha^\vee}^Y = t_{\alpha^\vee}^{-\frac{1}{2}} \frac{(1 - t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{\frac{1}{2}} Y^{\alpha^\vee})(1 + t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{-\frac{1}{2}} Y^{\alpha^\vee})}{(1 - Y^{2\alpha^\vee})} \quad \text{and} \quad \kappa_{\alpha^\vee}^Y = t_{\alpha^\vee}^{\frac{1}{2}} c_{\alpha^\vee}^Y.$$

$$\text{If } t_{\alpha^\vee}^{\frac{1}{2}} = u_{\alpha^\vee}^{\frac{1}{2}}, \quad \text{then} \quad c_{\alpha^\vee}^Y = \frac{t_{\alpha^\vee}^{-\frac{1}{2}} - t_{\alpha^\vee}^{\frac{1}{2}} Y^{\alpha^\vee}}{1 - Y^{\alpha^\vee}} \quad \text{and} \quad \kappa_{\alpha^\vee}^Y = \frac{1 - t_{\alpha^\vee} Y^{\alpha^\vee}}{1 - Y^{\alpha^\vee}}. \quad (16)$$

(More accurately, the function $c_{\alpha^\vee}^Y$ should be considered as a local factor of a c -function, see [22].) The expression $\kappa_{\alpha^\vee}^Y$ is a slightly renormalized version of the c -function $c_{\alpha^\vee}^Y$, which, although not technically necessary, is immensely helpful for making the formulas more palatable.

In general, for arbitrary $t_{\alpha^\vee}^{\frac{1}{2}}$ and $u_{\alpha^\vee}^{\frac{1}{2}}$,

$$\begin{aligned}
 c_{\alpha^\vee}^Y + c_{-\alpha^\vee}^Y &= \frac{(1 - t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{\frac{1}{2}} Y^{\alpha^\vee})(1 + t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{-\frac{1}{2}} Y^{\alpha^\vee})}{t_{\alpha^\vee}^{\frac{1}{2}}(1 - Y^{2\alpha^\vee})} + \frac{(1 - t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{\frac{1}{2}} Y^{-\alpha^\vee})(1 + t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{-\frac{1}{2}} Y^{-\alpha^\vee})}{t_{\alpha^\vee}^{\frac{1}{2}}(1 - Y^{-2\alpha^\vee})} \\
 &= \frac{(1 - t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{\frac{1}{2}} Y^{\alpha^\vee})(1 + t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{-\frac{1}{2}} Y^{\alpha^\vee})}{t_{\alpha^\vee}^{\frac{1}{2}}(1 - Y^{2\alpha^\vee})} + \frac{t_{\alpha^\vee}(1 - t_{\alpha^\vee}^{-\frac{1}{2}} u_{\alpha^\vee}^{-\frac{1}{2}} Y^{\alpha^\vee})(1 + t_{\alpha^\vee}^{-\frac{1}{2}} u_{\alpha^\vee}^{\frac{1}{2}} Y^{\alpha^\vee})}{t_{\alpha^\vee}^{\frac{1}{2}}(1 - Y^{2\alpha^\vee})} \\
 &= \frac{1 - t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{\frac{1}{2}} Y^{\alpha^\vee} + t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{-\frac{1}{2}} Y^{\alpha^\vee} - t_{\alpha^\vee} Y^{2\alpha^\vee} + t_{\alpha^\vee} - t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{-\frac{1}{2}} Y^{\alpha^\vee} + t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{\frac{1}{2}} Y^{\alpha^\vee} - Y^{2\alpha^\vee}}{t_{\alpha^\vee}^{\frac{1}{2}}(1 - Y^{2\alpha^\vee})} \\
 &= \frac{(1 + t_{\alpha^\vee})(1 - Y^{2\alpha^\vee})}{t_{\alpha^\vee}^{\frac{1}{2}}(1 - Y^{2\alpha^\vee})} = t_{\alpha^\vee}^{\frac{1}{2}} + t_{\alpha^\vee}^{-\frac{1}{2}}. \tag{17}
 \end{aligned}$$

Let $w \in W$ and let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced word for w . The *coroot sequence* of the reduced word $w = s_{i_1} \cdots s_{i_\ell}$ is

$$\text{the sequence } (\beta_k^\vee \mid k \in \{1, \dots, \ell\} \text{ and } i_k \neq \pi) \text{ given by } \beta_k^\vee = s_{i_\ell} \cdots s_{i_{k+1}} \alpha_{i_k}^\vee. \tag{18}$$

Then define

$$\kappa_w^Y = \prod_{\beta^\vee \in \text{Inv}(w)} \kappa_{\beta^\vee}^Y, \quad \text{where } \text{Inv}(w) = \{\beta_1^\vee, \dots, \beta_\ell^\vee\} \tag{19}$$

is the set of elements in a coroot sequence for a reduced word for w . If $w = uv$ with $\ell(u) + \ell(v) = \ell(w)$, then the coroot sequence of w is v^{-1} times the coroot sequence of u followed by the coroot sequence of v so that

$$\text{Inv}(uv) = v^{-1} \text{Inv}(u) \cup \text{Inv}(v).$$

The inversions of elements of W_{fin} come in two types: ‘droite’ and ‘standard’. The ‘droite’ inversions are inversions for which the coroot is in the W_{fin} -orbit of $\alpha_n^\vee$ [which is on the right end of the diagram in (15)], and the ‘standard’ inversions are inversions for which the coroot is in the W_{fin} -orbit of $\alpha_{n-1}^\vee$ [the middle part of the diagram in (15)]. Using the indicators d for ‘droite’ and s for ‘standard’, let

$$(S_{0,d}^\vee)^+ = \{\varepsilon_1^\vee, \dots, \varepsilon_n^\vee\} \quad \text{and} \quad (S_{0,s}^\vee)^+ = \left\{ \begin{array}{l} \varepsilon_i^\vee - \varepsilon_j^\vee, \\ \varepsilon_i^\vee + \varepsilon_j^\vee \end{array} \mid i, j \in \{1, \dots, n\} \text{ with } i < j \right\},$$

and for $v \in W_{\text{fin}}$, define

$$\text{Inv}_d(v) = \text{Inv}(v) \cap (S_{0,d}^\vee)^+ \quad \text{and} \quad \text{Inv}_s(v) = \text{Inv}(v) \cap (S_{0,s}^\vee)^+.$$

For $v \in W_{\text{fin}}$, define

$$\begin{aligned}\kappa_v^+ &= \left(\prod_{\beta^\vee \in \text{Inv}_s(v)} \kappa_{\beta^\vee}^Y \right) \left(\prod_{\beta^\vee \in \text{Inv}_d(v)} \kappa_{\beta^\vee}^Y \right), \quad \kappa_v^\pm = \left(\prod_{\beta^\vee \in \text{Inv}_s(v)} \kappa_{\beta^\vee}^Y \right) \left(\prod_{\beta^\vee \in \text{Inv}_d(v)} \kappa_{\beta^\vee}^{Y^{-1}} \right), \\ \kappa_v^- &= \left(\prod_{\beta^\vee \in \text{Inv}_s(v)} \kappa_{\beta^\vee}^{Y^{-1}} \right) \left(\prod_{\beta^\vee \in \text{Inv}_d(v)} \kappa_{\beta^\vee}^{Y^{-1}} \right), \quad \kappa_v^\mp = \left(\prod_{\beta^\vee \in \text{Inv}_s(w)} \kappa_{\beta^\vee}^{Y^{-1}} \right) \left(\prod_{\beta^\vee \in \text{Inv}_d(w)} \kappa_{\beta^\vee}^Y \right).\end{aligned}\tag{20}$$

Finally, define also

$$\kappa_{\text{dr}}^Y = \prod_{i=1}^n \kappa_{\varepsilon_i^Y}^Y = \prod_{i=1}^n \frac{(1 - t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} Y_i)(1 + t_n^{\frac{1}{2}} t_0^{-\frac{1}{2}} Y_i)}{(1 - Y_i^2)}$$

and

$$\kappa_{\text{st}}^Y = \prod_{1 \leq i < j \leq n} \kappa_{\varepsilon_i^Y - \varepsilon_j^Y}^Y \kappa_{\varepsilon_i^Y + \varepsilon_j^Y}^Y = \prod_{1 \leq i < j \leq n} \frac{(1 - t Y_i Y_j^{-1})}{(1 - Y_i Y_j^{-1})} \frac{(1 - t Y_i Y_j)}{(1 - Y_i Y_j)}\tag{21}$$

so that

$$\kappa_{w_0}^Y = \kappa_{\text{dr}}^Y \kappa_{\text{st}}^Y.\tag{22}$$

The product κ_{dr} is the product of the $\kappa_{\beta^\vee}$ over all ‘droite’ coroots (coroots that are in the W_{fin} -orbit of $\alpha_n^\vee$), and κ_{st} is the product of the $\kappa_{\beta^\vee}$ over all ‘standard’ coroots (coroots that are in the W_{fin} -orbit of $\alpha_{n-1}^\vee$).

This section has presented the c -functions and related functions $\kappa_{\beta^\vee}^Y$ in terms of the $\{Y_1, \dots, Y_n\}$ variables. We will also consider these notions in other sets of variables, such as $\{Y_1^{-1}, \dots, Y_n^{-1}\}$, $\{X_1, \dots, X_n\}$ and $\{X_1^{-1}, \dots, X_n^{-1}\}$, and use notations such as $\kappa_{\beta^\vee}^{Y^{-1}}$, κ_β^X and $\kappa_\beta^{X^{-1}}$, respectively. For example,

$$\kappa_{w_0}^{X^{-1}} = \left(\prod_{i=1}^n \frac{(1 - t_n^{\frac{1}{2}} u_n^{\frac{1}{2}} X_i^{-1})(1 + t_n^{\frac{1}{2}} u_n^{-\frac{1}{2}} X_i^{-1})}{(1 - X_i^{-2})} \right) \left(\prod_{1 \leq i < j \leq n} \frac{(1 - t X_i^{-1} X_j)(1 - t X_i^{-1} X_j^{-1})}{(1 - X_i^{-1} X_j)(1 - X_i^{-1} X_j^{-1})} \right).\tag{23}$$

3.4. The algebras $\tilde{H}_{\text{loc}}$ and $\tilde{H}_{\text{int}}$

The Koornwinder polynomials are elements of the polynomial ring $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, which are characterized, up to normalization, by the fact that they are eigenvectors for the Cherednik–Dunkl operators $Y_1, \dots, Y_n$ [see (46)]. However, the c -functions, which form the core calculus for working with Koornwinder polynomials, are elements of $\mathbb{K}(X)$, the field of fractions of the polynomial ring. Thus extending from $\mathbb{K}[X]$ to $\mathbb{K}(X)$ is necessary for handling the tools.

The Cherednik–Dunkl operators are elements of the DAHA $\tilde{H}_{\text{int}}$, which is formed by pasting the two polynomial rings $\mathbb{K}[X]$ and $\mathbb{K}[Y]$ together with a finite Hecke algebra H_{fin} . However, the right home for the c -functions and the operators for creating Koornwinder polynomials is a larger algebra $\tilde{H}_{\text{loc}}$, which extends the algebra $\tilde{H}_{\text{int}}$ by extending $\mathbb{K}[X]$ and $\mathbb{K}[Y]$ to the fraction fields $\mathbb{K}(X)$ and $\mathbb{K}(Y)$. In this section, we introduce the algebras $\tilde{H}_{\text{loc}}$ and $\tilde{H}_{\text{int}}$ by generators and relations.

Let $\mathbb{K}(X)$ and $\mathbb{K}(Y)$ be the fraction fields of $\mathbb{K}[X]$ and $\mathbb{K}[Y]$, respectively. Recall that $\mathbb{K}[X]$ is the group algebra of Q and $\mathbb{K}[Y]$ is the group algebra of $Q^\vee$, that

$$X^{\frac{1}{2}\delta} = q \quad \text{and} \quad Y^{-\frac{1}{2}K} = q^{\frac{1}{2}}$$

and that W_X acts on $Q^\vee$ and W_Y acts on Q by the formulas given in (5) and (8) so that

$$wX^\mu = X^{w\mu} \quad \text{and} \quad zY^{\lambda^\vee} = Y^{z\lambda^\vee}, \quad (24)$$

for $w \in W_Y, z \in W_X, \lambda^\vee \in Q^\vee$ and $\mu \in Q$.

Let $\tilde{H}_{\text{loc}}$ be the $\mathbb{K}$ -algebra generated by $\eta_{s_0}^\vee, \dots, \eta_{s_n}^\vee, \xi_{s_0}, \dots, \xi_{s_n}, T_1, \dots, T_n, T_{\alpha_0}, T_{\alpha_1}, \dots, T_{\alpha_n}$ and $T_{\alpha_0^\vee}, T_{\alpha_1^\vee}, \dots, T_{\alpha_n^\vee}$ and $T_{s_{e_1}}$ and $\mathbb{K}(X)$ and $\mathbb{K}(Y)$ with relations

$$\begin{array}{ccc} \eta_{s_0}^\vee & \eta_{s_1}^\vee & \eta_{s_2}^\vee & \dots & \eta_{s_{n-2}}^\vee & \eta_{s_{n-1}}^\vee & \eta_{s_n}^\vee & \eta_{s_i}^2 = 1, & \eta_{s_i} Y^{\lambda^\vee} = Y^{s_i \lambda^\vee} \eta_{s_i} \\ \xi_{s_0} & \xi_{s_1} & \xi_{s_2} & \dots & \xi_{s_{n-2}} & \xi_{s_{n-1}} & \xi_{s_n} & \xi_{s_i}^2 = 1, & \xi_{s_i} X^\mu = X^{s_i \mu} \xi_{s_i}, \end{array} \quad (25)$$

for $i \in \{0, 1, \dots, n\}, \lambda^\vee \in Q^\vee$ and $\mu \in Q$, and

$$T_{\alpha_i^\vee} + t_{\alpha_i^\vee}^{-\frac{1}{2}} = (1 + \eta_{s_i}^\vee) c_{\alpha_i^\vee}^\vee \quad \text{and} \quad T_{\alpha_i} + t_{\alpha_i}^{-\frac{1}{2}} = (1 + \xi_{s_i}) c_{\alpha_i}^{X^{-1}}, \quad \text{for } i \in \{0, 1, \dots, n\}, \quad (26)$$

$$T_j = T_{\alpha_j} = T_{\alpha_j^\vee} \quad \text{for } j \in \{1, \dots, n\} \quad \text{and} \quad T_{s_{e_1}} = T_1 \cdots T_{n-1} T_n T_{n-1} \cdots T_1, \quad (27)$$

$$Y^{\varepsilon_1^\vee} = T_{\alpha_0} T_{s_{e_1}} \quad X^{\varepsilon_1} = (T_{\alpha_0^\vee})^{-1} T_{s_{e_1}}^{-1}, \quad (28)$$

$$Y^{\varepsilon_{j+1}^\vee} = T_{\alpha_j}^{-1} Y^{\varepsilon_j^\vee} T_{\alpha_j}^{-1}, \quad \text{and} \quad X^{\varepsilon_{j+1}} = T_{\alpha_j} X^{\varepsilon_j} T_{\alpha_j}, \quad \text{for } j \in \{1, \dots, n-1\}, \quad (29)$$

$$Y^{-\varepsilon_1^\vee} X^{\varepsilon_1} = q^{\frac{1}{2}} \left(u_0^{\frac{1}{2}} - u_0^{-\frac{1}{2}} \right) T_{s_{e_1}}^{-1} + q T_{s_{e_1}}^{-1} X^{-\varepsilon_1} Y^{\varepsilon_1} T_{s_{e_1}}^{-1}. \quad (30)$$

This presentation of $\tilde{H}_{\text{loc}}$ is not minimal as there are many redundant generators and many redundant relations. It is designed to specify notations and list the relations that we will need, and to motivate the operators on polynomials, which are the main tools for working with Macdonald polynomials in general. To be precise, since

$$(1 + \xi_{s_i}) c_{\alpha_i}^{X^{-1}} = T_{\alpha_i} + t_{\alpha_i}^{-\frac{1}{2}} = (T_{\alpha_i} - t_{\alpha_i}^{\frac{1}{2}}) + t_{\alpha_i}^{-\frac{1}{2}} + t_{\alpha_i}^{\frac{1}{2}} = (T_{\alpha_i} - t_{\alpha_i}^{\frac{1}{2}}) + c_{\alpha_i}^{X^{-1}} + c_{-\alpha_i}^{X^{-1}}$$

then

$$T_{\alpha_i} - t^{\frac{1}{2}} = -c_{-\alpha_i}^{X^{-1}} + \xi_{s_i} c_{\alpha_i}^{X^{-1}} = -c_{-\alpha_i}^{X^{-1}} (1 - \xi_{s_i}) = -c_{\alpha_i}^X (1 - \xi_{s_i}), \quad (31)$$

and this is the formula used to define the action of the DAHA on $\mathbb{K}[X]$ in (44).

The DAHA is the subalgebra $\tilde{H}_{\text{int}}$ inside $\tilde{H}_{\text{loc}}$

generated by $X_1^{\pm 1}, \dots, X_n^{\pm 1}, Y_1^{\pm 1}, \dots, Y_n^{\pm 1}$, and $T_1, \dots, T_n$.

The algebra $\tilde{H}_{\text{int}}$ is an integral form of $\tilde{H}_{\text{loc}}$ (alternatively, the algebra $\tilde{H}_{\text{loc}}$ is a completion, or localized form, of $\tilde{H}_{\text{int}}$). A common definition of the DAHA uses the relations listed in the following proposition, which follows without difficulty from the defining relations of $\tilde{H}_{\text{loc}}$.

PROPOSITION 3.1. *Let $i \in \{0, 1, \dots, n\}, \lambda^\vee \in Q^\vee$ and $\mu \in Q$. Let $T_0^\# = q^{-\frac{1}{2}} T_{\alpha_0^\vee}^{-1} T_{s_{e_1}}^{-1} T_{\alpha_0}^{-1}$.*

Then

$$\begin{array}{ccc} T_{\alpha_0^\vee} & T_{\alpha_1^\vee} & T_{\alpha_2^\vee} & \dots & T_{\alpha_{n-2}^\vee} & T_{\alpha_{n-1}^\vee} & T_{\alpha_n^\vee} & T_{\alpha_0} & T_{\alpha_1} & T_{\alpha_2} & \dots & T_{\alpha_{n-2}} & T_{\alpha_{n-1}} & T_{\alpha_n} \end{array} \quad (32)$$

$$T_{\alpha_0^\vee} T_{\alpha_1}^{-1} T_{\alpha_0} T_{\alpha_1} = T_{\alpha_1}^{-1} T_{\alpha_0} T_{\alpha_1} T_{\alpha_0^\vee}, \quad \left(T_0^\# - u_0^{\frac{1}{2}}\right) \left(T_0^\# + u_0^{\frac{1}{2}}\right) = 0, \quad (33)$$

$$\begin{aligned} \left(T_{\alpha_i^\vee} - t_{\alpha_i^\vee}^{\frac{1}{2}}\right) \left(T_{\alpha_i^\vee} + t_{\alpha_i^\vee}^{-\frac{1}{2}}\right) = 0 \quad \text{and} \quad T_{\alpha_i^\vee} Y^{\lambda^\vee} = Y^{s_i^\vee \lambda^\vee} T_{\alpha_i^\vee} \\ + \left(c_{\alpha_i^\vee}^Y - t_{\alpha_i^\vee}^{-\frac{1}{2}}\right) \left(Y^{\lambda^\vee} - Y^{s_i^\vee \lambda^\vee}\right), \end{aligned} \quad (34)$$

$$\begin{aligned} \left(T_{\alpha_i} - t_{\alpha_i}^{\frac{1}{2}}\right) \left(T_{\alpha_i} + t_{\alpha_i}^{-\frac{1}{2}}\right) = 0 \quad \text{and} \quad T_{\alpha_i} X^{-\mu} = X^{-s_i \mu} T_{\alpha_i} \\ + \left(c_{\alpha_i}^{X^{-1}} - t_{\alpha_i}^{-\frac{1}{2}}\right) \left(X^{-\mu} - X^{-s_i \mu}\right). \end{aligned} \quad (35)$$

REMARK 3.1. In personal communication, J. Stokman insightfully points out that the algebra $\tilde{H}_{\text{loc}}$ defined in (25)–(30) is fishy. One can Ore-localize DAHA in either the X -elements (so that the normalized intertwiners ξ_{s_j} are in the localized algebra) or in the Y -elements (so that the η_{s_j} are in the ones in the localized algebra) but not simultaneously. This difficulty is alluded to in a different form in [4, Remark 3.5]. However, trying to set up an accurate formal framework for handling an X – Y -localized algebra would be distracting from the combinatorial perspective of this paper. We feel that, even if the right localization formalism is not in place, the concept of an algebra $\tilde{H}_{\text{loc}}$ that contains all the useful relations for computations with these operators is a healthy point of view and so we have chosen to include it. The algebra $\tilde{H}_{\text{loc}}$ is not absolutely necessary for the results in this paper as the operators on the polynomial representation that we use are well defined, and the proofs that we give are valid.

3.5. Symmetrizers

There are four ways of symmetrizing/antisymmetrizing in the Koornwinder polynomial context, corresponding to the four 1-dimensional representations of W_{fin} . These 1-dimensional representations are the analogues of the usual sign of a permutation. The four symmetrizers, and useful formulas for them, are presented in sections 3.5, 3.6 and 3.7. The symmetrizers will be used in section 5 to construct and manage the bosonic (symmetric), fermionic (antisymmetric) and the two mesonic (half symmetric–half antisymmetric) versions of the Koornwinder polynomials. These four symmetrized/antisymmetrized versions of Koornwinder polynomials, and the relations between them, turn out to be fundamental in the proof of the norm formulas and constant term formulas that are established in section 6.

The finite Hecke algebra H_{fin} is the $\mathbb{K}$ -subalgebra of $\tilde{H}_{\text{int}}$ generated by $T_1, \dots, T_{n-1}$ and T_n . The finite Hecke algebra

$$H_{\text{fin}} \quad \text{has } \mathbb{K}\text{-basis} \quad \{T_\nu \mid \nu \in W_{\text{fin}}\}, \quad (36)$$

where $T_\nu = T_{i_1} \dots T_{i_k}$ if $\nu = s_{i_1} \dots s_{i_k}$ is a reduced word for ν in W_{fin} . The four 1-dimensional representations of H_{fin} are

$$\chi^+ : H_{\text{fin}} \rightarrow \mathbb{K}, \quad \chi^\pm : H_{\text{fin}} \rightarrow \mathbb{K}, \quad \chi^\mp : H_{\text{fin}} \rightarrow \mathbb{K}, \quad \chi^- : H_{\text{fin}} \rightarrow \mathbb{K},$$

given by

$$\begin{aligned} \chi^+(T_i) &= \begin{cases} t^{\frac{1}{2}}, & \text{if } i \in \{1, \dots, n-1\}, \\ t_n^{\frac{1}{2}}, & \text{if } i = n, \end{cases} & \chi^\pm(T_i) &= \begin{cases} t^{\frac{1}{2}}, & \text{if } i \in \{1, \dots, n-1\}, \\ (-t_n)^{-\frac{1}{2}}, & \text{if } i = n, \end{cases} \\ \chi^\mp(T_i) &= \begin{cases} (-t)^{-\frac{1}{2}}, & \text{if } i \in \{1, \dots, n-1\}, \\ t_n^{\frac{1}{2}}, & \text{if } i = n, \end{cases} & \chi^-(T_i) &= \begin{cases} (-t)^{-\frac{1}{2}}, & \text{if } i \in \{1, \dots, n-1\}, \\ (-t_n)^{-\frac{1}{2}}, & \text{if } i = n. \end{cases} \end{aligned}$$

For $v \in W_{\text{fin}}$, define $\ell_s(v)$ and $\ell_d(v)$ by

$$\chi^+(T_v) = \left(t^{\frac{1}{2}}\right)^{\ell_s(v)} \left(t_n^{\frac{1}{2}}\right)^{\ell_d(v)}.$$

The Hecke symmetrizers are

the elements ε_+ , $\varepsilon_{\pm}$, $\varepsilon_{\mp}$ and ε_- of H_{fin} ,

which are defined such that, in terms of the basis in (36), the coefficient of T_{w_0} is 1 and for $w \in W_{\text{fin}}$,

$$T_w \varepsilon_+ = \chi^+(T_w) \varepsilon_+, \quad T_w \varepsilon_{\pm} = \chi^{\pm}(T_w) \varepsilon_{\pm}, \quad T_w \varepsilon_{\mp} = \chi^{\mp}(T_w) \varepsilon_{\mp}, \quad T_w \varepsilon_- = \chi^-(T_w) \varepsilon_-.$$

In other words, if $\Xi \in \{+, \pm, \mp, -\}$, then $T_w \varepsilon_{\Xi} = \chi^{\Xi}(T_w) \varepsilon_{\Xi}$.

