# Kostka–Foulkes polynomials and Macdonald spherical functions

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**Abstract**

Generalized Hall–Littlewood polynomials (Macdonald spherical functions) and generalized Kostka–Foulkes polynomials ($q$-weight multiplicities) arise in many places in combinatorics, representation theory, geometry, and mathematical physics. This paper attempts to organize the different definitions of these objects and prove the fundamental combinatorial results from "scratch", in a presentation which, hopefully, will be accessible and useful for both the nonexpert and researchers currently working in this very active field. The combinatorics of the affine Hecke algebra plays a central role. The final section of this paper can be read independently of the rest of the paper. It presents, with proof, Lascoux and Schützenberger's positive formula for the Kostka–Foulkes poynomials in the type A case.

## 0 Introduction

The classical theory of Hall–Littlewood polynomials and the Kostka–Foulkes polynomials appears in the monograph of I.G. Macdonald [11]. The Hall–Littlewood polynomials form a basis of the ring of symmetric functions and the Kostka–Foulkes polynomials are the entries of the transition matrix between the Hall–Littlewood polynomials and the Schur functions.

This theory enters in many different places in algebra, geometry and combinatorics. Many of these connections appear in [11].

(a) [11, Ch. II] explains how this theory describes the structure of the Hall algebra of finite $\mathfrak{o}$-modules, where $\mathfrak{o}$ is a discrete valuation ring.

(b) [11, Ch. IV] explains how the Hall–Littlewood polynomials enter into the representation theory of $GL_n(\mathbb{F}_q)$ where $\mathbb{F}_q$ is a finite field with $q$ elements.

(c) [11, Ch, V] shows that the Hall–Littlewood polynomials arise as spherical functions for $GL_n(\mathbb{Q}_p)$ where $\mathbb{Q}_p$ is the field of $p$-adic numbers.

(d) [11, Ch. III §6 Ex. 6] explains how the Kostka–Foulkes polynomials relate to the intersection cohomology of unipotent orbit closures for $GL_n(\mathbb{C})$ and [11, Ch. III §8 Ex. 8] explains how the Kostka–Foulkes polynomials describe the graded decomposition of the representations of the symmetric groups $S_n$ on the cohomology of Springer fibers.

(e) [11, Ch. I App. A §8 and Ch. III §6] shows that the Kostka–Foulkes polynomials are $q$-analogues of the weight multiplicities for representations of $GL_n(\mathbb{C})$.

(f) [11, Ch. III (6.5)] explains how the Kostka–Foulkes polynomials encode a subtle statistic on column strict Young tableaux.

Macdonald [12, (4.1.2)] showed that there is a formula for the spherical functions for the Chevalley group $G(\mathbb{Q}_p)$ which generalizes the formula for Hall–Littlewood symmetric functions. This combinatorial formula is in terms of the root system data of the Chevalley group $G$. In [10] Lusztig showed that Macdonald's spherical function formula can be seen in terms of the affine Hecke algebra and that the "$q$-weight multiplicities" or generalized Kostka–Foulkes polynomials coming from these spherical functions are Kazhdan–Lusztig polynomials for the affine Weyl group. Kato [5] proved the "partition function formula" for the $q$-weight multiplicities which was conjectured by Lusztig. The partition function formula has led to continuing analysis of the connection between the $q$-weight multiplicities, functions on nilpotent orbits, filtrations of weight spaces by the kernels of powers of a regular nilpotent element, and degrees in harmonic polynomials (see [4] and the references there).

The connection between Hall–Littlewood polynomials and $\mathfrak{o}$-modules has seen generalizations in the theory of representations of quivers, the classical case being the case where the quiver is a loop consisting of one vertex and one edge. This theory has been generalized extensively by Ringel, Lusztig, Nakajima and many others and is developing quickly; fairly recent references are [15] and [16].

The connection to Springer representations of Weyl groups and the representations of Chevalley groups over finite fields has been developed extensively by Lusztig, Shoji and others; a good survey of the current theory is in [18] and the recent papers [19] show how this theory is beginning to extend its reach outside Lie theory into the realm of complex reflection groups.

Since the theory of Macdonald spherical functions (the generalization of Hall–Littlewood polynomials) and $q$-weight multiplicities (the generalization of Kostka–Foulkes polynomials) appears in so many important parts of mathematics it seems appropriate to give a survey of the basics of this theory. This paper is an attempt to collect together the fundamental combinatorial results analogous to those which are found for the type A case in [11]. The presentation here centers on the role played by the affine Hecke algebra. Hopefully this will help to illustrate how and why these objects arise naturally from a combinatorial point of view and, at the same time, provide enough underpinning to the algebra of the underlying algebraic groups to be useful to researchers in representation theory.

Using the terms *Hall–Littlewood polynomial* and *Macdonald spherical function* interchangeably, and using the words *Kostka–Foulkes polynomial* and $q$-

*weight multiplicity* interchangeably, the results that we prove in this paper are as follows.

(1) The interpretation of the Hall–Littlewood polynomials as elements of the affine Hecke algebra (via the Satake isomorphism).

(2) Macdonald's spherical function formula.

(3) The expansion of the Hall Littlewood polynomial in terms of the standard basis of the affine Hecke algebra.

(4) The triangularity of transition matrices between Macdonald spherical functions and other bases of symmetric functions.

(5) The straightening rules for Hall–Littlewood polynomials.

(6) The orthogonality of Macdonald spherical functions.

(7) The raising operator formula for Kostka–Foulkes polynomials.

(8) The partition function formula for $q$-weight multiplicities.

(9) The identification of the Kostka–Foulkes polynomial as a Kazhdan–Lusztig polynomial.

All of these results are proved here in general Lie type. They are all previously known, spread throughout various parts of the literature. The presentation here is a unified one; some of the proofs may (or may not) be new.

Section 4 is designed so that it can be read independently of the rest of the paper. In Section 4 we give the proof of Lascoux-Schützenberger's positive combinatorial formula [9] (see also [11, Ch. III (6.5)]) for Kostka–Foulkes polynomials in type A. Versions of this proof have appeared previously in [17] and in [2]. This proof has a reputation for being difficult and obscure. After finally getting the courage to attack the literature, we have found, in the end, that the proof is not so difficult after all. Hopefully we have been able to explain it so that others will also find it so.

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# 1 Weyl groups, affine Weyl groups, and the affine Hecke algebra

This section sets up the definitions and notations. Good references for this preliminary material are [1], [20] and [14].

## 1.1 The root system and the Weyl group

Let $\mathfrak{h}_\mathbb{R}^*$ be a real vector space with a nondegenerate symmetric bilinear form $\langle\ ,\ \rangle$. The basic data is a reduced irreducible root system $R$ (defined below) in $\mathfrak{h}_\mathbb{R}^*$. Associated to $R$ are the *weight lattice*

$$P = \{\lambda \in \mathfrak{h}_\mathbb{R}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in R\}, \quad \text{where} \quad \alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}, \tag{1.1}$$

and the *Weyl group* $W = \langle s_\alpha \mid \alpha \in R \rangle$ generated by the reflections

$$\begin{array}{rccc} s_\alpha : & \mathfrak{h}_\mathbb{R}^* & \longrightarrow & \mathfrak{h}_\mathbb{R}^* \\ & \lambda & \longmapsto & \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \end{array} \tag{1.2}$$

in the hyperplanes

$$H_\alpha = \{x \in \mathfrak{h}_\mathbb{R}^* \mid \langle x, \alpha^\vee \rangle = 0\}, \qquad \alpha \in R. \tag{1.3}$$

With these definitions $R$ is a reduced irreducible root system if it is a subset of $\mathfrak{h}_\mathbb{R}^*$ such that

(a) $R$ is finite, $0 \notin R$ and $\mathfrak{h}_\mathbb{R}^* = \mathbb{R}\text{-span}(R)$,

(b) $W$ permutes the elements of $R$, that is, $w\alpha \in R$ for $w \in W$ and $\alpha \in R$,

(c) $W$ is finite,

(d) $R \subseteq P$,

(e) if $\alpha \in R$ then the only other multiple of $\alpha$ in $R$ is $-\alpha$,

(f) $\mathfrak{h}_\mathbb{R}^*$ is an irreducible $W$-module.

The choice of a fundamental region $C$ for the action of $W$ on $\mathfrak{h}_\mathbb{R}^*$ is equivalent to a choice of *positive roots* $R^+$ of $R$,

$$R^+ = \{\alpha \in R \mid \langle x, \alpha^\vee \rangle > 0 \text{ for all } x \in C\}$$

and

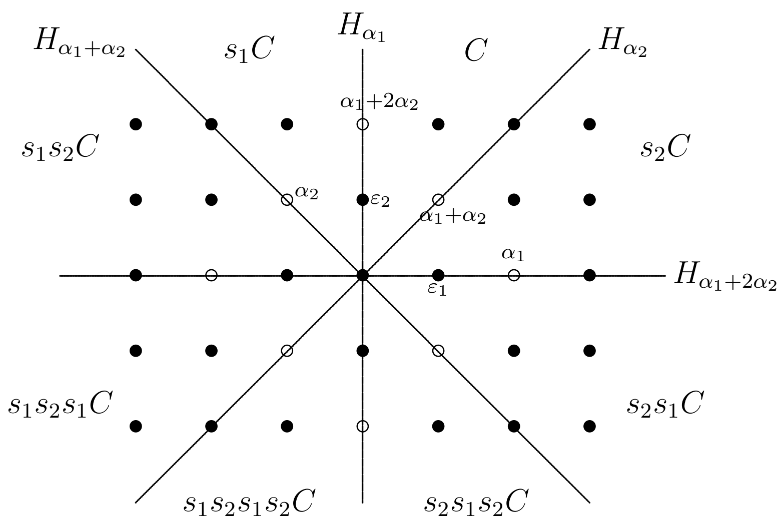
$$C = \{x \in \mathfrak{h}_\mathbb{R}^* \mid \langle x, \alpha^\vee \rangle > 0 \text{ for all } \alpha \in R^+\}.$$

**Example 1.1** If $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}^2$ with orthonormal basis $\varepsilon_1 = (1,0)$ and $\varepsilon_2 = (0,1)$, $P = \mathbb{Z}\text{-span}\{\varepsilon_1, \varepsilon_2\}$, and $W = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, s_1s_2s_1s_2\}$ is the dihedral group of order 8 generated by the reflections $s_1$ and $s_2$ in the hyperplanes $H_{\alpha_1}$ and $H_{\alpha_2}$, respectively, where

$$\begin{aligned} \alpha_1 &= 2\varepsilon_1, & \alpha_1^\vee &= \varepsilon_1, \\ \alpha_2 &= \varepsilon_2 - \varepsilon_1, & \alpha_2^\vee &= \alpha_2, \end{aligned}$$

then

$$R = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(\alpha_1+2\alpha_2)\}.$$




This is the root system of *type* $C_2$.

For each $\alpha \in R^+$ define the *raising operator* $R_\alpha\colon P \to P$ by $R_\alpha\mu = \mu + \alpha$. The *dominance order* on $P$ is given by

$$\mu \leqslant \lambda \qquad \text{if} \qquad \lambda = R_{\beta_1} \cdots R_{\beta_\ell}\mu \tag{1.4}$$

for some sequence of positive roots $\beta_1, \ldots, \beta_\ell \in R^+$.

The various fundamental chambers for the action of $W$ on $\mathfrak{h}_{\mathbb{R}}^*$ are the $w^{-1}C$, $w \in W$. The *inversion set* of an element $w \in W$ is

$$\begin{aligned} R(w) &= \{\alpha \in R^+ \mid H_\alpha \text{ is between } C \text{ and } w^{-1}C\}, \quad \text{and} \\ \ell(w) &= \mathrm{Card}(R(w)) \end{aligned} \tag{1.5}$$

is the *length* of $w$. If $R^- = -R^+ = \{-\alpha \mid \alpha \in R^+\}$ then

$$R = R^+ \cup R^- \quad \text{and} \quad R(w) = \{\alpha \in R^+ \mid w\alpha \in R^-\}, \qquad \text{for } w \in W.$$

The weight lattice, the set of *dominant integral weights*, and the set of *strictly dominant integral weights*, are

$$\begin{aligned} P &= \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in R\}, \\ P^+ = P \cap \overline{C} &= \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geqslant 0} \text{ for all } \alpha \in R^+\}, \\ P^{++} = P \cap C &= \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{>0} \text{ for all } \alpha \in R^+\}, \end{aligned} \tag{1.6}$$

where $\overline{C} = \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle x, \alpha^\vee \rangle \geqslant 0 \text{ for all } \alpha \in R^+\}$ is the closure of the fundamental chamber $C$.

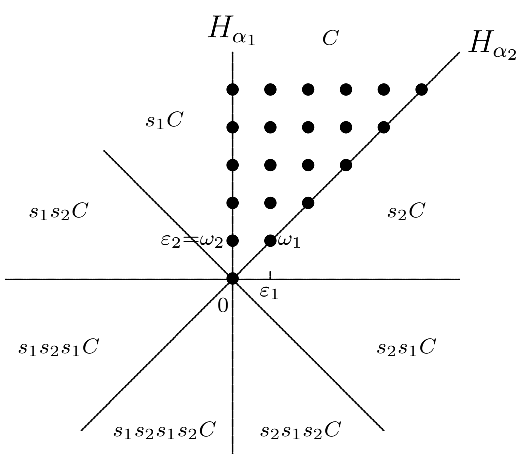
The *simple roots* are the positive roots $\alpha_1, \ldots, \alpha_n$ such that the hyperplanes $H_{\alpha_i}$, $1 \leqslant i \leqslant n$, are the *walls* of $C$. The *fundamental weights*, $\omega_1, \ldots, \omega_n \in P$, are given by $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$, $1 \leqslant i, j \leqslant n$, and

$$P = \sum_{i=1}^n \mathbb{Z}\omega_i, \qquad P^+ = \sum_{i=1}^n \mathbb{Z}_{\geqslant 0}\omega_i, \quad \text{and} \quad P^{++} = \sum_{i=1}^n \mathbb{Z}_{>0}\omega_i. \tag{1.7}$$

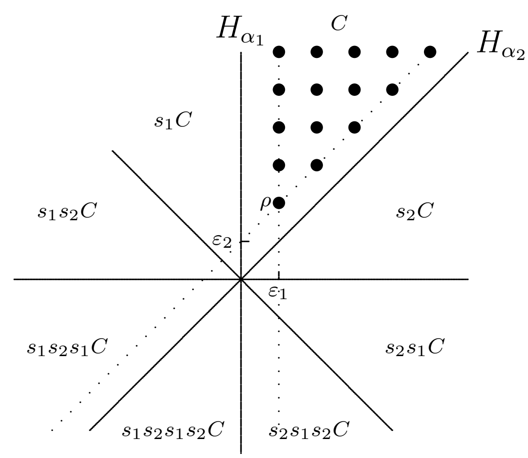
The set $P^+$ is an integral cone with vertex 0, the set $P^{++}$ is a integral cone with vertex

$$\rho = \sum_{i=1}^n \omega_i = \tfrac{1}{2} \sum_{\alpha \in R^+} \alpha, \quad \text{and the map} \quad \begin{array}{ccc} P^+ & \longrightarrow & P^{++} \\ \lambda & \longmapsto & \lambda + \rho \end{array} \tag{1.8}$$

is a bijection (see Proposition 2.3). In Example 1.1, with the root system of type $C_2$, the picture is




The set $P^+$




The set $P^{++}$

The *simple reflections* are $s_i = s_{\alpha_i}$, for $1 \leqslant i \leqslant n$. The Weyl group $W$ has a presentation by generators $s_1, \ldots, s_n$ and relations

$$\begin{aligned} s_i^2 &= 1, & &\text{for} \quad 1 \leqslant i \leqslant n, \\ \underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ factors}} &= \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ factors}}, & & i \neq j, \end{aligned} \tag{1.9}$$

where $\pi/m_{ij}$ is the angle between the hyperplanes $H_{\alpha_i}$ and $H_{\alpha_j}$. A *reduced word* for $w \in W$ is an expression $w = s_{i_1} \cdots s_{i_p}$ for $w$ as a product of simple reflections which has $p$ minimal. The following lemma describes the inversion set in terms of the simple roots and the simple reflections and shows that if $w = s_{i_1} \cdots s_{i_p}$ is a reduced expression for $w$ then $p = \ell(w)$.

