Combinatorics in affine flag varieties

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Dedicated to Gus Lehrer on the occasion of his 60^{th} *birthday*

Abstract

The Littelmann path model gives a realisation of the crystals of integrable representations of symmetrizable Kac-Moody Lie algebras. Recent work of Gaussent-Littelmann [GL] and others [BG] [GR] has demonstrated a connection between this model and the geometry of the loop Grassmanian. The alcove walk model is a version of the path model which is intimately connected to the combinatorics of the affine Hecke algebra. In this paper we define a refined alcove walk model which encodes the points of the affine flag variety. We show that this combinatorial indexing naturally indexes the cells in generalized Mirkovic-Vilonen intersections.

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1 Introduction

A *Chevalley group* is a group in which row reduction works. This means that it is a group with a special set of generators (the "elementary matrices") and relations which are generalisations of the usual row reduction operations. One way to efficiently encode these generators and relations is with a Kac-Moody Lie algebra $\mathfrak{g}$. From the data of the Kac-Moody Lie algebra and a choice of a commutative ring or field $\mathbb{F}$ the group $G(\mathbb{F})$ is built by generators and relations following Chevalley-Steinberg-Tits.

Of particular interest is the case where $\mathbb{F}$ is the field of fractions of $\mathfrak{o}$, the discrete valuation ring $\mathfrak{o}$ is the ring of integers in $\mathbb{F}$, $\mathfrak{p}$ is the unique maximal ideal in $\mathfrak{o}$ and $k = \mathfrak{o}/\mathfrak{p}$ is the residue field. The favourite examples are

$$\begin{array}{lll} \mathbb{F} = \mathbb{C}((t)) & \mathfrak{o} = \mathbb{C}[[t]] & k = \mathbb{C}, \\ \mathbb{F} = \mathbb{Q}_p & \mathfrak{o} = \mathbb{Z}_p & k = \mathbb{F}_p, \\ \mathbb{F} = \mathbb{F}_q((t)) & \mathfrak{o} = \mathbb{F}_q[[t]] & k = \mathbb{F}_q, \end{array}$$

where $\mathbb{Q}_p$ is the field of p-adic numbers, $\mathbb{Z}_p$ is the ring of p-adic integers, and $\mathbb{F}_q$ is the finite field with q elements. For clarity of presentation we shall work in the first case where $\mathbb{F} = \mathbb{C}((t))$. The diagram

$$\begin{array}{ccc} \mathbb{F} & & \\ \cup\!| & & \\ \mathfrak{o} & \stackrel{\mathrm{ev}_{t=0}}{\longrightarrow} & k = \mathfrak{o}/\mathfrak{p} \end{array} \qquad \text{gives} \qquad \begin{array}{ccccc} G & = & G(\mathbb{C}((t))) & & \\ \cup\!| & & \cup\!| & & \\ K & = & G(\mathbb{C}[[t]]) & \stackrel{\mathrm{ev}_{t=0}}{\longrightarrow} & G(\mathbb{C}) \\ \cup\!| & & \cup\!| & & \cup\!| \\ I & = & \mathrm{ev}_{t=0}^{-1}(B(\mathbb{C})) & \stackrel{\mathrm{ev}_{t=0}}{\longrightarrow} & B(\mathbb{C}) \end{array} \tag{1.1}$$

where $B(\mathbb{C})$ is the "Borel subgroup" of "upper triangular matrices" in $G(\mathbb{C})$. The *loop group* is $G = G(\mathbb{C}((t)))$, I is the standard *Iwahori subgroup* of G,

$$\begin{array}{c} G(\mathbb{C})/B(\mathbb{C}) \quad \text{is the \textit{flag variety}}, \\ G/I \quad \text{is the \textit{affine flag variety}}, \qquad \text{and} \qquad G/K \quad \text{is the \textit{loop Grassmanian}}. \end{array} \tag{1.2}$$

The primary tool for the study of these varieties (ind-schemes) are the following "classical" double coset decompositions, see [St, Ch. 8] and [Mac1, §(2.6)]

Theorem 1.1. *Let W be the Weyl group of $G(\mathbb{C})$, $\widetilde{W} = W \ltimes \mathfrak{h}_{\mathbb{Z}}$ the affine Weyl group, and U^- the subgroup of "unipotent lower triangular" matrices in $G(\mathbb{F})$ and $\mathfrak{h}_{\mathbb{Z}}^+$ the set of dominant elements of $\mathfrak{h}_{\mathbb{Z}}$. Then*

Bruhat decomposition	$G = \bigsqcup_{w \in W} BwB$	$K = \bigsqcup_{w \in W} IwI$	
Iwahori decomposition	$G = \bigsqcup_{w \in \widetilde{W}} IwI$	$G = \bigsqcup_{v \in \widetilde{W}} U^- vI$	
Cartan decomposition	$G = \bigsqcup_{\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}^+} Kt_{\lambda^\vee}K$	$G = \bigsqcup_{\mu^\vee \in \mathfrak{h}_{\mathbb{Z}}} U^- t_{\mu^\vee} K$	*Iwasawa decomposition*

It should be stressed that we have, intentionally, *not* given precise definitions of the objects in Theorem 1.1. Even in the classical case, the definition of $\mathfrak{h}_{\mathbb{Z}}$ in Theorem 1.1 is sensitive to small changes in the definition of G (center, completions, etc) and there are subtleties in making these definitions correctly in general. These issues are partly treated in [Ga1, Theorem 14.10, Lemma 16.14], [Ga2, Theorem 1.8], [GR, Remark 6.10] and [BF, Proposition 3.7].

In this paper we shall refine the Littelmann path model (in its alcove walk form, see [Ra]) by putting labels on the paths to provide a combinatorial indexing of the points in the affine flag variety. This combinatorial method of expressing the points of G/I gives detailed information about the structure of the intersections

$$U^- vI \cap IwI \qquad \text{with} \qquad v, w \in \widetilde{W}. \tag{1.3}$$

The corresponding intersections in G/K have arisen in many contexts. Most notably, the set of *Mirković-Vilonen cycles of shape* $\lambda^\vee$ *and weight* $\mu^\vee$ is the set of irreducible components of the closure of $U^- t_{\mu^\vee} K \cap K t_{\lambda^\vee} K$ in G/K,

$$MV(\lambda^\vee)_{\mu^\vee} = \mathrm{Irr}(\overline{U^- t_{\mu^\vee} K \cap K t_{\lambda^\vee} K}),$$

and

$$\text{when } k = \mathbb{F}_q, \qquad \mathrm{Card}_{G/K}(U^- t_{\mu^\vee} K \cap K t_{\lambda^\vee} K) \ \text{ is}$$

(up to some easily understood factors) the coefficient of the monomial symmetric function $m_{\mu^\vee}$ in the expansion of the Macdonald spherical function $P_{\lambda^\vee}$.

Sections 2-6 give elementary treatments of Borcherds-Kac-Moody Lie algebras, Chevalley groups, the flag variety, loop groups and affine flag varieties. With future developments in mind we have presented this material in the context of loop groups of symmetrizable Kac-Moody groups. In spite of the generality in Sections 2-6, the main results of this paper, given in Section 7, are only for loop groups of finite dimensional Chevalley groups. We do have some results in the more general case, but the restrictions of time and space have forced us to postpone the exposition of these results to a future paper.

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2 Borcherds-Kac-Moody Lie algebras

This section reviews definitions and sets notations for Borcherds-Kac-Moody Lie algebras. Standard references are the book of Kac [Kac], the books of Wakimoto [Wak1][Wak2], the survey article of Macdonald [Mac3] and the handwritten notes of Macdonald [Mac2]. Specifically, [Kac, Ch. 1] is a reference for §2.1, [Kac, Ch. 3 and 5] for §2.2, and [Kac, Ch. 2] for §2.3.

2.1 Constructing a Lie algebra from a matrix

Let $A = (a_{ij})$ be an $n \times n$ matrix. Let

$$r = \mathrm{rank}(A), \qquad \ell = \mathrm{corank}(A), \qquad \text{so that} \quad r + \ell = n. \tag{2.1}$$

By rearranging rows and columns we may assume that $(a_{ij})_{1\le i,j\le r}$ is nonsingular. Define a $\mathbb{C}$-vector space

$$\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{d}, \qquad \text{where} \qquad \begin{array}{l} \mathfrak{h}' \text{ has basis } h_1,\ldots,h_n, \quad \text{and} \\ \mathfrak{d} \text{ has basis } d_1,\ldots,d_\ell. \end{array} \tag{2.2}$$

Define $\alpha_1,\ldots,\alpha_n \in \mathfrak{h}^*$ by

$$\alpha_i(h_j) = a_{ij} \qquad \text{and} \qquad \alpha_i(d_j) = \delta_{i,r+j}, \tag{2.3}$$

and let

$$\bar{\mathfrak{h}}' = \mathfrak{h}'/\mathfrak{c}, \qquad \text{where} \qquad \mathfrak{c} = \{h \in \mathfrak{h}' \mid \alpha_i(h) = 0 \text{ for all } 1 \le i \le n\}. \tag{2.4}$$

Let $c_1 \ldots, c_\ell \in \mathfrak{h}'$ be a basis of $\mathfrak{c}$ so that $h_1,\ldots,h_r, c_1,\ldots,c_\ell, d_1,\ldots,d_\ell$ is another basis of $\mathfrak{h}$ and define $\kappa_1,\ldots,\kappa_\ell \in \mathfrak{h}^*$ by

$$\kappa_i(h_j) = 0, \qquad \kappa_i(c_j) = \delta_{ij}, \qquad \text{and} \qquad \kappa_i(d_j) = 0. \tag{2.5}$$

Then $\alpha_1,\ldots,\alpha_n,\kappa_1,\ldots,\kappa_\ell$ form a basis of $\mathfrak{h}^*$.

Let $\mathfrak{a}$ be the Lie algebra given by generators $\mathfrak{h}, e_1,\ldots,e_n, f_1,\ldots,f_n$ and relations

$$[h,h'] = 0, \qquad [e_i,f_j] = \delta_{ij}h_i, \qquad [h,e_i] = \alpha_i(h)e_i, \qquad [h,f_i] = -\alpha_i(h)f_i, \tag{2.6}$$

for $h,h' \in \mathfrak{h}$ and $1 \le i,j \le n$. The *Borcherds-Kac-Moody Lie algebra* of A is

$$\mathfrak{g} = \frac{\mathfrak{a}}{\mathfrak{r}}, \qquad \text{where} \qquad \mathfrak{r} \text{ is a the largest ideal of } \mathfrak{a} \text{ such that } \mathfrak{r} \cap \mathfrak{h} = 0. \tag{2.7}$$

The Lie algebra $\mathfrak{a}$ is graded by

$$Q = \sum_{i=1}^n \mathbb{Z}\alpha_i, \qquad \text{by setting} \quad \deg(e_i) = \alpha_i, \quad \deg(f_i) = -\alpha_i, \quad \deg(h) = 0, \tag{2.8}$$

for $h \in \mathfrak{h}$. Any ideal of $\mathfrak{a}$ is Q-graded and so $\mathfrak{g}$ is Q-graded (see [Mac2, (1.6)] or [Mac3, p. 81]),

$$\begin{array}{c} \mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right), \qquad \text{where} \quad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x\}, \quad \text{and} \\ R = \{\alpha \mid \alpha \ne 0 \text{ and } \mathfrak{g}_\alpha \ne 0\} \qquad \text{is the set of } \textit{roots} \text{ of } \mathfrak{g}. \end{array} \tag{2.9}$$

The *multiplicity* of a root $\alpha \in R$ is $\dim(\mathfrak{g}_\alpha)$ and the decomposition of $\mathfrak{g}$ in (2.9) is the decomposition of $\mathfrak{g}$ as an $\mathfrak{h}$-module (under the adjoint action). If

$$\begin{array}{l} \mathfrak{n}^+ \text{ is the subalgebra generated by } e_1,\ldots,e_n, \text{ and} \\ \mathfrak{n}^- \text{ is the subalgebra generated by } f_1,\ldots,f_n, \end{array}$$

then (see [Mac3, p. 83] or [Kac, §1.3])

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \qquad \text{and} \qquad \mathfrak{h} = \mathfrak{g}_0, \quad \mathfrak{n}^+ = \bigoplus_{\alpha\in R^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^- = \bigoplus_{\alpha \in R^+} \mathfrak{g}_{-\alpha}, \tag{2.10}$$

where

$$R^+ = Q^+ \cap R \qquad \text{with} \qquad Q^+ = \sum_{i=1}^n \mathbb{Z}_{\ge 0}\alpha_i. \tag{2.11}$$

Let $\mathfrak{c}$ and $\mathfrak{d}$ be as in (2.2) and (2.4). Then

$$\mathfrak{d} \text{ acts on } \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] \text{ by derivations}, \qquad \mathfrak{c} = Z(\mathfrak{g}) = Z(\mathfrak{g}'),$$

$$\begin{aligned} \mathfrak{g} &= \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ = \mathfrak{a}/\mathfrak{r} = \mathfrak{g}' \rtimes \mathfrak{d}, \\ \mathfrak{g}' &= \mathfrak{n}^- \oplus \mathfrak{h}' \oplus \mathfrak{n}^+ = [\mathfrak{g}, \mathfrak{g}], \\ \bar{\mathfrak{g}}' &= \mathfrak{n}^- \oplus \bar{\mathfrak{h}}' \oplus \mathfrak{n}^+ = \mathfrak{g}'/\mathfrak{c}, \end{aligned} \tag{2.12}$$

and $\mathfrak{g}'$ is the universal central extension of $\mathfrak{g}'$ (see [Kac, Ex. 3.14]).

