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Affine Hecke algebras and the Schubert calculus

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Abstract

Using a combinatorial approach that avoids geometry, this paper studies the structure of $K_T(G/B)$, the T-equivariant K-theory of the generalized flag variety G/B. This ring has a natural basis $\{[\mathcal{O}_{X_w}] \mid w \in W\}$ (the double Grothendieck polynomials), where $\mathcal{O}_{X_w}$ is the structure sheaf of the Schubert variety X_w. For rank two cases we compute the corresponding structure constants of the ring $K_T(G/B)$ and, based on this data, make a positivity conjecture for general G which generalizes the theorems of M. Brion (for $K(G/B)$) and W. Graham (for $H^*_T(G/B)$). Let $[X^\lambda] \in K_T(G/B)$ be the class of the homogeneous line bundle on G/B corresponding to the character of T indexed by λ. For general G we prove "Pieri–Chevalley formulas" for the products $[X^\lambda][\mathcal{O}_{X_w}]$, $[X^{-\lambda}][\mathcal{O}_{X_w}]$, $[X^{w_0\lambda}][\mathcal{O}_{X_w}]$, and $[\mathcal{O}_{X_{w_0 s_i}}][\mathcal{O}_{X_w}]$, where λ is dominant. By using the Chern character and comparing lowest degree terms the products which are computed in this paper also give results for the Grothendieck polynomials, double Schubert polynomials, and ordinary Schubert polynomials in, respectively $K(G/B)$, $H^*_T(G/B)$ and $H^*(G/B)$.

Keywords: Flag variety; K-theory; Affine Hecke algebras; Schubert varieties

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0. Introduction

Using a combinatorial approach which avoids geometry, this paper studies the ring structure of $K_T(G/B)$, the T-equivariant K-theory of the (generalized) flag variety G/B. Here, the data $G \supseteq B \supseteq T$ is a complex reductive algebraic group (or symmetrizable Kac–Moody group) G, a Borel subgroup B, and a maximal torus T, and $K_T(G/B)$

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is the Grothendieck group of T-equivariant coherent sheaves on G/B. Because of the T-equivariance the ring $K_T(G/B)$ is an R-algebra, where R is the representation ring of T. As explained by Grothendieck [6] (in the non-Kac–Moody case) and Kostant and Kumar [9] (in the general Kac–Moody case), the ring $K_T(G/B)$ has a natural R-basis $\{[\mathcal{O}_{X_w}] \mid w \in W\}$, where W is the Weyl group and $\mathcal{O}_{X_w}$ is the structure sheaf of the Schubert variety $X_w \subseteq G/B$. One of the main problems in the field is to understand the structure constants of the ring $K_T(G/B)$ with this basis, that is, the coefficients c_{wv}^z in the equations

$$[\mathcal{O}_{X_w}][\mathcal{O}_{X_v}] = \sum_{z\in W} c_{wv}^z [\mathcal{O}_{X_z}]. \tag{0.1}$$

Our approach is to work completely combinatorially and define $K_T(G/B)$ as a quotient of the affine nil-Hecke algebra. The fact that the combinatorial approach coincides with the geometric one is a consequence of the results of Kostant and Kumar [9] and Demazure [4]. In the combinatorial literature the elements $[\mathcal{O}_{X_w}]$ are often called (double) Grothendieck polynomials.

Let P be the weight lattice of G and, for $\lambda \in P$, let $[X^\lambda]$ be the homogeneous line bundle on G/B corresponding to the character of T indexed by λ. The theorem of Pittie [19] says that the ring $K_T(G/B)$ is generated by the $[X^\lambda]$, $\lambda \in P$. Steinberg [23] strengthened this result by displaying specific $[X^{-\lambda_w}]$, $w \in W$, which form an R-basis of $K_T(G/B)$. These results are often collectively known as the “Pittie–Steinberg theorem”.

The theorems which we prove in Section 2 are simply different points of view on the Pittie–Steinberg theorem. Though we are not aware of any reference which states these theorems in the generality which we consider, these theorems should be considered well known.

Let $s_1, \ldots, s_n$ be the simple reflections in W (determined by the data $(G \supseteq B \supseteq T)$), let w_0 be the longest element of W and let P^+ be the set of dominant weights in P. The Schubert varieties $X_{w_0 s_i}$ are the codimension one Schubert varieties in G/B. In Section 3 we prove “Pieri–Chevalley” formulas for the products

$$[X^\lambda][\mathcal{O}_{X_w}], \quad [X^{-\lambda}][\mathcal{O}_{X_w}], \quad [X^{w_0\lambda}][\mathcal{O}_{X_w}], \quad \text{and} \quad [\mathcal{O}_{X_{w_0 s_i}}][\mathcal{O}_{X_w}], \tag{0.2}$$

for $\lambda \in P^+$, $w \in W$ and $1 \le i \le n$. All of these Pieri–Chevalley formulas are given in terms of the combinatorics of the Littelmann path model [12–14]. The formula which we give for the first product in (0.2) is due to Pittie and Ram [20]. In this paper we provide more details of proof than appeared in [20]. The other formulas for the products in (0.2) follow by applying the duality theorem of Brion [1, Theorem 4] to the first formula. However, here we give an independent, combinatorial, proof and deduce Brion’s result as a consequence. The last formula is a consequence of the nice formula

$$[\mathcal{O}_{X_{w_0 s_i}}] = 1 - \mathrm{e}^{w_0\omega_i}[X^{-\omega_i}], \tag{0.3}$$

which is an easy consequence of the first two Pieri–Chevalley rules.

It is not difficult to “specialize” product formulas for $K_T(G/B)$ to corresponding product formulas for $K(G/B)$, $H_T^*(G/B)$, and $H^*(G/B)$ (by using the Chern character and comparing lowest degree terms, and ignoring the T-action). Thus the products which are computed in this paper also give results for ordinary Grothendieck polynomials, double Schubert polynomials, and ordinary Schubert polynomials. In Section 4 we explain how to do these conversions. For most of these cases the specialized versions of our Pieri–Chevalley rules are already very well known (see, for example, [3]).

In Section 5 we give explicitly

(a) two different kinds of formulas for $[\mathcal{O}_{X_w}]$ in terms of X^λ, and
(b) complete computations of the products in (0.1)

for the rank two root systems. This data allows us to make a "positivity conjecture" for the coefficients c^z_{wv} in (0.1). This conjecture generalizes the theorems of Brion [1, formula before Theorem 1] and Graham [7, Corollary 4.1], which treat the cases $K(G/B)$ and $H^*_T(G/B)$, respectively.

1. Preliminaries

Fix the following data and notation:

$\mathfrak{h}^*$	is a real vector space of dimension n,
R	is a reduced irreducible root system in $\mathfrak{h}^*$,
R^+	is a set of positive roots in R,
W	is the Weyl group of R,
$s_1, \ldots, s_n$	are the simple reflections in W,
m_{ij}	is the order of $s_i s_j$ in W, $i \neq j$,
$R(w) = \{\alpha \in R^+ \mid w\alpha \notin R^+\}$	is the inversion set of $w \in W$,
$\ell(w) = \mathrm{Card}(R(w))$	is the length of $w \in W$,
$\leq$	is the Bruhat–Chevalley order on W,
$\alpha_1, \ldots, \alpha_n$	are the simple roots in R^+,
$\omega_1, \ldots, \omega_n$	are the fundamental weights,
$P = \sum_{i=1}^n \mathbb{Z}\omega_i$	is the weight lattice,
$P^+ = \sum_{i=1}^n \mathbb{Z}_{\geq 0}\omega_i$	is the set of dominant integral weights.

For a brief, easy, introduction to root systems with lots of pictures for visualization see [18]. By [2, VI Section 1 no. 6 Corollary 2 to Proposition 17], if $w = s_{i_1} \cdots s_{i_p}$ is a reduced word for w, then

$$R(w) = \{\alpha_{i_p}, s_{i_p}\alpha_{i_{p-1}}, \ldots, s_{i_p} \cdots s_{i_2}\alpha_{i_1}\}. \tag{1.1}$$

The *affine nil-Hecke algebra* is the algebra $\tilde{H}$ given by generators $T_1, \ldots, T_n$ and X^λ, $\lambda \in P$, with relations

$$T_i^2 = T_i, \qquad \underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ factors}}, \qquad X^\lambda X^\mu = X^{\lambda+\mu}, \tag{1.2}$$

and

$$X^\lambda T_i = T_i X^{s_i\lambda} + \frac{X^\lambda - X^{s_i\lambda}}{1 - X^{-\alpha_i}}. \tag{1.3}$$

Let $T_w = T_{i_1} \cdots T_{i_p}$ for a reduced word $w = s_{i_1} \cdots s_{i_p}$. Then

$$\{X^\lambda T_w \mid w \in W, \lambda \in P\} \qquad \text{and} \qquad \{T_w X^\lambda \mid w \in W, \lambda \in P\} \tag{1.4}$$

are bases of $\tilde{H}$.

Both the *nil-Hecke algebra*,

$$H = \mathbb{Z}\text{-span}\,\{T_w \mid w \in W\}, \qquad \text{and} \qquad \mathbb{Z}[X] = \mathbb{Z}\text{-span}\,\{X^\lambda \mid \lambda \in P\} \tag{1.5}$$

are subalgebras of $\tilde{H}$. The action of W on $\mathbb{Z}[X]$ is given by defining

$$wX^\lambda = X^{w\lambda}, \qquad \text{for } w \in W, \lambda \in P, \tag{1.6}$$

and extending linearly. The proof of the following theorem is given in [22, Theorem 1.13 and Theorem 1.17]. The first statement of the theorem is due to Bernstein, Zelevinsky, and Lusztig [16, 8.1] and the second statement is due to Steinberg [23] and is known as the Pittie–Steinberg theorem.

Theorem 1.7. *Define*

$$\lambda_w = w^{-1} \sum_{s_i w < w} \omega_i, \qquad \textit{for } w \in W. \tag{1.8}$$

The center of $\tilde{H}$ *is* $Z(\tilde{H}) = \mathbb{Z}[X]^W$ *and each element* $f \in \mathbb{Z}[X]$ *has a unique expansion*

$$f = \sum_{w \in W} f_w X^{-\lambda_w}, \qquad \textit{with } f_w \in \mathbb{Z}[X]^W. \tag{1.9}$$

Let $\varepsilon_i = 1 - T_i$ and let $\varepsilon_w = \varepsilon_{i_1} \cdots \varepsilon_{i_p}$ for a reduced word $w = s_{i_1} \cdots s_{i_p}$. Then ε_w is well defined and independent of the reduced word for w since

$$\varepsilon_i^2 = \varepsilon_i, \qquad \text{and} \qquad \underbrace{\varepsilon_i \varepsilon_j \varepsilon_i \cdots}_{m_{ij}\ \text{factors}} = \underbrace{\varepsilon_j \varepsilon_i \varepsilon_j \cdots}_{m_{ij}\ \text{factors}}. \tag{1.10}$$

The second equality is a consequence of the formulas

$$\varepsilon_w = \sum_{v \le w} (-1)^{\ell(v)} T_v \qquad \text{and} \qquad T_w = \sum_{v \le w} (-1)^{\ell(v)} \varepsilon_v \tag{1.11}$$

which are straightforward to verify by induction on the length of w.