**Lemma 1.2** ([1, VI §1 no. 6 Cor. 2 to Prop. 17]) *Let $w = s_{i_1} \cdots s_{i_p}$ be a reduced word for $w$. Then*

$$R(w) = \{\alpha_{i_p}, s_{i_p}\alpha_{i_{p-1}}, \ldots, s_{i_p} \cdots s_{i_2}\alpha_{i_1}\}.$$

The *Bruhat order*, or *Bruhat-Chevalley order* (see [20, §8 App., p. 126]), is the partial order on $W$ such that $v \leqslant w$ if there is a reduced word for $v$, $v = s_{j_1} \cdots s_{j_k}$, which is a subword of a reduced word for $w$, $w = s_{i_1} \cdots s_{i_p}$, (that is, $s_{j_1}, \ldots, s_{j_k}$ is a subsequence of the sequence $s_{i_1}, \ldots, s_{i_p}$).

## 1.2 The affine Weyl group

For $\lambda \in P$, the *translation in $\lambda$* is

$$\begin{array}{rccc} t_\lambda : & \mathfrak{h}_{\mathbb{R}}^* & \longrightarrow & \mathfrak{h}_{\mathbb{R}}^* \\ & x & \longmapsto & x + \lambda. \end{array} \tag{1.10}$$

The *extended affine Weyl group* $\tilde{W}$ is the group

$$\tilde{W} = \{wt_\lambda \mid w \in W, \lambda \in P\}, \tag{1.11}$$

with multiplication determined by the relations

$$t_\lambda t_\mu = t_{\lambda+\mu}, \quad \text{and} \quad t_{w\lambda}w = wt_\lambda, \tag{1.12}$$

for $\lambda, \mu \in P$ and $w \in W$, and so $\tilde{W}$ is a semidirect product of $W$ and the group of translations $\{t_\lambda \mid \lambda \in P\}$. It is the group of transformations of $\mathfrak{h}_{\mathbb{R}}^*$ generated by the $s_\alpha$, $\alpha \in R^+$, and $t_\lambda$, $\lambda \in P$. The *affine Weyl group* $W_{\text{aff}}$ is the subgroup of $\tilde{W}$ generated by the reflections

$$\begin{aligned} &s_{\alpha,k} \colon \mathfrak{h}_{\mathbb{R}}^* \to \mathfrak{h}_{\mathbb{R}}^* \text{ in the hyperplanes} \\ &H_{\alpha,k} = \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle x, \alpha^\vee \rangle = k\}, \quad \alpha \in R^+, k \in \mathbb{Z}. \end{aligned} \tag{1.13}$$

The reflections $s_{\alpha,k}$ can be written as elements of $\tilde{W}$ via the formula

$$s_{\alpha,k} = t_{k\alpha}s_\alpha = s_\alpha t_{-k\alpha}. \tag{1.14}$$

The *highest short root* of $R$ is the unique element $\varphi \in R^+$ such that the *fundamental alcove*

$$A = C \cap \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle x, \varphi^\vee \rangle < 1\} \tag{1.15}$$

is a fundamental region for the action of $W_{\mathrm{aff}}$ on $\mathfrak{h}_{\mathbb{R}}^*$. The various fundamental chambers for the action of $W_{\mathrm{aff}}$ on $\mathfrak{h}_{\mathbb{R}}^*$ are $\tilde{w}^{-1}A$, $\tilde{w} \in W_{\mathrm{aff}}$. The *inversion set* of $\tilde{w} \in \tilde{W}$ is

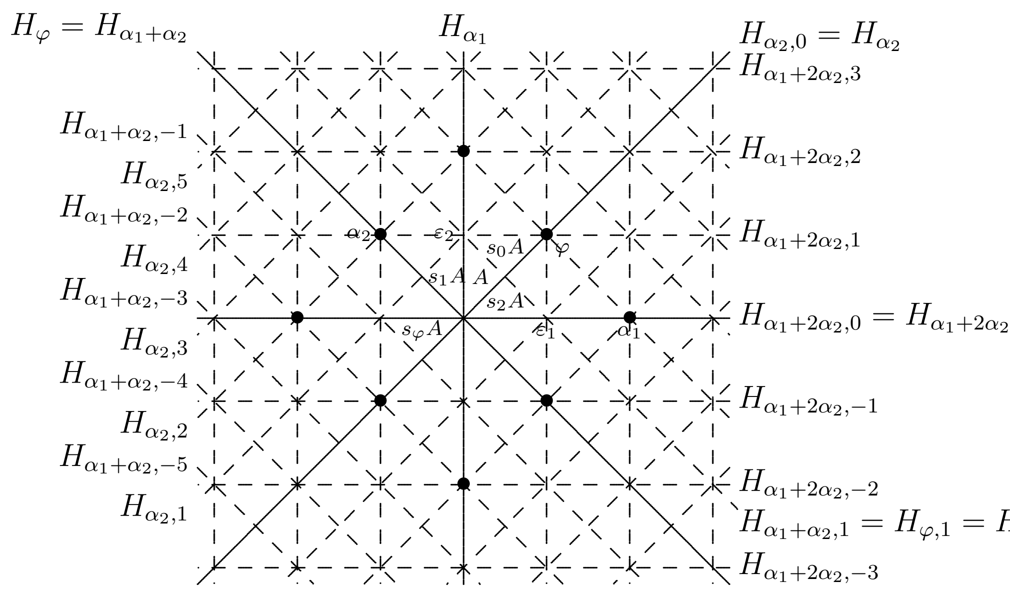
$$R(\tilde{w}) = \{H_{\alpha,k} \mid H_{\alpha,k} \text{ is between } A \text{ and } \tilde{w}^{-1}A\} \quad \text{and} \quad \ell(\tilde{w}) = \mathrm{Card}(R(\tilde{w}))$$

is the *length* of $\tilde{w}$. If $w \in W$ and $\lambda \in P$ then

$$\ell(wt_\lambda) = \sum_{\alpha \in R^+} |\langle \lambda, \alpha^\vee \rangle + \chi(w\alpha)|, \tag{1.16}$$

where, for a root $\beta \in R$, set $\chi(\beta) = 0$, if $\beta \in R^+$, and $\chi(\beta) = 1$, if $\beta \in R^-$.

Continuing Example 1.1, we have the picture




Let

$$H_{\alpha_0} = H_{\varphi,1} \quad \text{and} \quad s_0 = s_{\varphi,1} = t_\phi s_\phi = s_\phi t_{-\phi}, \tag{1.17}$$

and let $H_{\alpha_1}, \ldots, H_{\alpha_n}$ and $s_1, \ldots, s_n$ be as in (1.9). Then the walls of $A$ are the hyperplanes $H_{\alpha_0}, H_{\alpha_1}, \ldots, H_{\alpha_n}$ and the group $W_{\mathrm{aff}}$ has a presentation by generators $s_0, s_1, \ldots, s_n$ and relations

$$\begin{array}{rcll} s_i^2 &=& 1, & \text{for} \quad 0 \leqslant i \leqslant n, \\ \underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ factors}} &=& \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ factors}}, & \qquad i \neq j, \end{array} \tag{1.18}$$

where $\pi/m_{ij}$ is the angle between the hyperplanes $H_{\alpha_i}$ and $H_{\alpha_j}$.

Let $w_0$ be the longest element of $W$ and let $w_i$ be the longest element of the subgroup $W_{\omega_i} = \{w \in W \mid w\omega_i = \omega_i\}$. Let $\varphi^\vee = c_1\alpha_1^\vee + \cdots + c_n\alpha_n^\vee$. Then let

$$\Omega = \{g \in \tilde{W} \mid \ell(g) = 0\} = \{1\} \cup \{g_i \mid c_i = 1\}, \quad \text{where} \quad g_i = t_{\omega_i} w_i w_0, \tag{1.19}$$

(see [1, VI §2 no. 3 Prop. 6]). Each element $g \in \Omega$ sends the alcove $A$ to itself and thus permutes the walls $H_{\alpha_0}, H_{\alpha_1}, \ldots, H_{\alpha_n}$ of $A$. Denote the resulting permutation of $\{0, 1, \ldots, n\}$ also by $g$. Then

$$g s_i g^{-1} = s_{g(i)}, \quad \text{for } 0 \leqslant i \leqslant n, \tag{1.20}$$

and the group $\tilde{W}$ is presented by the generators $s_0, s_1, \ldots, s_n$ and $g \in \Omega$ with the relations (1.18) and (1.20). In the setting of Example 1.1, $W_{\omega_1} = \{1, s_2\}$, $W_{\omega_2} = \{1, s_1\}$, $w_1 = s_2$, $w_2 = s_1$ and $w_0 = s_1 s_2 s_1 s_2$, and $\varphi^\vee = 2\alpha_1^\vee + \alpha_2^\vee$ so that $c_1 = 2$, $c_2 = 1$ and $\Omega = \{1, g_2\} \cong \mathbb{Z}/2\mathbb{Z}$, where $g_2 = t_{\omega_2} s_2 s_1 s_2$.

## 1.3 The affine Hecke algebra

Let $q$ be an indeterminate and let $\mathbb{K} = \mathbb{Z}[q, q^{-1}]$. The affine Hecke algebra $\tilde{H}$ is the algebra over $\mathbb{K}$ given by generators $T_i$, $1 \leqslant i \leqslant n$, and $x^\lambda$, $\lambda \in P$, and relations

$$\begin{aligned}
&\underbrace{T_i T_j T_i \cdots}_{m_{ij}\ \text{factors}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij}\ \text{factors}}, && \text{for all } i \neq j, \\
&T_i^2 = (q - q^{-1}) T_i + 1, && \text{for all } 1 \leqslant i \leqslant n, \\
&x^\lambda x^\mu = x^\mu x^\lambda = x^{\lambda+\mu}, && \text{for all } \lambda, \mu \in P, \\
&x^\lambda T_i = T_i x^{s_i\lambda} + (q - q^{-1}) \frac{x^\lambda - x^{s_i\lambda}}{1 - x^{-\alpha_i}}, && \text{for all } 1 \leqslant i \leqslant n,\ \lambda \in P.
\end{aligned} \tag{1.21}$$

An alternative presentation of $\tilde{H}$ is by the generators $T_w$, $w \in \tilde{W}$, and relations

$$T_{w_1} T_{w_2} = T_{w_1 w_2}, \qquad \text{if } \ell(w_1 w_2) = \ell(w_1) + \ell(w_2),$$

$$T_{s_i} T_w = (q - q^{-1}) T_w + T_{s_i w}, \quad \text{if } \ell(s_i w) < \ell(w) \quad (0 \leqslant i \leqslant n).$$

With notations as in (1.10–1.20) the conversion between the two presentations is given by the relations

$$\begin{aligned}
T_w &= T_{i_1} \cdots T_{i_p}, && \text{if } w \in W_{\text{aff}} \text{ and } w = s_{i_1} \cdots s_{i_p} \text{ is a reduced word}, \\
T_{g_i} &= x^{\omega_i} T_{w_0 w_i}^{-1}, && \text{for } g_i \in \Omega \text{ as in (1.19)}, \\
x^\lambda &= T_{t_\mu} T_{t_\nu}^{-1}, && \text{if } \lambda = \mu - \nu \text{ with } \mu, \nu \in P^+, \\
T_{s_0} &= T_{s_\phi} x^{-\phi}, && \text{where } \phi \text{ is the highest short root of } R,
\end{aligned} \tag{1.22}$$

### 1.4 The Kazhdan–Lusztig basis

The algebra $\tilde{H}$ has bases

$$\{x^\lambda T_w \mid w \in W, \lambda \in P\} \quad \text{and} \quad \{T_w x^\lambda \mid w \in W, \lambda \in P\}.$$

The Kazhdan–Lusztig basis $\{C'_w \mid w \in \tilde{W}\}$ is another basis of $\tilde{H}$ which plays an important role. It is defined as follows.

The *bar involution* on $\tilde{H}$ is the $\mathbb{Z}$-linear automorphism $\overline{\phantom{x}}\colon \tilde{H} \to \tilde{H}$ given by

$$\overline{q} = q^{-1} \qquad \text{and} \qquad \overline{T_w} = T_{w^{-1}}^{-1}, \quad \text{for } w \in \tilde{W}.$$

For $0 \leqslant i \leqslant n$, $\overline{T_i} = T_i^{-1} = T_i - (q - q^{-1})$ and the bar involution is a $\mathbb{Z}$-algebra automorphism of $\tilde{H}$. If $w = s_{i_1} \cdots s_{i_p}$ is a reduced word for $w$ then, by the definition of the Bruhat order (defined after Lemma 1.2),

$$\begin{aligned} \overline{T_w} &= \overline{T_{i_1} \cdots T_{i_p}} = \overline{T_{i_1}} \cdots \overline{T_{i_p}} = T_{i_1}^{-1} \cdots T_{i_p}^{-1} \\ &= (T_{i_1} - (q - q^{-1})) \cdots (T_{i_p} - (q - q^{-1})) = T_w + \sum_{v < w} a_{vw} T_v, \end{aligned}$$

with $a_{vw} \in \mathbb{Z}[(q - q^{-1})]$.

Setting $\tau_i = qT_i$ and $t = q^2$, the second relation in (1.21)

$$T_i^2 = (q - q^{-1})T_i + 1 \quad \text{becomes} \quad \tau_i^2 = (t-1)\tau_i + t. \tag{1.23}$$

Let $\tau_w = q^{\ell(w)} T_w$ for $w \in \tilde{W}$. The *Kazhdan–Lusztig basis* $\{C'_w \mid \tilde{w} \in \tilde{W}\}$ of $\tilde{H}$ is defined [6] by

$$\overline{C}'_w = C'_w \quad \text{and} \quad C'_w = t^{-\ell(w)/2} \left( \sum_{y \leqslant w} P_{yw} \tau_y \right), \tag{1.24}$$

subject to $P_{yw} \in \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$, $P_{ww} = 1$, and $\deg_t(P_{yw}) \leqslant \frac{1}{2}(\ell(w) - \ell(y) - 1)$. If

$$p_{yw} = q^{-(\ell(w) - \ell(y))} P_{yw} \tag{1.25}$$

then

$$C'_w = q^{-\ell(w)} \sum_{y \leqslant w} P_{yw} q^{\ell(y)} T_y = \sum_{y \leqslant w} P_{yw} q^{-(\ell(w) - \ell(y))} T_y = \sum_{y \leqslant w} p_{yw} T_y, \tag{1.26}$$

with

$$p_{yw} \in \mathbb{Z}[q, q^{-1}], \quad p_{ww} = 1, \quad \text{and} \quad p_{yw} \in q^{-1}\mathbb{Z}[q^{-1}], \tag{1.27}$$

since $\deg_q(P_{yw}(q) q^{-(\ell(w) - \ell(y))}) \leqslant \ell(w) - \ell(y) - 1 - (\ell(w) - \ell(y)) = -1$. The following proposition establishes the existence and uniqueness of the $C'_w$ and the $p_{yw}$.