2.2 Cartan matrices, $\mathfrak{sl}_2$ subalgebras and the Weyl group

A *Cartan matrix* is an $n \times n$ matrix $A = (a_{ij})$ such that

$$a_{ij} \in \mathbb{Z}, \qquad a_{ii} = 2, \qquad a_{ij} \le 0 \text{ if } i \neq j, \qquad a_{ij} \neq 0 \text{ if and only if } a_{ji} \neq 0. \tag{2.13}$$

When A is a Cartan matrix the Lie algebra $\mathfrak{g}$ contains many subalgebras isomorphic to $\mathfrak{sl}_2$. For $1 \le i \le n$, the elements e_i and f_i act locally nilpotently on $\mathfrak{g}$ (see [Mac3, p. 85] or [Mac2, (1.19)] or [Kac, Lemma 3.5]),

$$\text{span}\{e_i, f_i, h_i\} \cong \mathfrak{sl}_2, \qquad \text{and} \qquad \tilde{s}_i = \exp(\text{ad}e_i)\exp(-\text{ad}f_i)\exp(\text{ad}e_i) \tag{2.14}$$

is an automorphism of $\mathfrak{g}$ (see [Kac, Lemma 3.8]). Thus $\mathfrak{g}$ has lots of symmetry.

The *simple reflections* $s_i\colon \mathfrak{h}^* \to \mathfrak{h}^*$ and $s_i\colon \mathfrak{h} \to \mathfrak{h}$ are given by

$$s_i\lambda = \lambda - \lambda(h_i)\alpha_i \qquad \text{and} \qquad s_ih = h - \alpha_i(h)h_i, \qquad \text{for } 1 \le i \le n, \tag{2.15}$$

$\lambda \in \mathfrak{h}^*$, $h \in \mathfrak{h}$, and

$$\tilde{s}_i\mathfrak{g}_\alpha = \mathfrak{g}_{s_i\alpha} \qquad \text{and} \qquad \tilde{s}_ih = s_ih, \qquad \text{for } \alpha \in R, \quad h \in \mathfrak{h}.$$

The *Weyl group* W is the subgroup of $GL(\mathfrak{h}^*)$ (or $GL(\mathfrak{h})$) generated by the simple reflections. The simple reflections on $\mathfrak{h}$ are reflections in the hyperplanes

$$\mathfrak{h}^{\alpha_i} = \{h \in \mathfrak{h} \mid \alpha_i(h) = 0\}, \qquad \text{and} \qquad \mathfrak{c} = \mathfrak{h}^W = \bigcap_{i=1}^n \mathfrak{h}^{\alpha_i}.$$

The representation of W on $\mathfrak{h}$ and $\mathfrak{h}^*$ are dual so that

$$\lambda(wh) = (w^{-1}\lambda)(h), \qquad \text{for } w \in W,\ \lambda \in \mathfrak{h}^*,\ h \in \mathfrak{h}.$$

The group W is presented by generators $s_1, \ldots, s_n$ and relations

$$s_i^2 = 1 \qquad \text{and} \qquad \underbrace{s_is_js_i\cdots}_{m_{ij}\text{ factors}} = \underbrace{s_js_is_j\cdots}_{m_{ij}\text{ factors}} \tag{2.16}$$

for pairs $i \neq j$ such that $a_{ij}a_{ji} < 4$, where $m_{ij} = 2, 3, 4, 6$ if $a_{ij}a_{ji} = 0, 1, 2, 3$, respectively (see [Mac2, (2.12)] or [Kac, Prop. 3.13]).

The *real roots* of $\mathfrak{g}$ are the elements of the set

$$R_{\text{re}} = \bigcup_{i=1}^n W\alpha_i, \qquad \text{and} \qquad R_{\text{im}} = R \backslash R_{\text{re}} \tag{2.17}$$

is the set of *imaginary roots* of $\mathfrak{g}$. If $\alpha = w\alpha_i$ is a real root then there is a subalgebra isomorphic to $\mathfrak{sl}_2$ spanned by

$$e_\alpha = \tilde{w}e_i, \quad f_\alpha = \tilde{w}f_i, \quad \text{and} \quad h_\alpha = \tilde{w}h_i, \tag{2.18}$$

and $s_\alpha = ws_iw^{-1}$ is a reflection in W acting on $\mathfrak{h}$ and $\mathfrak{h}^*$ by

$$s_\alpha\lambda = \lambda - \lambda(h_\alpha)\alpha \qquad \text{and} \qquad s_\alpha h = h - \alpha(h)h_\alpha, \qquad \text{respectively.} \tag{2.19}$$

Let $\mathfrak{h}_\mathbb{R} = \mathbb{R}\text{-span}\{h_1, \ldots, h_n, d_1, \ldots, d_\ell\}$. The group W acts on $\mathfrak{h}_\mathbb{R}$ and the *dominant chamber*

$$C = \{\lambda^\vee \in \mathfrak{h}_\mathbb{R} \mid \langle \alpha_i, \lambda^\vee \rangle \geq 0 \text{ for all } 1 \leq i \leq n\} \tag{2.20}$$

is a fundamental domain for the action of W on the *Tits cone*

$$X = \bigcup_{w \in W} wC = \{h \in \mathfrak{h}_\mathbb{R} \mid \langle \alpha, h \rangle < 0 \text{ for a finite number of } \alpha \in R^+\}. \tag{2.21}$$

$X = \mathfrak{h}_\mathbb{R}$ if and only if W is finite (see [Kac, Prop. 3.12] and [Mac2, (2.14)]).

2.3 Symmetrizable matrices and invariant forms

A *symmetrizable matrix* is a matrix $A = (a_{ij})$ such that there exists a diagonal matrix

$$\mathcal{E} = \text{diag}(\epsilon_1, \ldots, \epsilon_n), \quad \epsilon_i \in \mathbb{R}_{>0}, \qquad \text{such that} \qquad A\mathcal{E} \text{ is symmetric.} \tag{2.22}$$

If $\langle , \rangle \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is a $\mathfrak{g}$-invariant symmetric bilinear form then

$$\langle h_i, h \rangle = \langle [e_i, f_i], h \rangle = -\langle f_i, [e_i, h] \rangle = \langle f_i, \alpha_i(h)e_i \rangle = \alpha_i(h)\langle e_i, f_i \rangle,$$

so that

$$\langle h_i, h \rangle = \alpha_i(h)\epsilon_i, \qquad \text{where} \qquad \epsilon_i = \langle e_i, f_i \rangle. \tag{2.23}$$

Conversely, if A is a symmetrizable matrix then there is a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$ determined by the formulas in (2.23) (see [Mac2, (3.12)] or [Kac, Theorem 2.2]).

If A is a Cartan matrix and $\langle , \rangle \colon \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ is a W-invariant symmetric bilinear form then

$$\langle h_i, h \rangle = -\langle s_ih_i, h \rangle = -\langle h_i, s_ih \rangle = -\langle h_i, h - \alpha_i(h)h_i \rangle = -\langle h_i, h \rangle + \alpha_i(h)\langle h_i, h_i \rangle,$$

so that

$$\langle h_i, h \rangle = \alpha_i(h)\epsilon_i, \qquad \text{where} \qquad \epsilon_i = \tfrac{1}{2}\langle h_i, h_i \rangle. \tag{2.24}$$

In particular, $\alpha_i(h_j)\epsilon_i = \langle h_i, h_j \rangle = \langle h_j, h_i \rangle = \alpha_j(h_i)\epsilon_j$ so that A is symmetrizable. Conversely, if A is a symmetrizable Cartan matrix then there is a nondegenerate W-invariant symmetric bilinear form on $\mathfrak{h}$ determined by the formulas in (2.24) (see [Mac2, (2.26)]).

If $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$ then $[x_\alpha, y_\alpha] \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{g}_0 = \mathfrak{h}$ and $\langle h, [x_\alpha, y_\alpha] \rangle = -\langle [x_\alpha, h], y_\alpha \rangle = \alpha(h)\langle x_\alpha, y_\alpha \rangle$, so that

$$[x_\alpha, y_\alpha] = \langle x_\alpha, y_\alpha \rangle h_\alpha^\vee, \qquad \text{where } \langle h, h_\alpha^\vee \rangle = \alpha(h) \text{ for all } h \in \mathfrak{h} \tag{2.25}$$

determines $h_\alpha^\vee \in \mathfrak{h}$. If $\alpha \in R_{\text{re}}$ and $e_\alpha, f_\alpha, h_\alpha$ are as in (2.18) then

$$h_\alpha = [e_\alpha, f_\alpha] = \langle e_\alpha, f_\alpha \rangle h_\alpha^\vee \qquad \text{and} \qquad \langle e_\alpha, f_\alpha \rangle = \tfrac{1}{2}\langle h_\alpha, h_\alpha \rangle. \tag{2.26}$$

Let

$$\alpha^\vee = \langle e_\alpha, f_\alpha \rangle \alpha = \tfrac{1}{2}\langle h_\alpha, h_\alpha \rangle \alpha \qquad \text{so that} \qquad \alpha^\vee(h) = \langle h, h_\alpha \rangle. \tag{2.27}$$

Use the vector space isomorphism

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\sim} & \mathfrak{h}^* \\ h & \longmapsto & \langle h, \cdot \rangle \\ h_\alpha & \longmapsto & \alpha^\vee \\ h_\alpha^\vee & \longmapsto & \alpha \end{array} \qquad \text{to identify} \quad Q^\vee = \sum_{i=1}^n \mathbb{Z}h_i \quad \text{and} \quad Q^* = \sum_{i=1}^n \mathbb{Z}\alpha_i^\vee \tag{2.28}$$

and write

$$\langle \lambda^\vee, \mu \rangle = \mu(h_\lambda) \qquad \text{if} \quad \lambda^\vee = \lambda_1\alpha_1^\vee + \cdots + \lambda_n\alpha_n^\vee \quad \text{and} \quad h_\lambda = \lambda_1 h_1 + \cdots + \lambda_n h_n. \tag{2.29}$$

3 Steinberg-Chevalley groups

This section gives a brief treatment of the theory of Chevalley groups. The primary reference is [St] and the extensions to the Kac-Moody case are found in [Ti].