2. The ring $K_T(G/B)$

Let H and $\mathbb{Z}[X]$ be as in (1.5). The *trivial representation* of H is defined by the homomorphism $\mathbf{1} : H \to \mathbb{Z}$ given by $\mathbf{1}(T_i) = 1$. The first of the maps

$$\begin{array}{ccccc} \mathbb{Z}[X] & \overset{\sim}{\longrightarrow} & \tilde{H} T_{w_0} & \overset{\sim}{\longrightarrow} & \tilde{H} \otimes_H \mathbf{1} \\ f & \longmapsto & f T_{w_0} & \longmapsto & f \otimes \mathbf{1} \end{array}$$

is an $\tilde{H}$-module isomorphism if the action of $\tilde{H}$ on $\mathbb{Z}[X]$ is given by

$$T_i \cdot f = \frac{X^{\alpha_i} f - s_i f}{X^{\alpha_i} - 1}, \qquad \text{for } f \in \mathbb{Z}[X]. \tag{2.1}$$

The group algebra of P is

$$R = \mathbb{Z}\text{-span}\,\{e^{\lambda} \mid \lambda \in P\} \qquad \text{with } e^{\lambda}e^{\mu} = e^{\lambda+\mu}, \tag{2.2}$$

for $\lambda, \mu \in P$. Extend coefficients to R so that $\tilde{H}_R = R \otimes_{\mathbb{Z}} \tilde{H}$ and $R[X] = R \otimes_{\mathbb{Z}} \mathbb{Z}[X]$ are R-algebras. Define $K_T(G/B)$ to be the $\tilde{H}_R$-module

$$K_T(G/B) = R\text{-span}\,\{[\mathcal{O}_{X_w}] \mid w \in W\}, \tag{2.3}$$

so that the $[\mathcal{O}_{X_w}]$, $w \in W$, are an R-basis of $K_T(G/B)$, with $\tilde{H}_R$-action given by

$$X^{\lambda}[\mathcal{O}_{X_1}] = e^{\lambda}[\mathcal{O}_{X_1}], \qquad \text{and} \qquad T_i[\mathcal{O}_{X_w}] = \begin{cases} [\mathcal{O}_{X_{ws_i}}], & \text{if } ws_i > w, \\ [\mathcal{O}_{X_w}], & \text{if } ws_i < w. \end{cases} \tag{2.4}$$

If R is an $R[X]$-module via the R-algebra homomorphism given by

$$\begin{array}{rccc} e: & R[X] & \longrightarrow & R \\ & X^{\lambda} & \longmapsto & e^{\lambda} \end{array} \tag{2.5}$$

then, as $\tilde{H}_R$-modules, $K_T(G/B) \cong \tilde{H}_R \otimes_{R[X]} R_e$, where R_e is the R-rank 1 $R[X]$-module determined by the homomorphism e.

Let Q be the field of fractions of R and let $\overline{Q}$ be the algebraic closure of Q. For $w \in W$ let

$$b_w \quad \text{in } \overline{Q} \otimes_R K_T(G/B) \quad \text{be determined by} \quad X^{\lambda}b_w = e^{w\lambda}b_w, \qquad \text{for } \lambda \in P. \tag{2.6}$$

If the b_w exist, then they are a $\overline{Q}$-basis of $\overline{Q} \otimes_R K_T(G/B)$ since they are eigenvectors with distinct eigenvalues. If τ_i, $1 \le i \le n$, are the operators on $\overline{Q} \otimes_R K_T(G/B)$ given by

$$\tau_i = T_i - \frac{1}{1 - X^{-\alpha_i}}, \text{ then } b_1 = [\mathcal{O}_{X_1}] \text{ and } \tau_i b_w = b_{ws_i}, \qquad \text{for } ws_i > w, \tag{2.7}$$

because, a direct computation with relation (1.3) gives that $X^{\lambda}\tau_i b_w = \tau_i X^{s_i\lambda}b_w = \tau_i e^{ws_i\lambda}b_w = e^{ws_i\lambda}b_{ws_i}$. Thus the b_w, $w \in W$, exist and the form of the τ-operators shows that, in fact, they form a Q-basis of $Q \otimes_R K_T(G/B)$ (it was not really necessary to extend coefficients all the way to $\overline{Q}$). Eqs. (2.6) and (2.7) force

$$\underbrace{\tau_i\tau_j\tau_i\cdots}_{m_{ij}\text{ factors}} = \underbrace{\tau_j\tau_i\tau_j\cdots}_{m_{ij}\text{ factors}}, \qquad \text{and the equality} \qquad \tau_i^2 = \frac{1}{(X^{\alpha_i}-1)(X^{-\alpha_i}-1)}$$

is checked by direct computation using (1.3). Let $\tau_w = \tau_{i_1}\cdots\tau_{i_p}$ for a reduced word $w = s_{i_1}\cdots s_{i_p}$. Then, for $w \in W$,

$$b_w = \tau_{w^{-1}}b_1, \quad [\mathcal{O}_{X_w}] = T_{w^{-1}}[\mathcal{O}_{X_1}] \quad \text{and we define} \quad [\mathcal{I}_{X_w}] = \varepsilon_{w^{-1}}[\mathcal{O}_{X_1}], \tag{2.8}$$

where ε_w is as in (1.11). In terms of geometry, $[\mathcal{O}_{X_w}]$ is the class of the structure sheaf of the Schubert variety X_w in G/B and, up to a sign, $[\mathcal{I}_{X_w}]$ is class of the sheaf $\mathcal{I}_{X_w}$ determined by the exact sequence $0 \to \mathcal{I}_{X_w} \to \mathcal{O}_{X_w} \to \mathcal{O}_{\partial X_w} \to 0$, where $\partial X_w = \bigsqcup_{v<w} BvB$ (see [17, Theorem 2.1(ii)] and [15, Eq. (4)]). We are not aware of a good geometric characterization of the basis $\{[X^{-\lambda_w}] \mid w \in W\}$ of $K_T(G/B)$ which appears in the following theorem.

Theorem 2.9. *Let λ_w, $w \in W$, be as defined in Theorem 1.7 and let $[X^\lambda] = X^\lambda[\mathcal{O}_{X_{w_0}}] = X^\lambda T_{w_0}[\mathcal{O}_{X_1}]$ for $\lambda \in P$. Then the $[X^{-\lambda_w}]$, $w \in W$, form an R-basis of $K_T(G/B)$.*

Proof. Up to constant multiples, $[\mathcal{O}_{X_{w_0}}] = T_{w_0}[\mathcal{O}_{X_1}]$ is determined by the property

$$T_i[\mathcal{O}_{X_{w_0}}] = [\mathcal{O}_{X_{w_0}}], \qquad \text{for all } 1 \le i \le n. \tag{2.10}$$

If constants $c_w \in Q$ are given by

$$[\mathcal{O}_{X_{w_0}}] = \sum_{w\in W} c_w b_w,$$

then comparing coefficients of b_{ws_i}, for $ws_i > w$, on each side of (2.10) yields a recurrence relation for the c_w,

$$c_w = c_{ws_i}\left(\frac{1}{1-\mathrm{e}^{-w\alpha_i}}\right) \qquad \text{for } ws_i > w,$$

$$\text{which implies } c_{w_0v^{-1}} = \prod_{\alpha\in R(v)} \frac{1}{1-\mathrm{e}^{w_0\alpha}}, \tag{2.11}$$

via (1.1) and the fact that $c_{w_0} = 1$. Thus,

$$[X^{-\lambda_v}] = X^{-\lambda_v}[\mathcal{O}_{X_{w_0}}] = \sum_{w\in W} c_w \mathrm{e}^{-w\lambda_v} b_w,$$

and if C, M and A are the $|W| \times |W|$ matrices given by

$$C = \mathrm{diag}(c_w), \quad M = (\mathrm{e}^{-w\lambda_v}), \quad \text{and} \quad A = (a_{zw}), \quad \text{where } b_w = \sum_{z\in W} a_{zw}[\mathcal{O}_{X_z}],$$

then the transition matrix between the $X^{-\lambda_v}$ and the $[\mathcal{O}_{X_z}]$ is the product ACM. By (2.8) and the definition of the τ_i, the matrix A has determinant 1. Using the method of Steinberg [23] and subtracting row $\mathrm{e}^{-s_\alpha w\lambda_v}$ from row $\mathrm{e}^{-w\lambda_v}$ in the matrix M allows one to conclude that $\det(M)$ is divisible by

$$\prod_{\alpha\in R^+}(1-\mathrm{e}^{-\alpha})^{|W|/2} \quad \text{and identifying} \quad \prod_{w\in W} \mathrm{e}^{-w\lambda_w} = \prod_{i=1}^{n}\prod_{s_iw<w} \mathrm{e}^{-\omega_i} = (\mathrm{e}^{-\rho})^{|W|/2}$$

as the lowest degree term determines $\det(M)$ exactly. Thus,

$$\det(ACM) = 1 \cdot \left(\prod_{w\in W}\prod_{\alpha\in R(w)} \frac{1}{1-\mathrm{e}^{-\alpha}}\right)\left(\mathrm{e}^{\rho}\prod_{\alpha\in R^+}(1-\mathrm{e}^{-\alpha})\right)^{|W|/2} = (\mathrm{e}^{\rho})^{|W|/2}.$$

Since this is a unit in R, the transition matrix between the $[\mathcal{O}_{X_w}]$ and the $X^{-\lambda_v}$ is invertible. □

Theorem 2.12. *The composite map*

$$\begin{array}{ccccccccc} \Phi: & R[X] & \longrightarrow & \tilde{H}_R T_{w_0} & \hookrightarrow & \tilde{H}_R & \longrightarrow & K_T(G/B) \\ & f & \longmapsto & fT_{w_0} & & h & \longmapsto & h[\mathcal{O}_{X_1}] \end{array}$$

is surjective with kernel

$$\ker \Phi = \langle f - e(f) \mid f \in R[X]^W \rangle,$$

the ideal of the ring $R[X]$ *generated by the elements* $f - e(f)$ *for* $f \in R[X]^W$. *Hence*

$$K_T(G/B) \cong \frac{R[X]}{\langle f - e(f) \mid f \in R[X]^W \rangle}$$

has the structure of a ring.

Proof. Since $\Phi(X^\lambda) = X^\lambda T_{w_0}[\mathcal{O}_{X_1}] = X^\lambda[\mathcal{O}_{X_{w_0}}]$, it follows from Theorem 2.9 that Φ is surjective. Thus $K_T(G/B) \cong R[X]/\ker \Phi$. Let $I = \langle f - e(f) \mid f \in R[X]^W \rangle$. If $f \in R[X]^W$ then, for all $\lambda \in P$,

$$\begin{aligned}\Phi(X^\lambda(f - e(f))) &= X^\lambda(f - e(f))T_{w_0}[\mathcal{O}_{X_1}] = X^\lambda T_{w_0}(f - e(f))[\mathcal{O}_{X_1}] \\ &= X^\lambda T_{w_0}(e(f) - e(f))[\mathcal{O}_{X_1}] = 0,\end{aligned}$$

since $f - e(f) \in Z(\tilde{H}_R)$. Thus $I \subseteq \ker \Phi$. The ring $K_T(G/B) = R[X]/\ker \Phi$ is a free R-module of rank $|W|$ and, by Theorem 1.7, so is $R[X]/I$. Thus $\ker \Phi = I$. □

3. Pieri–Chevalley formulas

Recall that both

$$\{X^\lambda T_{w^{-1}} \mid \lambda \in P, w \in W\} \quad \text{and} \quad \{T_{z^{-1}} X^\mu \mid \mu \in P, z \in W\} \quad \text{are bases of } \tilde{H}.$$

If $c^{\mu,z}_{w,\lambda} \in \mathbb{Z}$ are the entries of the transition matrix between these two bases,

$$X^\lambda T_{w^{-1}} = \sum_{z \in W, \mu \in P} c^{\mu,z}_{w,\lambda} T_{z^{-1}} X^\mu, \tag{3.1}$$

then applying each side of (3.1) to $[\mathcal{O}_{X_1}]$ gives that

$$[X^\lambda][\mathcal{O}_{X_w}] = \sum_{z \in W, \mu \in P} c^{\mu,z}_{w,\lambda} e^\mu [\mathcal{O}_{X_z}], \qquad \text{in } K_T(G/B).$$

This is the most general form of “Pieri–Chevalley rule”. The problem is to determine the coefficients $c^{\mu,z}_{w,\lambda}$.