A reduced word for the longest element of W_{fin} is

$$w_0 = (s_1 \cdots s_n \cdots s_1)(s_2 \cdots s_n \cdots s_2) \cdots (s_{n-1} s_n s_{n-1}) s_n \quad \text{and} \quad t^{\frac{1}{2}\ell_s(w_0)} t_n^{\frac{1}{2}\ell_d(w_0)} = t^{\frac{1}{2}n(n-1)} t_n^{\frac{1}{2}n}.$$

In terms of the basis in (36), the symmetrizers are given explicitly by

$$\varepsilon_{\Xi} = \frac{1}{\chi^{\Xi}(T_{w_0})} \sum_{v \in W_{\text{fin}}} \chi^{\Xi}(T_v) T_v, \quad \text{for } \Xi \in \{+, \pm, \mp, -\}. \quad (37)$$

The Poincaré polynomial for W_{fin} is

$$W_0(t, t_n) = \sum_{w \in W_{\text{fin}}} t^{\ell_s(w)} t_n^{\ell_d(w)} = \sum_{w \in W_{\text{fin}}} \chi^+(T_w)^2. \quad (38)$$

Three alternate formulas for $W_0(t, t_n)$ are given in Proposition 5.2. Then

$$\begin{aligned} \varepsilon_+^2 &= \frac{1}{\chi^+(T_{w_0})} W_0(t, t_n) \varepsilon_+, & \varepsilon_{\pm}^2 &= \frac{1}{\chi^{\pm}(T_{w_0})} W_0(t, t_n^{-1}) \varepsilon_{\pm}, \\ \varepsilon_-^2 &= \frac{1}{\chi^-(T_{w_0})} W_0(t^{-1}, t_n^{-1}) \varepsilon_-, & \varepsilon_{\mp}^2 &= \frac{1}{\chi^{\mp}(T_{w_0})} W_0(t^{-1}, t_n) \varepsilon_{\mp}. \end{aligned} \quad (39)$$

EXAMPLE 3.1. For $\Xi = \pm$,

$$\varepsilon_{\pm} = \frac{1}{\chi^{\pm}(T_{w_0})} \sum_{w \in W_{\text{fin}}} \chi^{\pm}(T_w) T_w = t^{-\frac{1}{2}\ell_s(w_0)} \left(-t_n^{\frac{1}{2}}\right)^{\ell_d(w_0)} \sum_{w \in W_{\text{fin}}} t^{\frac{1}{2}\ell_s(w)} \left(-t_n^{-\frac{1}{2}}\right)^{\ell_d(w)} T_w,$$

and since $T_w \varepsilon_{\pm} = \chi^{\pm}(T_w) \varepsilon_{\pm}$, then

$$\varepsilon_{\pm}^2 = \frac{1}{\chi^{\pm}(T_{w_0})} \sum_{w \in W_{\text{fin}}} \chi^{\pm}(T_w)^2 \varepsilon_{\pm} = \frac{1}{\chi^{\pm}(T_{w_0})} W_0(t, t_n^{-1}) \varepsilon_{\pm}.$$

3.6. c-function formulas for symmetrizers

The definition of the Hecke symmetrizers and the formulas for them given in section 3.5 are purely in terms of the T_w in the DAHA $\tilde{H}_{\text{int}}$. However, the Koornwinder polynomials are more naturally constructed and managed with the η_w and ξ_w that are in $\tilde{H}_{\text{loc}}$ and so it becomes desirable to have expressions for the symmetrizers that are in terms of the η_w and ξ_w (and c -functions). Perhaps surprisingly, these conversion formulas, presented in Proposition 3.2, are compact and elegant (and useful!). We will use them in section 5 to provide formulas for the symmetric (bosonic), antisymmetric (fermionic) and half symmetric–half antisymmetric (mesonic) Koornwinder polynomials.

For $w \in W_{\text{fin}}$, let

$$\xi_w = \xi_{s_{i_1}} \cdots \xi_{s_{i_\ell}} \quad \text{and} \quad \eta_w = \eta_{s_{i_1}^\vee} \cdots \eta_{s_{i_\ell}^\vee},$$

if $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced word for w . There are four X -symmetrizers

$$\begin{aligned} e_+^X &= \sum_{w \in W_{\text{fin}}} \xi_w, & e_-^X &= \sum_{w \in W_{\text{fin}}} (-1)^{\ell_s(w) + \ell_d(w)} \xi_w, \\ e_\pm^X &= \sum_{w \in W_{\text{fin}}} (-1)^{\ell_d(w)} \xi_w, & e_\mp^X &= \sum_{w \in W_{\text{fin}}} (-1)^{\ell_s(w)} \xi_w, \end{aligned}$$

and four Y -symmetrizers

$$\begin{aligned} e_+^Y &= \sum_{w \in W_{\text{fin}}} \eta_w, & e_-^Y &= \sum_{w \in W_{\text{fin}}} (-1)^{\ell_s(w) + \ell_d(w)} \eta_w, \\ e_\pm^Y &= \sum_{w \in W_{\text{fin}}} (-1)^{\ell_d(w)} \eta_w, & e_\mp^Y &= \sum_{w \in W_{\text{fin}}} (-1)^{\ell_s(w)} \eta_w. \end{aligned}$$

The following proposition writes the Hecke symmetrizers in terms of the X -symmetrizers and the Y -symmetrizers.

PROPOSITION 3.2. *Let $\kappa_{\text{st}}^X, \kappa_{\text{dr}}^X$ be as defined in (21).*

$$\begin{aligned} \chi^+(T_{w_0})\varepsilon_+ &= e_+^X \kappa_{\text{st}}^{X^{-1}} \kappa_{\text{dr}}^{X^{-1}} = e_+^Y \kappa_{\text{st}}^Y \kappa_{\text{dr}}^Y & \text{and } \chi^+(T_{w_0})\varepsilon_- &= \kappa_{\text{st}}^X \kappa_{\text{dr}}^X e_-^X = \kappa_{\text{st}}^{Y^{-1}} \kappa_{\text{dr}}^{Y^{-1}} e_-^Y, \\ \chi^+(T_{w_0})\varepsilon_\pm &= \kappa_{\text{dr}}^X e_\pm^X \kappa_{\text{st}}^{X^{-1}} = \kappa_{\text{dr}}^{Y^{-1}} e_\pm^Y \kappa_{\text{st}}^Y & \text{and } \chi^+(T_{w_0})\varepsilon_\mp &= \kappa_{\text{st}}^X e_\mp^X \kappa_{\text{dr}}^{X^{-1}} = \kappa_{\text{st}}^{Y^{-1}} e_\mp^Y \kappa_{\text{dr}}^Y. \end{aligned}$$

Proof. Let us prove the formula $\chi^+(T_{w_0})\varepsilon_\pm = \kappa_{\text{dr}}^X e_\pm^X \kappa_{\text{st}}^{X^{-1}}$. The proof for the other cases is similar.

Let

$$R_\pm = \kappa_{\text{dr}}^X e_\pm^X \kappa_{\text{st}}^{X^{-1}}.$$

For $i \in \{1, \dots, n-1\}$, and using (31),

$$\begin{aligned} (T_i - t^{\frac{1}{2}})R_\pm &= (T_i - t^{\frac{1}{2}}) \kappa_{\text{dr}}^X e_\pm^X \kappa_{\text{st}}^{X^{-1}} = -c_{\alpha_i}^X (1 - \xi_{s_i}) \kappa_{\text{dr}}^X e_\pm^X \kappa_{\text{st}}^{X^{-1}} \\ &= -\kappa_{\text{dr}}^X c_{\alpha_i}^X (1 - \xi_{s_i}) e_\pm^X \kappa_{\text{st}}^{X^{-1}} = -\kappa_{\text{dr}}^X c_{\alpha_i}^X \cdot 0 \cdot \kappa_{\text{st}}^{X^{-1}} = 0, \quad \text{so that } T_i R_\pm = t^{\frac{1}{2}} R_\pm. \end{aligned}$$

Using (26),

$$\begin{aligned} (T_n + t_n^{-\frac{1}{2}})R_\pm &= (T_n + t_n^{-\frac{1}{2}}) \kappa_{\text{dr}}^X e_\pm^X \kappa_{\text{st}}^{X^{-1}} = (1 + \xi_{s_n}) c_{\alpha_n}^{X^{-1}} \kappa_{\text{dr}}^X e_\pm^X \kappa_{\text{st}}^{X^{-1}} = (1 + \xi_{s_n}) \kappa_{\text{dr}}^X c_{\alpha_n}^{X^{-1}} e_\pm^X \kappa_{\text{st}}^{X^{-1}} \\ &= \kappa_{\text{dr}}^X \left(1 + \frac{c_{\alpha_n}^{X^{-1}}}{c_{\alpha_n}^X} \xi_{s_n} \right) c_{\alpha_n}^{X^{-1}} e_\pm^X \kappa_{\text{st}}^{X^{-1}} = \kappa_{\text{dr}}^X \left(c_{\alpha_n}^{X^{-1}} + \frac{c_{\alpha_n}^{X^{-1}}}{c_{\alpha_n}^X} c_{\alpha_n}^X \xi_{s_n} \right) e_\pm^X \kappa_{\text{st}}^{X^{-1}} \\ &= \kappa_{\text{dr}}^X \left(c_{\alpha_n}^{X^{-1}} + c_{\alpha_n}^{X^{-1}} \xi_{s_n} \right) e_\pm^X \kappa_{\text{st}}^{X^{-1}} = \kappa_{\text{dr}}^X \left(c_{\alpha_n}^{X^{-1}} - c_{\alpha_n}^{X^{-1}} \right) e_\pm^X \kappa_{\text{st}}^{X^{-1}} = 0, \end{aligned}$$

so that $T_n R_\pm = -t_n^{-\frac{1}{2}} R_\pm$.

Since $T_i = \xi_{s_i} c_{\alpha_i}^{X^{-1}} + (c_{\alpha_i}^{X^{-1}} - t^{-\frac{1}{2}})$ and the coefficient of T_{w_0} in $\varepsilon_\pm$ is 1, then there are rational functions $a_w^{X^{-1}}$ such that

$$\begin{aligned} \chi^+(T_{w_0})\varepsilon_\pm &= \chi^+(T_{w_0}) \sum_{w \in W_{\text{fin}}} \xi_w a_w^{X^{-1}} = \chi^+(T_{w_0}) \xi_{w_0} c_{w_0}^{X^{-1}} + \chi^+(T_{w_0}) \sum_{w' < w_0} \xi_{w'} a_{w'}^{X^{-1}} \\ &= \kappa_{\text{dr}}^X \xi_{w_0} \kappa_{\text{st}}^{X^{-1}} + \chi^+(T_{w_0}) \sum_{w' < w_0} \xi_{w'} a_{w'}^{X^{-1}}. \end{aligned}$$

The element $\varepsilon_\pm \in H_{\text{fin}}$ is determined by the conditions that the coefficient of T_{w_0} is 1, and $T_n \varepsilon_\pm = -t_n^{-\frac{1}{2}} \varepsilon_\pm$ and $T_i \varepsilon_\pm = t^{\frac{1}{2}} \varepsilon_\pm$ for $i \in \{1, \dots, n-1\}$. So $\chi^+(T_{w_0})\varepsilon_\pm = R_\pm$. $\square$

3.7. Symmetrizers and stabilizers

The finite Weyl group W_{fin} acts on $\mathbb{Z}^n$ by the formulas in (4). Since the action of W_{fin} on $\mathbb{Z}^n$ is not free, there are elements of $\mathbb{Z}^n$ that have non-trivial stabilizers, and one is forced to confront these stabilizers. This section computes formulas for the symmetrizers that take into account, and allow us to manage, the cases when the stabilizer is non-trivial.

Let

$$(\mathbb{Z}_{\geq 0}^n)^+ = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \dots \geq \lambda_n \geq 0\}.$$

The set $(\mathbb{Z}_{\geq 0}^n)^+$ is a set of representatives of the W_{fin} -orbits on $\mathbb{Z}^n$. For $\lambda \in (\mathbb{Z}_{\geq 0}^n)^+$, let

$$W_\lambda = \{w \in W_{\text{fin}} \mid w\lambda = \lambda\} \quad \text{and} \quad W^\lambda = \left\{ \begin{array}{l} \text{minimal length representatives} \\ \text{of cosets in } W_{\text{fin}}/W_\lambda \end{array} \right\}.$$

Let w_λ be the longest element of W_λ , and let v_λ be the maximal element of W^λ . As in (3), then v_λ is the minimal length element of W_{fin} such that $v_\lambda \lambda$ is increasing with all entries ≤ 0 . Let

$$\varepsilon_\lambda^+ = \frac{1}{\chi^+(T_{w_\lambda})} \sum_{w \in W_\lambda} \chi^+(T_w) T_w \quad (40)$$

so that $T_w \varepsilon_\lambda^+ = \chi^+(T_w) \varepsilon_\lambda^+$, for $w \in W_\lambda$, and the coefficient of T_{w_λ} in ε_λ^+ is 1.

Define $\rho, \omega, \pi \in (\mathbb{Z}_{\geq 0}^n)^+$ by

$$\rho = (n, n-1, \dots, 2, 1), \quad \omega = (1, 1, \dots, 1), \quad \pi = (n-1, \dots, 2, 1, 0). \quad (41)$$

The statement of the following proposition is designed to stress the analogies between the four symmetrizers $\varepsilon_+, \varepsilon_\pm, \varepsilon_\mp$ and ε_- . As in Remark (53), in practice, there are simplifications since the stabilizers $W_{\lambda+\rho}$ and $W_{\lambda+\pi}$ have order 1 or 2. Specifically,

$$\begin{aligned} W_{\lambda+\rho} &= 1, \quad W^{\lambda+\rho} = W_{\text{fin}}, \quad \chi^+(T_{w_{\lambda+\rho}}) = 1 \quad \text{and} \quad \varepsilon_{\lambda+\rho}^+ = 1; \\ \text{if } \lambda_n &= 0 \quad \text{then} \quad W_{\lambda+\pi} = \{1, s_n\}, \quad \chi^+(T_{w_{\lambda+\pi}}) = t_n^{\frac{1}{2}} \quad \text{and} \quad \varepsilon_{\lambda+\pi}^+ = T_n + t_n^{-\frac{1}{2}}; \end{aligned}$$

and

$$\text{if } \lambda_n \neq 0 \quad \text{then} \quad W_{\lambda+\pi} = 1, \quad W^{\lambda+\pi} = W_{\text{fin}}, \quad \chi^+(T_{w_{\lambda+\pi}}) = 1 \quad \text{and} \quad \varepsilon_{\lambda+\pi}^+ = 1.$$

PROPOSITION 3.3. *Let $\lambda \in (\mathbb{Z}_{\geq 0}^n)^+$. Then*

$$\begin{aligned} \chi^+(T_{w_0}) \varepsilon_+ &= \chi^+(w_\lambda) \left(\sum_{z \in W^\lambda} \kappa_{v_\lambda z}^+ \eta_z \kappa_z^Y \right) \varepsilon_\lambda^+, \\ \chi^+(T_{w_0}) \varepsilon_\pm &= \chi^+(w_{\lambda+\omega}) \left(\sum_{z \in W^{\lambda+\omega}} (-1)^{\ell_d(z)} \kappa_{v_{\lambda+\omega} z}^\pm \eta_z \kappa_z^Y \right) \varepsilon_{\lambda+\omega}^+, \\ \chi^+(T_{w_0}) \varepsilon_\mp &= \chi^+(w_{\lambda+\pi}) \left(\sum_{z \in W^{\lambda+\pi}} (-1)^{\ell_s(z)} \kappa_{v_{\lambda+\pi} z}^\mp \eta_z \kappa_z^Y \right) \varepsilon_{\lambda+\pi}^+, \\ \chi^+(T_{w_0}) \varepsilon_- &= \chi^+(w_{\lambda+\rho}) \left(\sum_{z \in W^{\lambda+\rho}} (-1)^{\ell(z)} \kappa_{v_{\lambda+\rho} z}^- \eta_z \kappa_z^Y \right) \varepsilon_{\lambda+\rho}^+. \end{aligned}$$

Proof. We will prove the $\pm$ case. The proof for the other cases is similar.

Let

$$J_{\lambda+\omega} = \{\alpha \in (S^\vee)_{0,s}^+ \mid \alpha^\vee \notin \text{Inv}(w_{\lambda+\omega})\}$$

so that $J_{\lambda+\omega}$ is the complement of $\text{Inv}(w_{\lambda+\omega})$ in $S_{0,s}^+$. If $v \in W_{\lambda+\omega}$, then $vJ_{\lambda+\omega} = J_{\lambda+\omega}$, since v permutes the elements of $J_{\lambda+\omega}$.

$$\text{If } z \in W^{\lambda+\omega}, \quad \text{then} \quad \kappa_{\text{dr}}^{Y^{-1}} \eta_z \kappa_{J_{\lambda+\omega}}^Y = \kappa_{v_{\lambda+\omega}z}^{\pm} \eta_z \kappa_z^Y.$$

Then

$$\begin{aligned} \chi^+(T_{w_0})\varepsilon_{\pm} &= \kappa_{\text{dr}}^{Y^{-1}} e_{\pm}^Y \kappa_{\text{st}}^Y = \kappa_{\text{dr}}^{Y^{-1}} \left(\sum_{w \in W_{\text{fin}}} (-1)^{\ell_d(w)} \eta_w \right) \kappa_{\text{st}}^Y \\ &= \kappa_{\text{dr}}^{Y^{-1}} \left(\sum_{z \in W^{\lambda+\omega}} \sum_{v \in W_{\lambda+\omega}} (-1)^{\ell_d(z)} \eta_z \eta_v \right) \kappa_{\text{st}}^Y \\ &= \kappa_{\text{dr}}^{Y^{-1}} \left(\sum_{z \in W^{\lambda+\omega}} (-1)^{\ell_d(z)} \eta_z \right) \left(\sum_{v \in W_{\lambda+\omega}} \eta_v \right) \kappa_{J_{\lambda+\omega}}^Y \kappa_{W_{\lambda+\omega}}^Y \\ &= \kappa_{\text{dr}}^{Y^{-1}} \left(\sum_{z \in W^{\lambda+\omega}} (-1)^{\ell_d(z)} \eta_z \right) \kappa_{J_{\lambda+\omega}}^Y \left(\sum_{v \in W_{\lambda+\omega}} \eta_v \right) \kappa_{W_{\lambda+\omega}}^Y \\ &= \left(\sum_{z \in W^{\lambda+\omega}} (-1)^{\ell_d(z)} \kappa_{v_{\lambda+\omega}z}^{\pm} \eta_z \kappa_z^Y \right) \left(\sum_{v \in W_{\lambda+\omega}} \eta_v \right) \kappa_{W_{\lambda+\omega}}^Y \\ &= \left(\sum_{z \in W^{\lambda+\omega}} (-1)^{\ell_d(z)} \kappa_{v_{\lambda+\omega}z}^{\pm} \eta_z \kappa_z^Y \right) \chi^+(T_{w_{\lambda+\omega}}) \varepsilon_{\lambda+\omega}^+. \end{aligned}$$

□

REMARK 3.2. With $\omega = (1, 1, \dots, 1)$ and $\pi = (n-1, \dots, 2, 1, 0)$ and v_{ω} and v_{π} as defined in (3), then

$v_{\omega}(i) = -(n-i+1)$ so that $\text{Inv}_s(v_{\omega}) = \{\varepsilon_i + \varepsilon_j \mid i < j\}$ and $\text{Inv}_d(v_{\omega}) = \{\varepsilon_1, \dots, \varepsilon_n\}$ since, for example, $v_{\omega}(\varepsilon_1 - \varepsilon_n) = -\varepsilon_n + \varepsilon_1 = \varepsilon_1 - \varepsilon_n$. Then

$v_{\pi}(i) = \begin{cases} -i, & \text{if } i \neq n, \\ n, & \text{if } i = n, \end{cases}$ so that $\text{Inv}_s(v_{\pi}) = \{\varepsilon_i \pm \varepsilon_j\}$ and $\text{Inv}_d(v_{\pi}) = \{\varepsilon_1, \dots, \varepsilon_{n-1}\}$.

If $\lambda \in (\mathbb{Z}_{\geq 0}^n)^+$, then $W_{\lambda+\omega} \subseteq W_{\omega}$ and $W_{\lambda+\pi} \subseteq W_{\pi}$ giving

$$\text{Inv}_d(v_{\lambda+\omega}) \supseteq \text{Inv}_d(v_{\omega}) \quad \text{and} \quad \text{Inv}_s(v_{\lambda+\pi}) \supseteq \text{Inv}_s(v_{\pi}).$$

Thus,

$$\text{Inv}_d(v_{\lambda+\omega}) = \{\varepsilon_1, \dots, \varepsilon_n\} \quad \text{and} \quad \text{Inv}_s(v_{\lambda+\pi}) = \{\varepsilon_i \pm \varepsilon_j \mid i < j\}. \quad \square$$

4. ELECTRONIC MACDONALD-KOORNWINDER POLYNOMIALS

The electronic Koornwinder polynomials E_{μ} form a basis of the polynomial ring $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. They are simultaneous eigenvectors for the Cherednik–Dunkl operators $Y_1, \dots, Y_n$. This is analogy to the way that, in quantum mechanics, Hermite polynomials are eigenfunctions of a Hamiltonian operator. In this section, we set up the operators $Y_1, \dots, Y_n$ on the polynomial ring, characterize the electronic Macdonald polynomials E_{μ} as eigenvectors, and provide recursive formulas for computing them.

4.1. Operators on polynomials

Let $\mathbb{K}[x] = \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Let $\mathbb{Z}^n$ denote the set of length n sequences $\mu = (\mu_1, \dots, \mu_n)$ of integers. The ring

$\mathbb{K}[x]$ has basis $\{x^\mu \mid \mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n\}$, where $x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$.

Define operators $\xi_{s_0}, \xi_{s_1}, \dots, \xi_{s_n}$ on $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ by

$$\begin{aligned} (\xi_{s_0} f)(x_1, \dots, x_n) &= f(qx_1^{-1}, x_2, \dots, x_n), \\ (\xi_{s_i} f)(x_1, \dots, x_n) &= f(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n), \\ (\xi_{s_n} f)(x_1, \dots, x_n) &= f(x_1, \dots, x_{n-1}, x_n^{-1}), \end{aligned}$$

for $f \in \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and $i \in \{1, \dots, n-1\}$. Define operators $X_1, \dots, X_n$ on $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ by

$$X_j f = x_j f, \quad \text{for } j \in \{1, \dots, n\}. \quad (42)$$

Consider the induced representation

$$\text{Ind}_{H_Y}^{\tilde{H}}(\mathbf{1}_Y) = \tilde{H}_{\text{int}} \mathbf{1}_Y = \mathbb{K}\text{-span} \{X_1^{\mu_1} \cdots X_n^{\mu_n} \mathbf{1}_Y \mid \mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n\}$$

determined by

$$T_{\alpha_0} \mathbf{1}_Y = t_0^{\frac{1}{2}} \mathbf{1}_Y, \quad \text{and} \quad T_i \mathbf{1}_Y = t^{\frac{1}{2}} \mathbf{1}_Y, \quad \text{for } i \in \{1, \dots, n\}.$$

Then, the map

$$\begin{aligned} \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] &\longrightarrow \tilde{H}_{\text{int}} \mathbf{1}_Y \\ x_1^{\mu_1} \cdots x_n^{\mu_n} &\longmapsto X_1^{\mu_1} \cdots X_n^{\mu_n} \mathbf{1}_Y \end{aligned} \quad \text{is an } \tilde{H}_{\text{int}}\text{-module isomorphism.} \quad (43)$$

We shall often identify $\mathbb{K}[x] = \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and $\tilde{H}_{\text{int}} \mathbf{1}_Y$ and $\mathbb{K}[X] = \mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ via this isomorphism.