Let A be a Cartan matrix and let R_{re} be the real roots of the corresponding Borcherds-Kac-Moody Lie algebra $\mathfrak{g}$. Let U be the enveloping algebra of $\mathfrak{g}$. For each $\alpha \in R_{\mathrm{re}}$ fix a choice of e_α in (2.18) (a choice of $\tilde{w}$). Use the notation

$$x_\alpha(t) = \exp(te_\alpha) = 1 + e_\alpha + \frac{1}{2!}t^2e_\alpha^2 + \frac{1}{3!}t^3e_\alpha^3 + \cdots, \qquad \text{in } U[[t]].$$

Then

$$x_\alpha(t)x_\alpha(u) = x_\alpha(t+u) \qquad \text{in } U[[t,u]].$$

Following [Ti, 3.2], a *prenilpotent pair* is a pair of roots $\alpha, \beta \in R_{\mathrm{re}}$ such that there exists $w, w' \in W$ with

$$w\alpha, w\beta \in R_{\mathrm{re}}^+ \qquad \text{and} \qquad w'\alpha, w'\beta \in -R_{\mathrm{re}}^+.$$

This condition guarantees that the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ is nilpotent. Let α, β be a prenilpotent pair and let $e_\alpha \in \mathfrak{g}_\alpha$ and $e_\beta \in \mathfrak{g}_\beta$ be as in (2.18). By [St, Lemma 15] there are unique integers $C_{\alpha\beta}^{i,j}$ such that

$$x_\alpha(t)x_\beta(u) = x_\beta(u)x_\alpha(t)x_{\alpha+\beta}(C_{\alpha,\beta}^{1,1}tu)x_{2\alpha+\beta}(C_{\alpha,\beta}^{2,1}t^2u)x_{\alpha+2\beta}(C_{\alpha,\beta}^{1,2}ut^2)\cdots.$$

Let $\mathbb{F}$ be a commutative ring. The *Steinberg group*

$$\mathrm{St} \text{ is given by generators } x_\alpha(f) \text{ for } \alpha \in R_{\mathrm{re}},\ f \in \mathbb{F},$$

and relations

$$x_\alpha(f_1)x_\alpha(f_2) = x_\alpha(f_1+f_2), \qquad \text{for } \alpha \in R_{\mathrm{re}}, \quad \text{and} \tag{3.1}$$

$$x_\alpha(f_1)x_\beta(f_2) = x_\beta(f_2)x_\alpha(f_1)x_{\alpha+\beta}(C_{\alpha,\beta}^{1,1}f_1f_2)x_{2\alpha+\beta}(C_{\alpha,\beta}^{2,1}f_1^2f_2)x_{\alpha+2\beta}(C_{\alpha,\beta}^{1,2}f_1f_2^2)\cdots \tag{3.2}$$

for prenilpotent pairs α, β. In St define

$$n_\alpha(g) = x_\alpha(g)x_{-\alpha}(-g^{-1})x_\alpha(g), \quad n_\alpha = n_\alpha(1), \quad \text{and} \quad h_{\alpha^\vee}(g) = n_\alpha(g)n_\alpha^{-1}, \tag{3.3}$$

for $\alpha \in R_{\mathrm{re}}$ and $g \in \mathbb{F}^\times$.

Let $\mathfrak{h}_\mathbb{Z}$ be a $\mathbb{Z}$-lattice in $\mathfrak{h}$ which is stable under the W-action and such that

$$\mathfrak{h}_\mathbb{Z} \supseteq Q^\vee, \qquad \text{where} \quad Q^\vee = \mathbb{Z}\text{-span}\{h_1, \ldots, h_n\}$$

with $h_1, \dots, h_n$ as in (2.2). With

$$T \text{ given by generators } h_{\lambda^\vee}(g) \text{ for } \lambda^\vee \in \mathfrak{h}_\mathbb{Z},\ g \in \mathbb{F}^\times, \quad \text{and relations}$$

$$h_{\lambda^\vee}(g_1)h_{\lambda^\vee}(g_2) = h_{\lambda^\vee}(g_1g_2) \qquad \text{and} \qquad h_{\lambda^\vee}(g)h_{\mu^\vee}(g) = h_{\lambda^\vee+\mu^\vee}(g), \tag{3.4}$$

the *Tits group*

$$G \text{ is the group generated by } St \text{ and } T$$

with the relations coming from the third equation in (3.3) and the additional relations

$$h_{\lambda^\vee}(g)x_\alpha(f)h_{\lambda^\vee}(g)^{-1} = x_\alpha(g^{\langle\lambda^\vee,\alpha\rangle}f) \qquad \text{and} \qquad n_ih_{\lambda^\vee}(g)n_i^{-1} = h_{s_i\lambda^\vee}(g). \tag{3.5}$$

For $\alpha, \beta \in R_{\mathrm{re}}$ let $\epsilon_{\alpha\beta} = \pm 1$ be given by

$$\tilde{s}_\alpha(e_\beta) = \epsilon_{\alpha\beta}e_{s_\alpha\beta}, \qquad \text{where} \qquad \tilde{s}_\alpha = \exp(\mathrm{ad}e_\alpha)\exp(-\mathrm{ad}f_\alpha)\exp(\mathrm{ad}e_\alpha)$$

(see [CC, p.48] and [Ti, (3.3)]). By [St, Lemma 37] (see also [Ti, §3.7(a)])

$$n_\alpha(g)x_\beta(f)n_\alpha(g)^{-1} = x_{s_\alpha\beta}(\epsilon_{\alpha\beta}g^{-\langle\beta,\alpha^\vee\rangle}f), \qquad h_{\lambda^\vee}(g)x_\beta(f)h_{\lambda^\vee}(g)^{-1} = x_\beta(g^{\langle\beta,\lambda^\vee\rangle}f), \tag{3.6}$$

$$\text{and} \quad n_\alpha(g)h_{\lambda^\vee}(g')n_\alpha(g)^{-1} = h_{s_\alpha\lambda^\vee}(g'). \tag{3.7}$$

Thus G has a symmetry under the subgroup

$$N \text{ generated by } T \text{ and the } n_\alpha(g) \text{ for } \alpha \in R_{\mathrm{re}}, g \in \mathbb{F}^\times. \tag{3.8}$$

If $\mathbb{F}$ is big enough then N is the normalizer of T in G [St, Ex. (b) p. 36] and, by [St, Lemma 27], the homomorphism

$$\begin{matrix} N & \longrightarrow & W \\ n_\alpha(g) & \longmapsto & s_\alpha \end{matrix} \qquad \text{is surjective with kernel } T. \tag{3.9}$$

Remark 3.1. [Ti, §3.7(b)] If $\mathfrak{h}_\mathbb{Z} = Q^\vee$ and the first relation of (3.5) holds in St then there is a surjective homomorphism $\psi\colon \mathrm{St} \twoheadrightarrow G$. By [St, Lemma 22], the elements

$$n_\alpha h_{\lambda^\vee}(g)n_\alpha^{-1}h_{s_\alpha\lambda^\vee}(g)^{-1} \qquad \text{and} \qquad n_\alpha(g)n_\alpha^{-1}h_{\alpha^\vee}(g)^{-1}$$

automatically commute with each $x_\beta(f)$ so that $\ker(\psi) \subseteq Z(\mathrm{St})$. In many cases St is the universal central extension of G (see [Ti, 3.7(c)] and [St, Theorems 10,11,12]).

Remark 3.2. The algebra $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$ in (2.12) is generated by e_α, $\alpha \in R_{\mathrm{re}}$. A $\mathfrak{g}'$-module V is *integrable* if e_α, $\alpha \in R_{\mathrm{re}}$, act locally nilpotently so that

$$x_\alpha(c) = \exp(ce_\alpha), \qquad \text{for } \alpha \in R_{\mathrm{re}},\ c \in \mathbb{C}, \tag{3.10}$$

are well defined operators on V. The *Chevalley group* G_V is the subgroup of $GL(V)$ generated by the operators in (3.10). To do this integrally use a Kostant $\mathbb{Z}$-form and choose a lattice in the module V (see [Ti, §4.3-4] and [St, Ch. 1]). The *Kac-Moody group* is the group G_{KM} generated by symbols

$$x_\alpha(c), \quad \alpha \in R_{\mathrm{re}}, c \in \mathbb{C}, \qquad \text{with relations} \qquad x_\alpha(c_1)x_\alpha(c_2) = x_\alpha(c_1+c_2)$$

and the additional relations coming from forcing an element to be 1 if it acts by 1 on *every* integrable $\mathfrak{g}'$ module. This is essentially the Chevalley group G_V for the case when V is the adjoint representation and so $G_{KM} \subseteq \mathrm{Aut}(\mathfrak{g}')$. There are surjective homomorphisms

$$\mathrm{St}(\mathbb{C}) \twoheadrightarrow G_{KM} \twoheadrightarrow G_V.$$

See [Kac, Exercises 3.16-19] and [Ti, Proposition 1].

Remark 3.3. [St, Lemma 28] In the setting of Remark 3.2 let T_V be the subgroup of G_V generated by $h_{\alpha^\vee}(g)$ for $\alpha\in R_{\mathrm{re}}, g\in\mathbb{F}^\times$. Then

$$h_{\alpha_1^\vee}(g_1)\cdots h_{\alpha_n^\vee}(g_n)=1\quad\text{if and only if}\quad g_1^{\langle\mu,\alpha_1^\vee\rangle}\cdots g_n^{\langle\mu,\alpha_n^\vee\rangle}=1\quad\text{for all weights } \mu \text{ of } V,$$

$$Z(G_V)=\{h_{\alpha_1^\vee}(g_1)\cdots h_{\alpha_n^\vee}(g_n)\mid g_1^{\langle\beta,\alpha_1^\vee\rangle}\cdots g_n^{\langle\beta,\alpha_n^\vee\rangle}=1\quad\text{for all } \beta\in R\},$$

and if $\mathbb{F}$ is big enough

$$T_V=\{h_{\omega_1^\vee}(g_1)\cdots h_{\omega_n^\vee}(g_n)\mid g_1,\dots,g_n\in\mathbb{F}^\times\},$$

where $\omega_1^\vee,\dots,\omega_n^\vee$ is a $\mathbb{Z}$-basis of the $\mathbb{Z}$-span of the weights of V [St, Lemma 35].

4 Labeling points of the flag variety G/B

In this section we follow [St, Ch. 8] to show that the points of the flag variety are naturally indexed by labeled walks. This is the first step in making a precise connection between the points in the flag variety and the alcove walk theory in [Ra].

Let G be a Tits group as in (3.5) over the field $\mathbb{F}=\mathbb{C}$. The *root subgroups*

$$\mathcal{X}_\alpha=\{x_\alpha(c)\mid c\in\mathbb{C}\},\quad\text{for } \alpha\in R_{\mathrm{re}},\qquad\text{satisfy}\qquad w\mathcal{X}_\beta w^{-1}=\mathcal{X}_{w\beta},\tag{4.1}$$

for $w\in W$ and $\beta\in R_{\mathrm{re}}$, since $h_{\alpha^\vee}(c)\mathcal{X}_\beta h_{\alpha^\vee}(c)^{-1}=\mathcal{X}_\beta$ and $n_\alpha\mathcal{X}_\beta n_\alpha^{-1}=\mathcal{X}_{s_\alpha\beta}$. As a group $\mathcal{X}_\alpha$ is isomorphic to $\mathbb{C}$ (under addition).