3.1. The path model

A *path* in $\mathfrak{h}^*$ is a piecewise linear map $p : [0, 1] \to \mathfrak{h}^*$ such that $p(0) = 0$. For each $1 \le i \le n$ there are *root operators* e_i and f_i (see [14] Definitions 2.1 and 2.2) which act on the paths. If $\lambda \in P^+$ the *path model* for λ is

$$\mathcal{T}^\lambda = \{ f_{i_1} f_{i_2} \cdots f_{i_l} p_\lambda \},$$

the set of all paths obtained by applying the root operators to p_λ, where p_λ is the straight path from 0 to λ, that is, $p_\lambda(t) = t\lambda$, $0 \le t \le 1$. Each path p in $\mathcal{T}^\lambda$ is a concatenation of segments

$$p = p^{a_1}_{w_1\lambda} \otimes p^{a_2}_{w_2\lambda} \otimes \cdots \otimes p^{a_r}_{w_r\lambda} \qquad \text{with } w_1 \geq w_2 \geq \cdots \geq w_r$$
$$\text{and} \qquad a_1 + a_2 + \cdots + a_r = 1, \tag{3.2}$$

where, for $v \in W$ and $a \in (0, 1]$, $p^a_{v\lambda}$ is a piece of length a from the straight line path $p_{v\lambda} = vp_\lambda$. If $W_\lambda = \text{Stab}(\lambda)$ then the w_j should be viewed as cosets in W/W_λ and $\geq$ denotes the order on W/W_λ inherited from the Bruhat–Chevalley order on W. The total length of p is the same as the total length of p_λ which is assumed (or normalized) to be 1. For $p \in \mathcal{T}^\lambda$ let

$$p(1) = \sum_{i=1}^{r} a_i w_i \lambda \qquad \text{be the endpoint of } p,$$
$$\iota(p) = w_1, \qquad \text{the initial direction of } p, \qquad \text{and}$$
$$\phi(p) = w_r, \qquad \text{the final direction of } p.$$

If $h \in \mathcal{T}^\lambda$ is such that $e_i(h) = 0$ then h is the *head* of its i*-string*

$$S^\lambda_i(h) = \{h, f_i h, \ldots, f_i^m h\},$$

where m is the smallest positive integer such that $f_i^m h \neq 0$ and $f_i^{m+1} h = 0$. The full path model $\mathcal{T}^\lambda$ is the union of its i-strings. The endpoints and the inital and final directions of the paths in the i-string $S^\lambda_i(h)$ have the following properties:

$$\begin{aligned}
&(f_i^k h)(1) = h(1) - k\alpha_i, \qquad \text{for } 0 \leq k \leq m,\\
&\text{either} \qquad \iota(h) = \iota(f_i h) = \cdots = \iota(f_i^m h) < s_i \iota(h)\\
&\text{or} \qquad \iota(h) < \iota(f_i h) = \cdots = \iota(f_i^m h) = s_i \iota(h), \qquad \text{and}\\
&\text{either} \qquad s_i \phi(f_i^m h) < \phi(h) = \cdots = \phi(f_i^{m-1} h) = \phi(f_i^m h)\\
&\text{or} \qquad s_i \phi(f_i^m h) = \phi(h) = \cdots = \phi(f_i^{m-1} h) < \phi(f_i^m h).
\end{aligned} \tag{3.3}$$

The first property is [13, Lemma 2.1a], the second is [12, Lemma 5.3], and the last is a result of applying [13, Lemma 2.1e] to [12, Lemma 5.3]. All of these facts are really coming from the explicit form of the action of the root operators on the paths in $\mathcal{T}^\lambda$ which is given in [12, Proposition 4.2].

Let $\lambda \in P^+$, $w \in W$ and $z \in W/W_\lambda$, and let $p \in \mathcal{T}^\lambda$ be such that $\iota(p) \leq wW_\lambda$ and $\phi(p) \geq z$. Write p in the form (3.2) and let $\tilde{w}_1, \ldots, \tilde{w}_r, \tilde{z}$ be the maximal (in Bruhat order) coset representatives of the cosets $w_1, \ldots, w_r, z$ such that

$$w \geq \tilde{w}_1 \geq \tilde{w}_2 \geq \cdots \geq \tilde{w}_r \geq \tilde{z}. \tag{3.4}$$

Theorem 3.5. *Recall the notation* ε_v *from* (1.11). *Let* $\lambda \in P^+$ *and let* $W_\lambda = \text{Stab}(\lambda)$. *Let* $w \in W$. *Then, in the affine nil-Hecke algebra* $\tilde{H}$,

$$X^\lambda T_{w^{-1}} = \sum_{\substack{p \in \mathcal{T}^\lambda \\ \iota(p) \leq wW_\lambda}} T_{\phi(p)^{-1}} X^{p(1)} \qquad \textit{and}$$

$$X^{\lambda}\varepsilon_{w^{-1}} = \sum_{\substack{p\in\mathcal{T}^{\lambda}\\ \iota(p)=w}} \sum_{\substack{z\in W/W_{\lambda}\\ z\le\phi(p)}} (-1)^{\ell(w)+\ell(z)}\varepsilon_{\tilde{z}^{-1}}X^{p(1)},$$

where, if $W_{\lambda}\neq\{1\}$ *then* $T_{\phi(p)^{-1}} = T_{\tilde{w}_r^{-1}}$ *and* $\varepsilon_{z^{-1}} = \varepsilon_{\tilde{z}^{-1}}$ *with* $\tilde{w}_r$ *and* $\tilde{z}$ *as in* (3.4).

Proof. (a) The proof is by induction on $\ell(w)$. Let $w = s_i v$ where $s_i v > v$. Define

$$\mathcal{T}^{\lambda}_{\le w} = \{p\in\mathcal{T}^{\lambda} \mid \iota(p)\le wW_{\lambda}\}.$$

Assume $w = s_i v > v$. Then the facts in (3.3) imply that

(1) $\mathcal{T}^{\lambda}_{\le w}$ is a union of the strings $S_i(h)$ such that $h\in\mathcal{T}^{\lambda}_{\le v}$, and
(2) If $h\in\mathcal{T}^{\lambda}_{\le v}$ then either $S_i(h)\subseteq\mathcal{T}^{\lambda}_{\le v}$ or $S_i(h)\cap\mathcal{T}^{\lambda}_{\le v} = \{h\}$.

Using the facts in (3.3), a direct computation with the relation (1.3) establishes that, if $h\in\mathcal{T}^{\lambda}_{\le v}$ then

$$\sum_{p\in S_i(h)} T_{\phi(p)^{-1}}X^{\eta(1)} = T_{\phi(h)^{-1}}X^{h(1)}T_i, \qquad \text{and}$$

$$\sum_{p\in S_i(h)} T_{\phi(p)^{-1}}X^{\eta(1)} = \begin{cases} T_{\phi(h)^{-1}}X^{h(1)}T_i, & \text{if } S_i(h)\subseteq\mathcal{T}^{\lambda}_{\le v},\\ T_{\phi(h)^{-1}}X^{h(1)}T_i, & \text{if } S_i(h)\cap\mathcal{T}^{\lambda}_{\le v} = \{h\}.\end{cases}$$

Thus

$$\begin{aligned}
X^{\lambda}T_{w^{-1}} = X^{\lambda}T_{v^{-1}}T_i &= \left(\sum_{p\in\mathcal{T}^{\lambda}_{\le v}} T_{\phi(p)^{-1}}X^{p(1)}\right)T_i \qquad \text{(by induction)}\\
&= \sum_{\substack{h\in\mathcal{T}^{\lambda}_{\le v}\\ e_i(h)=0}}\left(\sum_{S_i(h)\subseteq\mathcal{T}^{\lambda}_{\le v}}\ \sum_{p\in S_i(h)} T_{\phi(p)^{-1}}X^{p(1)} + \sum_{S_i(h)\cap\mathcal{T}^{\lambda}_{\le v}=\{h\}} T_{\phi(h)^{-1}}X^{h(1)}\right)T_i\\
&= \sum_{\substack{h\in\mathcal{T}^{\lambda}_{\le w}\\ e_i(h)=0}}\left(\sum_{S_i(h)\subseteq\mathcal{T}^{\lambda}_{\le v}} T_{\phi(h)^{-1}}X^{h(1)}T_i + \sum_{S_i(h)\cap\mathcal{T}^{\lambda}_{\le v}=\{h\}} T_{\phi(h)^{-1}}X^{h(1)}\right)T_i\\
&= \sum_{\substack{h\in\mathcal{T}^{\lambda}_{\le w}\\ e_i(h)=0}}\left(\sum_{S_i(h)\subseteq\mathcal{T}^{\lambda}_{\le v}} T_{\phi(h)^{-1}}X^{h(1)}T_i + \sum_{S_i(h)\cap\mathcal{T}^{\lambda}_{\le v}=\{h\}}\ \sum_{p\in S_i(h)} T_{\phi(p)^{-1}}X^{p(1)}\right)\\
&= \sum_{p\in\mathcal{T}^{\lambda}_{\le w}} T_{\phi(p)^{-1}}X^{p(1)}.
\end{aligned}$$

(b) The proof is similar to case (a). For $w\in W$ let

$$\mathcal{T}^{\lambda}_{=w} = \{p\in\mathcal{T}^{\lambda} \mid \iota(p) = wW_{\lambda}\}.$$

Assume $w = s_i v > v$. Then the facts in (3.3) imply that

(1) $\mathcal{T}^\lambda_{=w}$ is a union of the strings $S_i(h)$ such that $h \in \mathcal{T}^\lambda_{=h}$, and
(2) If $h \in \mathcal{T}^\lambda_{=v}$ then either $S_i(h) \subseteq \mathcal{T}^\lambda_{=v}$ or $S_i(h) \cap \mathcal{T}^\lambda_{=v} = \{h\}$.

Let

$$\mathcal{E}_{\phi(p)} = \sum_{\substack{z\in W/W_\lambda \\ z\le\phi(p)}} (-1)^{\ell(z)} \varepsilon_{\tilde{z}^{-1}}. \tag{3.6}$$

Using (3.3), a direct computation with the relation (1.3) establishes that, if $h \in \mathcal{T}^\lambda_{=v}$ with $e_i h = 0$ then

$$\sum_{p\in S_i(h)} \mathcal{E}_{\phi(p)} X^{p(1)} T_i = 0, \qquad \text{and} \qquad \mathcal{E}_{\phi(h)} X^{h(1)} T_i = -\sum_{p\in S_i(h)-\{h\}} \mathcal{E}_{\phi(p)} X^{p(1)}.$$

Thus

$$\begin{aligned}
X^\lambda \varepsilon_{w^{-1}} = X^\lambda \varepsilon_{v^{-1}} \varepsilon_i &= (-1)^{\ell(v)} \left(\sum_{p\in\mathcal{T}^\lambda_{=v}} \mathcal{E}_{\phi(p)} X^{p(1)} \right) T_i \\
&= (-1)^{\ell(v)} \left(\sum_{S_i(h)\subseteq\mathcal{T}^\lambda_{=v}} \sum_{p\in S_i(h)} \mathcal{E}_{\phi(p)} X^{p(1)} + \sum_{S_i(h)\cap\mathcal{T}^\lambda_{=v}=\{h\}} \mathcal{E}_{\phi(h)} X^{h(1)} \right) T_i \\
&= (-1)^{\ell(v)} \left(0 - \sum_{S_i(h)\cap\mathcal{T}^\lambda_{=v}=\{h\}} \sum_{p\in S_i(h)-\{h\}} \mathcal{E}_{\phi(p)} X^{p(1)} \right) \\
&= (-1)^{\ell(w)} \left(\sum_{p\in\mathcal{T}^\lambda_{=w}} \mathcal{E}_{\phi(p)} X^{p(1)} \right). \quad \square
\end{aligned}$$