**Proposition 1.3** *Let $(\tilde{W}, \leqslant)$ be a partially ordered set such that for any $u, v \in \tilde{W}$ the interval $[u, v] = \{z \in \tilde{W} \mid u \leqslant z \leqslant v\}$ is finite. Let $M$ be a free $\mathbb{Z}[q, q^{-1}]$-module with basis $\{T_w \mid w \in \tilde{W}\}$ and with a $\mathbb{Z}$-linear involution $\overline{\phantom{x}}: M \to M$ such that*

$$\overline{q} = q^{-1} \quad \text{and} \quad \overline{T_w} = T_w + \sum_{v<w} a_{vw} T_v.$$

*Then there is a unique basis $\{C'_w \mid w \in \tilde{W}\}$ of $M$ such that*

(a) $\overline{C'_w} = C'_w$,

(b) $C'_w = T_w + \sum_{v<w} p_{vw} T_v$, *with $p_{vw} \in q^{-1}\mathbb{Z}[q^{-1}]$ for $v < w$.*

**Proof** The $p_{vw}$ are determined by induction as follows. Fix $v, w \in W$ with $v \leq w$. If $v = w$ then $p_{vw} = p_{ww} = 1$. For the induction step assume that $v < w$ and that $p_{zw}$ are known for all $v < z \leqslant w$.

The matrices $A = (a_{vw})$ and $P = (p_{vw})$ are upper triangular with 1's on the diagonal. The equations

$$\begin{aligned} T_w = \overline{\overline{T_w}} &= \sum_v \overline{a_{vw} T_v} = \sum_{u,v} a_{uv} \overline{a_{vw}} T_u \qquad \text{and} \\ \sum_u p_{uw} T_u = C'_w &= \overline{C'_w} = \sum_v \overline{p_{vw} T_v} = \sum_{u,v} \overline{p_{vw}} a_{uv} T_u, \end{aligned}$$

imply $A\overline{A} = \mathrm{Id}$ and $P = A\overline{P}$. Then

$$f = \sum_{u<z\leqslant w} a_{uz} \overline{p}_{zw} = ((A-1)\overline{P})_{uw} = (A\overline{P} - \overline{P})_{uw} = (P - \overline{P})_{uw} = p_{uw} - \overline{p}_{uw},$$

is a known element of $\mathbb{Z}[q, q^{-1}]$;

$$f = \sum_{k\in\mathbb{Z}} f_k q^k \quad \text{such that} \quad \overline{f} = \overline{(p_{uw} - \overline{p}_{uw})} = \overline{p}_{uw} - p_{uw} = -f.$$

Hence $f_k = -f_{-k}$ for all $k \in \mathbb{Z}$ and $p_{uw}$ is given by $p_{uw} = \sum_{k\in\mathbb{Z}_{<0}} f_k q^k$. □

The *finite Hecke algebra* $H$ and the *group algebra of* $P$ are the subalgebras of $\tilde{H}$ given, respectively, by

$$\begin{aligned} H &= (\text{subalgebra of } \tilde{H} \text{ generated by } T_1, \ldots, T_n), \\ \mathbb{K}[P] &= \mathbb{K}\text{-span } \{x^\lambda \mid \lambda \in P\}, \qquad \text{where } \mathbb{K} = \mathbb{Z}[q, q^{-1}], \end{aligned} \tag{1.28}$$

and $\mathbb{K}$-span$\{x^\lambda \mid \lambda \in P\}$ denotes the set of $\mathbb{K}$-linear combinations of elements $x^\lambda$ in $\tilde{H}$. The Weyl group $W$ acts on $\mathbb{K}[P]$ by

$$wf = \sum_{\mu\in P} c_\mu x^{w\mu}, \quad \text{for } w \in W \text{ and } f = \sum_{\mu\in P} c_\mu x^\mu \in \mathbb{K}[P]. \tag{1.29}$$

**Theorem 1.4** *The center of the affine Hecke algebra is the ring*

$$Z(\tilde{H}) = \mathbb{K}[P]^W = \{f \in \mathbb{K}[P] \mid wf = f \text{ for all } w \in W\}$$

*of symmetric functions in* $\mathbb{K}[P]$.

**Proof** If $z \in \mathbb{K}[P]^W$ then by the fourth relation in (1.21), $T_i z = (s_i z)T_i + (q - q^{-1})(1 - x^{-\alpha_i})^{-1}(z - s_i z) = zT_i + 0$, for $1 \leqslant i \leqslant n$, and by the third relation in (1.21), $zx^\lambda = x^\lambda z$, for all $\lambda \in P$. Thus $z$ commutes with all the generators of $\tilde{H}$ and so $z \in Z(\tilde{H})$.

Assume

$$z = \sum_{\lambda \in P, w \in W} c_{\lambda,w} x^\lambda T_w \in Z(\tilde{H}).$$

Let $m \in W$ be maximal in Bruhat order subject to $c_{\gamma,m} \neq 0$ for some $\gamma \in P$. If $m \neq 1$ there exists a dominant $\mu \in P$ such that $c_{\gamma+\mu-m\mu,m} = 0$ (otherwise $c_{\gamma+\mu-m\mu,m} \neq 0$ for every dominant $\mu \in P$, which is impossible since $z$ is a finite linear combination of $x^\lambda T_w$). Since $z \in Z(\tilde{H})$ we have

$$z = x^{-\mu} z x^\mu = \sum_{\lambda \in P, w \in W} c_{\lambda,w} x^{\lambda-\mu} T_w x^\mu.$$

Repeated use of the fourth relation in (1.21) yields

$$T_w x^\mu = \sum_{\nu \in P, v \in W} d_{\nu,v} x^\nu T_v$$

where $d_{\nu,v}$ are constants such that $d_{w\mu,w} = 1$, $d_{\nu,w} = 0$ for $\nu \neq w\mu$, and $d_{\nu,v} = 0$ unless $v \leqslant w$. So

$$z = \sum_{\lambda \in P, w \in W} c_{\lambda,w} x^\lambda T_w = \sum_{\lambda \in P, w \in W} \sum_{\nu \in P, v \in W} c_{\lambda,w} d_{\nu,v} x^{\lambda-\mu+\nu} T_v$$

and comparing the coefficients of $x^\gamma T_m$ gives $c_{\gamma,m} = c_{\gamma+\mu-m\mu,m} d_{m\mu,m}$. Since $c_{\gamma+\mu-m\mu,m} = 0$ it follows that $c_{\gamma,m} = 0$, which is a contradiction. Hence $z = \sum_{\lambda \in P} c_\lambda x^\lambda \in \mathbb{K}[P]$.

The fourth relation in (1.21) gives

$$zT_i = T_i z = (s_i z)T_i + (q - q^{-1})z'$$

where $z' \in \mathbb{K}[P]$. Comparing coefficients of $x^\lambda$ on both sides yields $z' = 0$. Hence $zT_i = (s_i z)T_i$, and therefore $z = s_i z$ for $1 \leqslant i \leqslant n$. So $z \in \mathbb{K}[P]^W$.

□

## 2 Symmetric and alternating functions and their q-analogues

Let $\mathbf{1}_0$ and $\varepsilon_0$ be the elements of the finite Hecke algebra $H$ which are determined by

$$\begin{array}{lll} \mathbf{1}_0^2 = \mathbf{1}_0 & \text{and} \quad T_i\mathbf{1}_0 = q\mathbf{1}_0, & \text{for all } 1 \leqslant i \leqslant n, \\ \varepsilon_0^2 = \varepsilon_0 & \text{and} \quad T_i\varepsilon_0 = (-q^{-1})\varepsilon_0, & \text{for all } 1 \leqslant i \leqslant n. \end{array}$$

In terms of the basis $\{T_w \mid w \in W\}$ of $H$ these elements have the explicit formulae

$$\mathbf{1}_0 = \frac{1}{W_0(q^2)} \sum_{w\in W} q^{\ell(w)} T_w\,, \quad \text{and} \quad \varepsilon_0 = \frac{1}{W_0(q^{-2})} \sum_{w \in W} (-q)^{-\ell(w)} T_w\,, \qquad (2.1)$$

where $W_0(t) = \sum_{w\in W} t^{\ell(w)}$. (To define these elements one should adjoin the element $W_0(q^2)^{-1}$ to $\mathbb{K}$ or to $H$.) The elements $\mathbf{1}_0$ and $\varepsilon_0$ are $q$-analogues of the elements in the group algebra of $W$ given by

$$\mathbf{1} = \frac{1}{|W|} \sum_{w\in W} w \quad \text{and} \quad \varepsilon = \frac{1}{|W|} \sum_{w\in W} (-1)^{\ell(w)} w, \qquad (2.2)$$

and the vector spaces $\mathbf{1}_0\tilde{H}\mathbf{1}_0$ and and $\varepsilon_0\tilde{H}\mathbf{1}_0$ are $q$-analogues of the vector spaces (more precisely, free $\mathbb{K} = \mathbb{Z}[q, q^{-1}]$-modules) of *symmetric functions* and *alternating functions*,

$$\begin{array}{rcl} \mathbb{K}[P]^W & = & \{f \in \mathbb{K}[P] \mid wf = f \text{ for all } w \in W\} = \mathbf{1}\,\mathbb{K}[P], \\ \mathcal{A} & = & \{f \in \mathbb{K}[P] \mid wf = (-1)^{\ell(w)} f \text{ for all } w \in W\} = \varepsilon\,\mathbb{K}[P], \end{array} \qquad (2.3)$$

respectively, where the action of $W$ on $\mathbb{K}[P]$ is as defined in 1.29.

For $\mu \in P$ let the orbit $W\mu$ and the stabilizer $W_\mu$ of $\mu$ be defined by

$$W\mu = \{w\mu \mid w \in W\} \qquad \text{and} \qquad W_\mu = \{w \in W \mid w\mu = \mu\}.$$

Then define

$$\begin{array}{ll} m_\mu = \displaystyle\sum_{\gamma\in W\mu} x^\gamma = \frac{|W|}{|W_\mu|}\,\mathbf{1}\,x^\mu\,, & a_\mu = \displaystyle\sum_{w\in W} (-1)^{\ell(w)} w x^\mu = |W|\,\varepsilon\,x^\mu\,, \\ M_\mu = \mathbf{1}_0 x^\mu \mathbf{1}_0\,, & A_\mu = \varepsilon_0 x^\mu \mathbf{1}_0\,. \end{array} \qquad (2.4)$$

Theorem 2.2 below shows that the elements in (2.4) which are indexed by elements of $P^+$ and $P^{++}$ form bases (over $\mathbb{K}$) of $\mathbb{K}[P]^W$, $\mathcal{A}$, $\mathbf{1}_0\tilde{H}\mathbf{1}_0$, and $\varepsilon_0\tilde{H}\mathbf{1}_0$. This will be a consequence of the following *straightening rules*. The straightening law for the $M_\mu$ given in the following Proposition is a generalization of [11, III §2 Ex. 2].

**Proposition 2.1** *For $\gamma \in P$ let $m_\gamma$, $a_\gamma$, $M_\gamma$, and $A_\gamma$ be as defined in (2.4). Let $\alpha_i$ be a simple root and let $\mu \in P$ be such that $d = \langle \mu, \alpha_i^\vee \rangle \geqslant 0$. Then*

$$m_{s_i\mu} = m_\mu, \qquad a_{s_i\mu} = -a_\mu, \qquad \text{and} \qquad A_{s_i\mu} = -A_\mu.$$

*Letting $t = q^{-2}$, $M_\mu = M_{s_i\mu}$ if $d = 0$, and if $d > 0$ then*

$$\begin{aligned} M_{s_i\mu} &= tM_\mu + \left( \sum_{j=1}^{\lfloor d/2-1 \rfloor} (t^2-1)t^{j-1} M_{\mu - j\alpha_i} \right) \\ &\quad + \begin{cases} (t-1)t^{d/2-1} M_{\mu-(d/2)\alpha_i}, & \text{if } d \text{ is even,} \\ 0, & \text{if } d \text{ is odd.} \end{cases} \end{aligned}$$

**Proof** The first two equalities follow from the definitions of $m_\lambda$ and $a_\mu$ and the fact that $\ell(s_i) = 1$.

Let $\mu \in P$ such that $d = \langle \mu, \alpha_i^\vee \rangle \geqslant 0$. Since $x^\mu + x^{s_i\mu}$ is in the center of the tiny little affine Hecke algebra generated by $T_i$ and the $x^\gamma$, $\gamma \in P$,

$$\begin{aligned} A_\mu + A_{s_i\mu} &= \varepsilon_0 (x^\mu + x^{s_i\mu}) \mathbf{1}_0 = q^{-1} \varepsilon_0 (x^\mu + x^{s_i\mu}) T_i \mathbf{1}_0 \\ &= q^{-1} \varepsilon_0 T_i (x^\mu + x^{s_i\mu}) \mathbf{1}_0 = -q^{-2} \varepsilon_0 (x^\mu + x^{s_i\mu}) \mathbf{1}_0 \\ &= -q^{-2} (A_\mu + A_{s_i\mu}). \end{aligned}$$

Thus $A_\mu + A_{s_i\mu} = 0$ which establishes the third statement.

If $d = 0$ then, by definition, $M_\mu = M_{s_i\mu}$. If $d > 0$ then multiplying the fourth relation in (1.21) by $\mathbf{1}_0$ on both the left and the right (and then multiplying by $q^{-1}$) gives

$$\mathbf{1}_0 (x^{s_i\mu} - x^\mu) \mathbf{1}_0 = q^{-1}(q - q^{-1}) \mathbf{1}_0 \left( \frac{x^{s_i\mu} - x^\mu}{1 - x^{-\alpha_i}} \right) \mathbf{1}_0.$$

Subtracting the same relation with $\mu$ replaced by $\mu - \alpha_i$ gives

$$\begin{aligned} &\mathbf{1}_0 (x^{s_i\mu} - x^\mu) \mathbf{1}_0 - \mathbf{1}_0 (x^{s_i\mu + \alpha_i} - x^{\mu - \alpha_i}) \mathbf{1}_0 \\ &\quad = (1 - q^{-2}) \mathbf{1}_0 \left( \frac{x^{s_i\mu} - x^\mu - x^{s_i\mu+\alpha_i} + x^{\mu-\alpha_i}}{1 - x^{-\alpha_i}} \right) \mathbf{1}_0 \\ &\quad = (1 - q^{-2}) \mathbf{1}_0 (-x^{s_i\mu + \alpha_i} - x^\mu) \mathbf{1}_0. \end{aligned}$$

So

$$\mathbf{1}_0 x^{s_i\mu} \mathbf{1}_0 = q^{-2} \mathbf{1}_0 x^\mu \mathbf{1}_0 - \mathbf{1}_0 x^{\mu - \alpha_i} \mathbf{1}_0 + q^{-2} \mathbf{1}_0 x^{s_i\mu + \alpha_i} \mathbf{1}_0.$$

Inductively applying this relation yields the result. The first cases are

$$M_{s_i\mu} = \begin{cases} M_\mu, & \text{if } \langle \mu, \alpha_i^\vee \rangle = 0, \\ q^{-2} M_\mu, & \text{if } \langle \mu, \alpha_i^\vee \rangle = 1, \\ q^{-2} M_\mu + (q^{-2} - 1) M_{\mu - \alpha_i}, & \text{if } \langle \mu, \alpha_i^\vee \rangle = 2, \\ q^{-2} M_\mu + (q^{-4} - 1) M_{\mu - \alpha_i}, & \text{if } \langle \mu, \alpha_i^\vee \rangle = 3, \\ q^{-2} M_\mu + (q^{-4} - 1) M_{\mu - \alpha_i} + q^{-2}(q^{-2} - 1) M_{\mu - 2\alpha_i}, & \text{if } \langle \mu, \alpha_i^\vee \rangle = 4. \end{cases}$$

□

Proposition 2.1 implies that, for all $\mu \in P$ and $w \in W$,

$$m_{w\mu} = m_\mu, \quad a_{w\mu} = (-1)^{\ell(w)} a_\mu, \quad \text{and} \quad A_{w\mu} = (-1)^{\ell(w)} A_\mu. \tag{2.5}$$

**Theorem 2.2** *Let $\mathbb{K} = \mathbb{Z}[q, q^{-1}]$. As free $\mathbb{K}$-modules*

$$\begin{array}{llll} \mathbb{K}[P]^W & \text{has basis } \{m_\lambda \mid \lambda \in P^+\}, & \mathbf{1}_0 \tilde{H} \mathbf{1}_0 & \text{has basis } \{M_\lambda \mid \lambda \in P^+\}, \\ \mathcal{A} & \text{has basis } \{a_\mu \mid \mu \in P^{++}\}, & \varepsilon_0 \tilde{H} \mathbf{1}_0 & \text{has basis } \{A_\mu \mid \mu \in P^{++}\}. \end{array}$$

**Proof** Since $\{x^\mu T_w \mid \mu \in P, w \in W\}$ form a basis of $\tilde{H}$ the elements $M_\mu = \mathbf{1}_0 x^\mu \mathbf{1}_0 = q^{-\ell(w)} \mathbf{1}_0 x^\mu T_w \mathbf{1}_0$, $\mu \in P$, span $\mathbf{1}_0 \tilde{H} \mathbf{1}_0$. By Proposition 2.1, if $\mu$ is on the negative side of a hyperplane $H_{\alpha_i}$, that is, if $\langle \mu, \alpha_i^\vee \rangle < 0$, then $M_\mu$ can be rewritten as a linear combination of $M_\gamma$ such that all terms have $\gamma$ on the nonnegative side of $H_{\alpha_i}$. By repeatedly applying the relation in Proposition 2.1, $M_\mu$ can be rewritten as a linear combination of $M_\gamma$ such that all terms have $\gamma$ on the nonnegative side of $H_{\alpha_1}, \ldots, H_{\alpha_n}$, that is, $\gamma \in P^+ = P \cap \overline{C}$, where $\overline{C} = \{x \in \mathbb{R}^n \mid \langle x, \alpha_i^\vee \rangle \geqslant 0 \text{ for all } 1 \leqslant i \leqslant n\}$.