The *flag variety* is G/B, where the subgroup

$$B \text{ is generated by } T \text{ and } x_\alpha(f) \text{ for } \alpha\in R^+_{\mathrm{re}},\ f\in\mathbb{C}.\tag{4.2}$$

Let $w\in W$. The *inversion set of* w is

$$R(w)=\{\alpha\in R^+_{\mathrm{re}}\mid w^{-1}\alpha\notin R^+_{\mathrm{re}}\}\qquad\text{and}\qquad \ell(w)=\mathrm{Card}(R(w))\tag{4.3}$$

is the *length of* w. View a reduced expression $\vec{w}=s_{i_1}\cdots s_{i_\ell}$ in the generators in (2.16) as a *walk* in W starting at 1 and ending at w,

$$1\quad\longrightarrow\quad s_{i_1}\quad\longrightarrow\quad s_{i_1}s_{i_2}\quad\longrightarrow\quad\cdots\quad\longrightarrow\quad s_{i_1}\cdots s_{i_\ell}=w.\tag{4.4}$$

Letting $x_i(c)=x_{\alpha_i}(c)$ and $n_i=n_{\alpha_i}(1)$, the following theorem shows that

$$BwB=\{x_{i_1}(c_1)n_{i_1}^{-1}x_{i_2}(c_2)n_{i_2}^{-1}\cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1}B\mid c_1,\dots,c_\ell\in\mathbb{C}\}\tag{4.5}$$

so that the G/B-points of BwB are in bijection with labelings of the edges of the walk by complex numbers $c_1,\dots,c_\ell$. The elements of $R(w)$ are

$$\beta_1=\alpha_{i_1},\quad \beta_2=s_{i_1}\alpha_{i_2},\quad \dots,\quad \beta_\ell=s_{i_1}\cdots s_{i_{\ell-1}}\alpha_{i_\ell},\tag{4.6}$$

and the first relation in (3.6) gives

$$x_{i_1}(c_1)n_{i_1}^{-1}x_{i_2}(c_2)n_{i_2}^{-1}\cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1}=x_{\beta_1}(\pm c_1)\cdots x_{\beta_\ell}(\pm c_\ell)n_w,\tag{4.7}$$

where $n_w=n_{i_1}^{-1}\cdots n_{i_\ell}^{-1}$.

Theorem 4.1. [St, Thm. 15 and Lemma 43] *Let $w \in W$ and let n_w be a representative of w in N. If*

$$R(w) = \{\beta_1, \ldots, \beta_\ell\} \qquad \text{then} \qquad \{x_{\beta_1}(c_1) \cdots x_{\beta_\ell}(c_\ell) n_w \mid c_1, \ldots, c_\ell \in \mathbb{C}\}$$

is a set of representatives of the B-cosets in BwB.

Proof. The conceptual reason for this is that

$$\begin{aligned}
BwB &= \left(\prod_{\alpha \in R^+_{\mathrm{re}}} \mathcal{X}_\alpha \right) n_w B = n_w \left(\prod_{w^{-1}\alpha \notin R^+_{\mathrm{re}}} \mathcal{X}_{w^{-1}\alpha} \right) \left(\prod_{w^{-1}\alpha \in R^+_{\mathrm{re}}} \mathcal{X}_{w^{-1}\alpha} \right) B \\
&= n_w \left(\prod_{w^{-1}\alpha \notin R^+_{\mathrm{re}}} \mathcal{X}_{w^{-1}\alpha} \right) B = \left(\prod_{\alpha \in R(w)} \mathcal{X}_\alpha \right) n_w B \\
&= \{x_{\beta_1}(c_1) \cdots x_{\beta_\ell}(c_\ell) n_w B \mid c_1, \ldots, c_\ell \in \mathbb{F}\}.
\end{aligned}$$

Since R^+_{re} may be infinite there is a subtlety in the decomposition and ordering of the product of $\mathcal{X}_\alpha$ in the second "equality" and it is necessary to proceed more carefully. Choose a reduced decomposition $w = s_{i_1} \cdots s_{i_\ell}$ and let $\beta_1, \ldots, \beta_\ell$ be the ordering of $R(w)$ from (4.6).
Step 1: Since $R(w) \subseteq R^+_{\mathrm{re}}$ there is an inclusion

$$\{x_{\beta_1}(c_1) \cdots x_{\beta_\ell}(c_\ell) n_w B \mid c_1, \ldots, c_\ell \in \mathbb{C}\} \subseteq BwB.$$

To prove equality proceed by induction on ℓ.

Base case: Suppose that $w = s_j$. Let $\alpha \in R^+_{\mathrm{re}}$ and $c, d \in \mathbb{C}$. If $c = 0$ or α, α_j is a prenilpotent pair then, by relation (3.2),

$$x_\alpha(d) x_{\alpha_j}(c) n_j^{-1} B = x_{\alpha_j}(c') n_j^{-1} B, \qquad \text{for some } c' \in \mathbb{C}. \tag{4.8}$$

If α, α_j is not a prenilpotent pair and $c \neq 0$ then $\alpha, -\alpha_j$ is a prenilpotent pair and, by (3.2),

$$x_\alpha(d) x_{\alpha_j}(c) n_j^{-1} B = x_\alpha(d) x_{-\alpha_j}(c^{-1}) B = x_{-\alpha_j}(c^{-1}) B = x_{\alpha_j}(c) n_j^{-1} B.$$

Thus $\{x_{\alpha_j}(c) n_j^{-1} B \mid c \in \mathbb{C}\}$ is B-invariant and so $Bs_jB = \{x_{\alpha_j}(c) n_j^{-1} B \mid c \in \mathbb{C}\}$.

Induction step: If $w = s_{i_1} \cdots s_{i_\ell}$ is reduced and if $\ell(ws_j) > \ell(w)$ then, by induction,

$$Bws_jB \subseteq BwB \cdot Bs_jB = \{x_{\beta_1}(c_1) \cdots x_{\beta_\ell}(c_\ell) x_{w\alpha_j}(c) n_w n_j^{-1} B \mid c_1, \ldots, c_\ell, c \in \mathbb{F}\},$$

so that $Bws_jB = \{x_{\beta_1}(c_1) \cdots x_{\beta_{\ell+1}}(c_{\ell+1}) n_{ws_j} B \mid c_1, \ldots, c_{\ell+1} \in \mathbb{C}\}$ with $\beta_{\ell+1} = w\alpha_j$.
Step 2: Prove that $\quad BwB = BvB$ if and only if $w = v \quad$ by induction on $\ell(w)$.

Base case: Suppose that $\ell(w) = 0$. Then $BwB = BvB$ implies that $v \in B$ so that there is a representative n_v of v such that $n_v \in B \cap N$. Then $vR^+_{\mathrm{re}} \subseteq R^+_{\mathrm{re}}$ since $n_v \mathcal{X}_\alpha n_v^{-1} = \mathcal{X}_{v\alpha} \in B$ for $\alpha \in R^+_{\mathrm{re}}$. So $\ell(v) = 0$. Thus, by (2.16), $v = 1$.

Induction step: Assume $BwB = BvB$ and s_j is such that $\ell(ws_j) < \ell(w)$. Since $BvB \cdot Bs_jB \subseteq BvB \cup Bvs_jB$ (see [St, Lemma 25],

$$Bws_jB \subseteq BwB \cdot Bs_jB = BvB \cdot Bs_jB \subseteq BvB \cup Bvs_jB = BwB \cup Bvs_jB.$$

Thus, by induction, $ws_j = w$ or $ws_j = vs_j$. Since $ws_j \neq w$, it follows that $w = v$.

Step 3: Let us show that if $x_{\alpha_{i_1}}(c_1)n_{i_1}^{-1}\cdots x_{\alpha_{i_\ell}}(c_\ell)n_{i_\ell}^{-1}B = x_{\alpha_{i_1}}(c_1')n_{i_1}^{-1}\cdots x_{\alpha_{i_\ell}}(c_\ell')n_{i_\ell}^{-1}B$, then $c_i = c_i'$ for $i = 1, 2, \ldots, \ell$. The left hand side of

$$x_{\alpha_2}(c_2)n_{i_2}^{-1}\cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1}B = n_{i_1}x_{i_1}(c_1'-c_1)n_{i_1}^{-1}\cdots x_{i_\ell}(c_\ell')n_{i_\ell}^{-1}B$$

is in $Bs_{i_2}\cdots s_{i_\ell}B$. If $c_1' \neq c_1$ then $n_{i_1}^{-1}x_{i_1}(c_1'-c_1)n_{i_1} \in Bs_{i_1}B$ and the right hand side is contained in

$$n_{i_1}^{-1}x_{i_1}(c_1'-c_1)n_{i_1}Bs_{i_2}\cdots s_{i_\ell}B \subseteq Bs_{i_1}B\cdot Bs_{i_2}\cdots s_{i_\ell}B = Bs_{i_1}\cdots s_{i_\ell}B.$$

By Step 2 this is impossible and so $c_1' = c_1$. Then, by induction, $c_i' = c_i$ for $i = 1, 2, \ldots, \ell$.

Step 4: From the definition of $R(w)$ it follows that if $\alpha, \beta \in R(w)$ and $\alpha + \beta \in R_{\mathrm{re}}$ then $\alpha+\beta \in R(w)$ and if $\alpha, \beta \in R(w)$ then α, β form a prenilpotent pair. Thus, by [St, Lemma 17], any total order on the set $R(w)$ can be taken in the statement of the theorem. □

Remark 4.2. Suppose that $\lambda \in \mathfrak{h}^*$ is dominant integral and $M(\lambda)$ is an (integrable) highest weight representation of G generated by a highest weight vector v_λ^+. Then the set $BwBv_\lambda^+$ contains the vector wv_λ^+ and is contained in the sum $\bigoplus_{\nu \geq w\lambda} M(\lambda)_\nu$ of the weight spaces with weights $\geq w\lambda$. This is another way to show that if $w \neq v$ then $BwB \neq BvB$ and accomplish Step 2 in the proof of Theorem 4.1.

5 Loop Lie algebras and their extensions

This section gives a presentation of the theory of loop Lie algebras. The main lines of the theory are exactly as in the classical case (see, for example, [Mac2, §4] and [Kac, ch. 7]) but, following recent trends (see [Ga2]. [GK], [GR] and [Rou]) we treat the more general setting of the loop Lie algebra of a Kac-Moody Lie algebra.

Let $\mathfrak{g}_0$ be a symmetrizable Kac-Moody Lie algebra with bracket $[,]_0\colon \mathfrak{g}_0 \otimes \mathfrak{g}_0 \to \mathfrak{g}_0$ and invariant form $\langle,\rangle_0\colon \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathbb{C}$. The *loop Lie algebra* is

$$\mathfrak{g}_0[t,t^{-1}] = \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}_0 \qquad \text{with bracket} \qquad [t^m x, t^n y]_0 = t^{m+n}[x,y]_0,$$

for $x, y \in \mathfrak{g}_0$. Let

$$\mathfrak{g} = \mathfrak{g}_0[t,t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d, \qquad \mathfrak{g}' = \mathfrak{g}_0[t,t^{-1}] \oplus \mathbb{C}c, \qquad \bar{\mathfrak{g}}' = \mathfrak{g}_0[t,t^{-1}] = \frac{\mathfrak{g}'}{\mathbb{C}c}$$

where the bracket on $\mathfrak{g}$ is given by

$$[t^m x, t^n y] = t^{m+n}[x,y]_0 + \delta_{m+n,0}m\langle x,y\rangle_0 c, \qquad c \in Z(\mathfrak{g}), \qquad [d, t^m x] = mt^m x. \tag{5.1}$$

By [Kac, Ex. 7.8], $\mathfrak{g}'$ is the universal central extension of $\bar{\mathfrak{g}}'$. An invariant symmetric form on $\mathfrak{g}$ is given by

$$\langle c,d\rangle = 1, \qquad \langle c, t^m y\rangle = \langle d, t^m y\rangle = 0, \qquad \langle c,c\rangle = \langle d,d\rangle = 0, \tag{5.2}$$

and

$$\langle t^m x, t^n y\rangle = \begin{cases} \langle x,y\rangle_0, & \text{if } m+n=0,\\ 0, & \text{otherwise,}\end{cases} \tag{5.3}$$

for $x, y \in \mathfrak{g}_0$, $m, n \in \mathbb{Z}$.