Corollary 3.7. *Let $\lambda, \mu \in P^+$ and let $w \in W$. Then, in the affine nil-Hecke algebra $\tilde{H}$,*

$$X^{-\lambda} T_{w^{-1}} = \sum_{\substack{p\in\mathcal{T}^{-w_0\lambda} \\ \phi(p)=ww_0}} \sum_{\substack{z\in W/W_{-w_0\lambda} \\ zw_0\ge\iota(p)}} (-1)^{\ell(w)+\ell(z)} T_{\tilde{z}^{-1}} X^{p(1)} \qquad \textit{and}$$

$$X^{w_0\mu} T_{w^{-1}} = \sum_{\substack{p\in\mathcal{T}^{\mu} \\ \phi(p)=ww_0}} \sum_{\substack{z\in W/W_{\mu} \\ zw_0\le\phi(p)}} (-1)^{\ell(w)+\ell(z)} T_{\tilde{z}^{-1}} X^{p(1)}.$$

Proof. The second identity is a restatement of the first with a change of variable $\mu = -w_0\lambda$. The first identity is obtained by applying the algebra involution

$$\begin{array}{ccc} \tilde{H} & \longrightarrow & \tilde{H} \\ T_w & \longmapsto & \varepsilon_w \\ X^\lambda & \longmapsto & X^{-\lambda} \end{array} \qquad \text{and the bijection} \qquad \begin{array}{ccc} \mathcal{T}^\lambda & \longrightarrow & \mathcal{T}^{-w_0\lambda} \\ p & \longrightarrow & p^* \end{array}$$

where p^* is the same path as p except translated so that its endpoint is at the origin. Representation theoretically, this bijection corresponds to the fact that $L(\lambda)^* \cong L(-w_0\lambda)$, if $L(\lambda)$ is the simple G-module of highest weight λ. Note that $p^*(1) = -p(1)$, $\iota(p^*) = \phi(p)w_0$, and $\phi(p^*) = \iota(p)w_0$. $\square$

Applying the identities from Theorem 3.5 and Corollary 3.7 to $[\mathcal{O}_{X_1}]$ yields the following product formulas in $K_T(G/B)$. In particular, this gives a combinatorial proof of the (T-equivariant extension) of the duality theorem of Brion [1, Theorem 4]. For $\lambda \in P$ and $w \in W$ let $[X^\lambda] = X^\lambda[\mathcal{O}_{X_{w_0}}] = X^\lambda T_{w_0}[\mathcal{O}_{X_1}]$ and let $c^z_{\lambda,w}$ be given by

$$[X^\lambda][\mathcal{O}_{X_w}] = \sum_{z\in W} c^z_{\lambda,w}[\mathcal{O}_{X_z}]. \tag{3.8}$$

Corollary 3.9. *Let* $\lambda \in P^+$, $w \in W$ *and* $W_\lambda = \mathrm{Stab}(\lambda)$. *Then, with notation as in* (3.8),

$$c^z_{\lambda,w} = \sum_{\substack{p\in\mathcal{T}^\lambda \\ wW_\lambda \geq \iota(p)\geq\phi(p)=zW_\lambda}} \mathrm{e}^{p(1)},$$

$$c^z_{w_0\lambda,w} = (-1)^{\ell(w)+\ell(z)} c^{ww_0}_{\lambda,zw_0}, \qquad \textit{and} \qquad c^z_{-\lambda,w} = (-1)^{\ell(w)+\ell(z)} c^{ww_0}_{-w_0\lambda,zw_0}.$$

Proposition 3.10. *For* $1 \leq i \leq n$, $[\mathcal{O}_{X_{w_0 s_i}}] = 1 - \mathrm{e}^{w_0\omega_i}[X^{-\omega_i}]$.

Proof. We shall show that

$$X^{-\omega_i}[\mathcal{O}_{X_{w_0}}] = \mathrm{e}^{-w_0\omega_i}([\mathcal{O}_{X_{w_0}}] - [\mathcal{O}_{X_{w_0 s_i}}]), \tag{3.11}$$

and the result will follow by solving for $[\mathcal{O}_{X_{s_i w_0}}]$. Let $\omega_j = -w_0\omega_i$. By Corollary 3.9,

$$c^z_{-\omega_i,w_0} = (-1)^{\ell(w_0)+\ell(z)} c^1_{\omega_j,zw_0} = (-1)^{\ell(w_0)+\ell(z)} \sum_{\substack{p\in\mathcal{T}^{\omega_j} \\ zw_0\geq\iota(p)\geq\phi(p)=1}} \mathrm{e}^{p(1)}.$$

The straight line path to ω_j, p_{ω_j}, has $\iota_{zw_0}(p_{\omega_j}) = \phi_{zw_0}(\omega_j)$ and is the unique path in $\mathcal{T}^{\omega_j}$ which may have final direction 1. Suppose $\phi_{zw_0}(p_{\omega_j}) = 1$. Then, since s_j is the only simple reflection which is not in $\mathrm{Stab}(\omega_j)$, it must be that $zw_0 \not\geq s_k$ for all $k \neq j$. Thus $zw_0 = 1$ or $zw_0 = s_j$ and so $c^z_{-\omega_i,w_0} \neq 0$ only if $z = w_0$ or $z = s_j w_0 = w_0 s_i$. Now (3.11) follows since p_{ω_j} has endpoint $\omega_j = -w_0\omega_i$. $\square$

Corollary 3.12. *Let* c^z_{wv} *be as in* (3.8). *Then, for*

$$c^\lambda_{w_0 s_i,w} = -(\mathrm{e}^{-(w\omega_i - w_0\omega_i)} - 1)$$

and

$$c^z_{w_0 s_i,w} = (-1)^{\ell(w)+\ell(z)+1} \sum_{\substack{p\in\mathcal{T}^{-w_0\omega_i} \\ zw_0\geq\iota(p)\geq\phi(p)=ww_0}} \mathrm{e}^{w_0\omega_i+p(1)}, \qquad \textit{for } z \neq w.$$

Proof. This follows from Proposition 3.10 and Corollary 3.9 and the fact that, in the case when $z = w$, there is a unique path p with $ww_0 = \iota(p) = \phi(p) = ww_0$ and endpoint $p(1) = ww_0(-w_0\omega_i) = -w\omega_i$. $\square$

4. Converting to $H_T^*(G/B)$

The *graded nil-Hecke algebra* is the algebra H_{gr} given by generators $t_1, \dots, t_n$ and x_λ, $\lambda \in P$, with relations

$$t_i^2 = 0, \quad \underbrace{t_i t_j t_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{t_j t_i t_j \cdots}_{m_{ij} \text{ factors}}, \quad x_{\lambda+\mu} = x_\lambda + x_\mu, \text{ and } x_\lambda t_i = t_i x_{s_i\lambda} + \langle \lambda, \alpha_i^\vee \rangle. \tag{4.1}$$

The subalgebra of H_{gr} generated by the x_λ is the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$, where $x_i = x_{\omega_i}$, and W acts on $\mathbb{Z}[x_1, \dots, x_n]$ by

$$w x_\lambda = x_{w\lambda} \text{ and } w(fg) = (wf)(wg), \qquad \text{for } w \in W, \lambda \in P, \ f, g \in \mathbb{Z}[x_1, \dots, x_n].$$

Then the last formula in (4.1) generalizes to

$$f t_i = t_i(s_i f) + \frac{f - s_i f}{\alpha_i}, \qquad \text{for } f \in \mathbb{Z}[x_1, \dots, x_n].$$

Let $t_w = t_{i_1} \cdots t_{i_p}$ for a reduced word $w = s_{i_1} \cdots s_{i_p}$ and let $\mathbb{Z}W_{\text{gr}}$ be the subalgebra of H_{gr} spanned by the t_w, $w \in W$. Then

$$\{x_1^{m_1} \cdots x_n^{m_n} t_w \mid w \in W, \ m_i \in \mathbb{Z}_{\geq 0}\} \quad \text{and} \quad \{t_w x_1^{m_1} \cdots x_n^{m_n} \mid w \in W, \ m_i \in \mathbb{Z}_{\geq 0}\}$$

are bases of H_{gr}.

Let $S = \mathbb{Z}[y_1, \dots, y_n]$ and extend coefficients to S so that $H_{\text{gr},S} = S \otimes_{\mathbb{Z}} H_{\text{gr}}$ and $S[x_1, \dots, x_n] = S \otimes_{\mathbb{Z}} \mathbb{Z}[x_1, \dots, x_n]$ are S-algebras. Define $H_T^*(G/B)$ to be the $H_{\text{gr},S}$ module

$$H_T^*(G/B) = S\text{-span}\,\{[X_w] \mid w \in W\}, \tag{4.2}$$

so that the $[X_w]$, $w \in W$, are an S-basis of $K_T(G/B)$, with $H_{\text{gr},S}$-action given by

$$x_i[X_1] = y_i[X_1], \qquad \text{and} \qquad t_i[X_w] = \begin{cases} [X_{ws_i}], & \text{if } ws_i > w, \\ 0, & \text{if } ws_i < w. \end{cases} \tag{4.3}$$

Let y be the S-algebra homomorphism given by

$$\begin{array}{rccc} y: & S[x_1, \dots, x_n] & \longrightarrow & S \\ & x_i & \longmapsto & y_i \end{array}$$

so that $H_T^*(G/B) \cong H_{\text{gr},S} \otimes_{S[x_1,\dots,x_n]} y$ as $H_{\text{gr},S}$-modules. Then, using analogous methods to the $K_T(G/B)$ case proves the following theorem, which gives the ring structure of $H_T^*(G/B)$ (see also the proof of [10, Prop. 2.9] for the same argument with (non-nil) graded Hecke algebras).

Theorem 4.4. *The composite map*

$$\begin{array}{rccccccc} \Phi: & S[x_1, \dots, x_n] & \longrightarrow & H_{\text{gr},S} t_{w_0} & \hookrightarrow & H_{\text{gr},S} & \longrightarrow & H_T^*(G/B) \\ & f & \longmapsto & f t_{w_0} & & h & \longmapsto & h[X_1] \end{array}$$

is surjective with kernel

$$\ker \Phi = \langle f - y(f) \mid f \in S[x_1, \dots, x_n]^W \rangle,$$

the ideal of the ring $S[x_1,\dots,x_n]$ *generated by the elements* $f - y(f)$ *for* $f \in S[x_1,\dots,x_n]^W$. *Hence*

$$H_T^*(G/B) \cong \frac{\mathbb{Z}[y_1,\dots,y_n,x_1,\dots,x_n]}{\langle f - y(f) \mid f \in S[x_1,\dots,x_n]^W\rangle}$$

has the structure of a ring.

As a vector space $H_{\mathrm{gr}} = \mathbb{Z}[x_1,\dots,x_n] \otimes \mathbb{Z}W_{\mathrm{gr}}$. Let $\widehat{H_{\mathrm{gr}}} = \mathbb{Q}[[x_1,\dots,x_n]] \otimes \mathbb{Q}W_{\mathrm{gr}}$ with multiplication determined by the relations in (4.1). Then $\widehat{H_{\mathrm{gr}}}$ is a completion of H_{gr} (this simply allows us to write infinite sums) and the elements of $\widehat{H_{\mathrm{gr}}}$ given by

$$\mathrm{ch}(X^\lambda) = \sum_{r\geq 0} \frac{1}{r!} x_\lambda^r \qquad \text{and} \qquad \mathrm{ch}(T_i) = t_i \cdot \frac{x_{\alpha_i}}{1-\mathrm{ch}(X^{\alpha_i})} \tag{4.5}$$

satisfy the relations of $\tilde{H}$ and thus ch extends to a ring homomorphism ch : $\tilde{H} \longrightarrow \widehat{H_{\mathrm{gr}}}$. It is this fact that really makes possible the transfer from K-theory to cohomology possible. Though it is not difficult to check that the elements in (3.5) satisfy the defining relations of $\tilde{H}$ it is helpful to realize that these formulas come from geometry. As explained in [21], the action of T_i on $K_T(G/B)$ and the action of t_i on $H_T^*(G/B)$ are, respectively, the push-pull operators $\pi_i^*(\pi_i)_!$ and $\pi_i^*(\pi_i)_*$, where if P_i is a minimal parabolic subgroup of G then $\pi_i : G/P_i \to G/B$ is the natural surjection. Then the first formula in (3.5) is the definition of the Chern character, and the second formula is the Grothedieck–Riemann–Roch theorem applied to the map π_i. The factor $X\alpha_i/(1-\mathrm{ch}(X^{\alpha_i}))$ is the Todd class of the bundle of tangents along the fibers of π_i (see [8, p. 91]).