If $\lambda \in P^+$, using the fourth relation in (1.21),

$$\begin{aligned} M_\lambda &= \mathbf{1}_0 x^\lambda \mathbf{1}_0 = \frac{1}{W_0(q^2)} \sum_{w \in W} q^{\ell(w)} T_w x^\lambda \mathbf{1}_0 = \frac{1}{W_0(q^2)} \sum_{\gamma, v, w} q^{\ell(w)} d_{v,\gamma} x^\gamma T_v \mathbf{1}_0 \\ &= \frac{1}{W_0(q^2)} \sum_{\gamma, v, w} q^{\ell(w)} d_{v,\gamma} x^\gamma q^{\ell(v)} \mathbf{1}_0 = \frac{1}{W_0(q^2)} \sum_{\gamma} d_\gamma x^\gamma \mathbf{1}_0, \end{aligned}$$

where $d_{v,\gamma}$ and $d_\gamma$ are some polynomials in $\mathbb{Z}[q, (q - q^{-1})]$ such that $d_{v,v\lambda} = 1$ so that $d_{w_0\lambda} = 1$. Furthermore $d_\gamma = 0$ unless $\gamma$ is in the convex hull of the points in the orbit $W\lambda$. Thus the coefficient of $x^{w_0\lambda}$ in $M_\lambda$ is $W_0(q^2)^{-1} q^{2\ell(w_0)}$ and the coefficient of $x^\gamma T_v$ can be nonzero only if $\gamma \geqslant w_0\lambda$. Thus the $M_\lambda$, $\lambda \in P^+$, are linearly independent.

The proof for the cases of $m_\mu$, $a_\mu$ and $A_\mu$ is easier, following directly from (2.5), the fact that $C = \{x \in \mathbb{R}^n \mid \langle x, \alpha_i^\vee \rangle > 0 \text{ for all } 1 \leqslant i \leqslant n\}$ is a fundamental chamber for the action of $W$, and that if $\mu \in P^+ \backslash P^{++}$ then $\langle \mu, \alpha_i^\vee \rangle = 0$ and $a_\mu = -a_{s_i\mu} = -a_\mu$, in which case $a_\mu = 0$ (similarly for $A_\mu$). $\square$

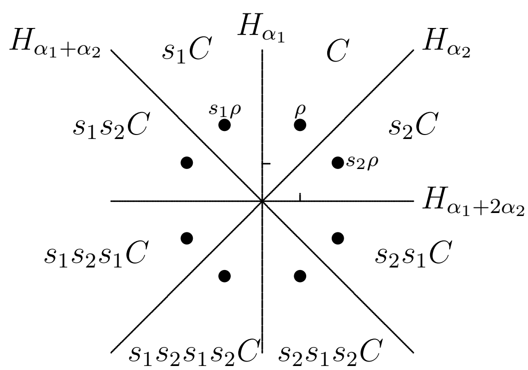
For $\lambda \in P$ define the *Schur function*, or *Weyl character*, by

$$s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho}, \quad \text{where} \quad \rho = \tfrac{1}{2} \sum_{\alpha \in R^+} \alpha. \tag{2.6}$$

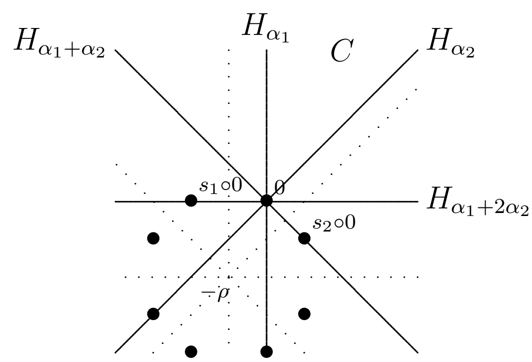
The straightening law for $a_\mu$ in (2.5) implies the following straightening law for the Schur functions. If $\mu \in P$ and $w \in W$ then, by (2.5) and the definition of $s_\mu$,

$$(-1)^{\ell(w)} s_\mu = \frac{(-1)^{\ell(w)} a_{\mu+\rho}}{a_\rho} = \frac{a_{w(\mu+\rho)-\rho+\rho}}{a_\rho} = s_{w \circ \mu}, \qquad \text{where} \quad w \circ \mu = w(\mu + \rho) - \rho. \tag{2.7}$$

The *dot action* of the Weyl group $W$ on $\mathfrak{h}_{\mathbb{R}}^*$ which is appearing here, $w \circ \mu = t_{-\rho} w t_\rho \mu = (t_\rho^{-1}) w t_\rho \mu$, is the ordinary action of $W$ on $\mathfrak{h}_{\mathbb{R}}^*$ except with the "center" shifted to $-\rho$. For the root system of type $C_2$, in Example 1.1, the picture is




the orbit $W\rho$




the orbit $W \circ 0$

The following proposition shows that the Weyl characters $s_\lambda$ are elements of $\mathbb{K}[P]^W$. The equality in part (a) is the *Weyl denominator formula*, a generalization of the factorization of the Vandermonde determinant $\det(x_i^{n-j}) = \prod_{1 \leqslant i,j \leqslant n}(x_i - x_j)$. In the remainder of this section we shall abuse language and use the term "vector space" in place of "free $\mathbb{K} = \mathbb{Z}[q, q^{-1}]$ module".

**Proposition 2.3** *Let $P^+$, $P^{++}$, $\mathbb{K}[P]^W$ and $\mathcal{A}$ be as in (1.7) and (2.4) and let $\rho$ be as in (1.8).*

(a) *If $f \in \mathcal{A}$ then $f$ is divisible by $a_\rho$ and*

$$a_\rho = x^\rho \prod_{\alpha \in R^+} (1 - x^{-\alpha})$$

(b) *The set $\{s_\lambda \mid \lambda \in P^+\}$ is a basis of $\mathbb{K}[P]^W$.*

(c) *The maps*

$$\begin{array}{ccc} P^+ & \longrightarrow & P^{++} \\ \lambda & \longmapsto & \lambda + \rho \end{array} \quad \textit{and} \quad \begin{array}{cccc} \Phi: & \mathbb{K}[P]^W & \longrightarrow & \mathcal{A} \\ & f & \longmapsto & a_\rho f \\ & s_\lambda & \longmapsto & a_{\lambda+\rho} \end{array}$$

*are a bijection and a vector space isomorphism, respectively.*

**Proof** Since $s_i$ takes $\alpha_i$ to $-\alpha_i$ and permutes the other elements of $R^+$,

$$\rho - \langle \rho, \alpha_i^\vee \rangle \alpha_i = s_i \rho = \rho - \alpha_i,$$

and so

$$\langle \rho, \alpha_i^\vee \rangle = 1, \quad \text{for all } 1 \leqslant i \leqslant n.$$

Thus the map $P^+ \to P^{++}$ given by $\lambda \mapsto \lambda + \rho$ is well defined and it is a bijection since it is invertible.

Let $d = x^\rho \prod_{\alpha\in R^+}(1 - x^{-\alpha}) = \prod_{\alpha\in R^+}(x^{\alpha/2} - x^{-\alpha/2})$. Since $s_i$ takes $\alpha_i$ to $-\alpha_i$ and permutes the other elements of $R^+$, $s_i d = -d$ for all $1 \leqslant i \leqslant n$ and so $wd = (-1)^{\ell(w)} d$ for all $w \in W$. Thus $d$ is an element of $\mathcal{A}$.

If $\alpha \in R^+$ and $f = \sum_{\mu\in P} c_\mu x^\mu \in \mathcal{A}$ then

$$\sum_{\mu\in P} c_\mu x^\mu = f = -s_\alpha f = \sum_{\mu\in P} -c_\mu x^{s_\alpha \mu},$$

and

$$f = \sum_{\langle\mu,\alpha^\vee\rangle\geqslant 0} c_\mu (x^\mu - x^{s_\alpha\mu}).$$

Since $(1 - x^{-\langle\mu,\alpha^\vee\rangle\alpha})$ is divisible by $(1 - x^{-\alpha})$ it follows that $x^\mu - x^{s_i\mu} = x^\mu(1 - x^{-\langle\mu,\alpha^\vee\rangle\alpha})$ is divisible by $(1 - x^{-\alpha})$ and thus that $f$ is divisible by $(1 - x^{-\alpha})$ for all $\alpha \in R^+$. Since the elements $(1 - x^{-\alpha})$ are relatively prime in the Laurent polynomial ring $\mathbb{K}[P]$ and $x^\rho$ is a unit in $\mathbb{K}[P]$, $f$ is divisible by $d$. Since both $f$ and $d$ are in $\mathcal{A}$, the quotient $f/d$ is an element of $\mathbb{K}[P]^W$.

The monomial $x^\rho$ appears in $a_\rho$ with coefficient 1 and it is the unique term $x^\mu$ in $a_\rho$ with $\mu \in P^+$. Since $d$ has highest term $x^\rho$ with coefficient 1 and $a_\rho$ is divisible by $d$ it follows that $a_\rho/d = 1$. Thus $a_\rho = d$, the inverse of the map $\Phi$ in (c) is well defined, and $\Phi$ is an isomorphism.

Since $\{a_{\lambda+\rho} \mid \lambda \in P^+\}$ is a basis of $\mathcal{A}$ and the map $\Phi$ is an isomorphism it follows that $\{s_\lambda \mid \lambda \in P^+\}$ is a $\mathbb{K}$-basis of $\mathbb{K}[P]^W$. □

## 2.1 The Satake isomorphism

The following theorem establishes a $q$-analogue of the isomorphism $\Phi$ from Proposition 2.3(c). The map $\Phi_1$ in the following theorem is the *Satake isomorphism*. We shall continue to abuse language and use the term "vector space" in place of "free $\mathbb{K} = \mathbb{Z}[q, q^{-1}]$ module".

**Theorem 2.4** *The vector space isomorphism $\Phi$ in Proposition 2.3(c) generalizes to a vector space isomorphism*

$$\begin{array}{rcccccc} \tilde{\Phi}: & Z(\tilde{H}) = \mathbb{K}[P]^W & \xrightarrow{\Phi_1} & Z(\tilde{H})\mathbf{1}_0 = \mathbf{1}_0\tilde{H}\mathbf{1}_0 & \xrightarrow{\Phi_2} & \varepsilon_0\tilde{H}\mathbf{1}_0 \\ & f & \longmapsto & f\mathbf{1}_0 & \longmapsto & A_\rho f\mathbf{1}_0 \\ & s_\lambda & \longmapsto & s_\lambda\mathbf{1}_0 & \longmapsto & A_{\lambda+\rho}. \end{array}$$

**Proof** Using the third equality in (2.5),

$$\varepsilon_0 a_\lambda \mathbf{1}_0 = \varepsilon_0 \left(\sum_{w\in W} (-1)^{\ell(w)} x^{w\lambda}\right)\mathbf{1}_0 = \sum_{w\in W} (-1)^{\ell(w)} A_{w\lambda} = |W|\, A_\lambda.$$

By Proposition 2.3(c) and Theorem 1.4, $s_\lambda \in \mathbb{K}[P]^W = Z(\tilde{H})$, and so

$$A_\rho s_\lambda \mathbf{1}_0 = \frac{1}{|W|}\varepsilon_0 a_\rho \mathbf{1}_0 s_\lambda \mathbf{1}_0 = \frac{1}{|W|}\varepsilon_0 a_\rho s_\lambda \mathbf{1}_0^2 = \frac{1}{|W|}\varepsilon_0 a_{\lambda+\rho}\mathbf{1}_0 = A_{\lambda+\rho}.$$

Since $\{s_\lambda \mid \lambda \in P^+\}$ is a basis of $\mathbb{K}[P]^W = Z(\tilde{H})$ and $\{A_{\lambda+\rho} \mid \lambda \in P^+\}$ is a basis of $\varepsilon_0 \tilde{H}\mathbf{1}_0$, the composite map

$$\begin{array}{ccccccc} Z(\tilde{H}) & \xrightarrow{\mathbf{1}_0} & Z(\tilde{H})\,\mathbf{1}_0 & \hookrightarrow & \mathbf{1}_0\tilde{H}\mathbf{1}_0 & \xrightarrow{A_\rho} & \varepsilon_0\tilde{H}\mathbf{1}_0 \\ f & \longmapsto & f\,\mathbf{1}_0 & \mapsto & f\,\mathbf{1}_0 & \longmapsto & A_\rho f\,\mathbf{1}_0 \\ s_\lambda & \longmapsto & s_\lambda\,\mathbf{1}_0 & \mapsto & s_\lambda\,\mathbf{1}_0 & \longmapsto & A_{\lambda+\rho} \end{array}$$

is a vector space isomorphism. □

If $\mu \in P$ let

$$W_\mu = \{w \in W \mid w\mu = \mu\} \quad \text{and} \quad W_\mu(t) = \sum_{w \in W_\mu} t^{\ell(w)}. \tag{2.8}$$

In particular, if $\mu = 0$, then $W_0 = W$ and $W_0(t)$ is the polynomial that appears in (2.1).

The *Hall–Littlewood polynomials*, or *Macdonald spherical functions*, are defined by

$$P_\mu(x;t) = \frac{1}{W_\mu(t)} \sum_{w\in W} w\Big(x^\mu \prod_{\alpha\in R^+} \frac{1-tx^{-\alpha}}{1-x^{-\alpha}}\Big), \quad \text{for } \mu \in P. \tag{2.9}$$

Then $m_\mu = P_\mu(x;1)$ and, using the Weyl denominator formula,

$$\begin{aligned} P_\mu(x;0) &= \sum_{w\in W} w\Big(\frac{x^\rho x^\mu}{x^\rho \prod_{\alpha\in R^+}(1-x^{-\alpha})}\Big) \\ &= \frac{1}{a_\rho}\sum_{w\in W}(-1)^{\ell(w)} w x^{\mu+\rho} = \frac{a_{\mu+\rho}}{a_\rho} = s_\mu, \end{aligned} \tag{2.10}$$

and so, conceptually, the spherical functions $P_\mu(x;t)$ interpolate between the Schur functions $s_\mu$ and the monomial symmetric functions $m_\mu$.