Fix a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ and let

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathbb{C}c \oplus \mathbb{C}d, \qquad \mathfrak{h}' = \mathfrak{h}_0 \oplus \mathbb{C}c, \qquad \bar{\mathfrak{h}}' = \mathfrak{h}_0. \tag{5.4}$$

As in (2.2), let $h_1, \ldots, h_n, d_1, \ldots, d_\ell$ be a basis of $\mathfrak{h}_0$ and let

$$\begin{array}{l}\{h_1, \ldots, h_n, d_1, \ldots, d_\ell, c, d\} \text{ be a basis of } \mathfrak{h} \text{ and} \\ \{\omega_1, \ldots, \omega_n, \delta_1, \ldots, \delta_\ell, \Lambda_0, \delta\} \text{ the dual basis in } \mathfrak{h}^*\end{array} \tag{5.5}$$

so that

$$\delta(\mathfrak{h}_0) = 0, \quad \delta(c) = 0, \quad \delta(d) = 1, \qquad \text{and} \qquad \Lambda_0(\mathfrak{h}_0) = 0, \quad \Lambda_0(c) = 1, \quad \Lambda_0(d) = 0. \tag{5.6}$$

Let R be as in (2.9). As an $\mathfrak{h}$-module

$$\mathfrak{g} = \left(\bigoplus_{\substack{\alpha\in R \\ k\in\mathbb{Z}}} \mathfrak{g}_{\alpha+k\delta} \right) \oplus \left(\bigoplus_{k\in\mathbb{Z}_{\neq 0}} \mathfrak{g}_{k\delta} \right) \oplus \mathfrak{h}, \quad \text{where} \quad \mathfrak{h} = \mathfrak{h}_0 \oplus \mathbb{C}c \oplus \mathbb{C}d, \tag{5.7}$$

$$\mathfrak{g}_{\alpha+k\delta} = t^k \mathfrak{g}_\alpha, \qquad \mathfrak{g}_{k\delta} = t^k \mathfrak{h}_0, \qquad \text{and} \qquad \tilde{R} = (R + \mathbb{Z}\delta) \cup \mathbb{Z}_{\neq 0}\delta \tag{5.8}$$

is the set of *roots* of $\mathfrak{g}$.

Let $\alpha \in R_{\mathrm{re}}$ with $\alpha = w\alpha_i$ and fix a choice of e_α, f_α and h_α in (2.18) (choose $\tilde{w}$). Then

$$e_{-\alpha+k\delta} = t^k f_\alpha, \qquad f_{-\alpha+k\delta} = t^{-k} e_\alpha, \qquad h_{-\alpha+k\delta} = -h_\alpha + k\langle e_\alpha, f_\alpha\rangle_0 c, \tag{5.9}$$

span a subalgebra isomorphic to $\mathfrak{sl}_2$. If $\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus \mathfrak{h}_0 \oplus \mathfrak{n}_0^+$ is the decomposition in (2.10) and

$\mathfrak{n}^+$ is the subalgebra generated by $\mathfrak{n}_0^+$ and $e_{-\alpha+k\delta}$ for $\alpha \in R_{\mathrm{re}}$, $k \in \mathbb{Z}_{>0}$, and
$\mathfrak{n}^-$ is the subalgebra generated by $\mathfrak{n}_0^-$ and $f_{-\alpha+k\delta}$ for $\alpha \in R_{\mathrm{re}}$, $k \in \mathbb{Z}_{>0}$,

then

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \quad \text{with} \quad \mathfrak{n}^+ = \mathfrak{n}_0^+ \oplus \left(\bigoplus_{\substack{\alpha\in R\cup\{0\} \\ k\in\mathbb{Z}_{>0}}} \mathfrak{g}_{\alpha+k\delta} \right) \quad \text{and} \quad \mathfrak{n}^- = \mathfrak{n}_0^- \oplus \left(\bigoplus_{\substack{\alpha\in R\cup\{0\} \\ k\in\mathbb{Z}_{<0}}} \mathfrak{g}_{\alpha+k\delta} \right).$$

The elements $e_{-\alpha+k\delta}$ and $f_{-\alpha+k\delta}$ in (5.9) act locally nilpotently on $\mathfrak{g}$ because f_α and e_α act locally nilpotently on $\mathfrak{g}_0$. Thus

$$\tilde{s}_{-\alpha+k\delta} = \exp(\mathrm{ad}\, t^k f_\alpha) \exp(-\mathrm{ad}\, t^{-k} e_\alpha) \exp(\mathrm{ad}\, t^k f_\alpha) \tag{5.10}$$

is a well defined automorphism of $\mathfrak{g}$ and

$$\tilde{s}_{-\alpha+k\delta} \mathfrak{g}_\beta = \mathfrak{g}_{s_{-\alpha+k\delta}\beta} \qquad \text{and} \qquad \tilde{s}_{-\alpha+k\delta} h = s_{-\alpha+k\delta} h, \tag{5.11}$$

for $h \in \mathfrak{h}$ and $\beta \in \tilde{R}$, where $s_{-\alpha+k\delta} \colon \mathfrak{h}^* \to \mathfrak{h}^*$ and $s_{-\alpha+k\delta} \colon \mathfrak{h} \to \mathfrak{h}$ are given by

$$s_{-\alpha+k\delta}\lambda = \lambda - \lambda(h_{-\alpha+k\delta})(-\alpha + k\delta) \quad \text{and} \quad s_{-\alpha+k\delta} h = h - (-\alpha + k\delta)(h) h_{-\alpha+k\delta}, \tag{5.12}$$

for $\lambda \in \mathfrak{h}^*$ and $h \in \mathfrak{h}$. The *Weyl group* of $\mathfrak{g}$ is the subgroup of $GL(\mathfrak{h}^*)$ (or $GL(\mathfrak{h})$) generated by the reflections $s_{-\alpha+k\delta}$,

$$W_{\mathrm{aff}} = \langle s_{-\alpha+k\delta} \mid \alpha \in R_{\mathrm{re}}, k \in \mathbb{Z} \rangle. \tag{5.13}$$

Noting that $\mathfrak{h}^* = \mathfrak{h}_0^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$ and $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathbb{C}c \oplus \mathbb{C}d$, use (5.12) to compute

$$\begin{array}{ll} s_{-\alpha+k\delta}(\bar{\lambda}) = \bar{\lambda} + \bar{\lambda}(h_\alpha)(-\alpha + k\delta), & s_{-\alpha+k\delta}(\bar{h}) = \bar{h} + \alpha(\bar{h})(-h_\alpha + k\langle e_\alpha, f_\alpha\rangle_0 c), \\ s_{-\alpha+k\delta}(\ell\Lambda_0) = \ell\Lambda_0 - k\ell\langle e_\alpha, f_\alpha\rangle_0(-\alpha + k\delta), & s_{-\alpha+k\delta}(mc) = mc, \\ s_{-\alpha+k\delta}(m\delta) = m\delta, & s_{-\alpha+k\delta}(\ell d) = \ell d - k\ell(-h_\alpha + k\langle e_\alpha, f_\alpha\rangle_0 c). \end{array}$$

for $\bar\lambda \in \mathfrak{h}_0^*$, $\bar h \in \mathfrak{h}_0$, $m, \ell \in \mathbb{C}$. For $\alpha \in R_{\rm re}$ and $k \in \mathbb{Z}$

$$\text{define } t_{k\alpha^\vee} \in W_{\rm aff} \text{ by} \qquad s_{-\alpha+k\delta} = t_{k\alpha^\vee} s_{-\alpha}, \tag{5.14}$$

and use (2.26) and (2.27) to compute

$$\begin{array}{ll} t_{k\alpha^\vee}(\bar\lambda) = \bar\lambda - \bar\lambda(kh_\alpha)\delta, & t_{k\alpha^\vee}(\bar h) = \bar h - k\alpha^\vee(\bar h)c, \\ t_{k\alpha^\vee}(\ell\Lambda_0) = \ell\Lambda_0 + \ell k\alpha^\vee - \ell\frac12\langle kh_\alpha, kh_\alpha\rangle_0\delta, & t_{k\alpha^\vee}(mc) = mc, \\ t_{k\alpha^\vee}(m\delta) = m\delta, & t_{k\alpha^\vee}(\ell d) = \ell d + \ell kh_\alpha - \ell\frac12\langle kh_\alpha, kh_\alpha\rangle_0 c. \end{array}$$

Then $t_{k\alpha^\vee}t_{j\beta^\vee}(\bar\lambda) = t_{kh_\alpha}(\bar\lambda - \bar\lambda(jh_\beta)\delta) = \bar\lambda - \bar\lambda(kh_\alpha + jh_\beta)\delta$, and

$$\begin{aligned} t_{k\alpha^\vee}t_{j\beta^\vee}(\ell\Lambda_0) &= t_{k\alpha^\vee}\left(\ell\Lambda_0 + \ell j\beta^\vee - \ell\tfrac12\langle jh_\beta, jh_\beta\rangle_0\delta\right) \\ &= \ell\Lambda_0 + \ell k\alpha^\vee - \ell\tfrac12\langle kh_\alpha, kh_\alpha\rangle_0\delta + \ell j\beta^\vee - \ell j\beta^\vee(kh_\alpha)\delta - \ell\tfrac12\langle jh_\beta, jh_\beta\rangle_0\delta \\ &= \ell\Lambda_0 + \ell(k\alpha^\vee + j\beta^\vee) - \ell\tfrac12\langle kh_\alpha + jh_\beta, kh_\alpha + jh_\beta\rangle_0\delta. \end{aligned}$$

This computation shows that $t_{k\alpha^\vee}t_{j\beta^\vee} = t_{j\alpha^\vee+k\beta^\vee}$. Thus, if W_0 is the Weyl group of $\mathfrak{g}_0$ and $Q^* = \mathbb{Z}\text{-span}\{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ then

$$W_{\rm aff} = \{t_{\lambda^\vee}w \mid \lambda^\vee \in Q^*, w \in W_0\} \qquad \text{with} \qquad t_{\lambda^\vee}t_{\mu^\vee} = t_{\lambda^\vee+\mu^\vee} \quad \text{and} \quad wt_{\lambda^\vee} = t_{w\lambda^\vee}w, \tag{5.15}$$

for $w \in W_0$, $\lambda^\vee, \mu^\vee \in Q^*$.

Since $\mathbb{C}\delta$ is $W_{\rm aff}$-invariant, the group $W_{\rm aff}$ acts on $\mathfrak{h}^*/\mathbb{C}\delta$ and $W_{\rm aff}$ acts on the set

$$\begin{array}{ccc} (\mathfrak{h}_0^* + \Lambda_0 + \mathbb{C}\delta)/\mathbb{C}\delta & \xrightarrow{\sim} & \mathfrak{h}_0^* \\ \bar\lambda + \Lambda_0 + \mathbb{C}\delta & \longmapsto & \bar\lambda \end{array} \tag{5.16}$$

and the $W_{\rm aff}$-action on the right hand side is given by

$$s_\alpha(\bar\lambda) = \bar\lambda - \bar\lambda(h_\alpha)\alpha \qquad \text{and} \qquad t_{k\alpha^\vee}(\bar\lambda) = \bar\lambda + k\alpha^\vee, \qquad \text{for } \bar\lambda \in \mathfrak{h}_0. \tag{5.17}$$

Here $\mathfrak{h}_0^*$ is a set with a $W_{\rm aff}$-action, the action of $W_{\rm aff}$ is *not linear*.

6 Loop groups and the affine flag variety G/I

This section gives a short treatment of loop groups following [St, Ch. 8] and [Mac1, §2.5 and 2.6]. This theory is currently a subject of intense research as evidenced by the work in [Ga2], [GK], [Rem], [Rou], [GR].