Then $\widehat{H_T^*(G/B)_{\mathbb{Q}}} = \mathbb{Q}[[y_1,\dots,y_n]] \otimes_{\mathbb{Z}[y_1,\dots,y_n]} H_T^*(G/B)$ is the appropriate completion of $H_T^*(G/B)$ to use to transfer the ring homomorphism ch : $\tilde{H}_R \to \widehat{H_{\mathrm{gr}}}$ to a ring homomorphism

$$\mathrm{ch}\colon K_T(G/B) \longrightarrow \widehat{H_T^*(G/B)_{\mathbb{Q}}} \qquad \text{by setting } \mathrm{ch}(h[\mathcal{O}_{X_1}]) = \mathrm{ch}(h)[X_1], \tag{4.6}$$

for $h \in \tilde{H}_R$. The ring $\widehat{H_T^*(G/B)_{\mathbb{Q}}}$ is a graded ring with

$$\deg(y_i) = 1 \qquad \text{and} \qquad \deg([X_w]) = \ell(w_0) - \ell(w), \tag{4.7}$$

$$\text{and,} \quad \text{for } w \in W, \qquad \mathrm{ch}([\mathcal{O}_{X_w}]) = [X_w] + \text{higher degree terms.} \tag{4.8}$$

In summary, if $e_i = \mathrm{e}^{\omega_i}$, $X_i = X^{\omega_i}$, $y_i = y_{\omega_i}$, $x_i = x_{\omega_i}$,

$$R[X] = \mathbb{Z}[e_1^{\pm 1},\dots,e_n^{\pm 1}, X_1^{\pm 1},\dots,X_n^{\pm 1}], \qquad \mathbb{Z}[X] = \mathbb{Z}[X_1^{\pm 1},\dots,X_n^{\pm 1}],$$
$$\text{and} \qquad \widehat{S}[x_1,\dots,x_n] = \mathbb{Q}[[y_1,\dots,y_n]][x_1,\dots,x_n],$$

then there is a commutative diagram of ring homomorphisms

$$\begin{tikzcd}
K_T(G/B) = \frac{R[X]}{\langle f-e(f) \mid f\in R[X]^W\rangle} \arrow[r, "\mathrm{ch}"] \arrow[d, "e_i=1"] & H_T^*(G/B)_{\mathbb{Q}} = \frac{\widehat{S}[x_1,\dots,x_n]}{\langle f-y(f) \mid f\in \widehat{S}[x_1,\dots,x_n]^W\rangle} \arrow[d, "y_i=0"] \\
K(G/B) = \frac{\mathbb{Z}[X]}{\langle f-f(1) \mid f\in \mathbb{Z}[X]^W\rangle} \arrow[r, "\mathrm{ch}"] & H^*(G/B)_{\mathbb{Q}} = \frac{\mathbb{Q}[x_1,\dots,x_n]}{\langle f-f(0) \mid f\in \mathbb{Q}[x_1,\dots,x_n]^W\rangle}.
\end{tikzcd}$$

5. Rank two and a positivity conjecture

In this section we will give explicit formulas for the rank two root systems. The data supports the following positivity conjecture which generalizes the theorems of Brion [1, formula before Theorem 1] and Graham [7, Corollary 4.1].

Conjecture 5.9. *For $\beta \in R^+$ let $y_\beta = \mathrm{e}^{-\beta}$ and $\alpha_\beta = \mathrm{e}^{-\beta} - 1$ and let $d(w) = \ell(w_0) - \ell(w)$ for $w \in W$. Let c^z_{wv} be the structure constants of $K_T(G/B)$ with respect to the basis $\{[\mathcal{O}_{X_w}] \mid w \in W\}$ as defined in (0.1). Then*

$$c^z_{wv} = (-1)^{d(w)+d(v)-d(z)} f(\alpha, y), \qquad \textit{where } f(\alpha, y) \in \mathbb{Z}_{\geq 0}[\alpha_\beta, y_\beta \mid \beta \in R^+],$$

that is, $f(\alpha, y)$ is a polynomial in the variables α_β and y_β, $\beta \in R^+$, which has non-negative integral coefficients.

In the following, for brevity, use the following notations:

$$\begin{aligned}
&\text{in } K_T(G/B), \quad [w] = [\mathcal{O}_{X_w}], \quad \alpha_{rs} = \mathrm{e}^{-(r\alpha_1+s\alpha_2)} - 1, \quad \text{and} \quad y_{rs} = \mathrm{e}^{-(r\alpha_1+s\alpha_2)},\\
&\text{in } K(G/B), \quad [w] = [\mathcal{O}_{X_w}], \quad \alpha_{rs} = 0, \quad \text{and} \quad y_{rs} = 1,\\
&\text{in } H^*_T(G/B), \quad [w] = [X_w], \quad \alpha_{rs} = r\alpha_1 + s\alpha_2, \quad \text{and} \quad y_{rs} = 1,\\
&\text{in } H^*(G/B), \quad [w] = [X_w], \quad \alpha_{rs} = 0, \quad \text{and} \quad y_{rs} = 1,
\end{aligned}$$

and in $H^*_T(G/B)$ and in $H^*(G/B)$ the terms in { } brackets do not appear.

Type A_2. For the root system R of type A_2

$$\begin{aligned}
&\alpha_1 = -\omega_1 + 2\omega_2, \qquad \lambda_1 = \rho, \qquad \lambda_{s_1} = \omega_2 = \tfrac{1}{3}\alpha_1 + \tfrac{2}{3}\alpha_2,\\
&\qquad \lambda_{s_2s_1} = s_2\omega_2 = \tfrac{1}{3}\alpha_1 - \tfrac{1}{3}\alpha_2,\\
&\alpha_2 = 2\omega_1 - \omega_2, \lambda_{w_0} = 0, \qquad \lambda_{s_2} = \omega_1 = \tfrac{2}{3}\alpha_1 + \tfrac{1}{3}\alpha_2,\\
&\qquad \lambda_{s_1s_2} = s_1\omega_1 = -\tfrac{1}{3}\alpha_1 + \tfrac{1}{3}\alpha_2.
\end{aligned}$$

Formulas for the Schubert classes in terms of homogeneous line bundles can be given by

$$\begin{aligned}
&[s_1s_2s_1] = 1, \qquad [1] = (1 - \mathrm{e}^{s_1\omega_1}X^{-\omega_1})[s_1] = (1 - \mathrm{e}^{s_2\omega_2}X^{-\omega_2})[s_2],\\
&[s_2s_1] = 1 - \mathrm{e}^{-\omega_1}X^{-\omega_2}, \qquad [s_1s_2] = 1 - \mathrm{e}^{-\omega_2}X^{-\omega_1}\\
&[s_1] = (1 - \mathrm{e}^{s_2\omega_2}X^{-\omega_2})[s_2s_1], \qquad [s_2] = (1 - \mathrm{e}^{s_1\omega_1}X^{-\omega_1})[s_1s_2],
\end{aligned}$$

and

$$\begin{aligned}
&[s_1s_2s_1] = 1, \qquad [s_1s_2] = 1 - \mathrm{e}^{-\omega_2}X^{-\omega_1}, \qquad [s_2s_1] = 1 - \mathrm{e}^{-\omega_1}X^{-\omega_2},\\
&[s_1] = 1 - \mathrm{e}^{-\omega_2}X^{-s_1\omega_1} - \mathrm{e}^{-\omega_2}X^{-\omega_1} + \mathrm{e}^{-2\omega_2}X^{-\omega_2},\\
&[s_2] = 1 - \mathrm{e}^{-\omega_1}X^{-s_2\omega_2} - \mathrm{e}^{-\omega_1}X^{-\omega_2} + \mathrm{e}^{-2\omega_1}X^{-\omega_1},\\
&[1] = 1 - \mathrm{e}^{-\omega_2}X^{-s_1\omega_1} - \mathrm{e}^{-\omega_1}X^{-s_2\omega_2} + \mathrm{e}^{-2\omega_1}X^{-\omega_1} + \mathrm{e}^{-2\omega_2}X^{-\omega_2} - \mathrm{e}^{-\rho}X^{-\rho}.
\end{aligned}$$

The multiplication of the Schubert classes is given by

$$\begin{aligned}
&[1]^2 = -\alpha_{10}\alpha_{01}\alpha_{11}[1], \qquad [s_1]^2 = \alpha_{01}\alpha_{11}[s_1], \qquad [s_2]^2 = \alpha_{10}\alpha_{11}[s_2],\\
&[1][s_1] = \alpha_{01}\alpha_{11}[1], \qquad [s_1][s_2] = -\alpha_{11}[1], \qquad [s_2][s_1s_2] = -\alpha_{11}[s_2],\\
&[1][s_2] = \alpha_{10}\alpha_{11}[1], \qquad [s_1][s_1s_2] = y_{01}[1] - \alpha_{01}[s_1],\\
&\qquad [s_2][s_2s_1] = y_{10}[1] - \alpha_{10}[s_2],
\end{aligned}$$

$$
\begin{aligned}
&[1][s_1s_2] = -\alpha_{11}[1], \qquad [s_1][s_2s_1] = -\alpha_{11}[s_1],\\
&[1][s_2s_1] = -\alpha_{11}[1], \qquad [s_1s_2]^2 = y_{01}[s_2] - \alpha_{01}[s_1s_2],\\
&\qquad [s_2s_1]^2 = y_{10}[s_1] - \alpha_{10}[s_2s_1].\\
&[s_1s_2][s_2s_1] = \{-[1]\} + [s_1] + [s_2],
\end{aligned}
$$

Type B_2. For the root system R of type B_2

$$
\begin{aligned}
&\alpha_1 = 2\omega_1 - \omega_2, \qquad \lambda_1 = \rho = 2\alpha_1 + \tfrac{3}{2}\alpha_2, \qquad \lambda_{s_1} = \omega_2 = \alpha_1 + \alpha_2,\\
&\alpha_2 = -2\omega_1 + 2\omega_2, \qquad \lambda_{w_0} = 0, \qquad \lambda_{s_2} = \omega_1 = \alpha_1 + \tfrac{1}{2}\alpha_2,\\
&\lambda_{s_2s_1} = s_2\omega_2 = \alpha_1, \qquad \lambda_{s_1s_2s_1} = s_1s_2\omega_2 = -\alpha_1,\\
&\lambda_{s_1s_2} = s_1\omega_1 = \tfrac{1}{2}\alpha_2, \qquad \lambda_{s_2s_1s_2} = s_2s_1\omega_1 = -\tfrac{1}{2}\alpha_2.
\end{aligned}
$$