4.2. The operators $T_{\alpha_0}, \dots, T_{\alpha_n}$ and $Y_1, \dots, Y_n$

Define operators $T_{\alpha_0}, T_{\alpha_1}, \dots, T_{\alpha_n}$ by

$$T_{\alpha_i} = t_{\alpha_i}^{\frac{1}{2}} - c_{\alpha_i}^X (1 - \xi_{s_i}), \quad \text{for } i \in \{0, 1, \dots, n\}. \quad (44)$$

Define

$$Y_j = T_{\alpha_{j-1}}^{-1} \cdots T_{\alpha_1}^{-1} T_{\alpha_0} T_{\alpha_1} \cdots T_{\alpha_n} \cdots T_{\alpha_j}, \quad \text{for } j \in \{1, \dots, n\}.$$

Using (12) and (14),

$$\begin{aligned} t_0^{\frac{1}{2}} T_{\alpha_0} &= t_0 - \left(\frac{(x_1 + q^{\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{-\frac{1}{2}})(x_1 - q^{\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{\frac{1}{2}})}{x_1^2 - q} \right) (1 - \xi_{s_0}), \\ t_n^{\frac{1}{2}} T_{\alpha_n} &= t_n - \left(\frac{(1 + t_n^{\frac{1}{2}} u_n^{-\frac{1}{2}} x_n)(1 - t_n^{\frac{1}{2}} u_n^{\frac{1}{2}} x_n)}{1 - x_n^2} \right) (1 - \xi_{s_n}), \\ t^{\frac{1}{2}} T_{\alpha_i} &= t - \frac{x_{i+1} - t x_i}{x_{i+1} - x_i} (1 - \xi_{s_i}), \quad \text{for } i \in \{1, \dots, n-1\} \end{aligned} \quad (45)$$

(see [14, §3] and [18, (13)] and [2, (73)]).

4.3. Electronic Macdonald polynomials E_μ

For $\mu \in \mathbb{Z}^n$, let $v_\mu \in W_{\text{fin}}$ be the minimal length signed permutation such that $v_\mu \mu$ is weakly increasing with all entries ≤ 0 . The *electronic Macdonald polynomials*

$$E_\mu \left(x_1, \dots, x_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) \in \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \quad \text{are indexed by } \mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$$

and are defined by the eigenvalue conditions

$$Y_j E_\mu = q^{-\mu_j} t^{-v_\mu(j)} \left(t_0^{\frac{1}{2}} t_n^{\frac{1}{2}} t^n \right)^{\text{sgn}(v_\mu(j))} E_\mu, \quad (46)$$

for $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ and $j \in \{1, \dots, n\}$. The normalization of E_μ is such that the coefficient of x^μ in E_μ is 1.

Let $\mathbb{K}(Y)$ be the field of fractions of $\mathbb{K}[Y]$. For $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$, define homomorphisms $\text{ev}_\mu^t : \mathbb{K}[Y] \rightarrow \mathbb{K}[Y]$ by

$$\text{ev}_\mu^t(Y_i) = q^{-\mu_i} t^{-v_\mu(j)} \left(t_0^{\frac{1}{2}} t_n^{\frac{1}{2}} t^n \right)^{\text{sgn}(v_\mu(j))}, \quad \text{for } i \in \{1, \dots, n\}. \quad (47)$$

Extend ev_μ^t to those elements of the field $\mathbb{K}(Y)$ for which the evaluated denominator is non-zero. By (46),

$$f E_\mu = \text{ev}_\mu^t(f) E_\mu, \quad \text{for } f \in \mathbb{K}[Y] \text{ and } \mu \in \mathbb{Z}^n. \quad (48)$$

4.4. The recursion for the E_μ

Although the eigenvalue conditions together with the normalization completely characterize the Koornwinder polynomials E_μ , computing them by solving directly for eigenvectors is not efficient. Fortunately, the operators $\tau_i^\vee = \eta_{s_i^\vee} c_{\alpha_i^\vee}^Y$ from the algebra $\tilde{H}_{\text{loc}}$ provide a very nice recursive way of computing the E_μ . This is analogous to the way that, in Schubert calculus, the Schubert polynomials are constructed recursively using divided-difference operators.

Define operators $T_{\alpha_0^\vee}, T_{\alpha_1^\vee}, \dots, T_{\alpha_n^\vee}$ on $\mathbb{K}[X]$ by

$$(T_{\alpha_0^\vee})^{-1} = X_1 T_{\alpha_1} \cdots T_{\alpha_n} \cdots T_{\alpha_1} \quad \text{and} \quad T_{\alpha_i^\vee} = T_{\alpha_i} \quad \text{for } i \in \{1, \dots, n\}. \quad (49)$$

and define

$$\tau_i^\vee = T_{\alpha_i^\vee} + \left(t_{\alpha_i^\vee}^{-\frac{1}{2}} - c_{\alpha_i^\vee}^Y \right) = (T_{\alpha_i^\vee})^{-1} + \left(t_{\alpha_i^\vee}^{\frac{1}{2}} - c_{\alpha_i^\vee}^Y \right), \quad \text{for } i \in \{0, 1, \dots, n\}. \quad (50)$$

The intertwining operators $\tau_0^\vee, \dots, \tau_n^\vee$ are quite different from the notations τ_i and τ_i' appearing in Remark 1.1 and should not be confused. By (26), $\tau_i^\vee = \eta_{s_i^\vee} c_{\alpha_i^\vee}^Y$ so that

$$\tau_i^\vee Y^{\lambda^\vee} = Y^{s_i^\vee \lambda^\vee} \tau_i^\vee, \quad \text{for } i \in \{0, 1, \dots, n\}. \quad (51)$$

The group W_X (generated by $s_0^\vee, \dots, s_n^\vee$) acts on $\mathbb{Z}^n$ by

$$\begin{aligned} s_0^\vee(\mu_1, \dots, \mu_n) &= (-\mu_1 + 1, \mu_2, \dots, \mu_n), \\ s_i^\vee(\mu_1, \dots, \mu_n) &= (\mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \mu_i, \mu_{i+2}, \dots, \mu_n), \quad \text{for } i \in \{1, \dots, n-1\}, \text{ and} \\ s_n^\vee(\mu_1, \dots, \mu_n) &= (\mu_1, \dots, \mu_{n-1}, -\mu_n). \end{aligned}$$

The relation (51) is the reason that the electronic Macdonald polynomials E_μ are equivalently defined by the following recursive relations:

- (E0) $E_{(0, \dots, 0)} = 1$;
- (E1) if $\mu_1 \leq 0$, then $E_{s_0^\vee \mu} = t^{n-1} t_n^{\frac{1}{2}} \tau_0^\vee E_\mu$;
- (E2) if $i \in \{1, \dots, n-1\}$ and $\mu_i > \mu_{i+1}$, then $E_{s_i^\vee \mu} = t^{\frac{1}{2}} \tau_i^\vee E_\mu$; and
- (E3) if $\mu_n > 0$, then $E_{s_n^\vee \mu} = t_n^{\frac{1}{2}} \tau_n^\vee E_\mu$.

4.5. The creation formula for E_μ

The recursion of the previous section can be packaged nicely as a single formula for creating the Koornwinder polynomial E_μ . This is the creation formula in (52).

Let $\mu \in \mathbb{Z}^n$ and let $h_\mu \in W_X$ denote the corresponding translation. Let $u_\mu \in W_X$ and $v_\mu \in W_{\text{fin}}$ be as defined in (3), so that

$$u_\mu \in W_X \quad \text{and} \quad v_\mu \in W_{\text{fin}} \quad \text{and} \quad h_\mu = u_\mu v_\mu, \quad \text{with} \quad \ell(h_\mu) = \ell(u_\mu) + \ell(v_\mu).$$

Using the identification of $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ with $\tilde{H}_{\text{int}} \mathbf{1}_Y$ as in (43), the creation formula for E_μ is

$$E_\mu = \frac{1}{\chi^+(T_{v_\mu^{-1}})} \tau_{u_\mu}^\vee \mathbf{1}_Y, \quad \text{where} \quad \tau_{u_\mu}^\vee = \tau_{i_1}^\vee \cdots \tau_{i_\ell}^\vee \quad (52)$$

if $u_\mu = s_{i_1} \cdots s_{i_\ell}$ is a reduced word for u_μ .

Proof. Using

$$\begin{aligned} Y_i \mathbf{1}_Y &= T_{\alpha_{i-1}}^{-1} \cdots T_{\alpha_1}^{-1} T_{\alpha_0} T_{\alpha_1} \cdots T_{\alpha_n} \cdots T_{\alpha_i} \mathbf{1}_Y = t^{-\frac{1}{2}(i-1)} t_0^{\frac{1}{2}} t^{\frac{1}{2}(n-1)} t_n^{\frac{1}{2}} t^{\frac{1}{2}(n-i)} \mathbf{1}_Y \\ &= t^{-i} \left(t_0^{\frac{1}{2}} t_n^{\frac{1}{2}} t^n \right) \mathbf{1}_Y, \end{aligned}$$

and $h_\mu^{-1} \varepsilon_j^\vee = h_{-\mu} \varepsilon_j^\vee = \varepsilon_j^\vee - \langle \mu, -\varepsilon_j^\vee \rangle K = \varepsilon_j^\vee + \mu_j K$ gives

$$\begin{aligned} Y_j \tau_{u_\mu}^\vee \mathbf{1}_Y &= Y^{\varepsilon_j^\vee} \tau_{u_\mu}^\vee \mathbf{1}_Y = Y^{\varepsilon_j^\vee} \tau_{u_\mu}^\vee \mathbf{1}_Y = \tau_{u_\mu}^\vee Y^{u_\mu^{-1} \varepsilon_j^\vee} \mathbf{1}_Y = \tau_{u_\mu}^\vee Y^{v_\mu h_\mu^{-1} \varepsilon_j^\vee} \mathbf{1}_Y \\ &= \tau_{u_\mu}^\vee Y^{v_\mu (\varepsilon_j^\vee + \mu_j K)} \mathbf{1}_Y = q^{-\mu_j} \tau_{u_\mu}^\vee Y^{\varepsilon_{v_\mu(j)}^\vee} \mathbf{1}_Y \\ &= q^{-\mu_j} \tau_{u_\mu}^\vee Y_{v_\mu(j)} \mathbf{1}_Y = q^{-\mu_j} t^{-v_\mu(j)} \left(t_0^{\frac{1}{2}} t_n^{\frac{1}{2}} t^n \right)^{\text{sgn}(v_\mu(j))} \tau_{u_\mu}^\vee \mathbf{1}_Y. \end{aligned}$$

Thus, $\tau_{u_\mu}^\vee \mathbf{1}_Y$ is an eigenvector of Y_j with eigenvalue $q^{-\mu_j} t^{-v_\mu(j)} (t_0^{\frac{1}{2}} t_n^{\frac{1}{2}} t^n)^{\text{sgn}(v_\mu(j))}$.

Using the formulas (50), the product $\tau_{u_\mu}^\vee$ can be expanded in terms of the elements

$$\{X^\gamma T_v \mid \gamma \in \mathbb{Z}^n, v \in W_{\text{fin}}\}.$$

Since $u_\mu = t_\mu v_\mu^{-1}$, then the top term in this expansion is $X^\mu T_{v_\mu^{-1}}$ and

$$X^\mu T_{v_\mu^{-1}} \mathbf{1}_Y = X^\mu \chi^+(T_{v_\mu^{-1}}) \mathbf{1}_Y, \quad \text{where} \quad \chi^+(T_{v_\mu^{-1}}) = t^{\frac{1}{2} \ell_s(v_\mu^{-1})} t_n^{\frac{1}{2} \ell_d(v_\mu^{-1})}.$$

Thus multiplying $\tau_{u_\mu}^\vee \mathbf{1}_Y$ by $\chi^+(T_{v_\mu^{-1}})^{-1}$ makes the coefficient of X^μ equal to 1. $\square$

5. BOSONIC, FERMIONIC AND MESONIC

The Weyl character formula is the formula that expresses the Schur function (a symmetric polynomial) as a quotient of two determinants (antisymmetric polynomials). There are Weyl character formulas in the Koornwinder context as well (see section 6.6). However, in the Koornwinder context, one finds that there are *four* Weyl character formulas, corresponding to the four symmetrizers ε_+ , $\varepsilon_\pm$, $\varepsilon_\mp$, ε_- introduced in section 3.5.

This section sets up the components for Weyl character formulas in the Koornwinder context. There are four types of symmetrized Koornwinder polynomials: the bosonic (symmetric) Koornwinder polynomials, the fermionic (antisymmetric) Koornwinder polynomials and two types of mesonic (half symmetric–half antisymmetric) Koornwinder polynomials.

The denominator in the classical Weyl character formula is the Vandermonde determinant, an antisymmetric polynomial with a magical factorization. In the Koornwinder case, the Weyl denominators also have magical factorizations. These Weyl denominators are presented in section 5.2.

Every antisymmetric function can be obtained by multiplying a symmetric function by the Weyl denominators. In [4, §4.3], we viewed this correspondence between symmetric functions and antisymmetric functions as an analogue of the Boson–Fermion correspondence relating the symmetric algebra realization of Fock space and exterior algebra realization of Fock space (a representation of a Heisenberg algebra, see [8, §14.10]). In the Koornwinder context, there are four spaces: the bosonic space (symmetric functions), fermionic space (antisymmetrized functions) and two mesonic spaces (half

symmetric–half antisymmetric functions). These four spaces are all isomorphic as vector spaces, the isomorphisms being given by multiplying by the different Weyl denominators. This structure is explained in section 5.3.

In sections 5.4, 5.5 and 5.6, we use the symmetrizers to give formulas for the Poincaré polynomial of W_{fin} , for the E -expansions of bosonic, fermionic and mesonic Koornwinder polynomials and formulas for the principal specializations. These results are Koornwinder analogues of the formulas in [4, Propositions 4.6 and 4.7 and Theorem 5.1]. All of these formulas are given, in an even more general setting, in [13, (5.5,16), (5.7,8), (5.2.14), (5.3.9)].

5.1. Bosonic, fermionic and mesonic Macdonald–Koornwinder polynomials

Let

$$(\mathbb{Z}_{\geq 0}^n)^+ = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \dots \geq \lambda_n \geq 0\}.$$

For $\lambda \in (\mathbb{Z}_{\geq 0}^n)^+$, let

$$W_\lambda = \{v \in W_{\text{fin}} \mid v\lambda = \lambda\} \quad \text{and} \quad W_\lambda(t, t_n) = \sum_{v \in W_\lambda} t^{\ell_s(v)} t_n^{\ell_d(v)} = \sum_{v \in W_\lambda} \chi^+(T_v)^2.$$

Let $\rho, \omega, \pi \in (\mathbb{Z}_{\geq 0}^n)^+$ be as defined in (41). Then, for $\lambda \in (\mathbb{Z}_{\geq 0}^n)^+$, define the *bosonic and fermionic Macdonald–Koornwinder polynomials* are

$$P_\lambda = \frac{\chi^+(T_{w_0})}{W_\lambda(t, t_n)} \varepsilon_{+E_\lambda} \quad \text{and} \quad A_{\lambda+\rho} = \frac{\chi^+(T_{w_0})}{W_{\lambda+\rho}(t, t_n)} \varepsilon_{-E_{\lambda+\rho}}, \quad (53)$$

and the *mesonic Koornwinder polynomials* are

$$A_{\lambda+\omega}^\pm = \frac{\chi^+(T_{w_0})}{W_{\lambda+\omega}(t, t_n)} \varepsilon_{\pm E_{\lambda+\omega}} \quad \text{and} \quad A_{\lambda+\pi}^\mp = \frac{\chi^+(T_{w_0})}{W_{\lambda+\pi}(t, t_n)} \varepsilon_{\mp E_{\lambda+\pi}}. \quad (54)$$

REMARK 5.1. The notation in (53) and (54) displays the parallelism among the expressions.

For computation, it is useful to note that the denominators can be given very explicitly:

$$W_{\lambda+\rho}(t, t_n) = 1, \quad W_{\lambda+\omega}(t, t_n) = \prod_i \frac{(1 - t^{m_i})}{1 - t}, \quad W_{\lambda+\pi}(t, t_n) = \begin{cases} t_n + 1, & \text{if } \lambda_n = 0, \\ 1, & \text{if } \lambda_n \neq 0, \end{cases}$$

where m_i is the number of parts of size i in $\lambda = (\lambda_1, \dots, \lambda_n)$. In particular, $W_{\lambda+\pi}(t, t_n)$ depends only on t_n , and $W_{\lambda+\omega}(t, t_n)$ depends only on t . The factor $\chi^+(T_{w_0})$ guarantees that the coefficient of $X^{w_0\lambda} = X^{-\lambda}$ is equal to 1 in P_λ . Similarly, the coefficient of $X^{-(\lambda+\rho)}$ is equal to 1 in $A_{\lambda+\rho}$, the coefficient of $X^{-(\lambda+\omega)}$ is equal to 1 in $A_{\lambda+\omega}^\pm$ and the coefficient of $X^{-(\lambda+\pi)}$ is equal to 1 in $A_{\lambda+\pi}^\mp$. $\square$

5.2. Weyl denominators

Define $a_\omega^\pm, a_\pi^\mp, a_\rho, A_\omega^\pm, A_\pi^\mp, A_\rho \in \mathbb{K}[X]$ by

$$\begin{aligned} a_\omega^\pm &= x^{-\omega} \prod_{i=1}^n (1 - x_i^2), \quad A_\omega^\pm = x^{-\omega} \left(\prod_{i=1}^n (1 - t_n^{\frac{1}{2}} u_n^{\frac{1}{2}} x_i) (1 + t_n^{\frac{1}{2}} u_n^{-\frac{1}{2}} x_i) \right), \\ a_\pi^\mp &= x^{-\pi} \prod_{1 \leq i < j \leq n} (1 - x_i x_j^{-1}) (1 - x_i x_j), \\ A_\pi^\mp &= x^{-\pi} \left(\prod_{1 \leq i < j \leq n} (1 - t x_i x_j) (1 - t x_i x_j^{-1}) \right), \\ a_\rho &= a_\pi^\mp a_\omega^\pm, \quad A_\rho = A_\pi^\mp A_\omega^\pm. \end{aligned} \quad (55)$$

Then

$$\begin{aligned} a_\omega^\pm &= \frac{1}{n!} e_\pm^X x^\omega, \quad a_\pi^\mp = \frac{1}{2} e_\mp^X x^\pi, \quad a_\rho = e_- x^\rho, \\ A_\omega^\pm &= \frac{t^{\frac{1}{2}n(n-1)} t_n^{\frac{1}{2}n}}{[n]!} \varepsilon_\pm^X x^\omega, \quad A_\pi^\mp = \frac{t^{\frac{1}{2}n(n-1)} t_n^{\frac{1}{2}n}}{(1+t_n)} \varepsilon_\mp^X x^\pi, \\ A_\rho &= t^{\frac{1}{2}n(n-1)} t_n^{\frac{1}{2}n} \varepsilon_- x^\rho, \end{aligned}$$

where

$$t^{\frac{1}{2}n(n-1)} t_n^{\frac{1}{2}n} = \chi^+(T_{w_0}), \quad [n]! = \prod_{i=1}^n \frac{(1-t^i)}{1-t} = W_\omega(t, t_n), \quad 1+t_n = W_\pi(t, t_n).$$

By (21),

$$\frac{A_\omega^\pm}{a_\omega^\pm} = \kappa_{\text{dr}}^X, \quad \frac{A_\pi^\mp}{a_\pi^\mp} = \kappa_{\text{st}}^X, \quad \frac{A_\rho}{a_\rho} = \kappa_{w_0}^X.$$

Since

$$\begin{aligned} A_\rho &= \frac{\chi^+(T_{w_0})}{W_{\lambda+\rho}(t, t_n)} \varepsilon_- x^\rho = \frac{\chi^+(T_{w_0})}{W_{\lambda+\rho}(t, t_n)} \varepsilon_- E_\rho, \\ A_\omega^\pm &= \frac{\chi^+(T_{w_0})}{W_\omega(t, t_n)} \varepsilon_\pm x^\omega = \frac{\chi^+(T_{w_0})}{W_\omega(t, t_n)} \varepsilon_\pm E_\omega, \quad \text{and} \quad A_\pi^\mp = \frac{\chi^+(T_{w_0})}{W_\pi(t, t_n)} \varepsilon_\mp x^\pi = \frac{\chi^+(T_{w_0})}{W_\pi(t, t_n)} \varepsilon_\mp E_\pi, \end{aligned}$$

there is no conflict of notation with the mesonic Macdonald polynomials introduced in (54).

5.3. Bosonic, fermionic and mesonic spaces

The polynomial ring $\mathbb{K}[X]$ is a module for the action of $\mathbb{K}[X]^{W_{\text{fin}}}$ and the structure of $\mathbb{K}[X]$ as a $\mathbb{K}[X]^{W_{\text{fin}}}$ -module is of classical importance in the theory of reflection groups. In fact, there are two commuting actions on $\mathbb{K}[X]$, the action of W_{fin} and the action of $\mathbb{K}[X]^{W_{\text{fin}}}$. The part of this picture that is captured by the X -symmetrizers can be stated as follows.

Define

$$\begin{aligned} \mathbb{K}[X]^{W_{\text{fin}}} &= \{f \in \mathbb{K}[X] \mid \text{if } w \in W_{\text{fin}}, \text{ then } wf = f\}, \\ \mathbb{K}[X]^\pm &= \{f \in \mathbb{K}[X] \mid \text{if } w \in W_{\text{fin}}, \text{ then } wf = (-1)^{\ell_d(w)} f\}, \\ \mathbb{K}[X]^\mp &= \{f \in \mathbb{K}[X] \mid \text{if } w \in W_{\text{fin}}, \text{ then } wf = (-1)^{\ell_s(w)} f\}, \\ \mathbb{K}[X]^{\text{det}} &= \{f \in \mathbb{K}[X] \mid \text{if } w \in W_{\text{fin}}, \text{ then } wf = (-1)^{\ell_s(w) + \ell_d(w)} f\}. \end{aligned}$$

Then

$$\begin{aligned} e_+^X \mathbb{K}[X] &= \mathbb{K}[X]^{W_{\text{fin}}}, & e_{\pm}^X \mathbb{K}[X] &= \mathbb{K}[X]^{\pm} = a_{\omega}^{\pm} \mathbb{K}[X]^{W_{\text{fin}}}, \\ e_-^X \mathbb{K}[X] &= \mathbb{K}[X]^{\text{det}} = a_{\rho} \mathbb{K}[X]^{W_{\text{fin}}}, & e_{\mp}^X \mathbb{K}[X] &= \mathbb{K}[X]^{\mp} = a_{\pi}^{\mp} \mathbb{K}[X]^{W_{\text{fin}}}. \end{aligned} \quad (56)$$

Now, we proceed to a t -analogue of the equalities in (56). In this case, Hecke algebra H_{fin} replaces the finite Weyl group, and the actions of H_{fin} and $\mathbb{K}[X]^{W_{\text{fin}}}$ are commuting actions on $\mathbb{K}[X]$. The part of this picture captured by the Hecke symmetrizers is the following.

The bosonic, fermionic and mesonic spaces are

$$\begin{aligned} \mathbb{K}[X]^{\text{Bos}} &= \{f \in \mathbb{K}[X] \mid T_n f = t_n^{\frac{1}{2}} f \text{ and } T_i f = t^{\frac{1}{2}} f, \text{ for } i \in \{1, \dots, n\}\}, \\ \mathbb{K}[X]^{\text{Fer}} &= \{f \in \mathbb{K}[X] \mid T_n f = -t_n^{-\frac{1}{2}} f \text{ and } T_i f = -t^{-\frac{1}{2}} f, \text{ for } i \in \{1, \dots, n\}\}, \\ \mathbb{K}[X]^{\text{Mes}\pm} &= \{f \in \mathbb{K}[X] \mid T_n f = -t_n^{-\frac{1}{2}} f \text{ and } T_i f = t^{\frac{1}{2}} f, \text{ for } i \in \{1, \dots, n\}\}, \\ \mathbb{K}[X]^{\text{Mes}\mp} &= \{f \in \mathbb{K}[X] \mid T_n f = t_n^{\frac{1}{2}} f \text{ and } T_i f = -t^{-\frac{1}{2}} f, \text{ for } i \in \{1, \dots, n\}\}, \end{aligned}$$

With these definitions, the following proposition establishes t -analogues of the equalities in (56). The Weyl denominators of section 5.2 are a key part of the structure.