The double cosets in $W\backslash\tilde{W}/W$ are $Wt_\lambda W$, $\lambda \in P^+$. If $\lambda \in P^+$ let $n_\lambda$ and $m_\lambda$ be the maximal and minimal length elements of $Wt_\lambda W$, respectively. Theorem 2.9 below will show that under the Satake isomorphism the Weyl characters $s_\lambda$ correspond to Kazhdan Lusztig basis elements $C'_{n_\lambda}$ and the polynomials $P_\mu(x;q^{-2})$ correspond to the elements $M_\mu = \mathbf{1}_0 x^\mu \mathbf{1}_0$. More precisely, we have the following diagram:

$$\begin{array}{cccc} \Phi_1 : & Z(\tilde{H}) = \mathbb{K}[P]^W & \longrightarrow & Z(\tilde{H})\mathbf{1}_0 = \mathbf{1}_0\tilde{H}\mathbf{1}_0 \\ & f & \longmapsto & f\mathbf{1}_0 \\ & q^{-\ell(w_0)}W_0(q^2)\,s_\lambda & \longmapsto & C'_{n_\lambda} \\ & \dfrac{W_\mu(q^{-2})}{W_0(q^{-2})}\,P_\mu(x;q^{-2}) & \longmapsto & M_\mu \end{array} \tag{2.11}$$

where $w_0$ is the longest element of $W$. The following three lemmas (of independent interest) are used in the proof of Theorem 2.9.

**Lemma 2.5** *Let $t_\alpha$, $\alpha \in R^+$, be commuting variables indexed by the positive roots. For $\lambda \in P^+$ let $P_\lambda(x;t)$ be as in (2.9), $W_\lambda$ as in (2.8), and define*

$$R_\lambda(x;t_\alpha) = \sum_{w\in W} w\left(x^\lambda \prod_{\alpha\in R^+} \frac{1-t_\alpha x^{-\alpha}}{1-x^{-\alpha}}\right)$$

*and*

$$W_\lambda(t_\alpha) = \sum_{w\in W_\lambda}\Big(\prod_{\alpha\in R(w)} t_\alpha\Big),$$

*where, as in (1.5), $R(w) = \{\alpha \in R^+ \mid w\alpha < 0\}$ is the inversion set of $w$. Then*

(a) $R_\lambda(x;t_\alpha) = \sum_{\mu\in P^+} u_{\lambda\mu} s_\mu,$
*with $u_{\lambda\mu} \in \mathbb{Z}[t_\alpha]$, $u_{\lambda\mu} = 0$ unless $\mu \leqslant \lambda$, and $u_{\lambda\lambda} = W_\lambda(t_\alpha)$.*

(b) $P_\lambda(x;t) = \sum_{\mu\in P^+} c_{\lambda\mu} s_\mu$, *with $c_{\lambda\mu} \in \mathbb{Z}[t]$, $c_{\lambda\mu} = 0$ unless $\mu \leqslant \lambda$, and $c_{\lambda\lambda} = 1$.*

**Proof** (a) If $E \subseteq R^+$ let

$$t_E = \prod_{\alpha\in E} t_\alpha \quad \text{and} \quad \alpha_E = \sum_{\alpha\in E} \alpha,$$

and let $a_\mu$ be as defined in (2.4). Using the Weyl denominator formula, Proposition 2.3(a), and the second equality in (2.5),

$$\begin{aligned} R_\lambda &= \sum_{w\in W} w\left(x^\lambda \prod_{\alpha\in R^+} \frac{1-t_\alpha x^{-\alpha}}{1-x^{-\alpha}}\right) \\ &= \sum_{w\in W} w\left(\frac{x^{\lambda+\rho}\prod_{\alpha\in R^+}(1-t_\alpha x^{-\alpha})}{x^\rho\prod_{\alpha\in R^+}(1-x^{-\alpha})}\right) \\ &= \frac{1}{a_\rho}\sum_{w\in W}(-1)^{\ell(w)} w\left(x^{\lambda+\rho}\prod_{\alpha\in R^+}(1-t_\alpha x^{-\alpha})\right) \\ &= \frac{1}{a_\rho}\sum_{w\in W}(-1)^{\ell(w)} w\left(\sum_{E\subseteq R^+}(-1)^{|E|} t_E x^{\lambda+\rho-\alpha_E}\right) \\ &= \frac{1}{a_\rho}\sum_{E\subseteq R^+}(-1)^{|E|} t_E a_{\lambda+\rho-\alpha_E} = \sum_{E\subseteq R^+}(-1)^{|E|} t_E s_{\lambda-\alpha_E}, \end{aligned}$$

which shows that $R_\lambda$ is a symmetric function and $u_{\lambda\mu} \in \mathbb{Z}[t_\alpha]$.

By the straightening law for Weyl characters (2.7), $s_{\lambda-\alpha_E}=0$ or $s_{\lambda-\alpha_E}=(-1)^{\ell(v)}s_\mu$ with

$$v\in W \text{ and } \mu\in P^+ \text{ such that } \quad \mu+\rho=v^{-1}(\lambda+\rho-\alpha_E).$$

Let $E^c$ denote the complement of $E$ in $R^+$. Since $v$ permutes the elements of $R^+$,

$$\begin{aligned} v^{-1}(\lambda+\rho-\alpha_E) &= v^{-1}\lambda+v^{-1}\left(\tfrac{1}{2}\sum_{\alpha\in E^c}\alpha-\tfrac{1}{2}\sum_{\alpha\in E}\alpha\right) \\ &= v^{-1}\lambda+\left(\tfrac{1}{2}\sum_{\alpha\in F^c}\alpha-\tfrac{1}{2}\sum_{\alpha\in F}\alpha\right)=v^{-1}\lambda+\rho-\alpha_F, \end{aligned}$$

for some subset $F\subseteq R^+$ (which could be determined explicitly in terms of $E$ and $v$). Hence

$$\mu=v^{-1}\lambda+\rho-\alpha_F-\rho=v^{-1}\lambda-\alpha_F\leqslant v^{-1}\lambda\leqslant\lambda. \tag{2.12}$$

This proves that $u_{\lambda\mu}=0$ unless $\mu\leqslant\lambda$.

In (2.12), $\mu=\lambda$ only if $v^{-1}\lambda=\lambda$ and $\rho=\rho-\alpha_F=v^{-1}(\rho-\alpha_E)$ in which case

$$\rho-\alpha_E=v\left(\tfrac{1}{2}\sum_{\alpha\in R^+}\alpha\right)=\rho-\sum_{\alpha\in R(v)}\alpha \quad \text{and} \quad E=R(v).$$

Thus

$$u_{\lambda\lambda}(t_\alpha)=\sum_{v^{-1}\in W_\lambda}t_{R(v)}.$$

(b) Set $t_\alpha=t$ for all $\alpha\in R^+$. Applying (a) with $\lambda=0$,

$$R_0(x;t)=\sum_{w\in W}w\left(\prod_{\alpha\in R^+}\frac{1-tx^{-\alpha}}{1-x^{-\alpha}}\right)=W_0(t). \tag{2.13}$$

Let $W^\lambda$ be a set of minimal length coset representatives of the cosets in $W/W_\lambda$. Every element $w\in W$ can be written uniquely as $w=uv$ with $u\in W^\lambda$ and $v\in W_\lambda$ (see [1, IV §1 Ex. 3]). Let

$$Z(\lambda)=\{\alpha\in R^+ \mid \langle\lambda,\alpha^\vee\rangle=0\},$$

and let $Z(\lambda)^c$ be the complement of $Z(\lambda)$ in $R^+$. Then $v\in W_\lambda$ permutes the elements of $Z(\lambda)^c$ among themselves and so

$$\begin{aligned} R_\lambda(x;t) &= \sum_{u\in W^\lambda}u\left(x^\lambda\Big(\prod_{\alpha\in Z(\lambda)^c}\frac{1-tx^{-\alpha}}{1-x^{-\alpha}}\Big)\sum_{v\in W_\lambda}v\Big(\prod_{\alpha\in Z(\lambda)}\frac{1-tx^{-\alpha}}{1-x^{-\alpha}}\Big)\right) \\ &= \sum_{u\in W^\lambda}u\left(x^\lambda\Big(\prod_{\alpha\in Z(\lambda)^c}\frac{1-tx^{-\alpha}}{1-x^{-\alpha}}\Big)W_\lambda(t)\right), \end{aligned}$$

where the last equality follows from (2.13). Thus there is an element $P_\lambda(x;t) \in \mathbb{F}[P]$ where $\mathbb{F}$ is the field of fractions of $\mathbb{Z}[t_\alpha]$ such that

$$R_\lambda(x;t) = W_\lambda(t) \sum_{u \in W^\lambda} u \left( x^\lambda \prod_{\alpha \in Z(\lambda)^c} \frac{1 - tx^{-\alpha}}{1 - x^{-\alpha}} \right) = W_\lambda(t) P_\lambda(x;t).$$

Since $R_\lambda$ is a symmetric polynomial (an element of $\mathbb{Z}[t][P]^W$), $P_\lambda(x;t) \in \mathbb{F}[P]^W$. Since $t$ only occurs in the numerators of the terms in the sum defining $P_\lambda$ in fact $P_\lambda$ is a symmetric polynomial with coefficients in $\mathbb{Z}[t]$. It follows that all the $u_{\lambda\mu}$ appearing in part (a) are divisible by $W_\lambda(t)$ and

$$P_\lambda(x;t) = \sum_{\mu \in P} c_{\lambda\mu} s_\mu, \quad \text{where } c_{\lambda\mu} = \frac{1}{W_\lambda(t)} u_{\lambda\mu}$$

are polynomials in $\mathbb{Z}[t]$ such that $c_{\lambda\lambda} = 1$ and $c_{\lambda\mu} = 0$ unless $\mu \leqslant \lambda$. □

Lemma 2.5 has the following interesting (and useful) corollary, see [13].

**Corollary 2.6** *Let $\rho$ and $\alpha^\vee$ be as in (1.8) and (1.1), respectively, and let $W_0(t)$ be as defined in (2.8).*

*(a)* $$\sum_{w \in W} w \left( \prod_{\alpha \in R^+} \frac{1 - tx^{-\alpha}}{1 - x^{-\alpha}} \right) = W_0(t).$$

*(b)* $$\prod_{\alpha \in R^+} \frac{1 - t^{\langle \rho, \alpha^\vee \rangle + 1}}{1 - t^{\langle \rho, \alpha^\vee \rangle}} = W_0(t).$$

**Proof** Part (a) follows from Lemma 2.5 (a) by setting $\lambda = 0$ and specializing $t_\alpha = t$ for all $\alpha \in R^+$.

(b) Applying the homomorphism

$$\begin{array}{ccc} \mathbb{Z}[t^{\pm 1}][P] & \longrightarrow & \mathbb{Z}[t^{\pm 1}] \\ x^\lambda & \longmapsto & t^{\langle -\rho, \lambda \rangle} \end{array}$$

to both sides of the identity in (a) for the root system $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$ gives

$$W_0(t) = \sum_{w \in W} \prod_{\alpha \in R^+} \left( \frac{1 - t^{\langle \rho, w\alpha^\vee \rangle + 1}}{1 - t^{\langle \rho, w\alpha^\vee \rangle}} \right). \tag{2.14}$$

If $w \in W$, $w \neq 1$, and $w = s_{i_1} \cdots s_{i_p}$ is a reduced word for $w$ then $w^{-1}(-\alpha_{i_1}) = (s_{i_1} w)^{-1} \alpha_{i_1} \in R(w)$ and so

$$\text{there is an } \alpha \in R^+ \text{ such that } w\alpha^\vee = -\alpha_{i_1}^\vee.$$

Then

$$\prod_{\alpha\in R^+}\frac{1-t^{\langle\rho,w\alpha^\vee\rangle+1}}{1-t^{\langle\rho,w\alpha^\vee\rangle}} = \frac{1-t^{\langle\rho,-\alpha_{i_1}^\vee\rangle+1}}{1-t^{\langle\rho,-\alpha_{i_1}^\vee\rangle}}\prod_{\substack{\alpha\in R^+\\ w\alpha\neq-\alpha_{i_1}}}\frac{1-t^{\langle\rho,w\alpha^\vee\rangle+1}}{1-t^{\langle\rho,w\alpha^\vee\rangle}}$$

$$= \frac{1-t^{-1+1}}{1-t^{-1}}\prod_{\substack{\alpha\in R^+\\ w\alpha\neq-\alpha_{i_1}}}\frac{1-t^{\langle\rho,w\alpha^\vee\rangle+1}}{1-t^{\langle\rho,w\alpha^\vee\rangle}} = 0\,.$$

Thus the only nonzero term on the right hand side of (2.14) occurs for $w=1$. □

**Lemma 2.7** *For $\lambda\in P^+$ let $t_\lambda\in\tilde W$ be the translation in $\lambda$ and let $n_\lambda$ be the maximal length element in the double coset $Wt_\lambda W$. Let $M_\lambda=\mathbf{1}_0x^\lambda\mathbf{1}_0$, as in (2.4). Then*

$$q^{-\ell(w_0)}W_0(q^2)\cdot\frac{W_0(q^{-2})}{W_\lambda(q^{-2})}\cdot M_\lambda=\sum_{x\in Wt_\lambda W}q^{\ell(x)-\ell(n_\lambda)}T_x,$$

*in the affine Hecke algebra $\tilde H$.*

**Proof** Let $\lambda\in P^+$. Let $W_\lambda=\mathrm{Stab}(\lambda)$ and let $w_0$ and $w_\lambda$ be the maximal length elements in $W$ and $W_\lambda$, respectively. Let $m_\lambda$ and $n_\lambda$ be the minimal and maximal length elements respectively in the double coset $Wt_\lambda W$. For each positive root $\alpha$ the hyperplanes $H_{\alpha,i}$, $1\leqslant i\leqslant\langle\lambda,\alpha^\vee\rangle$, are between the fundamental alcove $A$ and the alcove $t_\lambda A$ and so

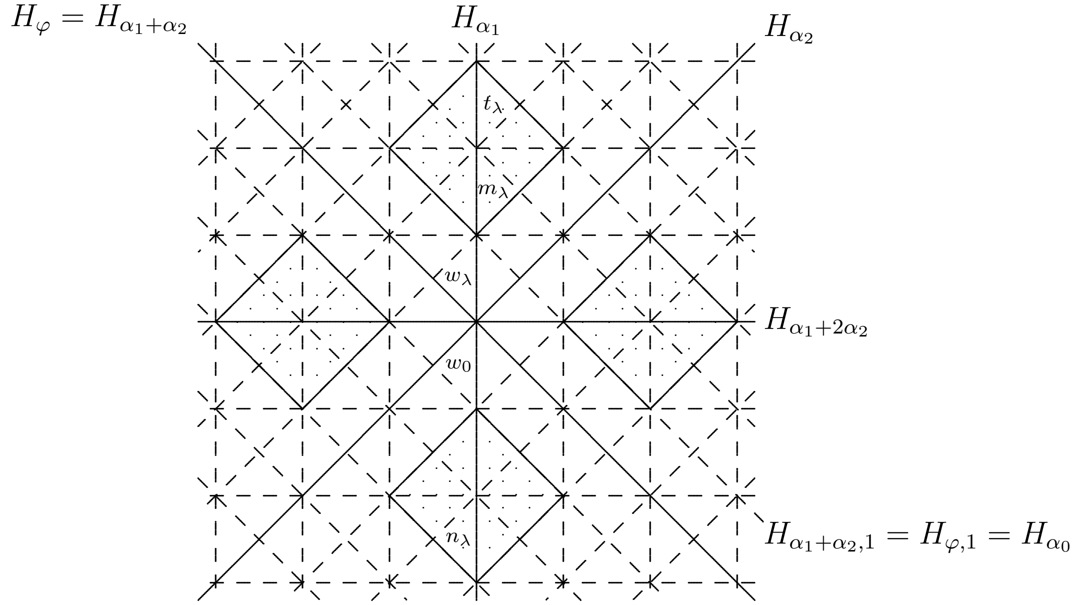
$$\ell(t_\lambda)=\sum_{\alpha\in R^+}\langle\lambda,\alpha^\vee\rangle=2\langle\lambda,\rho^\vee\rangle,\quad\text{where }\rho^\vee=\tfrac12\sum_{\alpha\in R^+}\alpha^\vee. \tag{2.15}$$

Since $m_\lambda=t_\lambda(w_\lambda w_0)$ and $n_\lambda=t_{w_0\lambda}w_0$,

$$\begin{aligned}\ell(m_\lambda)&=\ell(t_\lambda)-\ell(w_0w_\lambda)=\ell(t_\lambda)-(\ell(w_0)-\ell(w_\lambda)),\quad\text{and}\\ \ell(n_\lambda)&=\ell(t_\lambda)+\ell(w_0)=\ell(m_\lambda)+\ell(w_0)-\ell(w_\lambda)+\ell(w_0).\end{aligned} \tag{2.16}$$

For example, in the setting of Example 1.1, if $\lambda=2\omega_2$ in type $C_2$, then $W_\lambda=\{1,s_1\}$, $w_\lambda=s_1$, $w_0=s_1s_2s_1s_2$, $\ell(t_\lambda)=6$, $\ell(m_\lambda)=3$, and $\ell(n_\lambda)=10$. Labeling the alcove $wA$ by the element $w$, the 32 alcoves $wA$ with $w\in Wt_\lambda W$

make up the four shaded diamonds.