Formulas for the Schubert classes in terms of homogeneous line bundles can be given by

$$
\begin{aligned}
&[s_1s_2s_1s_2] = 1, \qquad [1] = (1 - \mathrm{e}^{s_1\omega_1}X^{-\omega_1})[s_1] = (1 - \mathrm{e}^{s_2\omega_2}X^{-\omega_2})[s_2],\\
&[s_1s_2s_1] = 1 - \mathrm{e}^{-\omega_2}X^{-\omega_2}, \qquad [s_2s_1s_2] = 1 - \mathrm{e}^{-\omega_1}X^{-\omega_1},\\
&[s_2s_1] = (1 - \mathrm{e}^{-\omega_1}X^{-s_1\omega_1})[s_2s_1s_2], \qquad [s_1s_2] = (1 - \mathrm{e}^{s_2s_1\omega_1}X^{-\omega_1})[s_2s_1s_2],\\
&[s_1] = (1 - \mathrm{e}^{s_2\omega_2}X^{-\omega_2})[s_2s_1], \qquad [s_2] = (1 - \mathrm{e}^{s_1\omega_1}X^{-\omega_1})[s_1s_2],
\end{aligned}
$$

and

$$
\begin{aligned}
&[s_1s_2s_1s_2] = 1, \qquad [s_1s_2s_1] = 1 - \mathrm{e}^{-\omega_2}X^{-\omega_2}, \qquad [s_2s_1s_2] = 1 - \mathrm{e}^{-\omega_1}X^{-\omega_1},\\
&[s_1s_2] = (1 - \mathrm{e}^{-\omega_2}) - \mathrm{e}^{-\omega_2}X^{-\omega_2} - \mathrm{e}^{-\omega_2}X^{-s_2\omega_2} + (\mathrm{e}^{-\rho} + \mathrm{e}^{-s_1\rho})X^{-\omega_1},\\
&[s_2s_1] = 1 - \mathrm{e}^{-\omega_1}X^{-\omega_1} - \mathrm{e}^{-\omega_1}X^{-s_1\omega_1} + \mathrm{e}^{-2\omega_1}X^{-\omega_2},\\
&[s_1] = (1 - \mathrm{e}^{-\omega_2}) + (\mathrm{e}^{-\rho} + \mathrm{e}^{-s_1\rho})X^{-s_1\omega_1} + (\mathrm{e}^{-\rho} + \mathrm{e}^{-s_1\rho})X^{-\omega_1}\\
&\qquad - \mathrm{e}^{-\omega_2}X^{-s_1s_2\omega_2} - \mathrm{e}^{-\omega_2}X^{-s_2\omega_2} - (\mathrm{e}^{-2\omega_2} + \mathrm{e}^{-\omega_2})X^{-\omega_2},\\
&[s_2] = (1 + \mathrm{e}^{-2\omega_1}) + \mathrm{e}^{-2\omega_1}X^{-s_2\omega_2} + \mathrm{e}^{-2\omega_1}X^{-\omega_2}\\
&\qquad - \mathrm{e}^{-\omega_1}X^{-s_2s_1\omega_1} - \mathrm{e}^{-\omega_1}X^{-s_1\omega_1} - (\mathrm{e}^{-3\omega_1} + \mathrm{e}^{-\omega_1})X^{-\omega_1},\\
&[1] = (1 + \mathrm{e}^{-2\omega_1}) - \mathrm{e}^{-\omega_1}X^{-s_2s_1\omega_1} + (\mathrm{e}^{-\rho} + \mathrm{e}^{-s_1\rho})X^{-s_1\omega_1} - (\mathrm{e}^{-3\omega_1} + \mathrm{e}^{-\omega_1})X^{-\omega_1}\\
&\qquad - \mathrm{e}^{-\omega_2}X^{-s_1s_2\omega_2} + \mathrm{e}^{-2\omega_1}X^{-s_2\omega_2} - (\mathrm{e}^{-2\omega_2} + \mathrm{e}^{-\omega_2})X^{-\omega_2} + \mathrm{e}^{-\rho}X^{-\rho}.
\end{aligned}
$$

The multiplication of the Schubert classes is given by

$$
\begin{aligned}
&[1]^2 = \alpha_{10}\alpha_{01}\alpha_{11}\alpha_{21}[1], \quad \begin{matrix} [s_1s_2s_1]^2 = \{-y_{11}[s_1]\} + (y_{01} + y_{11})[s_2s_1] - \alpha_{01}[s_1s_2s_1], \\ [s_1s_2s_1][s_2s_1s_2] = \{[1] - [s_1] - [s_2]\} + [s_1s_2] + [s_2s_1], \end{matrix}\\
&[1][s_1] = -\alpha_{01}\alpha_{11}\alpha_{21}[1],\\
&[1][s_2] = -\alpha_{10}\alpha_{11}\alpha_{21}[1],\\
&[1][s_1s_2] = \alpha_{11}\alpha_{21}[1], \qquad \begin{matrix} [s_2s_1s_2]^2 = y_{10}[s_1s_2] - \alpha_{10}[s_2s_1s_2], \\ [s_2s_1]^2 = -\alpha_{21}y_{10}[s_1] + \alpha_{10}\alpha_{21}[s_2s_1], \end{matrix}\\
&[1][s_2s_1] = \alpha_{11}\alpha_{21}[1],\\
&[1][s_1s_2s_1] = -\alpha_{11}(1 + y_{11})[1],\\
&[1][s_2s_1s_2] = -\alpha_{21}[1], \quad \begin{matrix} [s_2s_1][s_1s_2s_1] = y_{21}[s_1] - \alpha_{21}[s_2s_1], \\ [s_2s_1][s_2s_1s_2] = \{-y_{10}[1]\} + y_{10}[s_1] + y_{10}[s_2] - \alpha_{10}[s_2s_1], \end{matrix}\\
&[s_1]^2 = -\alpha_{01}\alpha_{11}\alpha_{21}[s_1], \qquad [s_2]^2 = -\alpha_{10}\alpha_{11}\alpha_{21}[s_2],\\
&[s_1][s_2] = \alpha_{11}\alpha_{21}[1],
\end{aligned}
$$

$$
\begin{aligned}
&[s_1][s_1s_2] = -\alpha_{11}(y_{01}+y_{11})[1] + \alpha_{01}\alpha_{11}[s_1], && [s_2][s_1s_2] = \alpha_{11}\alpha_{21}[s_2], \\
& && [s_2][s_2s_1] = -\alpha_{21}y_{10}[1] + \alpha_{10}\alpha_{21}[s_2], \\
&[s_1][s_2s_1] = \alpha_{11}\alpha_{21}[s_1], \\
&[s_1][s_1s_2s_1] = -\alpha_{11}(1+y_{11})[s_1], && [s_2][s_1s_2s_1] = y_{21}[1] - \alpha_{21}[s_2], \\
& && [s_2][s_2s_1s_2] = -\alpha_{21}[s_2], \\
&[s_1][s_2s_1s_2] = y_{11}[1] - \alpha_{11}[s_1],
\end{aligned}
$$

$$
\begin{aligned}
&[s_1s_2]^2 = -\alpha_{11}(y_{01}+y_{11})[s_2] + \alpha_{01}\alpha_{11}[s_1s_2], \\
&[s_1s_2][s_2s_1] = (\{\alpha_{11}\} + y_{21})[1] - \alpha_{11}[s_1] - \alpha_{21}[s_2], \\
&[s_1s_2][s_1s_2s_1] = \{-(y_{01}+y_{11})[1]\} + y_{01}[s_1] + (y_{11}+y_{12})[s_2] - \alpha_{01}[s_1s_2], \\
&[s_1s_2][s_2s_1s_2] = y_{11}[s_2] - \alpha_{11}[s_1s_2],
\end{aligned}
$$

$$
\begin{aligned}
&[s_2s_1]^2 = -\alpha_{21}y_{10}[s_1] + \alpha_{10}\alpha_{21}[s_2s_1], \\
&[s_2s_1][s_1s_2s_1] = y_{21}[s_1] - \alpha_{21}[s_2s_1], \\
&[s_2s_1][s_2s_1s_2] = \{-y_{10}[1]\} + y_{10}[s_1] + y_{10}[s_2] - \alpha_{10}[s_2s_1],
\end{aligned}
$$

Type G_2. For the root system R of type G_2

$$
\begin{aligned}
&\lambda_1 = \rho = 5\alpha + 3\alpha_2, && \lambda_{s_1s_2s_1} = s_1s_2\omega_2 = \alpha_2, \\
&\lambda_{s_1} = \omega_2 = 3\alpha_1 + 2\alpha_2, && \lambda_{s_2s_1s_2s_1} = s_2s_1s_2\omega_2 = -\alpha_2, \\
&\lambda_{s_2} = \omega_1 = 2\alpha_1 + \alpha_2, && \lambda_{s_1s_2s_1s_2} = s_1s_2s_1\omega_1 = -\alpha_1, \\
&\lambda_{s_2s_1} = s_2\omega_2 = 3\alpha_1 + \alpha_2, && \lambda_{s_1s_2s_1s_2s_1} = s_1s_2s_1s_2\omega_2 = -3\alpha_1 - \alpha_2, \\
&\lambda_{s_1s_2} = s_1\omega_1 = \alpha_1 + \alpha_2, && \lambda_{s_2s_1s_2s_1s_2} = s_2s_1s_2s_1\omega_1 = -\alpha_1 - \alpha_2, \\
&\lambda_{s_2s_1s_2} = s_2s_1\omega_1 = \alpha_1, && \lambda_{w_0} = 0.
\end{aligned}
$$

Formulas for the Schubert classes in terms of homogeneous line bundles can be given by

$$
\begin{aligned}
&[s_1s_2s_1s_2s_1s_2] = 1, \qquad [1] = (1 - \mathrm{e}^{s_1\omega_1}X^{-\omega_1})[s_1] = (1 - \mathrm{e}^{s_2\omega_2}X^{-\omega_2})[s_2], \\
&[s_1s_2s_1s_2s_1] = 1 - \mathrm{e}^{-\omega_2}X^{-\omega_2}, \qquad [s_2s_1s_2s_1s_2] = 1 - \mathrm{e}^{-\omega_1}X^{-\omega_1}, \\
&[s_2s_1s_2s_1] = (1 - \mathrm{e}^{-\omega_1}X^{-s_1\omega_1})[s_2s_1s_2s_1s_2], \\
&[s_1s_2s_1s_2] = (1 - \mathrm{e}^{-s_1\omega_1}X^{-\omega_1})[s_2s_1s_2s_1s_2], \\
&[s_1s_2s_1] = \text{see below}, \qquad [s_2s_1s_2] = \frac{1 - \mathrm{e}^{-s_2s_2\omega_1}X^{-\omega_1}}{1 + X^{-\omega_1}}[s_1s_2s_1s_2], \\
&[s_2s_1] = (1 - \mathrm{e}^{-\omega_1}X^{-s_1s_2s_1\omega_1})[s_2s_1s_2], \qquad [s_1s_2] = (1 - \mathrm{e}^{s_2s_1\omega_1}X^{-\omega_1})[s_1s_2], \\
&[s_1] = (1 - \mathrm{e}^{s_2\omega_2}X^{-\omega_2})[s_2s_1], \qquad [s_2] = (1 - \mathrm{e}^{s_1\omega_1}X^{-\omega_1})[s_1s_2], \\
&[s_1s_2s_1] = \frac{(1 - \mathrm{e}^{-\alpha_2}X^{-\omega_2})[s_2s_1s_2s_1] + \mathrm{e}^{-\alpha_2}(1 + \mathrm{e}^{\omega_1}X^{-\omega_2})[s_2s_1]}{1 + \mathrm{e}^{-\alpha_2}},
\end{aligned}
$$

and

$$
\begin{aligned}
&[w_0] = 1, \qquad [s_2s_1s_2s_1s_2] = 1 - y_{21}X^{-\omega_1}, \qquad [s_1s_2s_1s_2s_1] = 1 - y_{32}X^{-\omega_2}, \\
&[s_2s_1s_2s_1] = 1 - y_{21}X^{-\omega_1} - y_{21}X^{-s_1\omega_1} + y_{42}X^{-\omega_2}, \\
&[s_1s_2s_1s_2] = (1 - y_{32}) + (y_{22} + y_{42} + y_{43} + y_{53})X^{-\omega_1} - y_{32}X^{-s_1\omega_1} \\
&\qquad\qquad - y_{32}X^{-s_2s_1\omega_1} - y_{32}X^{-\omega_2} - y_{32}X^{-s_2\omega_2},
\end{aligned}
$$