PROPOSITION 5.1. *Let ε_+ , $\varepsilon_{\pm}$, $\varepsilon_{\mp}$, ε_- be the symmetrizers defined in (37).*

$$\begin{aligned} \mathbb{K}[X]^{\text{Bos}} &= \varepsilon_+ \mathbb{K}[X] = \mathbb{K}[X]^{W_{\text{fin}}}, & \mathbb{K}[X]^{\text{Fer}} &= \varepsilon_- \mathbb{K}[X] = A_{\rho} \mathbb{K}[X]^{W_{\text{fin}}}, \\ \mathbb{K}[X]^{\text{Mes}\pm} &= \varepsilon_{\pm} \mathbb{K}[X] = A_{\omega}^{\pm} \mathbb{K}[X]^{W_{\text{fin}}}, & \mathbb{K}[X]^{\text{Mes}\mp} &= \varepsilon_{\mp} \mathbb{K}[X] = A_{\pi}^{\mp} \mathbb{K}[X]^{W_{\text{fin}}}, \end{aligned}$$

Moreover, with P_{λ} , $A_{\lambda+\omega}^{\pm}$, $A_{\lambda+\pi}^{\mp}$ and $A_{\lambda+\rho}$ as in (53) and (54),

$$\begin{aligned} \{P_{\lambda} \mid \lambda \in (\mathbb{Z}_{\geq 0}^n)^+\} &\text{ is a basis of } \varepsilon_+ \mathbb{K}[X], & \{A_{\lambda+\rho} \mid \lambda \in (\mathbb{Z}_{\geq 0}^n)^+\} &\text{ is a basis of } \varepsilon_- \mathbb{K}[X], \\ \{A_{\lambda+\omega}^{\pm} \mid \lambda \in (\mathbb{Z}_{\geq 0}^n)^+\} &\text{ is a basis of } \varepsilon_{\pm} \mathbb{K}[X], & \{A_{\lambda+\pi}^{\mp} \mid \lambda \in (\mathbb{Z}_{\geq 0}^n)^+\} &\text{ is a basis of } \varepsilon_{\mp} \mathbb{K}[X], \end{aligned}$$

Proof. We will give the proof for the $\pm$ case. The proofs for the other cases are similar.

Assume $f \in \varepsilon_{\pm} \mathbb{K}[X]$. Then, there exists $g \in \mathbb{K}[X]$ such that $f = \varepsilon_{\pm} g$ and

$$T_n f = T_n \varepsilon_{\pm} g = -t_n^{-\frac{1}{2}} \varepsilon_{\pm} g = -t_n^{-\frac{1}{2}} f \quad \text{and} \quad T_i f = T_i \varepsilon_{\pm} g = t^{\frac{1}{2}} \varepsilon_{\pm} g = t^{\frac{1}{2}} f,$$

for $i \in \{1, \dots, n-1\}$. So $f \in \mathbb{K}[X]^{\text{Mes}\pm}$ and $\varepsilon_{\pm} \mathbb{K}[X] \subseteq \mathbb{K}[X]^{\text{Mes}\pm}$.

If $f \in \mathbb{K}[X]^{\text{Mes}\pm}$, then

$$f = \frac{\chi^{\pm}(T_{w_0})}{W_0(t, t_n^{-1})} \varepsilon_{\pm} f = \frac{\chi^{\pm}(T_{w_0})}{W_0(t, t_n^{-1})} c_{\text{dr}}^X e_{\pm}^X c_{\text{st}}^{X^{-1}} f = \frac{\chi^{\pm}(T_{w_0})}{W_0(t, t_n^{-1})} \frac{A_{\omega}^{\pm}}{a_{\omega}^{\pm}} e_{\pm}^X c_{\text{st}}^{X^{-1}} f \in \frac{A_{\omega}^{\pm}}{a_{\omega}^{\pm}} \mathbb{K}[X]^{\pm}.$$

Since $\frac{A_{\omega}^{\pm}}{a_{\omega}^{\pm}} \mathbb{K}[X]^{\pm} = \frac{A_{\omega}^{\pm}}{a_{\omega}^{\pm}} a_{\omega}^{\pm} \mathbb{K}[X]^{W_{\text{fin}}} = A_{\omega}^{\pm} \mathbb{K}[X]^{W_{\text{fin}}}$, then $\mathbb{K}[X]^{\text{Mes}\pm} \subseteq A_{\omega}^{\pm} \mathbb{K}[X]^{W_{\text{fin}}}$.

Assume $f \in A_{\omega}^{\pm} \mathbb{K}[X]^{W_{\text{fin}}}$. Then, there exists $g \in \mathbb{K}[X]^{W_{\text{fin}}}$ such that $f = A_{\omega}^{\pm} g$. Then

$$f = A_{\omega}^{\pm} g = \left(\frac{\chi^{\pm}(T_{w_0})}{W_{\omega}(t, t_n)} \varepsilon_{\pm} x^{\omega} \right) g = \varepsilon_{\pm} \left(\frac{\chi^{\pm}(T_{w_0})}{W_{\omega}(t, t_n)} x^{\omega} g \right) \in \varepsilon_{\pm} \mathbb{K}[X].$$

So $A_{\omega}^{\pm} \mathbb{K}[X]^{W_{\text{fin}}} \subseteq \varepsilon_{\pm} \mathbb{K}[X]$. This completes the proof that $\mathbb{K}[X]^{\text{Mes}\pm} = \varepsilon_{\pm} \mathbb{K}[X] = A_{\omega}^{\pm} \mathbb{K}[X]^{W_{\text{fin}}}$.

Define

$$M_{\mu}^{\pm} = \varepsilon_{\pm} E_{\mu} \quad \text{for } \mu \in \mathbb{Z}^n.$$

Let $i \in \{1, \dots, n\}$. If $\mu_n = 0$, then $s_n \mu = \mu$ and $T_n E_{\mu} = t_n^{\frac{1}{2}} E_{\mu}$ so that

$$M_{\mu}^{\pm} = \varepsilon_{\pm} T_{\alpha_n^{\vee}} E_{\mu} = (-t_n)^{\frac{1}{2}} \varepsilon_{\pm} T_{\alpha_n^{\vee}} E_{\mu} = -M_{\mu}, \quad \text{which forces } M_{\mu}^{\pm} = 0 \quad \text{when } \mu_n = 0.$$

If $i \in \{1, \dots, n\}$ and $s_i \mu > \mu$, then

$$\begin{aligned} M_{s_i \mu}^\pm &= \varepsilon_\pm E_{s_i \mu} = \varepsilon_\pm t^{\frac{1}{2}} \tau_i^\vee E_\mu = \varepsilon_\pm t^{\frac{1}{2}} (T_{\alpha_i^\vee} + (t^{-\frac{1}{2}} - c_{\alpha_i^\vee}^Y) E_\mu \\ &= \varepsilon_\pm t^{\frac{1}{2}} (t^{\frac{1}{2}} + t^{-\frac{1}{2}} - c_{\alpha_i^\vee}^Y) E_\mu = \varepsilon_\pm t^{\frac{1}{2}} c_{-\alpha_i^\vee}^Y E_\mu = \text{ev}_\mu^t(\kappa_{-\alpha_i^\vee}^Y) M_\mu, \end{aligned}$$

so that $M_\mu^\pm$ and $M_{s_i \mu}^\pm$ are linearly dependent. It follows that

$$\begin{aligned} \varepsilon_\pm \mathbb{K}[X] &= \mathbb{K}\text{-span}\{M_\lambda^\pm \mid \lambda \in \mathbb{Z}^n\} = \mathbb{K}\text{-span}\{M_{\lambda+\omega}^\pm \mid \lambda \in (\mathbb{Z}_{\geq 0}^n)^+\} \\ &= \mathbb{K}\text{-span}\{A_{\lambda+\omega}^\pm \mid \lambda \in (\mathbb{Z}_{\geq 0}^n)^+\} \end{aligned}$$

Since $A_{\lambda+\omega}^\pm$ has top coefficient $x^{-(\lambda+\omega)}$ (in the DBlex order, see section 6.3) and the monomials are linear independent in $\mathbb{K}[X]$, then the set $\{A_{\lambda+\omega}^\pm \mid \lambda \in (\mathbb{Z}_{\geq 0}^n)^+\}$ is linearly independent. $\square$

5.4. Formulas for the Poincaré polynomial

Recall from (38) and (47) that the Poincaré polynomial for W_{fin} is

$$W_0(t, t_n) = \sum_{w \in W_{\text{fin}}} \chi^+(T_w)^2 \quad \text{and} \quad \text{ev}_0^t(Y_i) = t^{n-i} t_0^{\frac{1}{2}} t_n^{\frac{1}{2}}$$

defines the evaluation homomorphism $\text{ev}_0^t : \mathbb{K}[Y] \rightarrow \mathbb{K}$. The following Proposition gives four ways of looking at the Poincaré polynomial: as a sum, as a product, as an evaluation of $\kappa_{w_0}^Y$ and as a symmetrization of $\kappa_{w_0}^{X^{-1}}$.

PROPOSITION 5.2. *The group W_{fin} acts on $\mathbb{K}[X]$ as in (24). Let w_0 denote the longest element of W_{fin} and let $\kappa_{w_0}^Y$ and $\kappa_{w_0}^{X^{-1}}$ be the normalized c-functions given in (22) and (23). Then*

$$W_0(t, t_n) = \prod_{i=1}^n \frac{(1-t^i)(1+t_n t^{i-1})}{(1-t)} = \text{ev}_0^t(\kappa_{w_0}^Y) = \sum_{w \in W_{\text{fin}}} w(\kappa_{w_0}^{X^{-1}}).$$

Proof. Since $\chi^+(T_w)^2 = t^{\ell_s(w)} t_n^{\ell_d(w)}$, then

$$\chi^+(T_{w_0}) \varepsilon_+ \mathbf{1}_Y = \sum_{w \in W_{\text{fin}}} \chi^+(T_w) T_w \mathbf{1}_Y = \sum_{w \in W_{\text{fin}}} (\chi^+(T_w))^2 \mathbf{1}_Y = W_0(t, t_n) \mathbf{1}_Y.$$

Using the first formula in Proposition 3.2,

$$\chi^+(T_{w_0}) \varepsilon_+ \mathbf{1}_Y = e_+^X \kappa_{w_0}^{X^{-1}} \mathbf{1}_Y = \sum_{w \in W_{\text{fin}}} \xi_w \kappa_{w_0}^{X^{-1}} \mathbf{1}_Y = \sum_{w \in W_{\text{fin}}} w(\kappa_{w_0}^{X^{-1}}).$$

Using the second formula in Proposition 3.2, $\varepsilon_+ \mathbf{1}_Y = e_+^Y \kappa_{w_0}^Y \mathbf{1}_Y$ and

$$\begin{aligned} \chi^+(T_{w_0}) \varepsilon_+ \mathbf{1}_Y &= \left(\sum_{w \in W_{\text{fin}}} \eta_w \right) \kappa_{w_0}^Y \mathbf{1}_Y = \text{ev}_0^t(\kappa_{w_0}^Y) \left(1 + \sum_{w \in W_{\text{fin}}, w \neq 1} \eta_w \right) \mathbf{1}_Y \\ &= \text{ev}_0^t(\kappa_{w_0}^Y) (1+0) \mathbf{1}_Y = \text{ev}_0^t(\kappa_{w_0}^Y) \mathbf{1}_Y. \end{aligned}$$

Finally,

$$\begin{aligned}
\text{ev}_0^t(\kappa_{w_0}^Y) &= \text{ev}_0^t \left(\prod_{i < j} \frac{(1 - tY_i Y_j^{-1})(1 - tY_i Y_j)}{(1 - Y_i Y_j^{-1})(1 - Y_i Y_j)} \right) \text{ev}_0^t \left(\prod_{i=1}^n \frac{(1 - t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} Y_i)(1 + t_n^{\frac{1}{2}} t_0^{-\frac{1}{2}} Y_i)}{(1 - Y_i^2)} \right) \\
&= \left(\prod_{i < j} \frac{(1 - t t^{n-i} t_0^{\frac{1}{2}} t_n^{\frac{1}{2}} t^{-(n-j)} t_n^{-\frac{1}{2}} t_0^{-\frac{1}{2}})(1 - t t^{n-i} t_0^{\frac{1}{2}} t_n^{\frac{1}{2}} t^{n-j} t_0^{\frac{1}{2}} t_n^{\frac{1}{2}})}{(1 - t^{n-i} t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} t^{-(n-j)} t_n^{-\frac{1}{2}} t_0^{-\frac{1}{2}})(1 - t^{n-i} t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} t^{n-j} t_0^{\frac{1}{2}} t_n^{\frac{1}{2}})} \right) \\
&\quad \left(\prod_{i=1}^n \frac{(1 - t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} t_n^{\frac{1}{2}} t^{n-i})(1 + t_n^{\frac{1}{2}} t_0^{-\frac{1}{2}} t_n^{\frac{1}{2}} t^{n-i})}{(1 - t_0 t_n t^{2(n-i)})} \right) \\
&= \left(\prod_{i < j} \frac{(1 - t^{j-i+1})(1 - t^{2n-j-i+1} t_n t_0)}{(1 - t^{j-i})(1 - t^{2n-j-i} t_n t_0)} \right) \left(\prod_{i=1}^n \frac{(1 - t_n t_0 t^{n-i})(1 + t_n t^{n-i})}{(1 - t_0 t_n t^{2(n-i)})} \right) \\
&= \left(\prod_{j=2}^n \frac{(1 - t^j)(1 - t^{2n-j} t_n t_0)}{(1 - t)(1 - t^{2(n-j)+1} t_n t_0)} \right) \left(\prod_{i=1}^n \frac{(1 - t_n t_0 t^{n-i})(1 + t_n t^{n-i})}{(1 - t_0 t_n t^{2(n-i)})} \right) \\
&= \left(\prod_{i=1}^n \frac{(1 - t^i)(1 + t_n t^i)}{(1 - t)} \right) \left(\prod_{j=2}^n \frac{(1 - t^{2n-j} t_n t_0)}{(1 - t^{2(n-j)+1} t_n t_0)} \right) \left(\prod_{i=1}^n \frac{(1 - t_n t_0 t^{n-i})}{(1 - t_0 t_n t^{2(n-i)})} \right) \\
&= \left(\prod_{i=1}^n \frac{(1 - t^i)(1 + t_n t^{i-1})}{(1 - t)} \right) \left(\prod_{i=1}^{2n} \frac{(1 - t^{i-1} t_n t_0)}{(1 - t^{i-1} t_n t_0)} \right) = \prod_{i=1}^n \frac{(1 - t^i)(1 + t_n t^{i-1})}{(1 - t)}. \quad \square
\end{aligned}$$

5.5. E-expansions

The following proposition uses the formulas for symmetrizers in terms of c -functions from Proposition 3.3 to give explicit expansion of the bosonic, fermionic and mesonic Koornwinder polynomials in terms of the E_μ . The coefficients in these expansions are evaluations of c -functions. This is an example of how the c -functions (which live in the field of fractions) appear in the structure even when doing expansions of polynomials that live in $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

PROPOSITION 5.3. (*E-expansion formulas*) Let $\lambda \in (\mathbb{Z}_{\geq 0}^n)^+$ and let $P_\lambda, A_{\lambda+\omega}^\pm, A_{\lambda+\pi}^\mp$ and $A_{\lambda+\rho}$ be as defined in (53) and (54). Let ev_μ^t be the evaluation homomorphisms defined in (47) and let $\kappa_{v_\mu}^\Xi$ be the normalized c -functions defined in (20) and (3). Then

$$\begin{aligned}
P_\lambda &= \sum_{\mu \in W\lambda} \text{ev}_\mu^t(\kappa_{v_\mu}^+) E_\mu, \quad A_{\lambda+\omega}^\pm = \sum_{\mu \in W(\lambda+\omega)} (-1)^{\ell_d(v_\mu)} \text{ev}_\mu^t(\kappa_{v_\mu}^\pm) E_\mu, \\
A_{\lambda+\rho} &= \sum_{\mu \in W(\lambda+\rho)} (-1)^{\ell(v_\mu)} \text{ev}_\mu^t(\kappa_{v_\mu}^-) E_\mu, \quad A_{\lambda+\pi}^\mp = \sum_{\mu \in W(\lambda+\pi)} (-1)^{\ell_s(v_\mu)} \text{ev}_\mu^t(\kappa_{v_\mu}^\mp) E_\mu.
\end{aligned}$$

Proof. Let us do the case $A_{\lambda+\omega}^\pm$. The other cases are similar.

Since $\tau_i^\vee = \eta_{s_i}^Y c_{\alpha_i^\vee}^Y$ and $t^{\frac{1}{2}} \tau_i^\vee E_\mu = E_{s_i^\vee \mu}$, then

$$E_{s_i^\vee \mu} = t^{\frac{1}{2}} \tau_i^\vee E_\mu = t^{\frac{1}{2}} \eta_{s_i}^Y c_{\alpha_i^\vee}^Y E_\mu = \eta_{s_i}^Y \kappa_{\alpha_i}^Y E_\mu = \kappa_{\alpha_i}^{Y^{-1}} \eta_{s_i}^Y E_\mu.$$

If $z \in W^{\lambda+\omega}$, then

$$E_z(\lambda+\omega) = \eta_{s_{i_1}} \kappa_{\alpha_{i_1}}^Y \cdots \eta_{s_{i_\ell}} \kappa_{\alpha_{i_\ell}}^Y E_{\lambda+\omega} = \eta_z \kappa_z^Y E_{\lambda+\omega} = \kappa_{z^{-1}}^{Y^{-1}} \eta_z E_{\lambda+\omega}.$$

If $w \in W_{\lambda+\omega}$, then

$$T_w E_{\lambda+\omega} = \chi^+(T_w) E_{\lambda+\omega} \quad (\text{in the same way that } T_w \mathbf{1}_Y = \chi^+(T_w) \mathbf{1}_Y \text{ for } w \in W_{\text{fin}}).$$

This gives that

$$\varepsilon_{\lambda+\omega}^+ E_{\lambda+\omega} = \frac{1}{\chi^+(T_{w_{\lambda+\omega}})} \sum_{w \in W_{\lambda+\omega}} \chi^+(T_w)^2 E_{\lambda+\omega} = \frac{1}{\chi^+(T_{w_{\lambda+\omega}})} W_{\lambda+\omega}(t, t_n) E_{\lambda+\omega}.$$

Using Proposition (13) gives

$$\begin{aligned} A_{\lambda+\omega}^\pm &= \frac{\chi^+(T_{w_0})}{W_{\lambda+\omega}(t, t_n)} \varepsilon_\pm E_{\lambda+\omega} = \frac{1}{W_{\lambda+\omega}(t, t_n)} \\ &\quad \times \left(\sum_{v \in W^{\lambda+\omega}} (-1)^{\ell_d(z)} \kappa_{v_{\lambda+\omega} z}^\pm \eta_z^Y \kappa_z^Y \right) \chi^+(T_{w_{\lambda+\omega}}) \varepsilon_{\lambda+\omega}^+ E_{\lambda+\omega} \\ &= \left(\sum_{v \in W^{\lambda+\omega}} (-1)^{\ell_d(z)} \kappa_{v_{\lambda+\omega} z}^\pm \eta_z^Y \kappa_z^Y \right) E_{\lambda+\omega} = \sum_{v \in W^{\lambda+\omega}} (-1)^{\ell_d(z)} \kappa_{v_{\lambda+\omega} z}^\pm E_{z(\lambda+\omega)} \\ &= \sum_{z \in W_{\text{fin}}} (-1)^{\ell_d(z)} \text{ev}_z^t(\lambda+\omega) (\kappa_{v_{\lambda+\omega} z}^\pm) E_{z(\lambda+\omega)}. \end{aligned}$$

□

5.6. Principal specializations

One of the most pleasing combinatorial miracles in Lie theory is that principal specializations of Schur functions and Weyl characters factor as products (see [8, §10.9] and [10, Ch. I §3 Ex. 1]). This feature extends to Macdonald–Koornwinder polynomials, and the result in this section shows that the principal specializations of Macdonald–Koornwinder polynomials are evaluations of c -functions, which come naturally out of the recursive construction of the electronic Macdonald polynomial E_μ .

Define ring homomorphisms $\text{ev}_0^t : \mathbb{K}[Y] \rightarrow \mathbb{K}$ and $\text{ev}_0^{t^{-1}} : \mathbb{K}[Y] \rightarrow \mathbb{K}$ by

$$\text{ev}_0^t(Y_i) = t^{n-i} t_0^{\frac{1}{2}} t_n^{\frac{1}{2}} \quad \text{and} \quad \text{ev}_0^{t^{-1}}(Y_i^{-1}) = t^{-(n-i)} t_0^{-\frac{1}{2}} t_n^{-\frac{1}{2}}, \quad \text{for } i \in \{1, \dots, n\}.$$

THEOREM 5.1. *Let $\mu, \lambda \in \mathbb{Z}^n$ with $\lambda_1 \geq \dots \geq \lambda_n \geq 0$. Let u_μ and h_λ be as defined in (3) and the normalized c -functions $\kappa_{u_\mu}^{Y^{-1}}$ and $\kappa_{h_\lambda}^{Y^{-1}}$ as in (19). Let*

$$a_1 = t^{-(n-1)} (t_0 t_n)^{-\frac{1}{2}}, \quad a_2 = t^{-(n-2)} (t_0 t_n)^{-\frac{1}{2}}, \quad \dots, \quad a_n = (t_0 t_n)^{-\frac{1}{2}}.$$

Then

$$E_\mu(a_1, \dots, a_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}) = \frac{1}{\chi^+(T_{v_\mu^{-1}})} \text{ev}_0^t(c_{u_\mu}^{Y^{-1}}) \quad \text{and}$$

$$P_\lambda(a_1, \dots, a_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}) = \text{ev}_0^{t^{-1}}(c_{h_\lambda}^{Y^{-1}}).$$

Proof. For this proof, use the realization of the polynomial representation $\mathbb{K}[X]$ as an induced module $\tilde{H}\mathbf{1}_Y$ via the $\tilde{H}$ -module isomorphism of (43). Let $\mathbf{1}_X$ be a formal symbol, which satisfies $\mathbf{1}_X T_j = t^{\frac{1}{2}} \mathbf{1}_X$ and $\mathbf{1}_X T_0^\vee = t_0^{\frac{1}{2}} \mathbf{1}_X$. Using $X_1 = (T_0^\vee)^{-1} T_1^{-1} \dots T_n^{-1} \dots T_1^{-1}$ and $X_{i+1} = T_i X_i T_i$ gives

$$\mathbf{1}_X X_i = t^{-(n-1)} t_n^{-\frac{1}{2}} t_0^{-\frac{1}{2}} t^{i-1}, \quad \text{for } i \in \{1, \dots, n\}.$$

Thus, if $\mu \in \mathbb{Z}^n$, then

$$\mathbf{1}_X E_\mu \left(x_1, \dots, x_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) = \mathbf{1}_X E_\mu \left(a_1, \dots, a_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right).$$

For $i \in \{0, 1, \dots, n\}$,

$$\mathbf{1}_X \tau_i^\vee = \mathbf{1}_X \left(T_{\alpha_i^\vee} + \left(c_{\alpha_i^\vee}^{Y^{-1}} - t^{\frac{1}{2}} \right) \right) = \mathbf{1}_X \left(t^{\frac{1}{2}} + \left(c_{\alpha_i^\vee}^{Y^{-1}} - t^{\frac{1}{2}} \right) \right) = \mathbf{1}_X c_{\alpha_i^\vee}^{Y^{-1}}.$$

By (48),

$$c_{\alpha_i^\vee}^{Y^{-1}} \mathbf{1}_Y = \text{ev}_0^t(c_{\alpha_i^\vee}^{Y^{-1}}) \mathbf{1}_Y.$$

If $w \in W$ and $\ell(s_i w) > \ell(w)$, then

$$\mathbf{1}_X \tau_i^\vee \tau_w^\vee \mathbf{1}_Y = \mathbf{1}_X c_{\alpha_i^\vee}^{Y^{-1}} \tau_w^\vee \mathbf{1}_Y = \mathbf{1}_X \tau_w^\vee c_{w^{-1}\alpha_i^\vee}^{Y^{-1}} \mathbf{1}_Y = \text{ev}_0^t \left(c_{w^{-1}\alpha_i^\vee}^{Y^{-1}} \right) \mathbf{1}_X \tau_w^\vee \mathbf{1}_Y.$$

This is the induction step giving that if $w \in W$ and $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced word for w , then

$$\mathbf{1}_X \tau_w^\vee \mathbf{1}_Y = \mathbf{1}_X \tau_{i_1}^\vee \cdots \tau_{i_\ell}^\vee \mathbf{1}_Y = \mathbf{1}_X \text{ev}_0^t \left(c_w^{Y^{-1}} \right) \mathbf{1}_Y = \text{ev}_0^t \left(c_w^{Y^{-1}} \right) \mathbf{1}_X \mathbf{1}_Y.$$

Thus

$$\mathbf{1}_X E_\mu \mathbf{1}_Y = \frac{1}{\chi^+(T_{v_\mu^{-1}})} \mathbf{1}_X \tau_{u_\mu}^\vee \mathbf{1}_Y = \frac{1}{\chi^+(T_{v_\mu^{-1}})} \text{ev}_0^t(c_{u_\mu}^{Y^{-1}}) \mathbf{1}_X \mathbf{1}_Y.$$