Let $\mathfrak{g}_0$ be a symmetrizable Kac-Moody Lie algebra and let $\mathfrak{h}_\mathbb{Z}$ be a $\mathbb{Z}$-lattice in $\mathfrak{h}_0$ that contains $Q^\vee = \mathbb{Z}\text{-span}\{h_1, \ldots, h_n\}$.

$$\text{The } \textit{loop group} \text{ is the Tits group } G = G_0(\mathbb{C}((t))) \tag{6.1}$$

over the field $\mathbb{F} = \mathbb{C}((t))$. Let $K = G_0(\mathbb{C}[[t]])$ and $G_0(\mathbb{C})$ be the Tits groups of $\mathfrak{g}_0$ and $\mathfrak{h}_\mathbb{Z}$ over the rings $\mathbb{C}[[t]]$ and $\mathbb{C}$, respectively, and let $B(\mathbb{C})$ be the standard *Borel subgroup* of $G_0(\mathbb{C})$ as defined in (4.2). Let

$$U^- \text{ be the subgroup of } G \text{ generated by } x_{-\alpha}(f) \text{ for } \alpha \in R_{\rm re}^+ \text{ and } f \in \mathbb{C}((t)), \tag{6.2}$$

and define the standard *Iwahori subgroup* I of G by

$$\begin{array}{ccccc} G & = & G_0(\mathbb{C}((t))) & & \\ \cup\!| & & \cup\!| & & \\ K & = & G_0(\mathbb{C}[[t]]) & \stackrel{\mathrm{ev}_{t=0}}{\longrightarrow} & G_0(\mathbb{C}) \\ \cup\!| & & \cup\!| & & \cup\!| \\ I & = & \mathrm{ev}_{t=0}^{-1}(B(\mathbb{C})) & \stackrel{\mathrm{ev}_{t=0}}{\longrightarrow} & B(\mathbb{C}). \end{array} \tag{6.3}$$

The *affine flag variety* is G/I.

For $\alpha+j\delta\in R_{\mathrm{re}}+\mathbb{Z}\delta$ and $c\in\mathbb{C}$, define

$$x_{\alpha+j\delta}(c)=x_\alpha(ct^j) \qquad \text{and} \qquad t_{\lambda^\vee}=h_{\lambda^\vee}(t^{-1}), \tag{6.4}$$

and, for $c\in\mathbb{C}^\times$, define

$$n_{\alpha+j\delta}(c)=x_{\alpha+j\delta}(c)x_{-\alpha-j\delta}(-c^{-1})x_{\alpha+j\delta}(c), \tag{6.5}$$

$$n_{\alpha+j\delta}=n_{\alpha+j\delta}(1), \quad \text{and} \quad h_{(\alpha+j\delta)^\vee}(c)=n_{\alpha+j\delta}(c)n_{\alpha+j\delta}^{-1} \tag{6.6}$$

analogous to (3.3).

The group

$$\widetilde{W}=\{t_{\lambda^\vee}w \mid \lambda^\vee\in\mathfrak{h}_{\mathbb{Z}}, w\in W_0\} \qquad \text{with} \quad t_{\lambda^\vee}t_{\mu^\vee}=t_{\lambda^\vee+\mu^\vee} \quad \text{and} \quad wt_{\lambda^\vee}=t_{w\lambda^\vee}w, \tag{6.7}$$

acts on $\mathfrak{h}_0^*\oplus\mathbb{C}\delta$ by

$$v(\mu+k\delta)=v\mu+k\delta \qquad \text{and} \qquad t_{\lambda^\vee}(\mu+k\delta)=\mu+(k-\langle\lambda^\vee,\mu\rangle)\delta \tag{6.8}$$

for $v\in W_0$, $\lambda^\vee\in\mathfrak{h}_{\mathbb{Z}}$, $\mu\in\mathfrak{h}_{\mathbb{Z}}^*$, and $k\in\mathbb{Z}$. Then $n_{\alpha+j\delta}(c)=t_{-j\alpha^\vee}n_\alpha(c)=n_\alpha(ct^j)$,

$$n_\alpha x_{\beta+k\delta}(c)n_\alpha^{-1}=n_\alpha x_\beta(ct^k)n_\alpha^{-1}=x_{s_\alpha\beta}(\epsilon_{\alpha,\beta}ct^k)=x_{s_\alpha(\beta+k\delta)}(\epsilon_{\alpha,\beta}c)$$

for $\alpha\in R_{\mathrm{re}}$, and, for $\lambda^\vee\in\mathfrak{h}_{\mathbb{Z}}$,

$$t_{\lambda^\vee}x_{\beta+k\delta}(c)t_{\lambda^\vee}^{-1}=x_{\beta+k\delta}(t^{-\langle\lambda^\vee,\beta\rangle}c)=x_{t_{\lambda^\vee}(\beta+k\delta)}(c).$$

Thus the *root subgroups*

$$\mathcal{X}_{\alpha+j\delta}=\{x_{\alpha+j\delta}(c)\mid c\in\mathbb{C}\} \qquad \text{satisfy} \qquad w\mathcal{X}_{\alpha+j\beta}w^{-1}=\mathcal{X}_{w(\alpha+j\delta)} \tag{6.9}$$

for $w\in\widetilde{W}$ and $\alpha+j\delta\in R_{\mathrm{re}}+\mathbb{Z}\delta$. These relations are a reflection of the symmetry of the group G under the group defined in (3.8):

$$\widetilde{N}=N(\mathbb{C}((t))) \quad \text{generated by } n_\alpha(g),\, h_{\lambda^\vee}(g), \text{ for } g\in\mathbb{C}((t))^\times, \tag{6.10}$$

$\alpha\in R_{\mathrm{re}}$, and $\lambda^\vee\in\mathfrak{h}_{\mathbb{Z}}$. The homomorphism $\widetilde{N}\to W_0$ from (3.9) lifts to a surjective homomorphism (see [Mac1, p. 26 and p. 28])

$$\begin{array}{ccc} \widetilde{N} & \longrightarrow & \widetilde{W} \\ n_{\alpha+j\delta} & \longmapsto & t_{-j\alpha^\vee}s_\alpha \\ t_{\lambda^\vee} & \longmapsto & t_\lambda^\vee \end{array} \qquad \text{with kernel } H \text{ generated by } h_\lambda(d),\ d\in\mathbb{C}[[t]]^\times.$$

Define

$$\tilde{R}_{\mathrm{re}}^I=(R_{\mathrm{re}}^+ +\mathbb{Z}_{\geq 0}\delta)\sqcup(-R_{\mathrm{re}}^+ +\mathbb{Z}_{>0}\delta) \quad \text{and} \quad \tilde{R}_{\mathrm{re}}^U=-R_{\mathrm{re}}^+ +\mathbb{Z}\delta \tag{6.11}$$

so that

$$\begin{array}{llll} \mathcal{X}_{\alpha+j\delta}\subseteq I & \text{if and only if} & \alpha+j\delta\in\tilde{R}_{\mathrm{re}}^I & \text{and} \\ \mathcal{X}_{\alpha+j\delta}\subseteq U^- & \text{if and only if} & \alpha+j\delta\in\tilde{R}_{\mathrm{re}}^U. & \end{array} \tag{6.12}$$

Note that $\tilde{R}_{\mathrm{re}}^I\sqcup(-\tilde{R}_{\mathrm{re}}^I)=\tilde{R}_{\mathrm{re}}^U\sqcup(-\tilde{R}_{\mathrm{re}}^U)=R_{\mathrm{re}}+\mathbb{Z}\delta$.

7 The folding algorithm and the intersections $U^-vI \cap IwI$

In this section we prove our main theorem, which gives a precise connection between the alcove walks in [Ra] and the points in the affine flag variety. The algorithm here is essentially that which is found in [BD] and, with our setup from the earlier sections, it is the 'obvious one'. The same method has, of course, been used in other contexts, see, for example, [C].

A special situation in the loop group theory is when $\mathfrak{g}_0$ is finite dimensional. In this case, the extended loop Lie algebra $\mathfrak{g}$ defined in (5.1) is also a Kac-Moody Lie algebra. If G_0 is the Tits group of $\mathfrak{g}_0$ and $G = G_0(\mathbb{C}((t)))$ is the corresponding loop group then the subgroup I defined in (6.3) differs from the Borel subgroup of the Kac-Moody group G_{KM} for $\mathfrak{g}$ only by elements of T, and the affine flag variety of G coincides with the flag variety of G_{KM}. Thus, in this case, Theorem 4.1 provides a labeling of the points of the affine flag variety.

Suppose that $\mathfrak{g}_0$ is a finite dimensional complex semisimple Lie algebra presented as a Kac-Moody Lie algebra with generators $e_1, \ldots, e_n, f_1, \ldots, f_n, h_1, \ldots, h_n$ and Cartan matrix $A = (\alpha_i(h_j))_{1\le i,j,\le n}$. Let φ be the highest root of R (the highest weight of the adjoint representation), fix

$$e_\varphi \in \mathfrak{g}_\varphi, \quad f_\varphi \in \mathfrak{g}_{-\varphi} \quad \text{such that} \quad \langle e_\varphi, f_\varphi\rangle_0 = 1,$$

and let

$$e_0 = e_{-\varphi+\delta} = tf_\varphi, \quad f_0 = f_{-\varphi+\delta} = t^{-1}e_\varphi, \qquad h_0 = [e_0, f_0] = [tx_{-\varphi}, t^{-1}x_\varphi] = -h_\varphi + c,$$

as in (5.9). The magical fact is that, in this case, $\mathfrak{g} = \mathfrak{g}_0[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ is a Kac-Moody Lie algebra with generators $e_0, \ldots, e_n$, $f_0, \ldots, f_n$, $h_0, \ldots, h_n, d$ and Cartan matrix

$$A^{(1)} = (\alpha_i(h_j))_{0\le i,j\le n}, \qquad \text{where} \quad \alpha_0 = -\varphi + \delta \quad \text{and} \quad h_0 = -h_\varphi + c, \tag{7.1}$$

where δ is as in (5.6) (see [Kac, Thm. 7.4]).

The *alcoves* are the open connected components of

$$\mathfrak{h}_\mathbb{R} \setminus \bigcup_{-\alpha+j\delta\in\tilde{R}^I_{\mathrm{re}}} H_{-\alpha+j\delta}, \qquad \text{where} \quad H_{-\alpha+j\delta} = \{x^\vee \in \mathfrak{h}_\mathbb{R} \mid \langle x^\vee, \alpha\rangle = j\}.$$

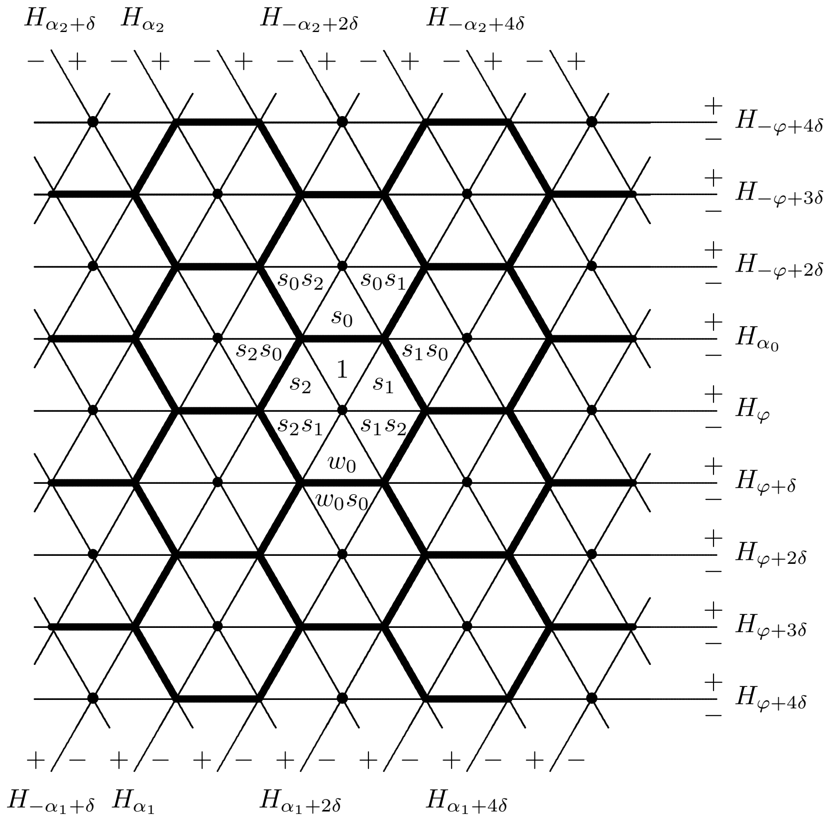
Under the map in (5.16) the chambers wC of the Tits cone X (see (2.20) and (2.21)) become the alcoves. Each alcove is a fundamental region for the action of W_{aff} on $\mathfrak{h}_\mathbb{R}$ given by (5.17) and W_{aff} acts simply transitively on the set of alcoves (see [Kac, Prop. 6.6]). Identify $1 \in W_{\mathrm{aff}}$ with the *fundamental alcove*