$$
\begin{aligned}
[s_2s_1s_2] = {} & (1 - y_{21} + y_{42}) + (y_{42} - y_{21} - y_{52} - y_{53} - y_{63})X^{-\omega_1} \\
& + (y_{42} - y_{21})X^{-s_1\omega_1} + (y_{42} - y_{21})X^{-s_2s_1\omega_1} + y_{42}X^{-\omega_2} + y_{42}X^{-s_2\omega_2}, \\
[s_1s_2s_1] = {} & (1 - 2y_{32}) + (y_{22} + y_{42} + y_{43} + y_{53})X^{-\omega_1} \\
& + (y_{22} + y_{42} + y_{43} + y_{53})X^{-s_1\omega_1} - y_{32}X^{-s_2s_1\omega_1} - y_{32}X^{-s_1s_2s_1\omega_1} \\
& - (y_{32} + y_{43} + y_{53})X^{-\omega_2} - y_{32}X^{-s_2\omega_2} - y_{32}X^{-s_1s_2\omega_2}, \\
[s_2s_1] = {} & (1 - y_{21} + 2y_{42}) + (y_{42} - y_{21} - y_{52} - y_{53} - y_{63})X^{-\omega_1} \\
& + (y_{42} - y_{21} - y_{32} - y_{53} - y_{63})X^{-s_1\omega_1} + (y_{42} - y_{21})X^{-s_2s_1\omega_1} \\
& + (y_{42} - y_{21})X^{-s_1s_2s_1\omega_1} + (y_{42} + y_{63})X^{-\omega_2} + y_{42}X^{-s_2\omega_2} + y_{42}X^{-s_1s_2\omega_2}, \\
[s_1s_2] = {} & 1 - y_{11} - y_{21} - y_{32} - y_{43} - y_{53} + (y_{22} + y_{32})(1 + y_{10} + y_{20})X^{-\omega_1} \\
& + (y_{22} + y_{32} + y_{42})X^{-s_1\omega_1} + (y_{22} + y_{32} + y_{42})X^{-s_2s_1\omega_1} \\
& - (y_{32} + y_{43} + y_{53})X^{-\omega_2} - (y_{32} + y_{43} + y_{53})X^{-s_2\omega_2} - y_{32}X^{-s_1s_2\omega_2} \\
& - y_{32}X^{-s_2s_1s_2\omega_2}, \\
[s_2] = {} & (1 + y_{31} + y_{32} + 2y_{42} + y_{63}) - (y_{21} + y_{52} + y_{53} + y_{84})X^{-\omega_1} \\
& - (y_{21} + y_{52} + y_{53})X^{-s_1\omega_1} - (y_{21} + y_{52} + y_{53})X^{-s_2s_1\omega_1} - y_{21}X^{-s_1s_2s_1\omega_1} \\
& - y_{21}X^{-s_2s_1s_2s_1\omega_1} + (y_{42} + y_{63})X^{-\omega_2} + (y_{42} + y_{63})X^{-s_2\omega_2} \\
& + y_{42}X^{-s_1s_2\omega_2} + y_{42}X^{-s_2s_1s_2\omega_2}, \\
[s_1] = {} & 1 - (y_{11} + y_{21} + y_{32} + 2y_{43} + 2y_{53}) + (y_{22} + y_{54})(1 + y_{10} + y_{20})X^{-\omega_1} \\
& + (y_{22} + y_{54})(1 + y_{10} + y_{20})X^{-s_1\omega_1} + (y_{22} + y_{32} + y_{42})X^{-s_2s_1\omega_1} \\
& + (y_{22} + y_{32} + y_{42})X^{-s_1s_2s_1\omega_1} - (y_{32} + y_{43} + y_{53} + y_{64})X^{-\omega_2} \\
& - (y_{32} + y_{43} + y_{53})X^{-s_2\omega_2} - (y_{32} + y_{43} + y_{53})X^{-s_1s_2\omega_2} - y_{32}X^{-s_2s_1s_2\omega_2} \\
& - y_{32}X^{-s_1s_2s_1s_2\omega_2}, \\
[1] = {} & (1 + y_{31} + y_{42} + y_{63} - y_{53} - y_{43}) - y_{21}(1 + y_{32})^2X^{-\omega_1} \\
& + y_{22}(1 + y_{10} + y_{20})(1 + y_{21} + y_{31})X^{-s_1\omega_1} - (y_{21} + y_{52} + y_{53})X^{-s_2s_1\omega_1} \\
& + y_{22}X^{-s_1s_2s_1\omega_1} - y_{21}X^{-s_2s_1s_2s_1\omega_1} - y_{32}(1 + y_{11})(1 + y_{21})X^{-\omega_2} \\
& + (y_{42} + y_{63})X^{-s_2\omega_2} - (y_{32} + y_{43} + y_{53})X^{-s_1s_2\omega_2} + y_{42}X^{-s_2s_1s_2\omega_2} \\
& - y_{32}X^{-s_1s_2s_1s_2\omega_2} + y_{53}X^{-\rho}.
\end{aligned}
$$

The multiplication of the Schubert classes is given by

$$
\begin{aligned}
& [1]^2 = \alpha_{10}\alpha_{01}\alpha_{11}\alpha_{21}\alpha_{31}\alpha_{32}[1], && [1][s_2s_1s_2] = -\alpha_{21}\alpha_{31}\alpha_{32}[1], \\
& [1][s_1] = -\alpha_{01}\alpha_{11}\alpha_{21}\alpha_{31}\alpha_{32}[1], && [1][s_1s_2s_1s_2] = \alpha_{21}\alpha_{32}(1 + y_{21})[1], \\
& [1][s_2] = -\alpha_{10}\alpha_{11}\alpha_{21}\alpha_{31}\alpha_{32}[1], && [1][s_2s_1s_2s_1] = \alpha_{21}\alpha_{32}(1 + y_{21})[1], \\
& [1][s_1s_2] = \alpha_{11}\alpha_{21}\alpha_{31}\alpha_{32}[1], && [1][s_1s_2s_1s_2s_1] = -\alpha_{32}(1 + y_{32})[1], \\
& [1][s_2s_1] = \alpha_{11}\alpha_{21}\alpha_{31}\alpha_{32}[1], && [1][s_2s_1s_2s_1s_2] = -\alpha_{21}(1 + y_{21})[1], \\
& [1][s_1s_2s_1] = -\alpha_{11}\alpha_{21}\alpha_{32}(1 + y_{11} + y_{21})[1], &&
\end{aligned}
$$

$$
\begin{aligned}
& [s_1]^2 = -\alpha_{01}\alpha_{11}\alpha_{21}\alpha_{31}\alpha_{32}[s_1] \\
& [s_1][s_2] = \alpha_{11}\alpha_{21}\alpha_{31}\alpha_{32}[1] \\
& [s_1][s_1s_2] = -\alpha_{11}\alpha_{21}\alpha_{32}(y_{01} + y_{11} + y_{21})[1] + \alpha_{01}\alpha_{11}\alpha_{21}\alpha_{32}[s_1]
\end{aligned}
$$

$$[s_1][s_2s_1] = \alpha_{11}\alpha_{21}\alpha_{31}\alpha_{32}[s_1]$$

$$[s_1][s_1s_2s_1] = -\alpha_{11}\alpha_{21}\alpha_{32}(1 + y_{11} + y_{21})[s_1]$$

$$[s_1][s_2s_1s_2] = \alpha_{21}\alpha_{32}(y_{11} + y_{21})[1] - \alpha_{11}\alpha_{21}\alpha_{32}[s_1]$$

$$[s_1][s_1s_2s_1s_2] = -\alpha_{32}(y_{22} + y_{32})[1] + \alpha_{11}\alpha_{32}(1 + y_{11})[s_1]$$

$$[s_1][s_2s_1s_2s_1] = \alpha_{21}\alpha_{32}(1 + y_{21})[s_1]$$

$$[s_1][s_1s_2s_1s_2s_1] = -\alpha_{32}(1 + y_{32})[s_1]$$

$$[s_1][s_2s_1s_2s_1s_2] = y_{32}[1] - \alpha_{32}[s_1]$$

$$[s_2]^2 = -\alpha_{10}\alpha_{11}\alpha_{21}\alpha_{31}\alpha_{32}[s_2]$$

$$[s_2][s_1s_2] = \alpha_{11}\alpha_{21}\alpha_{31}\alpha_{32}[s_2]$$

$$[s_2][s_2s_1] = -\alpha_{21}\alpha_{31}\alpha_{32}y_{10}[1] + \alpha_{10}\alpha_{21}\alpha_{31}\alpha_{32}[s_2]$$

$$[s_2][s_1s_2s_1] = \alpha_{21}\alpha_{32}(y_{21} + y_{31})[1] - \alpha_{21}\alpha_{31}\alpha_{32}[s_2]$$

$$[s_2][s_2s_1s_2] = -\alpha_{21}\alpha_{31}\alpha_{32}[s_2]$$

$$[s_2][s_1s_2s_1s_2] = \alpha_{21}\alpha_{32}(1 + y_{21})[s_2]$$

$$[s_2][s_2s_1s_2s_1] = -\alpha_{21}(y_{31} + y_{52})[1] + \alpha_{21}\alpha_{31}(1 + y_{21})[s_2]$$

$$[s_2][s_1s_2s_1s_2s_1] = y_{63}[1] - \alpha_{21}(1 + y_{21} + y_{42})[s_2]$$

$$[s_2][s_2s_1s_2s_1s_2] = -\alpha_{21}(1 + y_{21})[s_2]$$

$$[s_1s_2]^2 = -\alpha_{11}\alpha_{21}\alpha_{32}(y_{01} + y_{11} + y_{21})[s_2] + \alpha_{01}\alpha_{11}\alpha_{21}\alpha_{32}[s_1s_2]$$

$$[s_1s_2][s_2s_1] = \alpha_{21}\alpha_{32}(y_{11} + y_{21} + \alpha_{31})[1] - \alpha_{11}\alpha_{21}\alpha_{32}[s_1] - \alpha_{21}\alpha_{31}\alpha_{32}[s_2]$$

$$\begin{aligned}[s_1s_2][s_1s_2s_1] = {} & -\alpha_{32}(y_{32} + y_{42}\{+\alpha_{11}(y_{01} + 2y_{11} + y_{21})\})[1] \\ & + \alpha_{11}\alpha_{32}(y_{01} + y_{11})[s_1] + (\alpha_{31}\alpha_{32}y_{11} \\ & + \alpha_{11}\alpha_{32}(y_{01} + y_{11} + y_{21}))[s_2] - \alpha_{01}\alpha_{11}\alpha_{32}[s_1s_2]\end{aligned}$$

$$[s_1s_2][s_2s_1s_2] = \alpha_{21}\alpha_{32}(y_{11} + y_{21})[s_2] - \alpha_{11}\alpha_{21}\alpha_{32}[s_1s_2]$$

$$[s_1s_2][s_1s_2s_1s_2] = -\alpha_{32}(y_{22} + y_{32})[s_2] + \alpha_{11}\alpha_{32}(1 + y_{11})[s_1s_2]$$

$$\begin{aligned}[s_1s_2][s_2s_1s_2s_1] = {} & (y_{63}\{+\alpha_{32}(y_{11} + y_{21})\})[1] - \alpha_{32}y_{11}[s_1] - (\alpha_{32}(y_{11} + y_{21}) \\ & + \alpha_{31}y_{32})[s_2] + \alpha_{11}\alpha_{32}[s_1s_2]\end{aligned}$$