Using $\mathbf{1}_X \varepsilon_+ = \frac{1}{\chi^+(T_{w_0})} W_0(t, t_n) \mathbf{1}_X$ from (39) gives

$$\begin{aligned} P_\lambda \left(a_1, \dots, a_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) \mathbf{1}_X \mathbf{1}_Y &= \mathbf{1}_X P_\lambda \mathbf{1}_Y = \mathbf{1}_X \frac{\chi^+(T_{w_0})}{W_\lambda(t, t_n)} \varepsilon_+ E_\lambda \mathbf{1}_Y \\ &= \frac{W_0(t, t_n)}{W_\lambda(t, t_n)} \mathbf{1}_X E_\lambda \mathbf{1}_Y = \frac{1}{\chi^+(T_{v_\lambda^{-1}})} \frac{W_0(t, t_n)}{W_\lambda(t, t_n)} \text{ev}_0^t(c_{u_\lambda}^{Y^{-1}}) \mathbf{1}_X \mathbf{1}_Y. \end{aligned}$$

Let

$$w_\lambda \in W_{\text{fin}} \quad \text{be the longest element of } W_\lambda = \{v \in W_{\text{fin}} \mid v\lambda = \lambda\}.$$

Then by Proposition 5.2,

$$v_\lambda^{-1} = (w_0 w_\lambda)^{-1} = w_\lambda w_0 \quad \text{and} \quad W_\lambda(t, t_n) = \text{ev}_0^t(\kappa_{w_\lambda}^Y) = \text{ev}_0^{t^{-1}}(\kappa_{w_\lambda}^{Y^{-1}}).$$

Since $h_\lambda = u_\lambda v_\lambda$, then $u_\lambda = h_\lambda v_\lambda^{-1} = v_\lambda^{-1} h_{v_\lambda \lambda} = v_\lambda^{-1} h_{v_\lambda \lambda} = v_\lambda^{-1} h_{w_0 \lambda}$. Using this and $\text{ev}_0^t(Y_i) = \text{ev}_0^{t^{-1}}(Y_i^{-1}) = \text{ev}_0^{t^{-1}}(Y_{-i}) = \text{ev}_0^{t^{-1}}(Y_{w_0(i)})$ gives

$$\text{ev}_0^t(c_{u_\lambda}^{Y^{-1}}) = \text{ev}_0^{t^{-1}}(v_\lambda^{-1} c_{u_\lambda}^{Y^{-1}}).$$

Therefore,

$$\begin{aligned} \frac{1}{\chi^+(T_{v_\lambda^{-1}})} \frac{W_0(t, t_n)}{W_\lambda(t, t_n)} \text{ev}_0^t(c_{u_\lambda}^{Y^{-1}}) &= \frac{\chi^+(T_{w_\lambda})}{\chi^+(T_{w_0})} \text{ev}_0^{t^{-1}} \left(\frac{\kappa_{w_0}^{Y^{-1}}}{\kappa_{w_\lambda}^{Y^{-1}}} \right) \text{ev}_0^t(c_{u_\lambda}^{Y^{-1}}) \\ &= \text{ev}_0^{t^{-1}} \left(\frac{c_{w_0}^{Y^{-1}}}{c_{w_\lambda}^{Y^{-1}}} \right) \text{ev}_0^{t^{-1}}(v_\lambda^{-1} c_{u_\lambda}^{Y^{-1}}) \\ &= \text{ev}_0^{t^{-1}}(c_{v_\lambda}^{Y^{-1}}) \text{ev}_0^{t^{-1}}(v_\lambda^{-1} c_{u_\lambda}^{Y^{-1}}) = \text{ev}_0^{t^{-1}}(c_{u_\lambda v_\lambda}^{Y^{-1}}) = \text{ev}_0^{t^{-1}}(c_{h_\lambda}^{Y^{-1}}), \end{aligned}$$

which completes the proof of the second statement. $\square$

REMARK 5.2. The principal specializations of the fermionic and mesonic Macdonald polynomials are

$$\begin{aligned} A_{\lambda+\rho} \left(a_1, \dots, a_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) &= 0, \\ A_{\lambda+\omega}^{\pm} \left(a_1, \dots, a_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) &= 0, \\ \text{and} \\ A_{\lambda+\pi}^{\mp} \left(a_1, \dots, a_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) &= 0. \end{aligned}$$

To establish this for $A_{\lambda+\omega}^{\pm}$, use $\mathbf{1}_X \varepsilon_{\pm} = 0$ to get

$$\begin{aligned} A_{\lambda+\omega}^{\pm} \left(a_1, \dots, a_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) \mathbf{1}_X \mathbf{1}_Y \\ = \mathbf{1}_X A_{\lambda+\omega}^{\pm} \mathbf{1}_Y = \frac{\chi^+(T_{w_0})}{W_{\lambda+\omega}(t, t_n)} \mathbf{1}_X \varepsilon_{\pm} E_{\lambda+\omega} \mathbf{1}_Y = 0. \end{aligned}$$

The proof for the other cases is similar. $\square$

6. ORTHOGONALITY

In this section, we study the Koornwinder polynomials as a family of orthogonal polynomials for a specific inner product. The inner product $(\cdot, \cdot)_+$ is defined via multiplication by a kernel and taking the constant term (for those with an analytic bent, taking the constant term is an integral, and the kernel is what defines the measure for the integral). The Macdonald–Koornwinder inner product is defined in sections 6.1 and 6.2. The kernel is a huge product of c -functions, one for each positive root in the affine root system of type $(C_n^{\vee}, C_n)$.

The Koornwinder polynomials are characterized by orthogonality with respect to this inner product and a triangular expansion in terms of monomials. In order to use the Hecke algebra as a tool in the inner product setting, it is crucial to establish that the adjoints of operators that come from the Hecke algebra are tractable. This is done in section 6.4. In particular, we find that the symmetrizers are self-adjoint operators.

The proof of the Weyl character formulas and the norm formulas for Koornwinder polynomials rely on a shift of parameters coming from multiplying by the Weyl denominators. These going-up-a-level formulas, derived in section 6.5, are the key to establishing recursive relations for computing norms. The recursive relations are derived in section 6.7, and the norm formula for $(P_{\lambda}, P_{\lambda})_+$ is established in section 6.9.

In the same way that there are four symmetrizers, there are four going-up-a-level formulas, four Weyl character formulas and four types of recursion relations. In each case, one of the four formulas is usually a triviality, but we have included these trivial formulas in our exposition each time in order to highlight the underlying symmetry of the structures. In the end, the various formulas combine and complement each other to provide the inductive structure for computing norms in terms of c -functions.

6.1. The kernel Δ_{CC}^+

For an affine root β , define

$$\kappa_{\beta}^X = t_{\beta}^{\frac{1}{2}} c_{\beta}^X = \frac{(1 - t_{\beta}^{\frac{1}{2}} u_{\beta}^{\frac{1}{2}} X^{\beta})(1 + t_{\beta}^{\frac{1}{2}} u_{\beta}^{-\frac{1}{2}} X^{\beta})}{(1 - X^{2\beta})}. \quad (57)$$

More specifically, if $i, j \in \{1, \dots, n\}$ with $i < j$ and $r \in \mathbb{Z}_{\geq 0}$, then

$$\kappa_{\varepsilon_i - \varepsilon_j + (r+1)\delta}^X = \frac{1 - tq^{r+1}X_iX_j^{-1}}{1 - q^{r+1}X_iX_j^{-1}}, \quad \kappa_{\varepsilon_i + (r+1)\delta}^X = \frac{(1 - q^{r+1}t_n^{\frac{1}{2}}u_n^{\frac{1}{2}}X_i)(1 + q^{r+1}t_n^{\frac{1}{2}}u_n^{-\frac{1}{2}}X_i)}{1 - q^{2r+2}X_i^2},$$

$$\kappa_{\varepsilon_i + \varepsilon_j + (r+1)\delta}^X = \frac{1 - tq^{r+1}X_iX_j}{1 - q^{r+1}X_iX_j}, \quad \kappa_{\varepsilon_i + (r+\frac{1}{2})\delta}^X = \frac{(1 - q^{r+\frac{1}{2}}t_0^{\frac{1}{2}}u_0^{\frac{1}{2}}X_i)(1 + q^{r+\frac{1}{2}}t_0^{\frac{1}{2}}u_0^{-\frac{1}{2}}X_i)}{1 - q^{2r+1}X_i^2}.$$

We shall group roots of the affine root system into three types: ‘gauche’, ‘droite’ and ‘standard’. The ‘droite’ roots are elements of the W_Y -orbit of α_n (which is on the right end of the diagram in (9)), the ‘gauche’ roots are roots in the W_Y -orbit of α_0 (which is on the left end of the diagram (9)) and the ‘standard’ inversions are roots in the W_{fin} -orbit of $\alpha_{n-1}^\vee$ (the middle part of the diagram in (9)). In other words, let S^+ be the set of positive roots for the affine root system so that

$$S^+ = S_{g,+}^+ \cup S_{g,-}^+ \cup S_{s,+}^+ \cup S_{s,-}^+ \cup S_{d,+}^+ \cup S_{d,-}^+ \cup S_{0,s}^+ \cup S_{0,d}^+,$$

where

$$S_{s,+}^+ = \left\{ (\varepsilon_i - \varepsilon_j) + (r+1)\delta \mid i, j \in \{1, \dots, n\}, \right. \\ \left. (\varepsilon_i + \varepsilon_j) + (r+1)\delta \mid i < j \text{ and } r \in \mathbb{Z}_{\geq 0} \right\},$$

$$S_{s,-}^+ = \left\{ -(\varepsilon_i - \varepsilon_j) + (r+1)\delta, \right. \\ \left. -(\varepsilon_i + \varepsilon_j) + (r+1)\delta \mid i < j \text{ and } r \in \mathbb{Z}_{\geq 0} \right\},$$

$$S_{g,+}^+ = \left\{ \varepsilon_i + (r + \frac{1}{2})\delta \mid i \in \{1, \dots, n\}, \right. \\ \left. r \in \mathbb{Z}_{\geq 0} \right\}, \quad S_{d,+}^+ = \left\{ \varepsilon_i + (r+1)\delta \mid i \in \{1, \dots, n\}, \right. \\ \left. r \in \mathbb{Z}_{\geq 0} \right\},$$

$$S_{g,-}^+ = \left\{ -\varepsilon_i + (r + \frac{1}{2})\delta \mid i \in \{1, \dots, n\}, \right. \\ \left. r \in \mathbb{Z}_{\geq 0} \right\}, \quad S_{d,-}^+ = \left\{ -\varepsilon_i + (r+1)\delta \mid i \in \{1, \dots, n\}, \right. \\ \left. r \in \mathbb{Z}_{\geq 0} \right\},$$

$$S_{0,s}^+ = \{\varepsilon_i \pm \varepsilon_j \mid i, j \in \{1, \dots, n\} \text{ with } i < j\}, \quad S_{0,d}^+ = \{\varepsilon_i \mid i \in \{1, \dots, n\}\}.$$

Then, define

$$\Delta_{\text{CC}}^+ = \prod_{\beta \in S^+} \kappa_\beta^X. \quad (58)$$

and

$$\Delta_g^X = \prod_{\beta \in S_{g,+}^+} \kappa_\beta^X, \quad \Delta_s^X = \prod_{\beta \in S_{s,+}^+} \kappa_\beta^X, \quad \Delta_d^X = \prod_{\beta \in S_{d,+}^+} \kappa_\beta^X, \quad \Delta_{0,s}^X = \prod_{\beta \in S_{0,s}^+} \kappa_\beta^X,$$

$$\Delta_g^{X^{-1}} = \prod_{\beta \in S_{g,-}^+} \kappa_\beta^X, \quad \Delta_s^{X^{-1}} = \prod_{\beta \in S_{s,-}^+} \kappa_\beta^X, \quad \Delta_d^{X^{-1}} = \prod_{\beta \in S_{d,-}^+} \kappa_\beta^X, \quad \Delta_{0,d}^X = \prod_{\beta \in S_{0,d}^+} \kappa_\beta^X,$$

so that

$$\Delta_{\text{CC}}^+ = \Delta_g^X \Delta_g^{X^{-1}} \Delta_s^X \Delta_s^{X^{-1}} \Delta_d^X \Delta_d^{X^{-1}} \Delta_{0,s}^X \Delta_{0,d}^X.$$

REMARK 6.1. In terms of the Askey–Wilson parameters a, b, c, d [see (1)], the expression Δ_{CC}^+ used to define the inner product is

$$\Delta_{\text{CC}}^+ = \Delta^{(1)} \Delta^{(2)},$$

where

$$\Delta^{(2)} = \prod_{1 \leq i < j \leq n} \frac{(tx_i x_j^{-1}; q)_\infty (tx_i x_j; q)_\infty (tx_i^{-1} x_j; q)_\infty (tx_i^{-1} x_j^{-1}; q)_\infty}{(x_i x_j^{-1}; q)_\infty (x_i x_j; q)_\infty (qx_i^{-1} x_j; q)_\infty (qx_i^{-1} x_j^{-1}; q)_\infty}$$

and

$$\Delta^{(1)} = \prod_{i=1}^n \frac{(cx_i; q)_\infty (qcx_i^{-1}; q)_\infty (dx_i; q)_\infty (qdx_i^{-1}; q)_\infty (ax_i; q)_\infty (ax_i^{-1}; q)_\infty (bx_i; q)_\infty (bx_i^{-1}; q)_\infty}{(x_i^2; q)_\infty (qx_i^{-2}; q)_\infty}.$$

This is verified by noting that $\Delta^{(2)} = \Delta_s^X \Delta_{0,s}^X \Delta_s^{X^{-1}}$, and that $\Delta^{(1)} = \Delta_g^X \Delta_g^{X^{-1}} \Delta_d^X \Delta_d^{X^{-1}} \Delta_{0,d}^X$ since

$$\begin{aligned} \prod_{i=1}^n \prod_{r \in \mathbb{Z}_{\geq 0}} \frac{(cx_i; q)_{\infty} (dx_i; q)_{\infty}}{(x_i^2; q^2)_{\infty}} &= \prod_{i=1}^n \prod_{r \in \mathbb{Z}_{\geq 0}} \frac{(1 - t_n^{\frac{1}{2}} u_n^{\frac{1}{2}} q^r x_i) (1 + t_n^{\frac{1}{2}} u_n^{-\frac{1}{2}} q^r x_i)}{(1 - q^{2r} x_i^2)} = \Delta_d^X \Delta_{0,d}^X, \\ \prod_{i=1}^n \prod_{r \in \mathbb{Z}_{\geq 0}} \frac{(qcx_i^{-1}; q)_{\infty} (qdx_i^{-1}; q)_{\infty}}{(q^2 x_i^{-2}; q^2)_{\infty}} &= \prod_{i=1}^n \prod_{r \in \mathbb{Z}_{\geq 0}} \frac{(1 - t_n^{\frac{1}{2}} u_n^{\frac{1}{2}} q^{r+1} x_i^{-1}) (1 + t_n^{\frac{1}{2}} u_n^{-\frac{1}{2}} q^{r+1} x_i^{-1})}{(1 - q^{2r+2} x_i^{-2})} = \Delta_d^{X^{-1}}, \\ \prod_{i=1}^n \prod_{r \in \mathbb{Z}_{\geq 0}} \frac{(ax_i; q)_{\infty} (bx_i; q)_{\infty}}{(qx_i; q^2)_{\infty}} &= \prod_{i=1}^n \prod_{r \in \mathbb{Z}_{\geq 0}} \frac{(1 - t_0^{\frac{1}{2}} u_0^{\frac{1}{2}} q^{r+\frac{1}{2}} x_i) (1 + t_0^{\frac{1}{2}} u_0^{-\frac{1}{2}} q^{r+\frac{1}{2}} x_i)}{(1 - q^{2r+1} x_i^2)} = \Delta_g^X, \\ \prod_{i=1}^n \prod_{r \in \mathbb{Z}_{\geq 0}} \frac{(ax_i^{-1}; q)_{\infty} (bx_i^{-1}; q)_{\infty}}{(qx_i^{-2}; q^2)_{\infty}} &= \prod_{i=1}^n \prod_{r \in \mathbb{Z}_{\geq 0}} \frac{(1 - t_0^{\frac{1}{2}} u_0^{\frac{1}{2}} q^{r+\frac{1}{2}} x_i^{-1}) (1 + t_0^{\frac{1}{2}} u_0^{-\frac{1}{2}} q^{r+\frac{1}{2}} x_i^{-1})}{(1 - q^{2r+1} x_i^{-2})} = \Delta_g^{X^{-1}}. \end{aligned}$$

□

6.2. Definition of the inner product

Let $\mathbb{K}[x] = \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Define an involution $\bar{} : \mathbb{K}[x] \rightarrow \mathbb{K}[x]$ by

$$\bar{f}(x_1, \dots, x_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}) = f(x_1^{-1}, \dots, x_n^{-1}; q^{-1}, t^{-1}, t_0^{-\frac{1}{2}}, u_0^{-\frac{1}{2}}, t_n^{-\frac{1}{2}}, u_n^{-\frac{1}{2}}). \quad (59)$$

Let Δ_{CC}^+ be as defined in (58). Define a scalar product $(\ , \)_+ : \mathbb{K}[x] \times \mathbb{K}[x] \rightarrow \mathbb{K}$ by

$$(f_1, f_2)_+ = \text{ct} \left(\frac{f_1 \bar{f}_2}{\Delta_{CC}^+} \right), \quad \text{where} \quad \text{ct}(f) = (\text{constant term in } f), \quad \text{for } f \in \mathbb{K}[x]. \quad (60)$$

6.3. The inner product characterization of E_{μ} and P_{λ}

Define

$$(\mathbb{Z}^n)^+ = \{(\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n \mid \gamma_1 \geq \dots \geq \gamma_n \geq 0\}.$$

The elements of $(\mathbb{Z}^n)^+$ are partially ordered by the dominance order: For $\lambda, \mu \in (\mathbb{Z}_{\geq 0}^n)^+$,

$$\lambda < \mu \quad \text{if} \quad \lambda_1 + \dots + \lambda_i \leq \mu_1 + \dots + \mu_i, \quad \text{for } i \in \{1, \dots, n\}.$$

The elements of $\mathbb{Z}^n$ are partially ordered by the DBlex order: For $\lambda, \mu \in \mathbb{Z}^n$,

$$\begin{aligned} \lambda^+ &< \mu^+ \text{ in dominance order} \\ \lambda &\leq \mu \quad \text{if} \quad \lambda^+ = \mu^+ \text{ and } z_{\lambda} < z_{\mu} \text{ in Bruhat order on } W_{\text{fin}}, \end{aligned}$$

where $\lambda^+ \in W_{\text{fin}} \lambda \cap (\mathbb{Z}_{\geq 0}^n)^+$ and z_{λ} is the minimal length element of W_{fin} so that $\lambda = z_{\lambda} \lambda^+$.

For $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$, write $x^{\mu} = x_1^{\mu_1} \dots x_n^{\mu_n}$, and for $\gamma \in (\mathbb{Z}^n)^+$, define the *monomial symmetric function* m_{γ} by

$$m_{\gamma} = \sum_{\mu \in W_{\text{fin}} \gamma} x^{\mu}, \quad \text{where the sum is over all elements of the orbit } W_{\text{fin}}\text{-orbit of } \gamma.$$

With these definitions we have the following characterizations of the E_{μ} and the P_{λ} . The proofs of Propositions 6.1 and 6.2 are exactly as in [13, (5.2.1) and (5.3.1)] and [4, Prop. 6.2 and 6.3].

PROPOSITION 6.1. *Let $\mu \in \mathbb{Z}^n$. The electronic Macdonald polynomial E_{μ} is the unique element of $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ such that*

$$(a) \ E_{\mu} = x^{\mu} + (\text{lower terms});$$

(b) If $v \in \mathbb{Z}^n$ and $v < \mu$, then $(E_\mu, x^v)_+ = 0$.

PROPOSITION 6.2. Let $\lambda \in (\mathbb{Z}^n)^+$. The bosonic Macdonald polynomial P_λ is the unique element of $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{W_{\text{fin}}}$ such that

- (a) $P_\lambda = m_\lambda + (\text{lower terms})$;
 (b) If $\gamma \in (\mathbb{Z}^n)^+$ and $\gamma < \lambda$, then $(P_\lambda, m_\gamma)_+ = 0$.

6.4. Adjoints and orthogonality

For a linear operator $M : \mathbb{K}[X] \rightarrow \mathbb{K}[X]$, the *adjoint* of M is the linear operator $M^* : \mathbb{K}[X] \rightarrow \mathbb{K}[X]$ determined by

$$(Mf_1, f_2)_+ = (f_1, M^*f_2)_+, \quad \text{for } f_1, f_2 \in \mathbb{K}[X],$$

where the inner product on $\mathbb{K}[X]$ is as defined in (60).

The following Proposition computes the adjoints of operators on $\mathbb{K}[X]$, which come from $\tilde{H}_{\text{loc}}$.

PROPOSITION 6.3. Let $i \in \{1, \dots, n\}$ and $k \in \{0, 1, \dots, n\}$. Then, as operators on $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$,

$$x_i^* = x_i^{-1}, \quad T_{\alpha_k}^* = T_{\alpha_k}^{-1}, \quad Y_i^* = Y_i^{-1}, \quad \xi_{s_k}^* = \frac{\kappa_{\alpha_k}^X}{\kappa_{-\alpha_k}^X} \xi_{s_k}.$$

Proof. Let $J = \frac{1}{\Delta_{\text{CC}}^+}$.