The double coset $Wt_\lambda W$

Then

$$\begin{aligned}\mathbf{1}_0 x^\lambda \mathbf{1}_0 &= \mathbf{1}_0 T_{t_\lambda} \mathbf{1}_0 = \mathbf{1}_0 T_{m_\lambda w_0 w_\lambda} \mathbf{1}_0 = \mathbf{1}_0 T_{m_\lambda} T_{w_0 w_\lambda} \mathbf{1}_0 \\ &= q^{\ell(w_0)-\ell(w_\lambda)} \mathbf{1}_0 T_{m_\lambda} \mathbf{1}_0 \\ &= \frac{q^{\ell(w_0)-\ell(w_\lambda)-\ell(m_\lambda)}}{W(q^2)} \left( \sum_{w \in W} q^{\ell(w)} T_w \right) q^{\ell(m_\lambda)} T_{m_\lambda} \mathbf{1}_0 .\end{aligned}$$

Let $W^\lambda$ be a set of minimal length coset representatives of the cosets in $W/W_\lambda$. Every element $w \in W$ has a unique expression $w = uv$ with $u \in W^\lambda$ and $v \in W_\lambda$. If $v \in W_\lambda$ then

$$vm_\lambda = vt_\lambda w_\lambda w_0 = t_\lambda v w_\lambda w_0 = m_\lambda (w_\lambda w_0)^{-1} v w_\lambda w_0 = m_\lambda (w_0^{-1} w_\lambda^{-1} v w_\lambda w_0).$$

Since conjugation by $w_\lambda$ and conjugation by $w_0$ are automorphisms of $W_\lambda$ and $W$ respectively taking simple reflections to simple reflections,

$$\ell(v) = \ell(w_0^{-1} w_\lambda^{-1} v w_\lambda w_0) \ .$$

Thus

$$\begin{aligned}
\mathbf{1}_0 x^\lambda \mathbf{1}_0 &= \frac{q^{\ell(w_0)-\ell(w_\lambda)-\ell(m_\lambda)}}{W_0(q^2)} \sum_{u\in W^\lambda} q^{\ell(u)}T_u \sum_{v\in W_\lambda} q^{\ell(v)}T_v q^{\ell(m_\lambda)}T_{m_\lambda}\mathbf{1}_0 \\
&= \frac{q^{2\ell(w_0)-2\ell(w_\lambda)-\ell(t_\lambda)}}{W_0(q^2)} \left(\sum_{u\in W^\lambda} q^{\ell(u)}T_u q^{\ell(m_\lambda)}T_{m_\lambda}\right) \\
&\qquad\qquad \cdot\left(\sum_{v\in w_0^{-1}w_\lambda^{-1}W_\lambda w_\lambda w_0} q^{\ell(v)}T_v\right)\mathbf{1}_0 \\
&= \frac{q^{-2\ell(w_\lambda)-\ell(t_\lambda)}}{W_0(q^{-2})} \left(\sum_{u\in W^\lambda} q^{\ell(u)}T_u\right) q^{\ell(m_\lambda)}T_{m_\lambda} W_\lambda(q^2)\mathbf{1}_0 \\
&= \frac{q^{-2\ell(w_\lambda)-\ell(t_\lambda)}W_\lambda(q^2)}{W_0(q^2)W_0(q^{-2})} \left(\sum_{u\in W^\lambda} q^{\ell(u)}T_u\right) q^{\ell(m_\lambda)}T_{m_\lambda} \left(\sum_{w\in W} q^{\ell(w)}T_w\right) \\
&= \frac{q^{-\ell(t_\lambda)}W_\lambda(q^{-2})}{W_0(q^2)W_0(q^{-2})} \sum_{x\in Wt_\lambda W} q^{\ell(x)}T_x \\
&= \frac{q^{-\ell(t_\lambda)+\ell(n_\lambda)}W_\lambda(q^{-2})}{W_0(q^2)W_0(q^{-2})} \left(\sum_{x\in Wt_\lambda W} q^{\ell(x)-\ell(n_\lambda)}T_x\right) \\
&= \frac{q^{\ell(w_0)}}{W_0(q^2)}\frac{W_\lambda(q^{-2})}{W_0(q^{-2})} \left(\sum_{x\in Wt_\lambda W} q^{\ell(x)-\ell(n_\lambda)}T_x\right).
\end{aligned}$$

□

**Lemma 2.8** *Let $w_0$ be the longest element of $W$ and let $\lambda \in P$.*

(a) $\overline{x^\lambda} = T_{w_0}x^{w_0\lambda}T_{w_0}^{-1}$.

(b) $\overline{\mathbf{1}_0} = \mathbf{1}_0$ *and* $\overline{\varepsilon_0} = \varepsilon_0$.

(c) *If* $z \in \mathbb{Z}[P]^W$ *then* $\bar{z} = z$.

(d) $\overline{q^{-\ell(w_0)}A_{\lambda+\rho}} = q^{-\ell(w_0)}A_{\lambda+\rho}$.

**Proof** (a) If $\lambda \in P^+$ then $w_0t_\lambda = t_{w_0\lambda}w_0$, $\ell(w_0t_\lambda) = \ell(w_0)+\ell(t_\lambda)$ and $\ell(t_{w_0\lambda}w_0) = \ell(t_{w_0\lambda})+\ell(w_0)$. Thus,

$$T_{w_0}T_{t_\lambda} = T_{w_0t_\lambda} = T_{t_{w_0\lambda}w_0} = T_{t_{w_0\lambda}}T_{w_0}, \quad \text{for } \lambda \in P^+.$$

Let $\lambda \in P$ and write $\lambda = \mu - \nu$ with $\mu,\nu \in P^+$. Since $-w_0\mu \in P^+$ and $-w_0\nu \in P^+$,

$$\begin{aligned}
\overline{x^\lambda} &= \overline{T_{t_\mu}T_{t_\nu}^{-1}} = T_{t_{-\mu}}^{-1}T_{t_{-\nu}} = T_{w_0}T_{t_{-w_0\mu}}^{-1}T_{t_{-w_0\nu}}T_{w_0}^{-1} \\
&= T_{w_0}(x^{-w_0\lambda})^{-1}T_{w_0}^{-1} = T_{w_0}x^{w_0\lambda}T_{w_0}^{-1}.
\end{aligned}$$

(b) For $1 \leqslant i \leqslant n$,

$$\overline{\mathbf{1}_0^2} = \overline{\mathbf{1}_0}^2 \qquad \text{and} \qquad T_i\overline{\mathbf{1}_0} = \overline{T_i^{-1}\mathbf{1}_0} = \overline{q^{-1}\mathbf{1}_0} = q\overline{\mathbf{1}_0},$$

$$\overline{\varepsilon_0^2} = \overline{\varepsilon_0}^2 \qquad \text{and} \qquad T_i\overline{\varepsilon_0} = \overline{T_i^{-1}\varepsilon_0} = \overline{-q\varepsilon_0} = -q^{-1}\overline{\varepsilon_0}.$$

These are the defining properties (before 2.1) of $\mathbf{1}_0$ and $\varepsilon_0$ and so $\overline{\mathbf{1}_0} = \mathbf{1}_0$ and $\overline{\varepsilon_0} = \varepsilon_0$.

(c) If $z = \sum_{\mu\in P} c_\mu x^\mu \in \mathbb{Z}[P]^W$, then, since $c_\mu \in \mathbb{Z}$, $\overline{c_\mu} = c_\mu$ and, by (a),

$$\overline{z} = \sum_{\mu\in P} \overline{c_\mu}\,\overline{x^\mu} = \sum_{\mu\in P} c_\mu T_{w_0} x^{w_0\mu} T_{w_0}^{-1} = T_{w_0}\left(\sum_{\mu\in P} c_\mu x^{w_0\mu}\right) T_{w_0}^{-1} = T_{w_0} z T_{w_0}^{-1},$$

since $z \in \mathbb{Z}[P]^W$ is $W$-invariant. Finally, since $\mathbb{Z}[P]^W \subseteq Z(\tilde{H})$, $z$ is central, and $\overline{z} = T_{w_0} z T_{w_0}^{-1} = z$.

(d) By (a), (b) and the third equality in (2.5),

$$\begin{aligned}
\overline{q^{-\ell(w_0)}A_{\lambda+\rho}} &= q^{\ell(w_0)}\overline{\varepsilon_0 x^{\lambda+\rho}\mathbf{1}_0} = q^{\ell(w_0)}\varepsilon_0 T_{w_0} x^{w_0(\lambda+\rho)} T_{w_0}^{-1}\mathbf{1}_0 \\
&= q^{\ell(w_0)}(-q^{-1})^{\ell(w_0)}\varepsilon_0 x^{w_0(\lambda+\rho)}\mathbf{1}_0 q^{-\ell(w_0)} = (-q^{-1})^{\ell(w_0)} A_{w_0(\lambda+\rho)} \\
&= q^{-\ell(w_0)}A_{\lambda+\rho}.
\end{aligned}$$

□

The following theorem is due to Lusztig [10]. Part (a) was originally proved in a different formulation by Macdonald [12, (4.1.2)].

**Theorem 2.9** *If $\mu \in P$ let $W_\mu$ be the stabilizer of $\mu$ and let $W_\mu(t)$ be as in (2.8).*

(a) *Let $\mu \in P$. Let $P_\mu(x;t)$ be the Macdonald spherical function defined in (2.9) and define $M_\mu = \mathbf{1}_0 x^\mu \mathbf{1}_0$ as in (2.4). In the affine Hecke algebra $\tilde{H}$,*

$$\frac{W_\mu(q^{-2})}{W_0(q^{-2})} \cdot P_\mu(x;q^{-2})\mathbf{1}_0 = M_\mu.$$

(b) *For $\lambda \in P^+$ let $t_\lambda \in \tilde{W}$ be the translation in $\lambda$ and let $n_\lambda$ be the maximal length element in the double coset $Wt_\lambda W$. Let $s_\lambda$ be the Weyl character and let $C'_{n_\lambda}$ be the Kazhdan–Lusztig basis element as defined in (2.6) and (1.26), respectively. In the affine Hecke algebra $\tilde{H}$,*

$$q^{-\ell(w_0)}W_0(q^2) \cdot s_\lambda \mathbf{1}_0 = C'_{n_\lambda}.$$

**Proof** (a) By Theorem 2.4 there is an element $\tilde{P}_\lambda \in \mathbb{K}[P]^W$ such that $\tilde{P}_\lambda \mathbf{1}_0 = \mathbf{1}_0 x^\lambda \mathbf{1}_0$. To find $\tilde{P}_\lambda$ first do a rank 1 calculation,

$$\begin{aligned}
(q^{-1}+T_i)x^\lambda \mathbf{1}_0 &= \left(q^{-1}x^\lambda + x^{s_i\lambda}T_i + (q-q^{-1})\Big(\frac{x^\lambda - x^{s_i\lambda}}{1-x^{-\alpha_i}}\Big)\right)\mathbf{1}_0 \\
&= \frac{1}{1-x^{-\alpha_i}}\begin{pmatrix} q^{-1}x^\lambda(1-x^{-\alpha_i}) + qx^{s_i\lambda}(1-x^{-\alpha_i}) \\ +qx^\lambda - qx^{s_i\lambda} - q^{-1}x^\lambda + q^{-1}x^{s_i\lambda}\end{pmatrix}\mathbf{1}_0 \\
&= (1-x^{-\alpha_i})^{-1}(-q^{-1}x^{\lambda-\alpha_i} - qx^{s_i\lambda-\alpha_i} + qx^\lambda + q^{-1}x^{s_i\lambda})\mathbf{1}_0 \\
&= (1-x^{-\alpha_i})^{-1}(x^\lambda(q-q^{-1}x^{-\alpha_i}) + x^{s_i\lambda}(q^{-1}-qx^{-\alpha_i}))\mathbf{1}_0 \\
&= \left(\frac{q-q^{-1}x^{-\alpha_i}}{1-x^{-\alpha_i}}\cdot x^\lambda + \frac{x^{-\alpha_i}}{x^{-\alpha_i}}\cdot\frac{q^{-1}x^{\alpha_i}-q}{x^{\alpha_i}-1}\cdot x^{s_i\lambda}\right)\mathbf{1}_0 \\
&= (1+s_i)\left(\frac{q-q^{-1}x^{-\alpha_i}}{1-x^{-\alpha_i}}\,x^\lambda\right)\mathbf{1}_0.
\end{aligned}$$

Since $\mathbf{1}_0$ is a linear combination of products of $T_i$ it can also be written as a linear combination of products of $q^{-1}+T_i$. Thus $\mathbf{1}_0 x^\lambda \mathbf{1}_0$ can be written as a linear combination of terms of the form

$$(1+s_{i_1})\left(\frac{q-q^{-1}x^{-\alpha_{i_1}}}{1-x^{-\alpha_{i_1}}}\right)\cdots(1+s_{i_p})\left(\frac{q-q^{-1}x^{-\alpha_{i_p}}}{1-x^{-\alpha_{i_p}}}\right)x^\lambda\mathbf{1}_0.$$

Thus

$$\mathbf{1}_0 x^\lambda \mathbf{1}_0 = \tilde{P}_\lambda \mathbf{1}_0, \quad \text{where} \quad \tilde{P}_\lambda = \sum_{w\in W} x^{w\lambda} w c_w,$$

and the $c_w$ are some linear combinations of products of terms of the form $(q-q^{-1}x^\alpha)/(1-x^\alpha)$ for roots $\alpha \in R$. Since $\tilde{P}_\lambda$ is an element of $\mathbb{K}[P]^W$,

$$\tilde{P}_\lambda = \sum_{w\in W} w(x^{w_0\lambda} w_0 c_{w_0}),$$

where $w_0$ is the longest element of $W$. The coefficient $w_0 c_{w_0}$ comes from the highest term in the expansion of

$$\mathbf{1}_0 = \frac{1}{W_0(q^2)}(q^{2\ell(w_0)}T_{w_0} + \text{lower terms})$$

in terms of linear combination of products of the $(q^{-1}+T_i)$. If $w_0 = s_{i_1}\cdots s_{i_p}$ is a reduced word for $w_0$ then

$$\begin{aligned}
w_0 c_{w_0} &= \frac{q^{\ell(w_0)}}{W_0(q^2)} s_{i_1}\left(\frac{q-q^{-1}x^{-\alpha_{i_1}}}{1-x^{-\alpha_{i_1}}}\right)\cdots s_{i_p}\left(\frac{q-q^{-1}x^{-\alpha_{i_p}}}{1-x^{-\alpha_{i_p}}}\right) \\
&= \frac{q^{\ell(w_0)}}{W_0(q^2)} s_{i_1}\cdots s_{i_p}\left(\frac{q-q^{-1}x^{-s_{i_p}\cdots s_{i_2}\alpha_{i_1}}}{1-x^{-s_{i_p}\cdots s_{i_2}\alpha_{i_1}}}\right)\left(\frac{q-q^{-1}x^{-s_{i_p}\cdots s_{i_3}\alpha_{i_2}}}{1-x^{-s_{i_p}\cdots s_{i_2}\alpha_{i_2}}}\right) \\
&\qquad \cdots\left(\frac{q-q^{-1}x^{-\alpha_{i_p}}}{1-x^{-\alpha_{i_p}}}\right) \\
&= \frac{q^{\ell(w_0)}}{W_0(q^2)} w_0 \prod_{\alpha\in R^+}\frac{q-q^{-1}x^{-\alpha}}{1-x^{-\alpha}} = \frac{q^{2\ell(w_0)}}{W_0(q^2)} w_0 \prod_{\alpha\in R^+}\frac{1-q^{-2}x^{-\alpha}}{1-x^{-\alpha}},
\end{aligned}$$

by Lemma 1.2 and the fact that $\ell(w_0) = \mathrm{Card}(R^+)$. Thus, since $q^{-2\ell(w_0)}W_0(q^2) = W_0(q^{-2})$,

$$\tilde{P}_\lambda = \frac{1}{W_0(q^{-2})} \sum_{w \in W} w\left(x^\lambda \prod_{\alpha \in R^+} \frac{1 - q^{-2}x^{-\alpha}}{1 - x^{-\alpha}}\right).$$