$$A_0 = \{x^\vee \in \mathfrak{h}_\mathbb{R} \mid \langle x^\vee, \alpha_i\rangle > 0 \text{ for all } 0 \le i \le n\}$$

to make a bijection

$$W_{\mathrm{aff}} \longleftrightarrow \{\text{alcoves}\}.$$

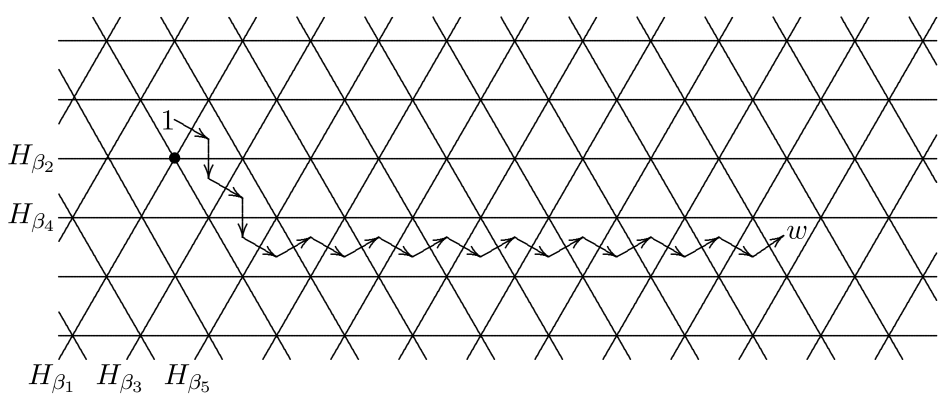
For example, when $\mathfrak{g}_0 = \mathfrak{sl}_3$,



(7.2)

The alcoves are the triangles and the (centres of) hexagons are the elements of $Q^\vee$.

Let $w \in W_{\mathrm{aff}}$. Following the discussion in (4.4)-(4.6), a reduced expression $\vec{w} = s_{i_1} \cdots s_{i_\ell}$ is a *walk* starting at 1 and ending at w,

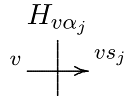


and the points of

$$IwI = \{x_{i_1}(c_1)n_{i_1}^{-1}x_{i_2}(c_2)n_{i_2}^{-1} \cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1}I \mid c_1, \ldots, c_\ell \in \mathbb{C}\} \tag{7.3}$$

are in bijection with labelings of the edges of the walk by complex numbers $c_1, \ldots, c_\ell$. The elements of $R(w) = \{\beta_1, \ldots, \beta_\ell\}$ are the elements of $\tilde{R}^I_{\mathrm{re}}$ corresponding to the sequence of hyperplanes crossed by the walk.

The labeling of the hyperplanes in (7.2) is such that neighboring alcoves have



with $v\alpha_j \in \tilde{R}^I_{\mathrm{re}}$ if v is closer to 1 than vs_j. (7.4)

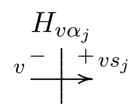
The *periodic orientation* (illustrated in (7.2)) is the orientation of the hyperplanes $H_{\alpha+k\delta}$ such that

(a) 1 is on the positive side of H_α for $\alpha \in R^+_{\mathrm{re}}$,

(b) $H_{\alpha+k\delta}$ and H_α have parallel orientiations.

This orientation is such that

$v\alpha_j \in \tilde{R}^U_{\mathrm{re}}$ if and only if



. (7.5)

Together, (7.4) and (7.5) provide a powerful combinatorics for analyzing the intersections $U^-vI \cap IwI$. We shall use the first identity in (3.3), in the form

$$x_\alpha(c)n_\alpha^{-1} = x_{-\alpha}(c^{-1})x_\alpha(-c)h_{\alpha^\vee}(c) \qquad \text{(main folding law)}, \tag{7.6}$$

to rewrite the points of IwI given in (7.3) as elements of U^-vI. Suppose that

$$x_{i_1}(c_1)n_{i_1}^{-1}\cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1} = x_{\gamma_1}(c_1')\cdots x_{\gamma_\ell}(c_\ell')n_v b, \qquad \text{where } b \in I, \tag{7.7}$$

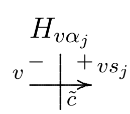
$v \in W_{\mathrm{aff}}$ and $n_v = n_{j_1}^{-1}\cdots n_{j_k}^{-1}$ if $v = s_{i_1}\cdots s_{i_k}$ is a reduced word, and $\gamma_1,\ldots,\gamma_\ell \in \tilde{R}^U_{\mathrm{re}}$ so that $x_{\gamma_1}(c_1')\cdots x_{\gamma_\ell}(c_\ell') \in U^-$. Then the procedure described in (7.8)-(7.10) will compute $c_{\ell+1}' \in \mathbb{C}$, $b' \in I$, $v' \in W_{\mathrm{aff}}$ and $\gamma_{\ell+1} \in \tilde{R}^U_{\mathrm{re}}$ so that

$$x_{i_1}(c_1)n_{i_1}^{-1}\cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1}x_j(c)n_j^{-1} = x_{\gamma_1}(c_1')\cdots x_{\gamma_\ell}(c_\ell')x_{\gamma_{\ell+1}}(c_{\ell+1})n_{v'}b'.$$

Keep the notations in (7.7). Since $bx_j(c)n_j^{-1} \in Is_jI$ there are unique $\tilde{c} \in \mathbb{C}$ and $b' \in I$ such that $bx_j(c)n_j^{-1} = x_j(\tilde{c})n_j^{-1}b'$ and

$$\begin{aligned} x_{i_1}(c_1)n_{i_1}^{-1}\cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1}x_j(c)n_j^{-1} &= x_{\gamma_1}(c_1')\cdots x_{\gamma_\ell}(c_\ell')n_v b x_j(c)n_j^{-1} \\ &= x_{\gamma_1}(c_1')\cdots x_{\gamma_\ell}(c_\ell')n_v x_j(\tilde{c})n_j^{-1}b'. \end{aligned}$$

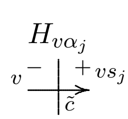
Case 1: If $v\alpha_j \in \tilde{R}^U_{\mathrm{re}}$,



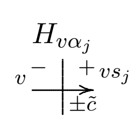
, then $x_{\gamma_1}(c_1')\cdots x_{\gamma_\ell}(c_\ell')n_v x_j(\tilde{c})n_j^{-1}b'$ is equal to

$$x_{\gamma_1}(c_1')\cdots x_{\gamma_\ell}(c_\ell')x_{v\alpha_j}(\pm\tilde{c})n_{vs_j}b' \in U^-vs_jI \cap Iws_jI.$$

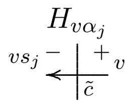
In this case, $\gamma_{\ell+1} = v\alpha_j$, $v' = vs_j$, and



becomes



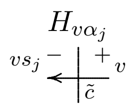
. (7.8)

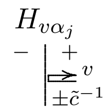
Case 2: If $v\alpha_j \notin \tilde{R}^U_{\mathrm{re}}$ and $\tilde{c} \neq 0$, 

then

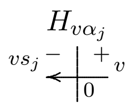
$$\begin{aligned} x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') n_v x_{\alpha_j}(\tilde{c}) n_j^{-1} b' &= x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') n_v x_{-\alpha_j}(\tilde{c}^{-1}) x_{\alpha_j}(-\tilde{c}) h_{\alpha_j^\vee}(\tilde{c}) b' \\ &= x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') n_v x_{-\alpha_j}(\tilde{c}^{-1}) b'' \\ &= x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') x_{\gamma_{\ell+1}}(\pm\tilde{c}^{-1}) n_v b'' \in U^- vI \cap Iws_jI, \end{aligned}$$

where $\gamma_{\ell+1} = -v\alpha_j$ and $b'' = x_{\alpha_j}(-\tilde{c}) h_{\alpha_j^\vee}(\tilde{c}) b'$. So



becomes


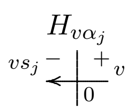
(7.9)

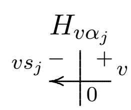
Case 3: If $v\alpha_j \notin \tilde{R}^U_{\mathrm{re}}$ and $\tilde{c} = 0$, 

then

$$\begin{aligned} x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') n_v x_{\alpha_j}(0) n_j^{-1} b' &= x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') n_v x_{-\alpha_j}(0) n_j^{-1} b' \\ &= x_{\gamma_1}(c_1') \cdots x_{\gamma_\ell}(c_\ell') x_{\gamma_{\ell+1}}(0) n_{vs_j} b' \in U^- vs_jI \cap Iws_jI, \end{aligned}$$

where $\gamma_{\ell+1} = -v\alpha_j$. So



becomes


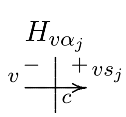
(7.10)

We have proved the following theorem.

Theorem 7.1. *If $w \in W_{\mathrm{aff}}$ and $\vec{w} = s_{i_1} \cdots s_{i_\ell}$ is a minimal length walk to w define*

$$\mathcal{P}(\vec{w})_v = \left\{ \begin{matrix} \textit{labeled folded paths } p \textit{ of type } \vec{w} \\ \textit{which end in } v \end{matrix} \right\} \qquad \textit{for } v \in W_{\mathrm{aff}},$$

where a labeled folded path of type $\vec{w}$ *is a sequence of steps of the form*



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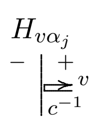

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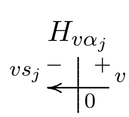


, *where the kth step has $j = i_k$.*

Viewing $U^- vI \cap IwI$ as a subset of G/I, there is a bijection

$$\mathcal{P}(\vec{w})_v \longleftrightarrow U^- vI \cap IwI.$$

Remark 7.2. The paths in $\mathcal{P}(\vec{w})_v$ indicate a decomposition of $U^- vI \cap IwI$ into "cells", where the cell associated to a nonlabeled path p is the set of points of $U^- vI \cap IwI$ which have the same underlying nonlabeled path. It would be very interesting to understand, combinatorially, the closure relations between these cells.