◦ Adjoint of multiplication by x_i :

$$(x_i f, g)_+ = \text{ct}(x_i f \cdot \bar{g} \cdot J) = \text{ct}(f \cdot \overline{x_i^{-1} g} \cdot J) = (f, x_i^{-1} g)_+.$$

◦ Adjoint of ξ_{s_k} : With κ_β as in (57),

$$\begin{aligned} (\kappa_\beta^X)^* &= \overline{\kappa_\beta^X} = \frac{(1 - t_\beta^{-\frac{1}{2}} u_\beta^{-\frac{1}{2}} X^{-\beta})(1 + t_\beta^{-\frac{1}{2}} u_\beta^{\frac{1}{2}} X^{-\beta})}{(1 - X^{-2\beta})} \\ &= \frac{t_\beta^{-\frac{1}{2}} u_\beta^{-\frac{1}{2}} X^{-\beta} t_\beta^{-\frac{1}{2}} u_\beta^{\frac{1}{2}} X^{-\beta} (1 - t_\beta^{\frac{1}{2}} u_\beta^{\frac{1}{2}} X^\beta)(1 + t_\beta^{\frac{1}{2}} u_\beta^{-\frac{1}{2}} X^\beta)}{X^{-2\beta} (1 - X^{2\beta})} = t_\beta^{-1} \kappa_\beta. \end{aligned}$$

Since $s_k S^+ = (S^+ - \{\alpha_k\}) \cup \{-\alpha_k\}$, we have

$$\xi_{s_k} \Delta_{\text{CC}}^+ = \xi_{s_k} \left(\prod_{\beta \in S^+} \kappa_\beta \right) = \left(\prod_{\beta \in S^+ - \{\alpha_k\}} \kappa_\beta \right) \kappa_{-\alpha_k} = \Delta_{\text{CC}}^+ \frac{\kappa_{-\alpha_k}}{\kappa_{\alpha_k}}. \quad (61)$$

Using $\text{ct}(\xi_{s_k} g) = \text{ct}(g)$ and the formula from (61),

$$\begin{aligned} (\xi_{s_k} f, g)_+ &= \text{ct}((\xi_{s_k} f) \bar{g} J) = \text{ct}(\xi_{s_k}(f(\xi_{s_k}(\bar{g} J)))) = \text{ct}(f(\xi_{s_k}(\bar{g} J))) \\ &= \text{ct}\left(f(\xi_{s_k} \bar{g}) J \frac{\kappa_{\alpha_k}}{\kappa_{-\alpha_k}}\right) = \text{ct}\left(f \frac{\overline{\kappa_{\alpha_k}}}{\kappa_{-\alpha_k}} (\xi_{s_k} g) J\right) = \left(f, \frac{\kappa_{\alpha_k}}{\kappa_{-\alpha_k}} (\xi_{s_k} g)\right)_+. \end{aligned}$$

- Adjoint of T_{α_k} : Using the formula for T_{α_k} in (26) and recalling that $(\ , \)_+$ is sesquilinear with respect to the involution $\overline{} : \mathbb{K} \rightarrow \mathbb{K}$,

$$\begin{aligned} (T_{\alpha_k})^* &= \left(-t_{\alpha_k}^{-\frac{1}{2}} + (1 + \xi_{s_k})c_{\alpha_k}^{X^{-1}} \right)^* = \left(-t_{\alpha_k}^{-\frac{1}{2}} + (1 + \xi_{s_k})t_{\alpha_k}^{-\frac{1}{2}}\kappa_{-\alpha_k} \right)^* \\ &= -t_{\alpha_k}^{\frac{1}{2}} + t_{\alpha_k}^{\frac{1}{2}}\overline{\kappa_{-\alpha_k}}(1 + \xi_{s_k}^*) \\ &= -t_{\alpha_k}^{\frac{1}{2}} + t_{\alpha_k}^{\frac{1}{2}}t_{\alpha_k}^{-1}\kappa_{-\alpha_k} \left(1 + \frac{\kappa_{\alpha_k}}{\kappa_{-\alpha_k}}\xi_{s_k} \right) = -t_{\alpha_k}^{\frac{1}{2}} + t_{\alpha_k}^{-\frac{1}{2}}(\kappa_{-\alpha_k} + \kappa_{\alpha_k}\xi_{s_k}) \\ &= -t_{\alpha_k}^{\frac{1}{2}} + (1 + \xi_{s_k})c_{-\alpha_k}^X = T_{\alpha_k}^{-1}. \end{aligned}$$

- Adjoint of Y_j :

$$Y_1^* = (T_0 T_1 \cdots T_n \cdots T_1)^* = T_1^{-1} \cdots T_n^{-1} \cdots T_1^{-1} T_0^{-1} = (T_0 T_1 \cdots T_n \cdots T_1)^{-1} = Y_1^{-1},$$

and if $j \in \{2, \dots, n\}$, then

$$Y_j^* = (T_{j-1}^{-1} Y_{j-1} T_{j-1})^* = T_{j-1} Y_{j-1}^{-1} T_{j-1} = (T_{j-1}^{-1} Y_{j-1} T_{j-1})^{-1} = Y_j^{-1}.$$

□

Since $T_j^{-1} \varepsilon_{\pm}^* = T_j^* \varepsilon_{\pm}^* = (\varepsilon_{\pm} T_j)^*$ for $j \in \{1, \dots, n\}$, then $T_n^{-1} \varepsilon_{\pm}^* = (-t_n)^{\frac{1}{2}} \varepsilon_{\pm}^*$ and $T_i^{-1} \varepsilon_{\pm}^* = t^{-\frac{1}{2}} \varepsilon_{\pm}^*$ for $i \in \{1, \dots, n-1\}$. Since

$$\varepsilon_{\pm}^* = T_{w_0}^{-1} + (\text{lower terms}) = T_{w_0} + (\text{lower terms}), \quad \text{then} \quad \varepsilon_{\pm}^* = \varepsilon_{\pm}.$$

A similar argument applies to the other symmetrizers to show that

$$\varepsilon_{\Xi}^* = \varepsilon_{\Xi}, \quad \text{for} \quad \Xi \in \{+, \pm, \mp, -\}. \quad (62)$$

The relations $Y_i^* = Y_i^{-1}$ in combination with the knowledge of the eigenvalues for the action of the Y_i on the E_{μ} give the following orthogonality relations for Macdonald polynomials. The proof is exactly as in [13, (5.7.11)] and [4, Proposition 7.2].

PROPOSITION 6.4. *Let $(\mathbb{Z}_{\geq 0}^n)^+ = \{\lambda = (\lambda_1, \dots, \lambda_n) \mid \lambda_1 \geq \dots \geq \lambda_n \geq 0\}$ and, for $\lambda \in (\mathbb{Z}_{\geq 0}^n)^+$, let $P_{\lambda}, A_{\lambda+\omega}^{\pm}, A_{\lambda+\pi}^{\mp}$ and $A_{\lambda+\rho}$ be as defined in (53) and (54).*

- (a) *Let $\lambda, \mu \in \mathbb{Z}^n$. If $\mu \neq \lambda$, then $(E_{\lambda}, E_{\mu})_+ = 0$.*
- (b) *Let $\lambda, \mu \in (\mathbb{Z}_{\geq 0}^n)^+$. If $\mu \neq \lambda$, then*

$$\begin{aligned} (P_{\lambda}, P_{\mu})_+ &= 0, & (A_{\lambda+\omega}^{\pm}, A_{\mu+\omega}^{\pm})_+ &= 0, \\ (A_{\lambda+\rho}, A_{\mu+\rho})_+ &= 0, & (A_{\lambda+\pi}^{\mp}, A_{\mu+\pi}^{\mp})_+ &= 0. \end{aligned}$$

6.5. Going up a level

We describe four slightly different collections of five parameters by the brief notations

$$\begin{aligned} t^+ &= \left(t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right), & t^{\pm} &= \left(t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right), \\ t^- &= \left(qt, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right), & t^{\mp} &= \left(qt, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right), \end{aligned}$$

and define

$$\begin{aligned} (f, g)_+ &= (f, g)_{q, t^+} & (f, g)_{\pm} &= (f, g)_{q, t^{\pm}} \\ (f, g)_{-} &= (f, g)_{q, t^{-}} & (f, g)_{\mp} &= (f, g)_{q, t^{\mp}}. \end{aligned}$$

The following proposition shows that the norms of polynomials in the fermionic, bosonic and mesonic spaces can be computed as norms of symmetric polynomials, but *with shifted parameters*. Alternatively, in the world of norms for symmetric polynomials, the shifted parameters are a residue arising from the effect of multiplying by the Weyl denominators $A_\rho, A_\omega^\pm$ and $A_\pi^\mp$.

PROPOSITION 6.5. (*Going up a level*) Let $f, g \in \mathbb{K}[X]^{W_{\text{fin}}}$ so that f and g are symmetric functions. If $P_0 = 1$ and $A_\omega^\pm, A_\pi^\mp$ and A_ρ are the Weyl denominators defined in (55), then

$$\begin{aligned} (f, g)_+ &= \frac{W_0(t, t_n)}{W_0(t, t_n)} (P_0 f, P_0 g)_+, & (f, g)_\pm &= \frac{W_0(t, qt_n)}{W_0(t, t_n^{-1})} (A_\omega^\pm f, A_\omega^\pm g)_+, \\ (f, g)_- &= \frac{W_0(qt, qt_n)}{W_0(t^{-1}, t_n^{-1})} (A_\rho f, A_\rho g)_+, & (f, g)_\mp &= \frac{W_0(qt, t_n)}{W_0(t^{-1}, t_n)} (A_\pi^\mp f, A_\pi^\mp g)_+. \end{aligned}$$

Proof. Let $t^\pm$ be the five-tuple of parameters $t^\pm = (t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}})$ and let

$$\begin{aligned} \Delta_{\text{CC}}^\pm &= \Delta_g^X \left(t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}} \right) \Delta_g^{X^{-1}} \left(t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}} \right) \Delta_s^X(t) \Delta_s^{X^{-1}}(t) \Delta_d^X \left(qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) \Delta_d^{X^{-1}} \\ &\quad \times \left(qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) \Delta_{0,s}^X(t) \Delta_{0,d}^X \left(qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right). \end{aligned}$$

Since

$$\begin{aligned} \Delta_d^X \left(t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) &= \Delta_d^X \left(qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) \cdot \left(\prod_{1 \leq i \leq n} \left(1 - qt_n^{\frac{1}{2}} u_n^{\frac{1}{2}} x_i \right) \left(1 + qt_n^{\frac{1}{2}} u_n^{-\frac{1}{2}} x_i \right) \right) \\ &= \Delta_d^X \left(qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) \cdot x^\omega A_\omega^\pm \left(x, qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right), \\ \Delta_{0,d}^X \left(t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) &= \Delta_{0,d}^X \left(qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) \cdot \left(\prod_{1 \leq i \leq n} \frac{(1 - t_n^{\frac{1}{2}} u_n^{\frac{1}{2}} x_i)(1 + t_n^{\frac{1}{2}} u_n^{-\frac{1}{2}} x_i)}{(1 - qt_n^{\frac{1}{2}} u_n^{\frac{1}{2}} x_i)(1 + qt_n^{\frac{1}{2}} u_n^{-\frac{1}{2}} x_i)} \right) \\ &= \Delta_{0,d}^X \left(qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) \frac{x^\omega A_\omega^\pm \left(x, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right)}{x^\omega A_\omega^\pm \left(x, qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right)}, \\ \Delta_d^{X^{-1}} \left(t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) &= \Delta_d^{X^{-1}} \left(qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) \cdot \left(\prod_{1 \leq i \leq n} \left(1 - qt_n^{\frac{1}{2}} u_n^{\frac{1}{2}} x_i^{-1} \right) \left(1 + qt_n^{\frac{1}{2}} u_n^{-\frac{1}{2}} x_i^{-1} \right) \right) \\ &= \Delta_d^{X^{-1}} \left(qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) x^{-\omega} A_\omega^\pm \left(x^{-1}, qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right), \end{aligned}$$

then

$$\frac{\Delta_{\text{CC}}^+}{\Delta_{\text{CC}}^-} = A_\omega^\pm \left(x, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) A_\omega^\pm \left(x^{-1}, qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right).$$

Using Proposition 5.2 and that $\text{Card}(W_{\text{fin}}) = 2^n n!$ gives that for $h \in \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{W_{\text{fin}}}$,

$$\begin{aligned} \text{ct} \left(h \kappa_{w_0}^{X^{-1}} \right) &= \frac{1}{2^n n!} \text{ct} \left(\sum_{w \in W_{\text{fin}}} w(h \kappa_{w_0}^{X^{-1}}) \right) = \frac{1}{2^n n!} \text{ct} \left(h \sum_{w \in W_{\text{fin}}} w(\kappa_{w_0}^{X^{-1}}) \right) \\ &= \frac{W_0(t, t_n)}{2^n n!} \text{ct}(h). \end{aligned}$$

Let $\Delta_\infty^X = \Delta_g^X \Delta_s^X \Delta_d^X$ and $\Delta_0^X = \Delta_{0,s}(t) \Delta_{0,d}^X$ so that

$$\Delta_{CC}^+ = \Delta_\infty^X \Delta_\infty^{X^{-1}} \Delta_0^X \quad \text{and} \quad \Delta_0^X = \frac{A_\omega^\pm(x, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}})}{a_\omega^\pm(x)} \frac{A_\pi^\mp(x, t)}{a_\pi^\mp(x)}.$$

Then

$$\begin{aligned} (A_\omega^\pm f, A_\omega^\pm g)_+ &= \text{ct} \left(\frac{f\bar{g}}{\Delta_{CC}^+} A_\omega^\pm \left(x, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) \overline{A_\omega^\pm \left(x, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right)} \right) \\ &= \text{ct} \left(\frac{f\bar{g}}{\Delta_\infty^X \Delta_\infty^{X^{-1}}} \frac{A_\omega^\pm(x, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}})}{\Delta_0^X} A_\omega^\pm(x^{-1}, t_n^{-\frac{1}{2}}, u_n^{-\frac{1}{2}}) \right) \\ &= \text{ct} \left(\frac{f\bar{g}}{\Delta_\infty^X \Delta_\infty^{X^{-1}}} \frac{a_\omega^\pm(x) a_\pi^\mp(x)}{A_\pi^\mp(x, t)} A_\omega^\pm \left(x^{-1}, t_n^{-\frac{1}{2}}, u_n^{-\frac{1}{2}} \right) \right) \\ &= \text{ct} \left(\frac{f\bar{g}}{\Delta_\infty^X \Delta_\infty^{X^{-1}}} \frac{a_\omega^\pm(x) a_\pi^\mp(x) a_\omega^\pm(x^{-1}) a_\pi^\mp(x^{-1})}{A_\pi^\mp(x, t) A_\pi^\mp(x^{-1}, t)} \cdot \frac{A_\omega^\pm(x^{-1}, t_n^{-\frac{1}{2}}, u_n^{-\frac{1}{2}}) A_\pi^\mp(x^{-1}, t)}{a_\omega^\pm(x^{-1}) a_\pi^\mp(x^{-1})} \right) \\ &= \text{ct} \left(\frac{f\bar{g}}{\Delta_\infty^X \Delta_\infty^{X^{-1}}} \frac{a_\omega^\pm(x) a_\pi^\mp(x) a_\omega^\pm(x^{-1}) a_\pi^\mp(x^{-1})}{A_\pi^\mp(x, t) A_\pi^\mp(x^{-1}, t)} \cdot \kappa_{w_0}^{X^{-1}}(t, t_n^{-\frac{1}{2}}, u_n^{-\frac{1}{2}}) \right) \\ &= \frac{W_0(t, t_n^{-1})}{2^n n!} \text{ct} \left(\frac{f\bar{g}}{\Delta_\infty^X \Delta_\infty^{X^{-1}}} \frac{a_\omega^\pm(x) a_\pi^\mp(x) a_\omega^\pm(x^{-1}) a_\pi^\mp(x^{-1})}{A_\pi^\mp(x, t) A_\pi^\mp(x^{-1}, t)} \right) \end{aligned}$$

and

$$\begin{aligned} (f, g)_\pm &= \text{ct} \left(\frac{f\bar{g}}{\Delta_{CC}^\pm} \right) = \text{ct} \left(\frac{f\bar{g}}{\Delta_{CC}^\pm} A_\omega^\pm \left(x, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) A_\omega^\pm \left(x^{-1}, qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) \right) \\ &= \text{ct} \left(\frac{f\bar{g}}{\Delta_\infty^X \Delta_\infty^{X^{-1}}} \frac{A_\omega^\pm \left(x, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right)}{\Delta_0^X} A_\omega^\pm \left(x^{-1}, qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) \right) \\ &= \text{ct} \left(\frac{f\bar{g}}{\Delta_\infty^X \Delta_\infty^{X^{-1}}} \frac{a_\omega^\pm(x) a_\pi^\mp(x)}{A_\pi^\mp(x, t)} A_\omega^\pm \left(x^{-1}, qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) \right) \\ &= \text{ct} \left(\frac{f\bar{g}}{\Delta_\infty^X \Delta_\infty^{X^{-1}}} \frac{a_\omega^\pm(x) a_\pi^\mp(x) a_\omega^\pm(x^{-1}) a_\pi^\mp(x^{-1})}{A_\pi^\mp(x, t) A_\pi^\mp(x^{-1}, t)} \cdot \frac{A_\omega^\pm \left(x^{-1}, qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \right) A_\pi^\mp(x^{-1}, t)}{a_\omega^\pm(x^{-1}) a_\pi^\mp(x^{-1})} \right) \\ &= \text{ct} \left(\frac{f\bar{g}}{\Delta_\infty^X \Delta_\infty^{X^{-1}}} \frac{a_\omega^\pm(x) a_\pi^\mp(x) a_\omega^\pm(x^{-1}) a_\pi^\mp(x^{-1})}{A_\pi^\mp(x, t) A_\pi^\mp(x^{-1}, t)} \cdot \kappa_{w_0}^{X^{-1}}(t, qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}) \right) \\ &= \frac{W_0(t, qt_n)}{2^n n!} \text{ct} \left(\frac{f\bar{g}}{\Delta_\infty^X \Delta_\infty^{X^{-1}}} \frac{a_\omega^\pm(x) a_\pi^\mp(x) a_\omega^\pm(x^{-1}) a_\pi^\mp(x^{-1})}{A_\pi^\mp(x, t) A_\pi^\mp(x^{-1}, t)} \right), \end{aligned}$$

which proves the $\pm$ case. The other cases are similar. $\square$

6.6. Weyl character formulas

As in (41), let $0, \omega, \pi, \rho \in \mathbb{Z}^n$ by $0 = (0, 0, \dots, 0)$,

$$\omega = (1, 1, \dots, 1), \quad \pi = (n-1, n-2, \dots, 2, 1, 0), \quad \rho = (n, n-1, \dots, 3, 2, 1).$$

THEOREM 6.1. (Weyl character formulas) Let $\lambda \in \mathbb{Z}^n$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0$. Then

$$\begin{aligned} P_\lambda(q, t^+) &= \frac{P_{\lambda+\omega}(q, t^+)}{P_0(q, t^+)}, & P_\lambda(q, t^\pm) &= \frac{A_{\lambda+\omega}^\pm(q, t^\pm)}{A_\omega^\pm(q, t^\pm)}, \\ P_\lambda(q, t^-) &= \frac{A_{\lambda+\rho}(q, t^+)}{A_\rho(q, t^+)}, & P_\lambda(q, t^\mp) &= \frac{A_{\lambda+\pi}^\mp(q, t^+)}{A_\pi^\mp(q, t^+)}, \end{aligned}$$

Proof. The following is the proof for the case $P_\lambda(q, t^\pm)$. The proof for the other cases is similar.

Since $A_{\lambda+\omega}^\pm = t^{\frac{1}{2}\ell(w_0)} \varepsilon_\pm E_{\lambda+\omega}$, then $A_{\lambda+\omega}^\pm \in \mathbb{C}[X]^{\text{Mes}^\pm}$. Thus, by Proposition 5.1,

$$\text{there exists } f \in \mathbb{K}[X]^{W_{\text{fin}}} \text{ such that } A_{\lambda+\omega}^\pm = A_\omega^\pm f.$$

Since $\lambda + \omega$ is dominant, then $w_0(\lambda + \omega) = -(\lambda + \omega)$ is antidominant (weakly increasing with all entries ≤ 0). If $\mu \in \mathbb{Z}^n$ is such that the coefficient of x^μ in $A_{\lambda+\omega}^\pm$ is non-zero, then $\mu \leq -(\lambda + \omega)$ in the DBlex order. Thus, using the notations of section 6.3,

$$f = m_\lambda + (\text{lower terms}).$$

The E -expansion for $A_{\lambda+\omega}^\pm$ in Proposition 5.3 gives that

$$A_{\lambda+\omega}^\pm = \sum_{\mu \in W_{\text{fin}}(\lambda+\pi)} d_{\lambda+\omega}^\mu E_\mu = E_{w_0(\lambda+\omega)} + (\text{lower terms})$$

and, from the definitions of $A_\omega^\pm$ and m_ν ,

$$A_\omega^\pm m_\nu = x^{w_0(\nu+\omega)} + (\text{lower terms}).$$

Since $(E_{w_0(\lambda+\omega)}, x^\gamma)_+ = 0$ for $\gamma < w_0(\lambda + \omega)$, then

$$(A_\omega^\pm f, A_\omega^\pm m_\nu)_+ = (A_{\lambda+\omega}^\pm, A_\omega^\pm m_\nu)_+ = 0, \quad \text{for } \nu \in (\mathbb{Z}^n)^+ \text{ with } \nu < \lambda.$$

Using Proposition 6.5, since $f \in \mathbb{K}[X]^{W_{\text{fin}}}$ and $m_\nu \in \mathbb{K}[X]^{W_{\text{fin}}}$, then

$$(f, m_\nu)_\pm = \frac{W_0(t, qt_n)}{W_0(t, t_n^{-1})} (A_\omega^\pm f, A_\omega^\pm m_\nu)_+ = 0, \quad \text{for } \nu \in (\mathbb{Z}^n)^+ \text{ with } \nu < \lambda.$$

Thus, by Proposition 6.2, $f = P_\lambda(q, t^\pm)$. □

6.7. Reductions for norms

The following proposition shows that the c -functions provide an effective framework for describing the differences between the norms of the various flavours of Koornwinder polynomials.

PROPOSITION 6.6.

(a) Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$. Then

$$\frac{(E_\mu, E_\mu)_+}{(1, 1)_+} = \text{ev}_0^t \left(c_{u_\mu}^Y c_{u_\mu}^{Y^{-1}} \right).$$

(b) Let $\lambda \in (\mathbb{Z}_{\geq 0}^n)^+$.

$$\begin{aligned}\frac{(P_\lambda, P_\lambda)_+}{(E_\lambda, E_\lambda)_+} &= \frac{W_0(t, t_n)}{W_\lambda(t, t_n)} \text{ev}_\lambda^t(\kappa_{v_\lambda}^+), \\ \frac{(A_{\lambda+\omega}^\pm, A_{\lambda+\omega}^\pm)_+}{(E_{\lambda+\omega}, E_{\lambda+\omega})_+} &= \frac{W_0(t, t_n^{-1})}{W_{\lambda+\omega}(t, t_n)} t_n^n \text{ev}_{\lambda+\omega}^t(\kappa_{v_{\lambda+\omega}}^\pm), \\ \frac{(A_{\lambda+\pi}^\mp, A_{\lambda+\pi}^\mp)_+}{(E_{\lambda+\pi}, E_{\lambda+\pi})_+} &= \frac{W_0(t^{-1}, t_n)}{W_{\lambda+\pi}(t, t_n)} t^{n(n-1)} \text{ev}_{\lambda+\pi}^t(\kappa_{v_{\lambda+\pi}}^\mp), \\ \frac{(A_{\lambda+\rho}, A_{\lambda+\rho})_+}{(E_{\lambda+\rho}, E_{\lambda+\rho})_+} &= \frac{W_0(t^{-1}, t_n^{-1})}{W_{\lambda+\rho}(t, t_n)} t^{n(n-1)} t_n^n \text{ev}_{\lambda+\rho}^t(\kappa_{v_{\lambda+\rho}}^-).\end{aligned}$$

(c) Let $\lambda \in (\mathbb{Z}_{\geq 0}^n)^+$.

$$\begin{aligned}\frac{(A_{\lambda+\omega}^\pm, A_{\lambda+\omega}^\pm)_+}{(P_{\lambda+\omega}, P_{\lambda+\omega})_+} &= \frac{W_0(t, t_n^{-1})}{W_0(t, t_n)} t_n^n \text{ev}_{\lambda+\omega}^t \left(\frac{\kappa_{v_{\lambda+\omega}}^\pm}{\kappa_{v_{\lambda+\omega}}^+} \right), \\ \frac{(A_{\lambda+\pi}^\mp, A_{\lambda+\pi}^\mp)_+}{(P_{\lambda+\pi}, P_{\lambda+\pi})_+} &= \frac{W_0(t^{-1}, t_n)}{W_0(t, t_n)} t^{n(n-1)} \text{ev}_{\lambda+\pi}^t \left(\frac{\kappa_{v_{\lambda+\pi}}^\mp}{\kappa_{v_{\lambda+\pi}}^+} \right), \\ \frac{(A_{\lambda+\rho}, A_{\lambda+\rho})_+}{(P_{\lambda+\rho}, P_{\lambda+\rho})_+} &= \frac{W_0(t^{-1}, t_n^{-1})}{W_0(t, t_n)} t^{n(n-1)} t_n^n \text{ev}_{\lambda+\rho}^t \left(\frac{\kappa_{w_0}^-}{\kappa_{w_0}^+} \right).\end{aligned}$$

(d) Let $\lambda \in (\mathbb{Z}_{\geq 0}^n)^+$. Then

$$\begin{aligned}\frac{(P_\lambda(q, t^\pm), P_\lambda(q, t^\pm))_\pm}{(P_{\lambda+\omega}(q, t^+), P_{\lambda+\omega}(q, t^+))_+} &= \frac{W_0(t, qt_n)}{W_0(t, t_n)} t_n^{\ell_d(w_0)} \text{ev}_{\lambda+\omega}^t \left(\frac{\kappa_{v_{\lambda+\omega}}^\pm}{\kappa_{v_{\lambda+\omega}}^+} \right) \\ \frac{(P_\lambda(q, t^\mp), P_\lambda(q, t^\mp))_\mp}{(P_{\lambda+\pi}(q, t^+), P_{\lambda+\pi}(q, t^+))_+} &= \frac{W_0(qt, t_n)}{W_0(t, t_n)} t^{\ell_s(w_0)} \text{ev}_{\lambda+\pi}^t \left(\frac{\kappa_{v_{\lambda+\pi}}^\mp}{\kappa_{v_{\lambda+\pi}}^+} \right) \\ \frac{(P_\lambda(q, t^-), P_\lambda(q, t^-))_-}{(P_{\lambda+\rho}(q, t^+), P_{\lambda+\rho}(q, t^+))_+} &= \frac{W_0(qt, qt_n)}{W_0(t, t_n)} t^{\ell_s(w_0)} t_n^{\ell_d(w_0)} \text{ev}_{\lambda+\rho}^t \left(\frac{\kappa_{w_0}^-}{\kappa_{w_0}^+} \right).\end{aligned}$$