$$\begin{aligned}[s_1s_2][s_1s_2s_1s_2s_1] = {} & \{-(y_{33} + y_{43} + y_{53})[1]\} + y_{33}[s_1] + (y_{33} + y_{43} + y_{53})[s_2] \\ & - \alpha_{11}(1 + y_{11} + y_{22})[s_1s_2]\end{aligned}$$

$$[s_1s_2][s_2s_1s_2s_1s_2] = y_{32}[s_2] - \alpha_{32}[s_1s_2]$$

$$[s_2s_1]^2 = -\alpha_{21}\alpha_{31}\alpha_{32}y_{10}[s_1] + \alpha_{10}\alpha_{21}\alpha_{31}\alpha_{32}[s_2s_1]$$

$$[s_2s_1][s_1s_2s_1] = \alpha_{21}\alpha_{31}(y_{21} + y_{31})[s_1] - \alpha_{21}\alpha_{31}\alpha_{32}[s_2s_1]$$

$$\begin{aligned}[s_2s_1][s_2s_1s_2] = {} & -\alpha_{21}(y_{51} + y_{52}\{+\alpha_{31}y_{10}\})[1] + \alpha_{21}(\alpha_{10}y_{31} + \alpha_{32}y_{10})[s_1] \\ & + \alpha_{21}\alpha_{31}(y_{10} + y_{21})[s_2] - \alpha_{10}\alpha_{21}\alpha_{31}[s_2s_1]\end{aligned}$$

$$\begin{aligned}[s_2s_1][s_1s_2s_1s_2] = {} & (y_{62}\{+\alpha_{31}(y_{21} + y_{31})\})[1] - (\alpha_{31}y_{21} + \alpha_{10}(y_{31} + y_{41}))[s_1] \\ & - (\alpha_{31}y_{21} + \alpha_{32}y_{31})[s_2] + \alpha_{21}\alpha_{31}[s_2s_1]\end{aligned}$$

$$[s_2s_1][s_2s_1s_2s_1] = -\alpha_{21}(y_{31} + y_{52})[s_1] + \alpha_{21}\alpha_{31}(1 + y_{21})[s_2s_1]$$

$$[s_2s_1][s_1s_2s_1s_2s_1] = y_{63}[s_1] - \alpha_{21}(1 + y_{21} + y_{42})[s_2s_1]$$

$$[s_2s_1][s_2s_1s_2s_1s_2] = \{-y_{31}[1]\} + y_{31}[s_1] + y_{31}[s_2] - \alpha_{31}[s_2s_1]$$

$$\begin{aligned}[s_1s_2s_1]^2 = {}& -\alpha_{32}(y_{32} + y_{42}\{+\alpha_{11}(y_{11} + y_{21})\})[s_1] \\ & + (\alpha_{11}\alpha_{32}(y_{01} + y_{11} + y_{21}) + \alpha_{31}\alpha_{32}y_{11})[s_2s_1] - \alpha_{01}\alpha_{11}\alpha_{32}[s_1s_2s_1]\end{aligned}$$

$$\begin{aligned}[s_1s_2s_1][s_2s_1s_2] = {}& (1\{+\alpha_{11}(y_{11} + y_{22} + y_{33} + y_{31} + y_{42}) + \alpha_{31}(y_{21} + y_{32}) \\ & + \alpha_{32}y_{21}\})[1] - (\alpha_{11}(y_{21} + \alpha_{32}) + \alpha_{10}(y_{31} + y_{41} + y_{32} \\ & + y_{42}))[s_1] - (\alpha_{31}(y_{21} + y_{32}) + \alpha_{11}(y_{21} + y_{32} + y_{31} + \alpha_{42})[s_2] \\ & + \alpha_{11}\alpha_{32}[s_1s_2] + \alpha_{21}\alpha_{31}[s_2s_1]\end{aligned}$$

$$\begin{aligned}[s_1s_2s_1][s_1s_2s_1s_2] = {}& \{-(y_{33} + 2y_{43} + y_{53} + \alpha_{11}(y_{01} + y_{11}) + \alpha_{21}(y_{11} + y_{21}))[1]\} \\ & + (y_{33} + y_{43}\{+\alpha_{11}(y_{01} + y_{11}) + \alpha_{21}(y_{11} + y_{21})\})[s_1] \\ & ((y_{33} + y_{43} + y_{53})\{+\alpha_{11}(y_{01} + y_{11}) + \alpha_{21}(y_{11} + y_{21})\})[s_2] \\ & - \alpha_{11}(y_{01} + y_{11} + y_{22})[s_1s_2] - (\alpha_{11}(y_{01} + y_{11}) + \alpha_{21}(y_{11} \\ & + y_{21}))[s_2s_1] + \alpha_{01}\alpha_{11}[s_1s_2s_1]\end{aligned}$$

$$\begin{aligned}[s_1s_2s_1][s_2s_1s_2s_1] = {}& (y_{62}\{+\alpha_{32}y_{21}\})[s_1] - (\alpha_{31}y_{32} + \alpha_{32}(y_{11} + y_{21}))[s_2s_1] \\ & + \alpha_{11}\alpha_{32}[s_1s_2s_1]\end{aligned}$$

$$\begin{aligned}[s_1s_2s_1][s_1s_2s_1s_2s_1] = {}& \{-(y_{43} + y_{53})[s_1]\} + (y_{33} + y_{43} + y_{53})[s_2s_1] \\ & - \alpha_{11}(1 + y_{11} + y_{22})[s_1s_2s_1]\end{aligned}$$

$$\begin{aligned}[s_1s_2s_1][s_2s_1s_2s_1s_2] = {}& \{(y_{11} + y_{21})[1] - (y_{11} + y_{21})[s_1] - (y_{11} + y_{21})[s_2]\} \\ & + y_{11}[s_1s_2] + (y_{11} + y_{21})[s_2s_1] - \alpha_{11}[s_1s_2s_1]\end{aligned}$$

$$\begin{aligned}[s_2s_1s_2]^2 = {}& -\alpha_{21}(y_{21} + y_{42})[s_2] + (\alpha_{11}\alpha_{21}y_{31} + \alpha_{21}\alpha_{31}y_{10})[s_1s_2] \\ & - \alpha_{10}\alpha_{21}\alpha_{31}[s_2s_1s_2]\end{aligned}$$

$$[s_2s_1s_2][s_1s_2s_1s_2] = y_{53}[s_2] - (\alpha_{21}y_{31} + \alpha_{11}\alpha_{21}\alpha_{32}y_{21})[s_1s_2] + \alpha_{21}\alpha_{31}[s_2s_1s_2]$$

$$\begin{aligned}[s_2s_1s_2][s_2s_1s_2s_1] = {}& \{-(y_{51} + y_{52} + \alpha_{31}y_{10})[1]\} + (y_{41}\{+\alpha_{31}y_{10}\})[s_1] \\ & + (y_{42} + y_{52}\{+\alpha_{31}y_{10}\})[s_2] - (\alpha_{11}y_{31} + \alpha_{31}y_{10})[s_1s_2] \\ & - \alpha_{31}y_{10}[s_2s_1] + \alpha_{10}\alpha_{31}[s_2s_1s_2]\end{aligned}$$

$$\begin{aligned}[s_2s_1s_2][s_1s_2s_1s_2s_1] = {}& \{(y_{31} + y_{32} + y_{42})[1] - (y_{31} + y_{32})[s_1] \\ & - (y_{31} + y_{32} + y_{42})[s_2]\} + (y_{31} + y_{32})[s_1s_2] \\ & + y_{31}[s_2s_1] - \alpha_{31}[s_2s_1s_2]\end{aligned}$$

$$[s_2s_1s_2][s_2s_1s_2s_1s_2] = y_{31}[s_1s_2] - \alpha_{31}[s_2s_1s_2]$$

$$\begin{aligned}[s_1s_2s_1s_2]^2 = {}& \{-y_{43}[s_2]\} + (y_{32} + y_{42}\{+\alpha_{01}y_{21} + \alpha_{32}y_{11}\})[s_1s_2] \\ & - (\alpha_{01}(y_{11} + y_{21}) + \alpha_{31}(y_{01} + y_{11}))[s_2s_1s_2] + \alpha_{01}\alpha_{11}[s_1s_2s_1s_2]\end{aligned}$$

$$\begin{aligned}[s_1s_2s_1s_2][s_2s_1s_2s_1] = {}& \{(y_{21} + y_{31} + y_{32} + y_{42} + \alpha_{11})[1] - (y_{21} + y_{31} \\ & + y_{32} + \alpha_{11})[s_1] - (y_{21} + y_{31} + y_{32} + y_{42} + \alpha_{11})[s_2]\} \\ & + (y_{31} + y_{42}\{, +\alpha_{11}\})[s_1s_2] + (y_{21} + y_{31}\{+\alpha_{11}\})[s_2s_1] \\ & - \alpha_{11}[s_1s_2s_1] - \alpha_{31}[s_2s_1s_2]\end{aligned}$$

$$\begin{aligned}[s_1s_2s_1s_2][s_1s_2s_1s_2s_1] = {}& \{-(y_{01} + y_{11} + y_{21} + y_{22} + y_{32})[1] + (y_{01} + y_{11} \\ & + y_{21} + y_{22})[s_1] + (y_{01} + y_{11} + y_{21} + y_{22} + y_{32})[s_2] \\ & - (y_{01} + y_{11} + y_{21} + y_{22})[s_1s_2] - (y_{01} + y_{11} + y_{21})[s_2s_1]\} \\ & + y_{01}[s_1s_2s_1] + (y_{01} + y_{11} + y_{21})[s_2s_1s_2] - \alpha_{01}[s_1s_2s_1s_2]\end{aligned}$$

$$
\begin{aligned}
&[s_1s_2s_1s_2][s_2s_1s_2s_1s_2] = \{-y_{21}[s_1s_2]\} + (y_{11}+y_{21})[s_2s_1s_2] - \alpha_{11}[s_1s_2s_1s_2] \\
&[s_2s_1s_2s_1]^2 = \{-y_{52}[s_1] + (y_{42}+y_{52})[s_2s_1]\} - (\alpha_{11}y_{31}+\alpha_{31}y_{10})[s_1s_2s_1] \\
&\qquad + \alpha_{10}\alpha_{31}[s_2s_1s_2s_1] \\
&[s_2s_1s_2s_1][s_1s_2s_1s_2s_1] = \{y_{42}[s_1] - (y_{31}+y_{41})[s_2s_1]\} + (y_{31}+y_{32})[s_1s_2s_1] \\
&\qquad - \alpha_{31}[s_2s_1s_2s_1] \\
&[s_2s_1s_2s_1][s_2s_1s_2s_1s_2] = \{-y_{10}[1] + y_{10}[s_1] + y_{10}[s_2] - y_{10}[s_1s_2] - y_{10}[s_2s_1]\} \\
&\qquad + y_{10}[s_1s_2s_1] + y_{10}[s_2s_1s_2] - \alpha_{10}[s_2s_1s_2s_1] \\
&[s_1s_2s_1s_2s_1]^2 = \{-y_{32}[s_1] + (y_{22}+y_{32})[s_2s_1] - (y_{11}+y_{21}+y_{22})[s_1s_2s_1]\} \\
&\qquad + (y_{01}+y_{11}+y_{21})[s_2s_1s_2s_1] - \alpha_{01}[s_1s_2s_1s_2s_1] \\
&[s_1s_2s_1s_2s_1][s_2s_1s_2s_1s_2] = \{[1] - [s_1] - [s_2] + [s_1s_2] + [s_2s_1] \\
&\qquad - [s_1s_2s_1] - [s_2s_1s_2]\} + [s_1s_2s_1s_2] + [s_2s_1s_2s_1] \\
&[s_2s_1s_2s_1s_2]^2 = y_{10}[s_1s_2s_1s_2] - \alpha_{10}[s_2s_1s_2s_1s_2]
\end{aligned}
$$

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