8 An example

For the group $G = SL_3(\mathbb{C}((t)))$,

$$x_{\alpha_1}(c) = \begin{pmatrix}1 & c & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \quad h_{\alpha_1^\vee}(c) = \begin{pmatrix}c & 0 & 0\\ 0 & c^{-1} & 0\\ 0 & 0 & 1\end{pmatrix}, \quad n_1 = \begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$x_{\alpha_2}(c) = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & c\\ 0 & 0 & 1\end{pmatrix} \quad h_{\alpha_2^\vee}(c) = \begin{pmatrix}1 & 0 & 0\\ 0 & c & 0\\ 0 & 0 & c^{-1}\end{pmatrix}, \quad n_2 = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix},$$

$$x_{\alpha_0}(c) = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ ct & 0 & 1\end{pmatrix} \quad h_{\alpha_0^\vee}(c) = \begin{pmatrix}c^{-1} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & c\end{pmatrix}, \quad n_0 = \begin{pmatrix}0 & 0 & -t^{-1}\\ 0 & 1 & 0\\ t & 0 & 0\end{pmatrix}.$$

Let $w = s_2s_1s_0s_2s_0s_1s_0s_2s_0$ and $v = s_2s_1s_0s_2s_1s_2s_0$ so that

$$w = \begin{pmatrix}t^2 & 0 & 0\\ 0 & 0 & 1\\ 0 & -t^{-2} & 0\end{pmatrix} \qquad \text{and} \qquad v = \begin{pmatrix}0 & -1 & 0\\ t^2 & 0 & 0\\ 0 & 0 & t^{-2}\end{pmatrix}.$$

We shall use Theorem 7.1 to show that the points of $IwI \cap U^-vI$ are

$$x_2(c_1)n_2^{-1}x_1(c_2)n_1^{-1}x_0(c_3)n_0^{-1}x_2(c_4)n_2^{-1}x_0(c_5)n_0^{-1}x_1(c_6)n_1^{-1}x_0(c_7)n_0^{-1}x_2(c_8)n_2^{-1}x_0(c_9)n_0^{-1}I,$$

with $c_1, \ldots, c_9 \in \mathbb{C}$ such that

$$c_1 = 0, \quad c_2 = 0, \quad c_3 = 0, \quad c_4 = 0, \quad c_5 \neq 0, \quad c_6 = 0, \quad c_7 \neq 0, \quad c_9 = c_7^{-1}c_8. \tag{8.1}$$

Precisely,

$$x_2(0)n_2^{-1}x_1(0)n_1^{-1}x_0(0)n_0^{-1}x_2(0)n_2^{-1}x_0(c_5)n_0^{-1}x_1(0)n_1^{-1}x_0(c_7)n_0^{-1}x_2(c_8)n_2^{-1}x_0(c_7^{-1}c_8)n_0^{-1}$$

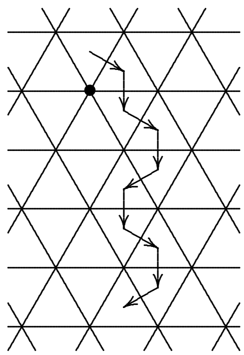
is equal to $u_9v_9b_9$, with $u_9 \in U^-$, $v_9 \in N$, $b_9 \in I$ given by

$$\begin{gathered} u_9 = \begin{pmatrix}1 & 0 & 0\\ c_5^{-1} - c_5^{-2}c_7^{-1}c_8t & 1 & 0\\ c_5^{-1}c_7^{-1}t^{-2} & 0 & 1\end{pmatrix}, \qquad v_9 = \begin{pmatrix}0 & 1 & 0\\ -t^2 & 0 & 0\\ 0 & 0 & t^{-2}\end{pmatrix} \\ b_9 = \begin{pmatrix}c_5^{-1} - c_5^{-2}c_7^{-1}c_8t & -c_5^{-2}c_7^{-1}c_8^2 & c_5^{-2}c_7^{-2}c_8^2\\ -t^2 & c_5c_7 + c_8t & -c_5 - c_7^{-1}c_8t\\ -c_5^{-1}c_7^{-1}t^2 & -c_5^{-1}c_7^{-1}c_8t & c_7^{-1} + c_5^{-1}c_7^{-2}c_8t\end{pmatrix}, \end{gathered} \tag{8.2}$$

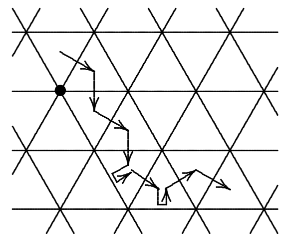
so that $u_9 = x_{-\alpha_2}(d_1)x_{-\varphi}(d_2)x_{-\alpha_2-\delta}(d_3)x_{-\varphi-\delta}(d_4)x_{-\alpha_1}(d_5)x_{-\alpha_2-2\delta}(d_6)x_{-\varphi-3\delta}(d_7)x_{-\alpha_1+\delta}(d_8)$ $\cdot x_{-\alpha_2-3\delta}(d_9)$ with

$$d_1 = d_2 = d_3 = d_4 = 0, \quad d_5 = c_5^{-1}, \quad d_6 = 0, \quad d_7 = c_5^{-1}c_7^{-1}, \quad d_8 = -c_5^{-2}c_7^{-1}c_8, \quad d_9 = 0. \tag{8.3}$$

Pictorially, the walk with labels $c_1, \dots, c_9$



becomes



,

the labeled folded path with labels $d_1, \dots, d_9$.

The step by step computation is as follows:

Step 1: If $c_1 = 0$ then

$$x_2(c_1)n_2^{-1} = x_{-\alpha_2}(0)n_2^{-1} = u_1 v_1 b_1, \qquad \text{with}$$

$$u_1 = x_{-\alpha_2}(0), \qquad v_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \text{and} \qquad b_1 = 1.$$

Step 2: If $c_2 = 0$ then, since $v_1 x_1(c_2) v_1^{-1} = x_\varphi(c_2)$,

$$u_1 v_1 b_1 x_1(c_2) n_1^{-1} = u_1 x_\varphi(c_2) v_1 n_1^{-1} b_1 = u_1 x_{-\varphi}(0) v_1 n_1^{-1} b_1 = u_2 v_2 b_2, \qquad \text{with}$$

$$u_2 = u_1 x_{-\varphi}(0), \qquad v_2 = v_1 n_1^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad b_2 = 1.$$

Step 3: If $c_3 = 0$ then, since $v_2 x_0(c_3) v_2^{-1} = x_{\alpha_2+\delta}(-c_3)$,

$$u_2 v_2 b_2 x_0(c_3) n_0^{-1} = u_2 x_{\alpha_2+\delta}(-c_3) v_2 n_0^{-1} b_2 = u_2 x_{-\alpha_2-\delta}(0) v_2 n_0^{-1} b_2 = u_3 v_3 b_3, \qquad \text{with}$$

$$u_3 = u_2 x_{-\alpha_2-\delta}(0), \qquad v_3 = v_2 n_0^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ t & 0 & 0 \\ 0 & 0 & t^{-1} \end{pmatrix}, \qquad \text{and} \qquad b_3 = 1.$$

Step 4: If $c_4 = 0$ then, since $v_3 x_2(c_4) v_3^{-1} = x_{\varphi+\delta}(-c_4)$,

$$u_3 v_3 b_3 x_2(c_4) n_2^{-1} = u_3 x_{\varphi+\delta}(-c_4) v_3 n_2^{-1} b_3 = u_3 x_{-\varphi-\delta}(0) v_3 n_2^{-1} b_3 = u_4 v_4 b_4, \qquad \text{with}$$

$$u_4 = u_3 x_{-\varphi-\delta}(0), \qquad v_4 = v_3 n_2^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ t & 0 & 0 \\ 0 & t^{-1} & 0 \end{pmatrix} \qquad \text{and} \qquad b_4 = 1.$$

Step 5: If $c_5 \neq 0$ then by the folding law and the fact that $v_4 x_{-\alpha_0}(c_5^{-1}) v_4^{-1} = x_{-\alpha_1}(c_5^{-1})$,

$$u_4 v_4 b_4 x_0(c_5) n_0^{-1} = u_4 v_4 x_{-\alpha_0}(c_5^{-1}) x_{\alpha_0}(-c_5) h_{\alpha_0^\vee}(c_5) b_4 = u_4 x_{-\alpha_1}(c_5^{-1}) v_4 b_5 = u_5 v_5 b_5,$$

where

$$u_5 = u_4 x_{-\alpha_1}(c_5^{-1}), \qquad v_5 = v_4, \qquad \text{and} \qquad b_5 = x_{\alpha_0}(-c_5) h_{\alpha_0^\vee}(c_5) b_4 = \begin{pmatrix} c_5^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ -t & 0 & c_5 \end{pmatrix}.$$

Step 6: If $c_5^{-1} c_6 = 0$ (so $c_6 = 0$) then

$$u_5 v_5 b_5 x_1(c_6) n_1^{-1} = u_5 v_5 x_1(c_5^{-1} c_6) n_1^{-1} b_5' = u_5 x_{-\alpha_2 - 2\delta}(0) v_5 n_1^{-1} b_5' = u_6 v_6 b_6,$$

with

$$u_6 = u_5 x_{-\alpha_2 - 2\delta}(0), \quad v_6 = v_5 n_1^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -t & 0 \\ t^{-1} & 0 & 0 \end{pmatrix} \quad \text{and} \quad b_6 = b_5' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_5^{-1} & 0 \\ -c_6 t & t & c_5 \end{pmatrix}$$

so that $b_5 x_1(c_6) n_1^{-1} = x_1(c_5^{-1} c_6) n_1^{-1} b_5'$.
Step 7: If $c_5 c_7 \neq 0$ then, since $v_6 x_{-\alpha_0}(c) v_6^{-1} = x_{-\varphi - 2\delta}(c)$,

$$\begin{aligned} u_6 v_6 b_6 x_0(c_7) n_0^{-1} &= u_6 v_6 x_0(c_5 c_7) n_0^{-1} b_6' = u_6 v_6 x_{-\alpha_0}(c_5^{-1} c_7^{-1}) x_{\alpha_0}(-c_5 c_7) h_{\alpha_0^\vee}(c_5 c_7) b_6' \\ &= u_6 x_{-\varphi - 2\delta}(c_5^{-1} c_7^{-1}) v_6 b_7 = u_7 v_7 b_7, \end{aligned}$$

where

$$u_7 = u_6 x_{-\varphi - 2\delta}(c_5^{-1} c_7^{-1}), \qquad v_7 = v_6, \qquad \text{and}$$

$$b_6' = \begin{pmatrix} c_5 & -1 & 0 \\ 0 & c_5^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b_7 = x_{\alpha_0}(-c_5 c_7) h_{\alpha_0^\vee}(c_5 c_7) b_6' = \begin{pmatrix} c_7^{-1} & -c_5^{-1} c_7^{-1} & 0 \\ 0 & c_5^{-1} & 0 \\ -c_5 t & t & c_5 c_7 \end{pmatrix},$$

so that $b_6 x_0(c_7) n_0^{-1} = x_0(c_5 c_7) n_0^{-1} b_6'$.
Step 8: No restrictions on $c_5^{-2} c_7^{-1} c_8$. Since $v_7 x_{\alpha_2}(c) v_7^{-1} = x_{-\alpha_1 + \delta}(-c)$,

$$u_7 v_7 b_7 x_2(c_8) n_2^{-1} = u_7 v_7 x_2(c_5^{-2} c_7^{-1} c_8) n_2^{-1} b_7' = u_7 x_{-\alpha_1 + \delta}(-c_5^{-2} c_7^{-1} c_8) v_7 n_2^{-1} b_7' = u_8 v_8 b_8,$$

with

$$u_8 = u_7 x_{-\alpha_1 + \delta}(-c_5^{-2} c_7^{-1} c_8), \qquad v_8 = v_7 n_2^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & t \\ t^{-1} & 0 & 0 \end{pmatrix}, \qquad \text{and}$$

$$b_8 = b_7' = \begin{pmatrix} c_7^{-1} & -c_5^{-1} c_7^{-1} c_8 & c_5^{-1} c_7^{-1} \\ -c_5 t & c_5 c_7 + c_8 t & -t \\ -c_5^{-1} c_7^{-1} c_8 t & c_5^{-2} c_7^{-1} c_8^2 t & c_5^{-1} - c_5^{-2} c_7^{-1} c_8 t \end{pmatrix},$$

so that $b_7 x_2(c_8) n_2^{-1} = x_2(c_5^{-2} c_7^{-1} c_8) n_2^{-1} b_7'$.
Step 9: If $c_5^{-1} c_7 c_9 - c_5^{-1} c_8 = 0$ (so $c_9 = c_7^{-1} c_8$) then

$$u_8 v_8 b_8 x_0(c_9) n_0^{-1} = u_8 v_8 x_0(c_5^{-1} c_7 c_9 - c_5^{-1} c_8) n_0^{-1} b_8' = u_8 x_{-\alpha_2 - 3\delta}(0) v_8 n_0^{-1} b_8' = u_9 v_9 b_9$$

with u_9, v_9 and b_9 as in (8.2).

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