Proof. (a) Using the creation formula for E_μ ,

$$\begin{aligned}(E_\mu, E_\mu)_+ &= \left(t^{-\frac{1}{2}\ell(v_\mu^{-1})} \tau_{u_\mu}^\vee \mathbf{1}_Y, t^{-\frac{1}{2}\ell(v_\mu^{-1})} \tau_{u_\mu}^\vee \mathbf{1}_Y \right)_+ = \left(\tau_{u_\mu^{-1}}^\vee \tau_{u_\mu}^\vee \mathbf{1}_Y, \mathbf{1}_Y \right)_+ \\ &= \left(c_{u_\mu}^Y c_{u_\mu}^{Y^{-1}} \mathbf{1}_Y, \mathbf{1}_Y \right)_+ = \text{ev}_0^t \left(c_{u_\mu}^Y c_{u_\mu}^{Y^{-1}} \right) \cdot (1, 1)_+.\end{aligned}$$

(b) Using Proposition 5.3 (E-expansion formulas),

$$A_{\lambda+\omega}^\pm = \frac{\chi^\pm(T_{w_0})}{W_{\lambda+\omega}(t, t_n)} \varepsilon_\pm E_{\lambda+\omega} = \sum_{\mu \in W_{\text{fin}}(\lambda+\omega)} d_{\lambda+\omega}^\mu E_\mu, \quad \text{with}$$

$$d_{\lambda+\omega}^{\lambda+\omega} = (-1)^{\ell_d(v_{\lambda+\omega})} \text{ev}_{\lambda+\omega}^t(\kappa_{v_{\lambda+\omega}}^\pm),$$

and (39)

$$\varepsilon_\pm^2 = \frac{1}{\chi^\pm(T_{w_0})} W_0(t, t_n^{-1}) \varepsilon_\pm,$$

gives

$$\begin{aligned}
(A_{\lambda+\omega}^{\pm}, A_{\lambda+\omega}^{\pm})_+ &= \left(\frac{\chi^+(T_{w_0})}{W_{\lambda+\omega}(t, t_n)} \varepsilon_{\pm} E_{\lambda+\omega}, \frac{\chi^+(T_{w_0})}{W_{\lambda+\omega}(t, t_n)} \varepsilon_{\pm} E_{\lambda+\omega} \right)_+ \quad (\text{by (54)}) \\
&= \frac{1}{W_{\lambda+\omega}(t, t_n) W_{\lambda+\omega}(t^{-1}, t_n^{-1})} (\varepsilon_{\pm}^2 E_{\lambda+\omega}, E_{\lambda+\omega})_+ \quad (\text{by (62)}) \\
&= \frac{1}{\chi^{\pm}(T_{w_0})} \frac{W_0(t, t_n^{-1})}{W_{\lambda+\omega}(t, t_n) W_{\lambda+\omega}(t^{-1}, t_n^{-1})} (\varepsilon_{\pm} E_{\lambda+\omega}, E_{\lambda+\omega})_+ \quad (\text{by (39)}) \\
&= \frac{\chi^+(T_{w_{\lambda+\omega}})^2}{\chi^{\pm}(T_{w_0}) \chi^+(T_{w_0})} \frac{W_0(t, t_n^{-1})}{W_{\lambda+\omega}(t, t_n)} (A_{\lambda+\omega}^{\pm}, E_{\lambda+\omega})_+ \quad (\text{by (54)}) \\
&= \frac{t^{\ell_d(w_0)}}{(-1)^{\ell_d(w_0)}} \frac{W_0(t, t_n^{-1})}{W_{\lambda+\omega}(t^{-1}, t_n^{-1})} d_{\lambda+\omega}^{\lambda+\omega}(E_{\lambda+\omega}, E_{\lambda+\omega})_+ \\
&= \frac{t_n^n}{(-1)^{\ell_d(w_0)}} \frac{W_0(t, t_n^{-1})}{W_{\lambda+\omega}(t, t_n)} (-1)^{\ell_d(v_{\lambda+\omega})} \text{ev}_{\lambda+\omega}^t(\kappa_{v_{\lambda+\omega}}^{\pm})(E_{\lambda+\omega}, E_{\lambda+\omega})_+ \\
&= \frac{W_0(t, t_n^{-1})}{W_{\lambda+\omega}(t, t_n)} t_n^n \text{ev}_{\lambda+\omega}^t(\kappa_{v_{\lambda+\omega}}^{\pm})(E_{\lambda+\omega}, E_{\lambda+\omega})_+,
\end{aligned}$$

because $(-1)^{\ell_d(v_{\lambda+\omega})} = (-1)^{\ell_d(w_0)}$. This proves the $\pm$ case. The proof for the other cases is similar.

(c) By part (b),

$$\begin{aligned}
\frac{(A_{\lambda+\omega}^{\pm}, A_{\lambda+\omega}^{\pm})_+}{(E_{\lambda+\omega}, E_{\lambda+\omega})_+} &= \frac{W_0(t, t_n^{-1})}{W_{\lambda+\omega}(t, t_n)} t_n^n \text{ev}_{\lambda+\omega}^t(\kappa_{v_{\lambda+\omega}}^{\pm}) \quad \text{and} \quad \frac{(P_{\lambda+\omega}, P_{\lambda+\omega})_+}{(E_{\lambda+\omega}, E_{\lambda+\omega})_+} \\
&= \frac{W_0(t, t_n)}{W_{\lambda+\omega}(t, t_n)} \text{ev}_{\lambda+\omega}^t(\kappa_{v_{\lambda+\omega}}^+)
\end{aligned}$$

gives the relation between $(A_{\lambda+\omega}^{\pm}, A_{\lambda+\omega}^{\pm})_+$ and $(P_{\lambda+\omega}, P_{\lambda+\omega})_+$. This proves the $\pm$ case. The proof for the other cases is similar.

(d) Using Propositions 6.5 (going up a level) and 6.1 (Weyl character formula) gives

$$\begin{aligned}
(P_{\lambda}(q, t^{\pm}), P_{\lambda}(q, t^{\pm}))_{\pm} &= \frac{W_0(t, qt_n)}{W_0(t, t_n^{-1})} (A_{\omega}^{\pm} P_{\lambda}(q, t^{\pm}), A_{\omega}^{\pm} P_{\lambda}(q, t^{\pm}))_+ \\
&= \frac{W_0(t, qt_n)}{W_0(t, t_n^{-1})} (A_{\lambda+\omega}^{\pm}(q, t^+), A_{\lambda+\omega}^{\pm}(q, t^+))_+ \\
&= \frac{W_0(t, qt_n)}{W_0(t, t_n)} t_n^n \text{ev}_{\lambda+\omega}^t \left(\frac{\kappa_{v_{\lambda+\omega}}^{\pm}}{\kappa_{v_{\lambda+\omega}}^+} \right) (P_{\lambda+\omega}(q, t^+), P_{\lambda+\omega}(q, t^+))_+
\end{aligned}$$

This proves the $\pm$ case. The proof for the other cases is similar. $\square$

6.8. The symmetric inner product

In this section, we define, for symmetric functions, a slightly different inner product $\langle \cdot, \cdot \rangle_+$ that has more symmetry than the original inner product $\langle \cdot, \cdot \rangle_+$. Fortunately, the difference between these inner products is only a factor of $W_0(t, t_n)$, which makes the reduction relations of the previous section even simpler. The symmetric inner product $\langle \cdot, \cdot \rangle_+$ has another very useful advantage: In terms of the Askey–Wilson parameters a, b, c, d , the inner product $\langle \cdot, \cdot \rangle_+$ is completely symmetric in a, b, c and d .

Define involutions $\bar{} : \mathbb{K}[X] \rightarrow \mathbb{K}[X]$ and ${}^\sigma : \mathbb{K}[X] \rightarrow \mathbb{K}[X]$ and ${}^t : \mathbb{K}[X] \rightarrow \mathbb{K}[X]$ by

$$\begin{aligned}\bar{f}\left(x_1, \dots, x_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}\right) &= f\left(x_1^{-1}, \dots, x_n^{-1}; q^{-1}, t^{-1}, t_0^{-\frac{1}{2}}, u_0^{-\frac{1}{2}}, t_n^{-\frac{1}{2}}, u_n^{-\frac{1}{2}}\right), \\ f^\sigma\left(x_1, \dots, x_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}\right) &= f\left(x_1^{-1}, \dots, x_n^{-1}; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}\right), \\ f^t\left(x_1, \dots, x_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}\right) &= f\left(x_1, \dots, x_n; q^{-1}, t^{-1}, t_0^{-\frac{1}{2}}, u_0^{-\frac{1}{2}}, t_n^{-\frac{1}{2}}, u_n^{-\frac{1}{2}}\right).\end{aligned}$$

Let $\kappa_{w_0}^{X^{-1}}$ be as given in (23) and define

$$\nabla_{CC} = \Delta_{CC}^+ \kappa_{w_0}^{X^{-1}}. \quad (63)$$

Then define a new scalar product $\langle , \rangle_+ : \mathbb{K}[X] \times \mathbb{K}[X] \rightarrow \mathbb{K}$ by

$$\langle f_1, f_2 \rangle_+ = \frac{1}{2^n n!} \text{ct} \left(f_1 \bar{f}_2 \frac{1}{\nabla_{CC}} \right), \quad \text{where } 2^n n! = \text{Card}(W_{\text{fin}}). \quad (64)$$

The following result provides a comparison of $\langle , \rangle_+$ and $\langle , \rangle_+$ as inner products on symmetric polynomials.

PROPOSITION 6.7. *Let $f, g \in \mathbb{K}[x_1^\pm, \dots, x_n^{\pm 1}]^{W_{\text{fin}}}$. Then*

$$\langle f, g \rangle_+ = \frac{1}{W_0(t, t_n)} (f, g^t)_+.$$

Proof. Let $f, g \in \mathbb{K}[x_1^\pm, \dots, x_n^{\pm 1}]^{W_{\text{fin}}}$. Since f, g^t and ∇_{CC} are all invariant under the action of W_{fin} , then

$$\begin{aligned}\langle f, g \rangle_+ &= \frac{1}{2^n n!} \text{ct} \left(f g^\sigma \frac{1}{\nabla_{CC}} \right) = \frac{1}{2^n n! \cdot W_0(t, t_n)} \text{ct} \left(f g^t \frac{1}{\nabla_{CC}} W_0(t, t_n) \right) \\ &= \frac{1}{2^n n! \cdot W_0(t, t_n)} \text{ct} \left(f g^t \frac{1}{\nabla_{CC}} \left(\sum_{w \in W_{\text{fin}}} w(\kappa_{w_0}^{X^{-1}}) \right) \right) \\ &= \frac{1}{2^n n! \cdot W_0(t, t_n)} \text{ct} \left(\sum_{w \in W_{\text{fin}}} w \left(f g^t \frac{1}{\nabla_{CC}} \kappa_{w_0}^{X^{-1}} \right) \right) \\ &= \frac{1}{2^n n! \cdot W_0(t, t_n)} \text{ct} \left(\sum_{w \in W_{\text{fin}}} w \left(f g^t \frac{1}{\Delta_{CC}^+} \right) \right) \\ &= \frac{1}{W_0(t, t_n)} \text{ct} \left(f g^t \frac{1}{\Delta_{CC}^+} \right) = \frac{1}{W_0(t, t_n)} (f, g^t)_+, \end{aligned}$$

where the third equality uses Proposition 5.2. □

The following corollary of Proposition 6.6 records the norm comparisons in terms of the symmetric inner product $\langle , \rangle_+$.

COROLLARY 6.1. Let $\lambda \in (\mathbb{Z}_{\geq 0}^n)^+$. Then

$$\begin{aligned}\frac{\langle P_\lambda(q, t^\pm), P_\lambda(q, t^\pm) \rangle_\pm}{\langle P_{\lambda+\omega}(q, t^+), P_{\lambda+\omega}(q, t^+) \rangle_+} &= t_n^{\ell_d(w_0)} \text{ev}_{\lambda+\omega}^t \left(\frac{\kappa_{v_{\lambda+\omega}}^\pm}{\kappa_{v_{\lambda+\omega}}^+} \right) \\ \frac{\langle P_\lambda(q, t^\mp), P_\lambda(q, t^\mp) \rangle_\mp}{\langle P_{\lambda+\pi}(q, t^+), P_{\lambda+\pi}(q, t^+) \rangle_+} &= t_n^{\ell_s(w_0)} \text{ev}_{\lambda+\pi}^t \left(\frac{\kappa_{v_{\lambda+\pi}}^\mp}{\kappa_{v_{\lambda+\pi}}^+} \right) \\ \frac{\langle P_\lambda(q, t^-), P_\lambda(q, t^-) \rangle_-}{\langle P_{\lambda+\rho}(q, t^+), P_{\lambda+\rho}(q, t^+) \rangle_+} &= t_n^{\ell_s(w_0)} t_n^{\ell_d(w_0)} \text{ev}_{\lambda+\rho}^t \left(\frac{\kappa_{w_0}^-}{\kappa_{w_0}^+} \right).\end{aligned}$$

REMARK 6.2. Using Remark 6.1, then

$$\begin{aligned}\nabla_{CC} &= \nabla^{(1)} \nabla^{(2)}, \quad \text{where} \\ \nabla^{(2)} &= \prod_{1 \leq i < j \leq n} \frac{(tx_i x_j^{-1}; q)_\infty (tx_i x_j; q)_\infty (tx_i^{-1} x_j; q)_\infty (tx_i^{-1} x_j^{-1}; q)_\infty}{(x_i x_j^{-1}; q)_\infty (x_i x_j; q)_\infty (x_i^{-1} x_j; q)_\infty (x_i^{-1} x_j^{-1}; q)_\infty}\end{aligned}$$

and

$$\nabla^{(1)} = \prod_{i=1}^n \frac{(cx_i; q)_\infty (cx_i^{-1}; q)_\infty (dx_i; q)_\infty (dx_i^{-1}; q)_\infty (ax_i; q)_\infty (ax_i^{-1}; q)_\infty (bx_i; q)_\infty (bx_i^{-1}; q)_\infty}{(x_i^2; q)_\infty (x_i^{-2}; q)_\infty}.$$

This formula shows that ∇_{CC} , and thus

$\langle \cdot, \cdot \rangle_+$ and $P_\lambda(q, t^+)$, are completely symmetric in the Askey–Wilson parameters a, b, c and d . $\square$

REMARK 6.3. Using Remark 3.2,

$$\frac{\kappa_{v_{\lambda+\omega}}^\pm}{\kappa_{v_{\lambda+\omega}}^+} = \frac{\kappa_{v_\omega}^\pm}{\kappa_{v_\omega}^+} = \prod_{i=1}^n \frac{\kappa_{\varepsilon_i}^{Y^{-1}}}{\kappa_{\varepsilon_i}^Y} = \frac{\kappa_{w_0}^\pm}{\kappa_{w_0}^+} \quad \text{and} \quad \frac{\kappa_{v_{\lambda+\pi}}^\mp}{\kappa_{v_{\lambda+\pi}}^+} = \frac{\kappa_{v_\pi}^\mp}{\kappa_{v_\pi}^+} = \prod_{1 \leq i < j \leq n} \frac{\kappa_{\varepsilon_i - \varepsilon_j}^{Y^{-1}} \kappa_{\varepsilon_i + \varepsilon_j}^{Y^{-1}}}{\kappa_{\varepsilon_i - \varepsilon_j}^Y \kappa_{\varepsilon_i + \varepsilon_j}^Y} = \frac{\kappa_{w_0}^\mp}{\kappa_{w_0}^+}$$

and

$$\frac{\kappa_{v_\omega}^\pm}{\kappa_{v_\omega}^+} \cdot \frac{\kappa_{v_\omega}^\mp}{\kappa_{v_\omega}^+} = \frac{\kappa_{w_0}^-}{\kappa_{w_0}^+} = \left(\prod_{i=1}^n \frac{\kappa_{\varepsilon_i}^{Y^{-1}}}{\kappa_{\varepsilon_i}^Y} \right) \left(\prod_{1 \leq i < j \leq n} \frac{\kappa_{\varepsilon_i - \varepsilon_j}^{Y^{-1}} \kappa_{\varepsilon_i + \varepsilon_j}^{Y^{-1}}}{\kappa_{\varepsilon_i - \varepsilon_j}^Y \kappa_{\varepsilon_i + \varepsilon_j}^Y} \right).$$

$\square$

6.9. The norm formula

In this section, we use the recursive relations of Corollary 6.1 to derive a c -function formula for $\langle P_\lambda(q, t^+), P_\lambda(q, t^+) \rangle_+$.

Introduce Y -versions of the c -function products from Remark 6.1 as follows:

$$\begin{aligned}\Delta_s^Y(t^+) \Delta_{0,s}^Y(t^+) &= \prod_{1 \leq i < j \leq n} \frac{(tY_i Y_j^{-1}; q)_\infty (tY_i Y_j; q)_\infty}{(Y_i Y_j^{-1}; q)_\infty (Y_i Y_j; q)_\infty}, \\ \Delta_d^Y(t^+) \Delta_{0,d}^Y(t^+) &= \prod_{i=1}^n \frac{(t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} Y_i; q)_\infty (-t_n^{\frac{1}{2}} t_0^{-\frac{1}{2}} Y_i; q)_\infty}{(Y_i^2; q^2)_\infty}, \\ \Delta_g^Y(t^+) &= \prod_{i=1}^n \frac{(u_n^{\frac{1}{2}} u_0^{\frac{1}{2}} q^{\frac{1}{2}} Y_i; q)_\infty (-u_n^{\frac{1}{2}} u_0^{-\frac{1}{2}} q^{\frac{1}{2}} Y_i; q)_\infty}{(qY_i^2; q^2)_\infty}, \\ \Delta_s^{Y^{-1}}((t^+)^{-1}) &= \prod_{1 \leq i < j \leq n} \frac{(t^{-1} qY_i^{-1} Y_j; q)_\infty (t^{-1} qY_i^{-1} Y_j^{-1}; q)_\infty}{(qY_i^{-1} Y_j; q)_\infty (qY_i^{-1} Y_j^{-1}; q)_\infty}, \\ \Delta_d^{Y^{-1}}((t^+)^{-1}) &= \prod_{i=1}^n \frac{(t_n^{-\frac{1}{2}} t_0^{-\frac{1}{2}} qY_i^{-1}; q)_\infty (-t_n^{-\frac{1}{2}} t_0^{\frac{1}{2}} qY_i^{-1}; q)_\infty}{(q^2 Y_i^{-2}; q^2)_\infty}, \\ \Delta_g^{Y^{-1}}((t^+)^{-1}) &= \prod_{i=1}^n \frac{(u_n^{-\frac{1}{2}} u_0^{-\frac{1}{2}} q^{\frac{1}{2}} Y_i^{-1}; q)_\infty (-u_n^{-\frac{1}{2}} u_0^{\frac{1}{2}} q^{\frac{1}{2}} Y_i^{-1}; q)_\infty}{(qY_i^{-2}; q^2)_\infty}.\end{aligned}$$

Define homomorphisms $\text{ev}_{q^{\lambda_t \rho}} : \mathbb{K}[Y] \rightarrow \mathbb{K}$ and $\text{ev}_{q^{-\lambda_t - \rho}} : \mathbb{K}[Y] \rightarrow \mathbb{K}$ by

$$\text{ev}_{q^{\lambda_t \rho}}(Y_i) = q^{\lambda_i} t^{n-i} t_0^{\frac{1}{2}} t_n^{\frac{1}{2}} \quad \text{and} \quad \text{ev}_{q^{-\lambda_t - \rho}}(Y_i) = q^{-\lambda_i} t^{-(n-i)} t_0^{-\frac{1}{2}} t_n^{-\frac{1}{2}} = \text{ev}_{q^{\lambda_t \rho}}(Y_i^{-1}).$$

The correspondence between the parameters $q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}$ and the Askey–Wilson parameters is given by

$$q = q, \quad a = q^{\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{\frac{1}{2}}, \quad b = -q^{\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{-\frac{1}{2}}, \quad c = t_n^{\frac{1}{2}} u_n^{\frac{1}{2}}, \quad d = -t_n^{\frac{1}{2}} u_n^{-\frac{1}{2}}.$$

To control the spacing of the formulas, introduce a notation

$$(z_1, z_2, \dots, z_k; q)_\infty = (z_1; q)_\infty (z_2; q)_\infty \cdots (z_k; q)_\infty. \quad (65)$$

Then, $\text{ev}_{q^{\lambda_t \rho}}(\Delta_d^Y(t^+) \Delta_{0,d}^Y(t^+) \Delta_g^Y(t^+))$ is equal to

$$\begin{aligned}& \prod_{i=1}^n \frac{(t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} q^{\lambda_i} t^{n-i} t_n^{\frac{1}{2}} t_0^{\frac{1}{2}}, -t_n^{\frac{1}{2}} t_0^{-\frac{1}{2}} q^{\lambda_i} t^{n-i} t_n^{\frac{1}{2}} t_0^{\frac{1}{2}}, u_n^{\frac{1}{2}} u_0^{\frac{1}{2}} q^{\frac{1}{2}} q^{\lambda_i} t^{n-i} t_n^{\frac{1}{2}} t_0^{\frac{1}{2}}, -u_n^{\frac{1}{2}} u_0^{-\frac{1}{2}} q^{\frac{1}{2}} q^{\lambda_i} t^{n-i} t_n^{\frac{1}{2}} t_0^{\frac{1}{2}}; q)_\infty}{(q^{2\lambda_i} t^{2(n-i)} t_n t_0, q q^{2\lambda_i} t^{2(n-i)} t_n t_0; q^2)_\infty} \\ &= \prod_{i=1}^n \frac{(t_n t_0 q^{\lambda_i} t^{n-i}, -t_n q^{\lambda_i} t^{n-i}, t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} u_n^{\frac{1}{2}} u_0^{\frac{1}{2}} q^{\frac{1}{2}} q^{\lambda_i} t^{n-i}, -t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} u_n^{\frac{1}{2}} u_0^{-\frac{1}{2}} q^{\frac{1}{2}} q^{\lambda_i} t^{n-i}; q)_\infty}{(q^{2\lambda_i} t^{2(n-i)} t_n t_0; q)_\infty} \\ &= \prod_{i=1}^n \frac{(q^{-1} abcd q^{\lambda_i} t^{n-i}, cd q^{\lambda_i} t^{n-i}, ac q^{\lambda_i} t^{n-i}, bc q^{\lambda_i} t^{n-i}; q)_\infty}{(q^{-1} abcd q^{2\lambda_i} t^{2(n-i)}; q)_\infty} \quad (66)\end{aligned}$$

and $\text{ev}_{q^{-\lambda_t - \rho}}(\Delta_d^{Y^{-1}}((t^+)^{-1}) \Delta_g^{Y^{-1}}((t^+)^{-1}))$ is equal to

$$\begin{aligned}& \prod_{i=1}^n \frac{(t_n^{-\frac{1}{2}} t_0^{-\frac{1}{2}} q q^{\lambda_i} t^{n-i} t_n^{\frac{1}{2}} t_0^{\frac{1}{2}}, -t_n^{-\frac{1}{2}} t_0^{\frac{1}{2}} q q^{\lambda_i} t^{n-i} t_n^{\frac{1}{2}} t_0^{\frac{1}{2}}, u_n^{-\frac{1}{2}} u_0^{-\frac{1}{2}} q^{\frac{1}{2}} q^{\lambda_i} t^{n-i} t_n^{\frac{1}{2}} t_0^{\frac{1}{2}}, -u_n^{-\frac{1}{2}} u_0^{\frac{1}{2}} q^{\frac{1}{2}} q^{\lambda_i} t^{n-i} t_n^{\frac{1}{2}} t_0^{\frac{1}{2}}; q)_\infty}{(q^2 q^{2\lambda_i} t^{2(n-i)} t_n t_0, q q^{2\lambda_i} t^{2(n-i)} t_n t_0; q^2)_\infty} \\ &= \prod_{i=1}^n \frac{(q q^{\lambda_i} t^{n-i}, -t_0 q q^{\lambda_i} t^{n-i}, t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} u_n^{-\frac{1}{2}} u_0^{-\frac{1}{2}} q^{\frac{1}{2}} q^{\lambda_i} t^{n-i}, -t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} u_n^{-\frac{1}{2}} u_0^{\frac{1}{2}} q^{\frac{1}{2}} q^{\lambda_i} t^{n-i}; q)_\infty}{(q q^{2\lambda_i} t^{2(n-i)} t_n t_0; q)_\infty} \\ &= \prod_{i=1}^n \frac{(q q^{\lambda_i} t^{n-i}, ab q^{\lambda_i} t^{n-i}, bd q^{\lambda_i} t^{n-i}, ad q^{\lambda_i} t^{n-i}; q)_\infty}{(abcd q^{2\lambda_i} t^{2(n-i)}; q)_\infty}. \quad (67)\end{aligned}$$