(b) Since $W_0(q^{-2}) = q^{-2\ell(w_0)}W_0(q^2)$, Lemma 2.8 gives

$$\overline{q^{-\ell(w_0)}W_0(q^2)s_\lambda \mathbf{1}_0} = q^{\ell(w_0)}W_0(q^{-2})\overline{s_\lambda}\mathbf{1}_0 = q^{-\ell(w_0)}W_0(q^2)s_\lambda \mathbf{1}_0.$$

By Lemma 2.5(b),

$$s_\lambda = \sum_{\mu \in P^+} K_{\lambda\mu}(t)P_\mu(x;t),$$

where $K_{\lambda\mu}(t) \in \mathbb{Z}[t]$, $K_{\lambda\mu}(t) = 0$ unless $\mu \leqslant \lambda$ and $K_{\lambda\lambda}(t) = 1$. Thus, by part (a) and Lemma 2.7

$$\begin{aligned} q^{-\ell(w_0)}W_0(q^2)s_\lambda \mathbf{1}_0 &= \sum_{\mu \in P^+} q^{-\ell(w_0)}W_0(q^2)K_{\lambda\mu}(q^{-2})P_\mu(x;q^{-2})\mathbf{1}_0 \\ &= \sum_{\mu \in P^+} \sum_{x \in Wt_\mu W} q^{\ell(x)-\ell(n_\mu)}K_{\lambda\mu}(q^{-2})T_x, \end{aligned}$$

where the polynomials $K_{\lambda\mu}(q^{-2}) \in \mathbb{Z}[q^{-2}]$ are 0 unless $\mu \leqslant \lambda$ and $K_{\lambda\lambda}(q^{-2}) = 1$. Hence $q^{-\ell(w_0)}W(q^2)s_\lambda \mathbf{1}_0$ is a bar invariant element of $\tilde{H}$ such that its expansion in terms of the basis $\{T_w \mid w \in \tilde{W}\}$ is triangular with coefficient of $T_{n_\lambda}$ equal to 1 and all other coefficients in $q^{-1}\mathbb{Z}[q^{-1}]$. These are the defining properties (1.26)–(1.27) of $C'_{n_\lambda}$. □

## 3 Orthogonality and formulae for Kostka–Foulkes polynomials

Let $\mathbb{K} = \mathbb{Z}[t]$. If $f = \sum_{\mu \in P} f_\mu x^\mu \in \mathbb{K}[P]$ let

$$\bar{f} = \sum_{\mu \in P} f_\mu x^{-\mu}, \quad \text{and} \quad [f]_1 = f_0 = (\text{coefficient of 1 in } f). \tag{3.1}$$

Define a symmetric bilinear form

$$\langle\ ,\ \rangle_t : \mathbb{K}[P] \times \mathbb{K}[P] \to \mathbb{K} \quad \text{by} \quad \langle f, g\rangle_t = \frac{1}{|W|}\left[f\bar{g}\prod_{\alpha \in R} \frac{1 - x^\alpha}{1 - tx^\alpha}\right]_1. \tag{3.2}$$

"Specializing" $t$ at the values 0 and 1 gives inner products

$$\langle\ ,\ \rangle_0 : \mathbb{K}[P] \times \mathbb{K}[P] \to \mathbb{K} \qquad \text{and} \qquad \langle\ ,\ \rangle_1 : \mathbb{K}[P] \times \mathbb{K}[P] \to \mathbb{K}$$

with

$$\langle f, g\rangle_0 = \frac{1}{|W|}\left[f\bar{g}\prod_{\alpha \in R}(1 - x^\alpha)\right]_1 \quad \text{and} \quad \langle f, g\rangle_1 = \frac{1}{|W|}[f\bar{g}]_1. \tag{3.3}$$

**Proposition 3.1** *Let* $\lambda$ *and* $\mu \in P^+$. *Then*

$$\langle m_\lambda, m_\mu \rangle_1 = \frac{1}{|W_\lambda|}\delta_{\lambda\mu}, \quad \langle s_\lambda, s_\mu \rangle_0 = \delta_{\lambda\mu}, \quad \text{and} \quad \langle P_\lambda, P_\mu \rangle_t = \frac{1}{W_\lambda(t)}\delta_{\lambda\mu}.$$

**Proof** Letting $W\lambda$ denote the $W$-orbit of $\lambda$, the first equality follows from

$$|W_\lambda|\langle m_\lambda, m_\mu \rangle_1 = \frac{|W_\lambda|}{|W|} \sum_{\gamma \in W\lambda, \nu \in W\mu} [x^\gamma x^{-\nu}]_1 = \delta_{\lambda\mu} \frac{|W_\lambda|}{|W|} \sum_{\gamma \in W\lambda} 1 = \delta_{\lambda\mu}.$$

If $\lambda, \mu \in P^+$,

$$\begin{aligned}
\langle s_\lambda, s_\mu \rangle_0 &= \frac{1}{|W|}[\overline{a_\rho s_\lambda} a_\rho s_\mu]_1 = \frac{1}{|W|}[\overline{a_{\lambda+\rho}} a_{\mu+\rho}]_1 \\
&= \frac{1}{|W|} \sum_{v,w \in W} (-1)^{\ell(v)} (-1)^{\ell(w)} [x^{-v(\lambda+\rho)} x^{w(\mu+\rho)}]_1 \\
&= \delta_{\lambda\mu} \frac{1}{|W|} \sum_{v \in W} (-1)^{\ell(v)} (-1)^{\ell(v)} \; = \; \delta_{\lambda\mu},
\end{aligned}$$

giving the second statement.

By Lemma 2.5(b) the matrix $K^{-1}$ given by the values $(K^{-1})_{\lambda\mu}$ in the equation

$$P_\lambda(x;t) = \sum_\mu (K^{-1})_{\lambda\mu} s_\mu,$$

has entries in $\mathbb{Z}[t]$ and is upper triangular with 1's on the diagonal, that is, $(K^{-1})_{\lambda\lambda} = 1$ and $(K^{-1})_{\lambda\mu} = 0$ unless $\mu \leqslant \lambda$. Since $P_\lambda(x;1) = m_\lambda$ the matrix $k^{-1}$ describing the change of basis

$$m_\lambda = \sum_\mu (k^{-1})_{\lambda\mu} s_\mu,$$

is the specialization of $K^{-1}$ at $t = 1$ and so $k^{-1}$ has entries in $\mathbb{Z}$ and is upper triangular with 1's on the diagonal. Then the matrix $A = K^{-1}k^{-1}$ giving the change of basis

$$P_\lambda(x;t) = \sum_{\nu \leqslant \lambda} A_{\lambda\nu} m_\mu, \tag{3.4}$$

has $A_{\lambda\mu} \in \mathbb{Z}[t]$, $A_{\lambda\lambda} = 1$, and $A_{\lambda\mu} = 0$ unless $\mu \leqslant \lambda$.

Let $Q^+$ be the set of nonnegative integral linear combinations of positive roots. Then

$$\begin{aligned}
P_\mu(x;t)W_\mu(t)\Big(\prod_{\alpha\in R}\frac{1-x^\alpha}{1-tx^\alpha}\Big) &= \sum_{w\in W} w\left(x^\mu \prod_{\alpha\in R^+}\frac{1-x^\alpha}{1-tx^\alpha}\right)\\
&= \sum_{w\in W} w\left(x^\mu \prod_{\alpha\in R^+}(1+\sum_{r>0} t^{r-1}(t-1)x^{r\alpha})\right)\\
&= \sum_{w\in W} w\left(\sum_{\nu\in Q^+} c_\nu x^{\mu+\nu}\right)\\
&= \sum_{\nu\in Q^+} c_\nu\left(\sum_{w\in W} w x^{\mu+\nu}\right),
\end{aligned}$$

where $c_\nu \in \mathbb{Z}[t]$ and $c_0 = 1$. Hence

$$P_\mu(x;t)W_\mu(t)\prod_{\alpha\in R}\frac{1-x^\alpha}{1-tx^\alpha} = |W_\mu| m_\mu + \sum_{\gamma>\mu} B_{\mu\gamma}m_\gamma = \sum_{\gamma\geqslant\mu} B_{\mu\gamma}m_\gamma, \qquad (3.5)$$

with $B_{\mu\gamma}\in\mathbb{Z}[t]$ and $B_{\mu\mu} = |W_\mu|$.

Assume that $\lambda \leqslant \mu$ if $\lambda$ and $\mu$ are comparable. Then, by using (3.4) and (3.5),

$$\begin{aligned}
\langle P_\lambda, P_\mu\rangle_t &= \frac{1}{W_\mu(t)}\left\langle P_\lambda, P_\mu W_\mu(t)\prod_{\alpha\in R}\frac{1-x^\alpha}{1-tx^\alpha}\right\rangle_1\\
&= \frac{1}{W_\mu(t)}\left\langle \sum_{\nu\leqslant\lambda} A_{\lambda\nu}m_\nu, \sum_{\gamma\geqslant\mu} B_{\mu\gamma}m_\gamma\right\rangle_1 .
\end{aligned}$$

Since $A_{\lambda\lambda} = 1$ and $B_{\mu\mu} = |W_\mu|$ the result follows from $\langle m_\lambda, m_\mu\rangle_1 = |W_\lambda|^{-1}\delta_{\lambda\mu}$. □

The following theorem shows that the spherical functions $P_\lambda(x,t)$ are uniquely determined by the triangularity in (3.4) and the orthogonality in the third equality of Proposition 3.1.

**Theorem 3.2** *Let $\mathbb{K} = \mathbb{Z}[t]$. The spherical functions $P_\lambda(x;t)$ are the unique elements of $\mathbb{K}[P]^W$ such that*

(a) $P_\lambda = m_\lambda + \sum_{\mu<\lambda} A_{\lambda\mu}m_\mu,$

(b) $\langle P_\lambda, P_\mu\rangle_t = 0$ *if* $\lambda\neq\mu$.

**Proof** Assume that the $P_\mu$ are determined for $\mu < \lambda$. Then the condition in (a) can be rewritten as

$$P_\lambda = m_\lambda + \sum_{\mu<\lambda} C_{\lambda\mu} P_\mu,$$

for some constants $C_{\lambda\mu}$. Take the inner product on each side with $P_\nu$, $\nu < \lambda$, and use property (b) to get the system of equations

$$0 = \langle m_\lambda, P_\nu\rangle_t + \sum_{\mu<\lambda} C_{\lambda\mu}\langle P_\mu, P_\nu\rangle_t = \langle m_\lambda, P_\nu\rangle_t + C_{\lambda\nu}\langle P_\nu, P_\nu\rangle_t.$$

Hence

$$C_{\lambda\nu} = \frac{-\langle m_\lambda, P_\nu\rangle_t}{\langle P_\nu, P_\nu\rangle_t}, \qquad \text{for each } \nu < \lambda,$$

and this determines $P_\lambda$. □

**Remark 3.3** (a) The inner product $\langle\ ,\ \rangle_t$ arises naturally in the context of $p$-adic groups. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and view the $x^\lambda$, $\lambda \in P$, as characters of

$$T = Hom(P, S^1) \quad \textit{via} \quad \begin{array}{cccc} x^\lambda : & T & \longrightarrow & \mathbb{C}^* \\ & s & \longmapsto & s(\lambda). \end{array} \tag{3.6}$$

Let $ds$ be the Haar measure on $T$ normalized so that

$$\langle x^\lambda, x^\mu\rangle = \int_T x^\lambda(s)\overline{x^\mu(s)}ds = \delta_{\lambda\mu}. \tag{3.7}$$

Letting $\mathbb{Q}_p$ be the field of $p$-adic numbers, Macdonald [12, (5.1.2)] showed that the Plancherel measure for the $p$-adic Chevalley group $G(\mathbb{Q}_p)$ corresponding to the root system $R$ is given by

$$d\mu(s) = \frac{W_0(p^{-1})}{|W|}\prod_{\alpha\in R}\frac{1-x^\alpha(s)}{1-p^{-1}x^\alpha(s)}. \tag{3.8}$$

The corresponding inner product is

$$W_0(p^{-1})\langle f, g\rangle_{p^{-1}} = \int_T f(s)\overline{g(s)}d\mu(s), \quad \textit{for } f, g \in C(T),$$

where $C(T)$ is the vector space of continuous functions on $T$.

(b) The inner product $\langle\ ,\ \rangle_t$ arises naturally in another representation theoretic context. The complex semisimple Lie algebra $\mathfrak{g}$ corresponding to the root system $R$ acts on $S(\mathfrak{g}^*)$, the ring of polynomials on $\mathfrak{g}$, by the (co-)adjoint action. As graded $\mathfrak{g}$-modules the characters of $S(\mathfrak{g}^*)$ and the subring of invariants $S(\mathfrak{g}^*)^{\mathfrak{g}}$ are

$$\begin{aligned} \mathrm{grch}(S(\mathfrak{g}^*)) &= \left(\prod_{i=1}^r \frac{1}{1-t}\right)\left(\prod_{\alpha\in R}\frac{1}{1-tx^\alpha}\right) \quad \textit{and} \\ \mathrm{grch}(S(\mathfrak{g}^*)^{\mathfrak{g}}) &= \prod_{i=1}^r \frac{1}{1-t^{d_i}} = \frac{1}{W_0(t)}\prod_{i=1}^r\frac{1}{1-t}, \end{aligned} \tag{3.9}$$

where $r$ is the rank of $\mathfrak{g}$ and $d_1, \ldots, d_r$ are the *degrees* of the Weyl group $W$. Let $\mathcal{H}$ denote the vector space of harmonic polynomials. An important theorem of Kostant [8, Theorem 0.2] gives

$$S(\mathfrak{g}^*) \cong S(\mathfrak{g}^*)^{\mathfrak{g}} \otimes \mathcal{H}, \quad \textit{and thus,} \quad \mathrm{grch}(\mathcal{H}) = W_0(t) \prod_{\alpha \in R} \frac{1}{1 - tx^{\alpha}}. \tag{3.10}$$

If $L(\lambda)$ denotes the finite dimensional irreducible $\mathfrak{g}$-module of highest weight $\lambda \in P^+$ then $L(\lambda)$ has character $s_\lambda$ and using the notation of (3.2),

$$\begin{aligned} &\sum_{k \geqslant 0} \dim(Hom_{\mathfrak{g}}(L(\lambda), L(\mu) \otimes \mathcal{H}^k) t^k \qquad (3.11) \\ &\quad = \left\langle s_\lambda, s_\mu W_0(t) \prod_{\alpha \in R} \frac{1}{1 - tx^{\alpha}} \right\rangle_0 \\ &\quad = W_0(t) \left[ s_\lambda \overline{s_\mu} \prod_{\alpha \in R} \frac{1 - x^{\alpha}}{1 - tx^{\alpha}} \right]_1 = W_0(t) \langle s_\lambda, s_\mu \rangle_t, \end{aligned}$$

where $\mathcal{H}^k$ is the vector space of degree $k$ harmonic polynomials.