THEOREM 6.2. Define $N_\lambda(q, t^+)$ to be the product

$$N_\lambda(q, t^+) = \text{ev}_{q^\lambda t^\rho} \left(\frac{1}{\Delta_g^Y(t^+) \Delta_s^Y(t^+) \Delta_d^Y(t^+) \Delta_{0,s}^Y(t^+) \Delta_{0,d}^Y(t^+)} \right) \\ \cdot \text{ev}_{q^{-\lambda} t^{-\rho}} \left(\frac{1}{\Delta_g^{Y^{-1}}((t^+)^{-1}) \Delta_s^{Y^{-1}}((t^+)^{-1}) \Delta_d^{Y^{-1}}((t^+)^{-1})} \right).$$

Then

$$\langle P_\lambda(q, t^+), P_\lambda(q, t^+) \rangle_+ = N_\lambda(q, t^+).$$

Proof. The proof is accomplished by verifying the following properties:

- (a) $N_\lambda(q, 1, 1, 1, 1, 1) = 1$.
- (b) $N_\lambda(q, t^+)$ is symmetric in the parameters a, b, c, d .
- (c) $N_\lambda(q, t^+)$ satisfies the recursions

$$\frac{N_\lambda(q, t^\pm)}{N_{\lambda+\omega}(q, t^+)} = t_n^n \text{ev}_{\lambda+\omega}^t \left(\frac{\kappa_{\nu_{\lambda+\omega}}^\pm}{\kappa_{\nu_{\lambda+\omega}}^+} \right) \quad \text{and} \quad \frac{N_\lambda(q, t^\mp)}{N_{\lambda+\pi}(q, t^+)} = t^{n(n-1)} \text{ev}_{\lambda+\pi}^t \left(\frac{\kappa_{\nu_{\lambda+\pi}}^\mp}{\kappa_{\nu_{\lambda+\pi}}^+} \right).$$

Let us explain how the result follows from these properties. To prove the statement for arbitrary parameters t, t_0, u_0, t_n, u_n , it suffices to prove the statement when $t_0 = q^{k_1}$, $u_0 = q^{k_2}$, $t_n = q^{k_3}$, $u_n = q^{k_4}$ and $t = q^{k_5}$ for arbitrary $k_1, k_2, k_3, k_4, k_5 \in \mathbb{Z}_{>0}$. The two recursions in (c)

$$\begin{aligned} & \text{write } N_\lambda(q, q^{k_5}, q^{k_1}, q^{k_2}, q^{k_3}, q^{k_4}) \text{ in terms of} \\ & N_{\lambda+\omega}(q, q^{k_5}, q^{k_1}, q^{k_2}, q^{k_3-1}, q^{k_4}) \quad \text{or} \quad N_{\lambda+\pi}(q, q^{k_5-1}, q^{k_1}, q^{k_2}, q^{k_3}, q^{k_4}) \end{aligned}$$

and using these successively, and using property (b) to rearrange k_1, k_2, k_3, k_4 as necessary to apply the first recursion in (c), the value $N_\lambda(q, q^{k_5}, q^{k_1}, q^{k_2}, q^{k_3}, q^{k_4})$ can be reduced to a product of factors multiplied by $N_{\lambda+\mu}(q, q^0, q^0, q^0, q^0, q^0)$, where μ is a non-negative integer combination of ω and π . By property (a), the $N_{\lambda+\mu}(q, q^0, q^0, q^0, q^0, q^0) = 1$. Since the factors in Corollary 6.1 are the same as the factors that appear in property (c), then the value

$$\langle P_\lambda(q, q^{k_5}, q^{k_1}, q^{k_2}, q^{k_3}, q^{k_4}), P_\lambda(q, q^{k_5}, q^{k_1}, q^{k_2}, q^{k_3}, q^{k_4}) \rangle_+ \quad \text{can be reduced by the same steps}$$

to the same product of factors multiplied by the norm of $P_{\lambda+\mu}(q, q^0, q^0, q^0, q^0, q^0)$ (which is equal to 1). In this way, Corollary 6.1 and the properties (a), (b) and (c) imply the statement of the theorem.

Property (a) follows from the fact that $\Delta_d^Y(1, 1, 1, 1, 1), \Delta_{0,d}^Y(1, 1, 1, 1, 1), \Delta_g^Y(1, 1, 1, 1, 1), \Delta_d^{Y^{-1}}(1, 1, 1, 1, 1)$ and $\Delta_g^{Y^{-1}}(1, 1, 1, 1, 1)$ are all equal to 1.

Property (b) follows from the fact that the product of the expressions in (66) and (67) is symmetric in the parameters a, b, c, d .

Property (c) Changing from the parameters $t^+ = (t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}})$ to the parameters $t^\pm = (t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}})$ replaces $t_n^{\frac{1}{2}}$ by $qt_n^{\frac{1}{2}}$ so that c gets replaced by qc and d gets replaced

by qd and a and b and t stay fixed. Write

$$N_{\lambda}^{gd}(q, t^+) = \text{ev}_{q^{\lambda}t^{\rho}} \left(\frac{1}{\Delta_g^Y(t^+) \Delta_d^Y(t^+) \Delta_{0,d}^Y(t^+)} \right) \cdot \text{ev}_{q^{-\lambda}t^{-\rho}} \left(\frac{1}{\Delta_g^{Y^{-1}}((t^+)^{-1}) \Delta_d^{Y^{-1}}((t^+)^{-1})} \right),$$

$$N_{\lambda}^s(q, t^+) = \text{ev}_{q^{\lambda}t^{\rho}} \left(\frac{1}{\Delta_s^Y(t^+) \Delta_{0,s}^Y(t^+)} \right) \cdot \text{ev}_{q^{-\lambda}t^{-\rho}} \left(\frac{1}{\Delta_s^{Y^{-1}}((t^+)^{-1})} \right),$$

so that $N_{\lambda}(q, t^+) = N_{\lambda}^{gd}(q, t^+) N_{\lambda}^s(q, t^+)$. Using the notation of (65),

$$N_{\lambda}^{gd}(q, t^{\pm}) = \prod_{i=1}^n \frac{(q^2 abcd q^{2\lambda_i} t^{2(n-i)}, q^2 q^{-1} abcd q^{2\lambda_i} t^{2(n-i)}; q)_{\infty}}{\left(qq^{\lambda_i} t^{n-i}, q^2 q^{-1} abcd q^{\lambda_i} t^{n-i}, abq^{\lambda_i} t^{n-i}, qacq^{\lambda_i} t^{n-i}, qadq^{\lambda_i} t^{n-i}, qbcq^{\lambda_i} t^{n-i}, qbdq^{\lambda_i} t^{n-i}, q^2 cdq^{\lambda_i} t^{n-i}; q \right)_{\infty}}$$

and

$$N_{\lambda+\omega}^{gd}(q, t^+) = \prod_{i=1}^n \frac{(abcd q^{2(\lambda_i+1)} t^{2(n-i)}, q^{-1} abcd q^{2(\lambda_i+1)} t^{2(n-i)}; q)_{\infty}}{\left(qq^{\lambda_i+1} t^{n-i}, q^{-1} abcd q^{\lambda_i+1} t^{n-i}, abq^{\lambda_i+1} t^{n-i}, acq^{\lambda_i+1} t^{n-i}, adq^{\lambda_i+1} t^{n-i}, bcq^{\lambda_i+1} t^{n-i}, bdq^{\lambda_i+1} t^{n-i}, cdq^{\lambda_i+1} t^{n-i}; q \right)_{\infty}}$$

and

$$\begin{aligned} \frac{N_{\lambda}^{gd}(q, t^{\pm})}{N_{\lambda+\omega}^{gd}(q, t^+)} &= \prod_{i=1}^n \frac{(1 - abcd q^{\lambda_i} t^{n-i})(1 - cdq^{\lambda_i+1} t^{n-i})}{(1 - qq^{\lambda_i} t^{n-i})(1 - abq^{\lambda_i} t^{n-i})} \\ &= \prod_{i=1}^n \frac{(1 - abcd q^{\lambda_i} t^{n-i})(1 - cdq^{\lambda_i+1} t^{n-i})}{(1 - q^{\lambda_i+1} t^{n-i})(1 - abq^{\lambda_i} t^{n-i})}. \end{aligned}$$

Since

$$\text{ev}_{q^{\lambda}t^{\rho}}(\Delta_s^Y(t^+) \Delta_{0,s}^Y(t^+)) = \prod_{1 \leq i < j \leq n} \frac{(q^{\lambda_i - \lambda_j} t^{j-i+1}; q)_{\infty} (q^{\lambda_i + \lambda_j} t^{2n-i-j+1} t_0 t_n; q)_{\infty}}{(q^{\lambda_i - \lambda_j} t^{j-i}; q)_{\infty} (q^{\lambda_i + \lambda_j} t^{2n-i-j} t_0 t_n; q)_{\infty}}$$

and

$$\text{ev}_{q^{-\lambda}t^{-\rho}}(\Delta_s^{Y^{-1}}(t^+)^{-1}) = \prod_{1 \leq i < j \leq n} \frac{(q^{\lambda_i - \lambda_j + 1} t^{j-i-1}; q)_{\infty} (q^{\lambda_i + \lambda_j + 1} t^{2n-i-j-1} t_0 t_n; q)_{\infty}}{(q^{\lambda_i - \lambda_j + 1} t^{j-i}; q)_{\infty} (q^{\lambda_i + \lambda_j + 1} t^{2n-i-j} t_0 t_n; q)_{\infty}},$$

then

$$N_{\lambda}^s(q, t^{\pm}) = \prod_{1 \leq i < j \leq n} \frac{\left(q^{\lambda_i - \lambda_j} t^{j-i}, q^{\lambda_i + \lambda_j} t^{2n-i-j} t_0 q^2 t_n, q^{\lambda_i - \lambda_j + 1} t^{j-i}, q^{\lambda_i + \lambda_j + 1} t^{2n-i-j} t_0 q^2 t_n; q \right)_{\infty}}{\left(q^{\lambda_i - \lambda_j} t^{j-i+1}, q^{\lambda_i + \lambda_j} t^{2n-i-j+1} t_0 q^2 t_n, q^{\lambda_i - \lambda_j + 1} t^{j-i-1}, q^{\lambda_i + \lambda_j + 1} t^{2n-i-j-1} t_0 q^2 t_n; q \right)_{\infty}}$$

and

$$N_{\lambda+\omega}^s(q, t^+) = \prod_{1 \leq i < j \leq n} \frac{\left(q^{\lambda_i - \lambda_j} t^{j-i}, q^{\lambda_i + \lambda_j + 2} t^{2n-i-j} t_0 t_n, q^{\lambda_i - \lambda_j + 1} t^{j-i}, q^{\lambda_i + \lambda_j + 3} t^{2n-i-j} t_0 t_n; q \right)_{\infty}}{\left(q^{\lambda_i - \lambda_j} t^{j-i+1}, q^{\lambda_i + \lambda_j + 2} t^{2n-i-j+1} t_0 t_n, q^{\lambda_i - \lambda_j + 1} t^{j-i-1}, q^{\lambda_i + \lambda_j + 3} t^{2n-i-j-1} t_0 t_n; q \right)_{\infty}},$$

since $\omega_i = 1$. So

$$\frac{N_{\lambda}^s(q, t^{\pm})}{N_{\lambda+\omega}^s(q, t^+)} = 1.$$

Since

$$\begin{aligned}
t_n^n \cdot \text{ev}_{\lambda+\omega}^t \left(\frac{\kappa_{\nu_{\lambda+\omega}}^{\pm}}{\kappa_{\nu_{\lambda+\omega}}^+} \right) &= \text{ev}_{\lambda+\omega}^t \left(\prod_{i=1}^n \frac{\kappa_{\varepsilon_i}^{Y^{-1}}}{\kappa_{\varepsilon_i}^Y} \right) \\
&= \prod_{i=1}^n \text{ev}_{\lambda+\omega}^t \left(t_n \frac{(1 - t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} Y_i^{-1})(1 + t_n^{\frac{1}{2}} t_0^{-\frac{1}{2}} Y_i^{-1})}{(1 - Y_i^{-2})} \frac{(1 - Y_i^2)}{(1 - t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} Y_i)(1 + t_n^{\frac{1}{2}} t_0^{-\frac{1}{2}} Y_i)} \right) \\
&= \prod_{i=1}^n \text{ev}_{\lambda+\omega}^t \left(\frac{(1 - t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} Y_i^{-1})(1 + t_n^{\frac{1}{2}} t_0^{-\frac{1}{2}} Y_i^{-1})}{(1 - t_n^{-\frac{1}{2}} t_0^{-\frac{1}{2}} Y_i^{-1})(1 + t_n^{-\frac{1}{2}} t_0^{\frac{1}{2}} Y_i^{-1})} \right) \\
&= \prod_{i=1}^n \left(\frac{(1 - t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} q^{\lambda_i+1} t^{n-i} t_0^{\frac{1}{2}} t_n^{\frac{1}{2}})(1 + t_n^{\frac{1}{2}} t_0^{-\frac{1}{2}} q^{\lambda_i+1} t^{n-i} t_0^{\frac{1}{2}} t_n^{\frac{1}{2}})}{(1 - t_n^{-\frac{1}{2}} t_0^{-\frac{1}{2}} q^{\lambda_i+1} t^{n-i} t_0^{\frac{1}{2}} t_n^{\frac{1}{2}})(1 + t_n^{-\frac{1}{2}} t_0^{\frac{1}{2}} q^{\lambda_i+1} t^{n-i} t_0^{\frac{1}{2}} t_n^{\frac{1}{2}})} \right) \\
&= \prod_{i=1}^n \left(\frac{(1 - t_n t_0 q^{\lambda_i+1} t^{n-i})(1 + t_n q^{\lambda_i+1} t^{n-i})}{(1 - q^{\lambda_i+1} t^{n-i})(1 + t_0 q^{\lambda_i+1} t^{n-i})} \right) \\
&= \prod_{i=1}^n \left(\frac{(1 - q^{-1} abcd q^{\lambda_i+1} t^{n-i})(1 - cd q^{\lambda_i+1} t^{n-i})}{(1 - q^{\lambda_i+1} t^{n-i})(1 - q^{-1} ab q^{\lambda_i+1} t^{n-i})} \right) \\
&= \prod_{i=1}^n \left(\frac{(1 - abcd q^{\lambda_i} t^{n-i})(1 - cd q^{\lambda_i+1} t^{n-i})}{(1 - q^{\lambda_i+1} t^{n-i})(1 - ab q^{\lambda_i} t^{n-i})} \right),
\end{aligned}$$

then

$$\frac{N_{\lambda}(q, t^{\pm})}{N_{\lambda+\omega}(q, t^+)} = \frac{N_{\lambda}^{gd}(q, t^{\pm})}{N_{\lambda+\omega}^{gd}(q, t^+)} \cdot \frac{N_{\lambda}^s(q, t^{\pm})}{N_{\lambda+\omega}^s(q, t^+)} = \frac{N_{\lambda}^{gd}(q, t^{\pm})}{N_{\lambda+\omega}^{gd}(q, t^+)} \cdot 1 = t_n^n \cdot \text{ev}_{\lambda+\omega}^t \left(\frac{\kappa_{\nu_{\lambda+\omega}}^{\pm}}{\kappa_{\nu_{\lambda+\omega}}^+} \right).$$

The proof of the second equality in (c) is similar as follows:

$$N_{\lambda}^{gd}(q, t^{\mp}) = \prod_{i=1}^n \frac{(q^2 abcd q^{2\lambda_i} (qt)^{2(n-i)}, q^2 q^{-1} abcd q^{2\lambda_i} (qt)^{2(n-i)}; q)_{\infty}}{\left(qq^{\lambda_i} (qt)^{n-i}, q^2 q^{-1} abcd q^{\lambda_i} (qt)^{n-i}, ab q^{\lambda_i} (qt)^{n-i}, qac q^{\lambda_i} (qt)^{n-i}, qad q^{\lambda_i} (qt)^{n-i}, qbc q^{\lambda_i} (qt)^{n-i}, qbd q^{\lambda_i} (qt)^{n-i}, q^2 cd q^{\lambda_i} (qt)^{n-i}; q \right)_{\infty}}$$

and

$$N_{\lambda+\pi}^{gd}(q, t^+) = \prod_{i=1}^n \frac{(abcd q^{2(\lambda_i+n-i)} t^{2(n-i)}, q^{-1} abcd q^{2(\lambda_i+n-i)} t^{2(n-i)}; q)_{\infty}}{\left(qq^{\lambda_i+n-i} t^{n-i}, q^{-1} abcd q^{\lambda_i+n-i} t^{n-i}, ab q^{\lambda_i+n-i} t^{n-i}, ac q^{\lambda_i+n-i} t^{n-i}, ad q^{\lambda_i+n-i} t^{n-i}, bc q^{\lambda_i+n-i} t^{n-i}, bd q^{\lambda_i+n-i} t^{n-i}, cd q^{\lambda_i+n-i} t^{n-i}; q \right)_{\infty}}$$

so that

$$\frac{N_{\lambda}^{gd}(q, t^{\mp})}{N_{\lambda}^{gd}(q, t^+)} = 1.$$

Then

$$N_{\lambda}^s(q, t^{\mp}) = \prod_{1 \leq i < j \leq n} \frac{\left(q^{\lambda_i - \lambda_j + j - i} t^{j-i}, q^{\lambda_i + \lambda_j + 2n - i - j} t^{2n - i - j} t_0 t_n, q^{\lambda_i - \lambda_j + j - i + 1} t^{j-i}, q^{\lambda_i + \lambda_j + 1 + 2n - i - j} t^{2n - i - j} t_0 t_n; q \right)_{\infty}}{\left(q^{\lambda_i - \lambda_j + j - i + 1} t^{j-i+1}, q^{\lambda_i + \lambda_j + 2n - i - j + 1} t^{2n - i - j + 1} t_0 t_n, q^{\lambda_i - \lambda_j + j - i} t^{j-i-1}, q^{\lambda_i + \lambda_j + 2n - i - j} t^{2n - i - j - 1} t_0 t_n; q \right)_{\infty}}$$

and

$$N_{\lambda+\pi}^s(q, t^+) = \prod_{1 \leq i < j \leq n} \frac{\left(q^{\lambda_i - \lambda_j + j - i} t^{j-i}, q^{\lambda_i + \lambda_j + 2n - i - j} t^{2n-i-j} t_0 t_n, q^{\lambda_i - \lambda_j + j - i + 1} t^{j-i}, q^{\lambda_i + \lambda_j + 1 + 2n - i - j} t^{2n-i-j} t_0 t_n; q \right)_{\infty}}{\left(q^{\lambda_i - \lambda_j + j - i} t^{j-i+1}, q^{\lambda_i + \lambda_j + 2n - i - j} t^{2n-i-j+1} t_0 t_n, q^{\lambda_i - \lambda_j + j - i + 1} t^{j-i-1}, q^{\lambda_i + \lambda_j + 1 + 2n - i - j} t^{2n-i-j-1} t_0 t_n; q \right)_{\infty}},$$

since $\pi_i = n - i$. So

$$\frac{N_{\lambda}^s(q, t^{\mp})}{N_{\lambda+\pi}^s(t^+)} = \prod_{1 \leq i < j \leq n} \frac{(1 - q^{\lambda_i - \lambda_j + j - i} t^{j-i+1})(1 - q^{\lambda_i + \lambda_j + 2n - i - j} t^{2n-i-j+1} t_0 t_n)}{(1 - q^{\lambda_i - \lambda_j + j - i} t^{j-i-1})(1 - q^{\lambda_i + \lambda_j + 2n - i - j} t^{2n-i-j-1} t_0 t_n)}$$

and

$$\frac{N_{\lambda}(q, t^{\mp})}{N_{\lambda+\pi}^s(t^+)} = \frac{N_{\lambda}^{gd}(q, t^{\mp})}{N_{\lambda+\pi}^{gd}(t^+)} \cdot \frac{N_{\lambda}^s(q, t^{\mp})}{N_{\lambda+\pi}^s(t^+)} = 1 \cdot \frac{N_{\lambda}^s(q, t^{\mp})}{N_{\lambda+\pi}^s(t^+)} = t^{n(n-1)} \text{ev}_{\lambda+\pi}^t \left(\frac{\kappa_{v_{\lambda+\pi}}^{\mp}}{\kappa_{v_{\lambda+\pi}}^+} \right),$$

since

$$\begin{aligned} t^{n(n-1)} \text{ev}_{\lambda+\pi}^t \left(\frac{\kappa_{v_{\lambda+\pi}}^{\mp}}{\kappa_{v_{\lambda+\pi}}^+} \right) &= \text{ev}_{\lambda+\pi}^t \left(\prod_{1 \leq i < j \leq n} t^2 \frac{\kappa_{\varepsilon_i - \varepsilon_j}^{Y^{-1}} \kappa_{\varepsilon_i + \varepsilon_j}^{Y^{-1}}}{\kappa_{\varepsilon_i - \varepsilon_j}^Y \kappa_{\varepsilon_i + \varepsilon_j}^Y} \right) \\ &= \text{ev}_{\lambda+\pi}^t \left(\prod_{1 \leq i < j \leq n} t^2 \frac{(1 - tY_i^{-1}Y_j)(1 - tY_i^{-1}Y_j^{-1})}{(1 - Y_i^{-1}Y_j)(1 - Y_i^{-1}Y_j^{-1})} \frac{(1 - Y_iY_j^{-1})(1 - Y_iY_j)}{(1 - tY_iY_j^{-1})(1 - tY_iY_j)} \right) \\ &= \text{ev}_{\lambda+\pi}^t \left(\prod_{1 \leq i < j \leq n} \frac{(1 - tY_i^{-1}Y_j)(1 - tY_i^{-1}Y_j^{-1})}{(1 - t^{-1}Y_i^{-1}Y_j)(1 - t^{-1}Y_i^{-1}Y_j^{-1})} \right) \\ &= \prod_{1 \leq i < j \leq n} \frac{(1 - tq^{\lambda_i + (n-i) - \lambda_j - (n-j)} t^{-(n-j) + (n-i)})(1 - tq^{\lambda_i + (n-i) + \lambda_j + (n-j)} t^{(n-i) + (n-j)} t_0 t_n)}{(1 - t^{-1}q^{\lambda_i + (n-i) - \lambda_j - (n-j)} t^{-(n-j) + (n-i)})(1 - t^{-1}q^{\lambda_i + (n-i) + \lambda_j + (n-j)} t^{(n-i) + (n-j)} t_0 t_n)} \\ &= \prod_{1 \leq i < j \leq n} \frac{(1 - tq^{\lambda_i - \lambda_j + j - i} t^{j-i})(1 - tq^{\lambda_i + \lambda_j + 2n - i - j} t^{2n-i-j} t_0 t_n)}{(1 - t^{-1}q^{\lambda_i - \lambda_j + j - i} t^{j-i})(1 - t^{-1}q^{\lambda_i + \lambda_j + 2n - i - j} t^{2n-i-j} t_0 t_n)}. \end{aligned}$$

□

6.10. The constant term

To get the constant term of $\frac{1}{\Delta_{CC}^+}$ specialize $\lambda = 0$. Then

$$N_0^{gd}(t^+) = \prod_{i=1}^n \frac{(abcdt^{2(n-i)}, q^{-1}abcdt^{2(n-i)}; q)_{\infty}}{(qt^{n-i}, q^{-1}abcdt^{n-i}, abt^{n-i}, act^{n-i}, adt^{n-i}, bct^{n-i}, bdt^{n-i}, cdt^{n-i}; q)_{\infty}}$$

and

$$N_0^s(t^+) = \prod_{1 \leq i < j \leq n} \frac{(t^{j-i}, t^{2n-i-j} t_0 t_n, qt^{j-i}, qt^{2n-i-j} t_0 t_n; q)_{\infty}}{(tt^{j-i}, tt^{2n-i-j} t_0 t_n, qt^{-1}t^{j-i}, qt^{-1}t^{2n-i-j} t_0 t_n; q)_{\infty}},$$

so that

$$W_0(t, t_n) \text{ct} \left(\frac{1}{\Delta_{CC}^+} \right) = W_0(t, t_n)(1, 1)_+ = \langle 1, 1 \rangle_+ = N_0(t^+) = N_0^{gd}(t^+) N_0^s(t^+).$$

ACKNOWLEDGEMENTS

We thank Zajj Daugherty, Stephen Griffeth, Bogdan Ion, Alejandro H. Morales and Martha Yip for the discussions, which greatly helped to solidify our understanding of the topics that we treat in this paper.

We thank O. Warnaar and J. Stokman for their helpful and insightful comments on a preliminary version of this paper.

FUNDING

L.C. was partially supported by a grant from the Simons Foundation (MPS-TSM-00007362).

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Received 9 October 2024; Revised 20 March 2025

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