### 3.1 Formulae for Kostka–Foulkes polynomials

For $\lambda \in P$ let $s_\lambda$ denote the Weyl character, as defined in (2.6). The *Kostka–Foulkes polynomials*, or *q-weight multiplicities*, $K_{\lambda\mu}(t)$, $\lambda, \mu \in P^+$, are defined by the change of basis formula

$$s_\lambda = \sum_{\mu \in P^+} K_{\lambda\mu}(t) P_\mu(x; t), \tag{3.12}$$

where the Macdonald spherical functions $P_\mu(x; t)$ are as in (2.9).

For each $\alpha \in R^+$ define the *raising operator* $R_\alpha \colon P \to P$ by

$$R_\alpha \lambda = \lambda + \alpha, \quad \text{and define} \quad (R_{\beta_1} \cdots R_{\beta_l}) s_\lambda = s_{R_{\beta_1} \cdots R_{\beta_l} \lambda}, \tag{3.13}$$

for any sequence $\beta_1, \ldots, \beta_l$ of positive roots. Using the straightening law for Weyl characters (2.7),

$$s_\mu = (-1)^{\ell(w)} s_{w \circ \mu}, \quad \text{where} \quad w \circ \mu = w(\mu + \rho) - \rho,$$

any $s_\mu$ is equal to 0 or to $\pm s_\lambda$ with $\lambda \in P^+$. Composing the action of raising operators on Weyl characters should be avoided. For example, if $\alpha_i$ is a simple root then (since $\langle \rho, \alpha_i^\vee \rangle = 1$) $s_{-\alpha_i} = -s_{s_i \circ (-\alpha_i)} = -s_{s_i(\rho - \alpha_i) - \rho} = -s_{-\alpha_i}$ giving that $s_{-\alpha_i} = 0$ and so

$$R_{\alpha_i}(R_{\alpha_i} s_{-2\alpha_i}) = R_{\alpha_i} s_{-\alpha_i} = R_{\alpha_i} \cdot 0 = 0, \text{ whereas } (R_{\alpha_i} R_{\alpha_i}) s_{-2\alpha_i} = s_0 = 1.$$

Let $Q^+$ be the set of nonnegative integral linear combinations of positive roots. Define the *q-analogue of the partition function* $F(\gamma;t)$, $\gamma\in P$, by

$$\prod_{\alpha\in R^+}\frac{1}{1-tx^\alpha}=\sum_{\gamma\in Q^+}F(\gamma;t)x^\gamma,\quad\text{and } F(\gamma;t)=0,\text{ if }\gamma\notin Q^+. \tag{3.14}$$

**Theorem 3.4** *Let $\lambda,\mu\in P^+$. Let $t_\mu$ be the translation in $\mu$ as defined in (1.10) and let $n_\mu$ be the longest element of the double coset $Wt_\mu W$. Let $W_\mu(t)$ be as in (2.8), $P_\mu(x;t)$ as in (2.9) and let $\langle\ ,\ \rangle_t$ be the inner product defined in (3.2). For $y,w\in\tilde W$ let $P_{yw}\in\mathbb{Z}[t^{\pm\frac12}]$ denote the Kazhdan–Lusztig polynomial defined in (1.26)-(1.27) and let $\rho^\vee=\frac12\sum_{\alpha\in R^+}\alpha^\vee$.*

(a) $K_{\lambda,\mu}(t)=W_\mu(t)\,\langle s_\lambda,P_\mu(x;t)\rangle_t$.

(b) $K_{\lambda\mu}(t)=$ *coefficient of* $s_\lambda$ *in* $\displaystyle\left(\prod_{\alpha\in R^+}\frac{1}{1-tR_\alpha}\right)s_\mu$.

(c) $K_{\lambda\mu}(t)=\displaystyle\sum_{w\in W}(-1)^{\ell(w)}F(w(\lambda+\rho)-(\mu+\rho);t)$.

(d) $K_{\lambda\mu}(t)=t^{\langle\lambda-\mu,\rho^\vee\rangle}P_{x,n_\lambda}(t^{-1})$, *for any* $x\in Wt_\mu W$.

**Proof** (a) This follows from the third equality in Proposition 3.1 and the definition of $K_{\lambda\mu}(t)$.
(b) Since

$$\begin{aligned}
&P_\mu(x;t)W_\mu(t)\prod_{\alpha\in R}\frac{1}{1-tx^\alpha}\\
&=\sum_{w\in W}w\left(x^\mu\prod_{\alpha\in R^+}\frac{1-tx^{-\alpha}}{1-x^{-\alpha}}\right)\prod_{\alpha\in R}\frac{1}{1-tx^\alpha}\\
&=\sum_{w\in W}w\left(x^{\mu+\rho}\frac{1}{x^\rho\prod_{\alpha\in R^+}(1-x^{-\alpha})(1-tx^\alpha)}\right)\\
&=\frac{1}{a_\rho}\sum_{w\in W}(-1)^{\ell(w)}w\left(\prod_{\alpha\in R^+}\left(\frac{1}{1-tx^\alpha}\right)x^{\mu+\rho}\right),
\end{aligned}$$

it follows that

$$\begin{aligned}
K_{\lambda\mu}(t)&=(\text{coefficient of }P_\mu(x;t)\text{ in }s_\lambda)=\langle s_\lambda,W_\mu(t)P_\mu(x;t)\rangle_t\\
&=\left\langle s_\lambda,W_\mu(t)P_\mu(x;t)\prod_{\alpha\in R}\frac{1}{1-tx^\alpha}\right\rangle_0\\
&=\text{coefficient of }s_\lambda\text{ in }\ \frac{1}{a_\rho}\sum_{w\in W}(-1)^{\ell(w)}w\left(\prod_{\alpha\in R^+}\left(\frac{1}{1-tx^\alpha}\right)x^{\mu+\rho}\right)\\
&=\text{coefficient of }s_\lambda\text{ in }\ \left(\prod_{\alpha\in R^+}\frac{1}{1-tR_\alpha}\right)s_\mu\,.
\end{aligned}$$

(c)

$$\begin{aligned} K_{\lambda\mu}(t) &= \text{coefficient of } s_\lambda \text{ in } \quad \frac{1}{a_\rho}\sum_{w\in W}(-1)^{\ell(w)}w\left(\prod_{\alpha\in R^+}\Big(\frac{1}{1-tx^\alpha}\Big)x^{\mu+\rho}\right) \\ &= \text{coefficient of } a_{\lambda+\rho} \text{ in } \quad \sum_{w\in W}(-1)^{\ell(w)}w\left(\Big(\sum_{\gamma\in Q^+}F(\gamma;t)x^\gamma\Big)x^{\mu+\rho}\right) \\ &= \text{coefficient of } x^{\lambda+\rho} \text{ in } \quad \sum_{w\in W}(-1)^{\ell(w)}w\left(\sum_{\gamma\in Q^+}F(\gamma;t)x^{\gamma+\mu+\rho}\right) \\ &= \sum_{w\in W}(-1)^{\ell(w)}F(w(\lambda+\rho)-(\mu+\rho);t), \end{aligned}$$

since $w^{-1}(\gamma+(\mu+\rho))=\lambda+\rho$ implies $\gamma=w(\lambda+\rho)-(\mu+\rho)$.

(d) Let $\lambda\in P^+$. By Theorem 2.9 and Lemma 2.7

$$\begin{aligned} \sum_{x\leqslant n_\lambda}q^{-(\ell(n_\lambda)-\ell(x))}P_{x,n_\lambda}(q^2)T_x &= C'_{n_\lambda}=q^{-\ell(w_0)}W_0(q^2)s_\lambda\mathbf{1}_0 \\ &= q^{-\ell(w_0)}W_0(q^2)\sum_{\mu\leqslant\lambda}K_{\lambda\mu}(q^{-2})P_\mu(x;q^{-2})\mathbf{1}_0 \\ &= q^{-\ell(w_0)}W_0(q^2)\sum_{\mu\leqslant\lambda}K_{\lambda\mu}(q^{-2})\frac{W_0(q^{-2})}{W_\mu(q^{-2})}M_\mu \\ &= \sum_{\mu\leqslant\lambda}K_{\lambda\mu}(q^{-2})\sum_{x\in Wt_\mu W}q^{\ell(x)-\ell(n_\mu)}T_x. \end{aligned}$$

Hence, for $\mu\leqslant\lambda$ and $x\in Wt_\mu W$,

$$K_{\lambda\mu}(q^{-2})=q^{\ell(n_\mu)-\ell(n_\lambda)}P_{x,n_\lambda}(q^2).$$

By (2.15) and (2.16),

$$\ell(n_\mu)-\ell(n_\lambda)=\ell(t_\mu)+\ell(w_0)-(\ell(t_\lambda)+\ell(w_0))=2\langle\mu,\rho^\vee\rangle-2\langle\lambda,\rho^\vee\rangle,$$

and the result follows on replacing $q^{-2}$ by $t$. ☐

With the notation of Remark 3.3(b), it follows from Theorem 3.4(a) and $s_0=P_0(x;t)$ that

$$\begin{aligned} K_{\lambda,0}(t) &= W_0(t)\langle s_\lambda,P_0(x;t)\rangle_t=W_0(t)\langle s_\lambda,s_0\rangle_t \\ &= \sum_{k\geqslant 0}\dim(\mathrm{Hom}_{\mathfrak{g}}(L(\lambda),\mathcal{H}^k)t^k. \end{aligned} \tag{3.15}$$

This is an important formula for the Kostka–Foulkes polynomial in the case that $\mu = 0$.

Let $J \subset \{\alpha_1, \ldots, \alpha_n\}$ be a subset of the set of simple roots and let

$$\mathfrak{h}_J^* = \mathbb{R}\text{-span}\{\alpha_j \in J\}, \qquad R_J = R \cap \mathfrak{h}_J^*, \qquad R_J^+ = R^+ \cap \mathfrak{h}_J^*, \tag{3.16}$$

$$W_J = \langle s_j \mid \alpha_j \in J\rangle, \qquad \text{and} \qquad P_J^+ = P^+ \cap \mathfrak{h}_J^*, \tag{3.17}$$

so that $R_J$ is a *parabolic subsystem* of the root system $R$, $R_J^+$ is the set of positive roots of $R_J$, $W_J$ is the Weyl group of $R_J$, and $P_J^+$ is the set of dominant integral weights for $R_J$. Let $\mathfrak{h}_J^\perp$ be the orthogonal complement to $\mathfrak{h}_J^*$ with respect to the inner product $\langle , \rangle$ so that

$$\mathfrak{h}^* = \mathfrak{h}_J^* \oplus \mathfrak{h}_J^\perp, \qquad \text{and write} \qquad \mu = \mu_J + \mu_J^\perp, \tag{3.18}$$

to denote the decomposition of an element $\mu \in \mathfrak{h}^*$ as a sum of $\mu_J \in \mathfrak{h}_J^*$ and $\mu_J^\perp \in \mathfrak{h}_J^\perp$.

**Proposition 3.5** *Let $J$ be a subset of the set of simple roots $\{\alpha_1, \ldots, \alpha_n\}$ and use notations as in (3.16-3.18). Then, for $\lambda, \mu \in P^+$,*

$$K_{\lambda\mu}(t) = \textit{coefficient of } s_\lambda \textit{ in } \left(\prod_{\alpha \in (R^+ \setminus R_J^+)} \frac{1}{1 - tR_\alpha}\right) \sum_{\lambda_J \in P_J^+} K_{\lambda_J \mu_J}(t) s_{\lambda_J + \mu_J^\perp},$$

*where $K_{\lambda_J\mu_J}(t)$ are Kostka-Foulkes polynomials for the root system $R_J$.*

**Proof** By the third equation in the proof of Theorem 3.4(b), $K_{\lambda\mu}(t)$ is the coefficient of $a_{\lambda+\rho}$ in

$$\begin{aligned}
&\sum_{w\in W} (-1)^{\ell(w)} w \left(\prod_{\alpha \in R^+} \left(\frac{1}{1 - tx^\alpha}\right) x^{\mu+\rho}\right) \\
&= \frac{1}{|W_J|} \sum_{w\in W} (-1)^{\ell(w)} w \sum_{v \in W_J} (-1)^{\ell(v)} v \\
&\qquad \left(\left(\prod_{\alpha \in (R^+ \setminus R_J^+)} \frac{1}{1 - tx^\alpha}\right)\left(\prod_{\alpha \in R_J^+} \frac{1}{1 - tx^\alpha}\right) x^{\mu_J + \rho_J} x^{\mu_J^\perp + \rho_J^\perp}\right) \\
&= \frac{1}{|W_J|} \sum_{w\in W} (-1)^{\ell(w)} w \left(\left(\prod_{\alpha \in (R^+ \setminus R_J^+)} \frac{1}{1 - tx^\alpha}\right) x^{\mu_J^\perp + \rho_J^\perp}\right. \\
&\qquad \left.\sum_{v \in W_J} (-1)^{\ell(v)} v \left(\prod_{\alpha \in R_J^+} \frac{1}{1 - tx^\alpha}\right) x^{\mu_J + \rho_J}\right) \\
&= \frac{1}{|W_J|} \sum_{w\in W} (-1)^{\ell(w)} w \left(\left(\prod_{\alpha \in (R^+ \setminus R_J^+)} \frac{1}{1 - tx^\alpha}\right) x^{\mu_J^\perp + \rho_J^\perp} \sum_{\lambda_J \in P_J^+} K_{\lambda_J\mu_J}(t) a_{\lambda_J + \rho_J}\right),
\end{aligned}$$

where the last equality follows from Theorem 3.4(b) applied to the root system $R_J$. Expanding $a_{\lambda_J+\rho_J}$ gives that $K_{\lambda\mu}(t)$ is the coeffient of $a_{\lambda+\rho}$ in

$$\begin{aligned}
&\frac{1}{|W_J|}\sum_{\lambda_J\in P_J^+} K_{\lambda_J\mu_J}(t)\sum_{w\in W}(-1)^{\ell(w)}w\Bigg(\Big(\prod_{\alpha\in(R^+\setminus R_J^+)}\frac{1}{1-tx^\alpha}\Big)x^{\mu_J^\perp+\rho_J^\perp} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \sum_{v\in W_J}(-1)^{\ell(v)}v\,x^{\lambda_J+\rho_J}\Bigg) \\
&= \frac{1}{|W_J|}\sum_{\lambda_J\in P_J^+} K_{\lambda_J\mu_J}(t)\sum_{w\in W}(-1)^{\ell(w)}w\sum_{v\in W_J}(-1)^{\ell(v)}v \\
&\qquad\qquad\qquad\qquad \Bigg(\Big(\prod_{\alpha\in(R^+\setminus R_J^+)}\frac{1}{1-tx^\alpha}\Big)x^{\mu_J^\perp+\rho_J^\perp+\lambda_J+\rho_J}\Bigg) \\
&= \sum_{\lambda_J\in P_J^+} K_{\lambda_J\mu_J}(t)\sum_{w\in W}(-1)^{\ell(w)}w\Bigg(\Big(\prod_{\alpha\in(R^+\setminus R_J^+)}\frac{1}{1-tx^\alpha}\Big)x^{\lambda_J+\mu_J^\perp+\rho}\Bigg)
\end{aligned}$$

from which the desired formula follows by dividing by $a_\rho$ and converting to raising operators (as in the proof of Theorem 3.4(b) above). □

## 4 The positive formula

In the type A case Lascoux and Schützenberger [9] have used the theory of column strict tableaux to give a positive formula for the Kostka–Foulkes polynomial. In this section we give a proof of this formula. Versions of this proof have appeared previously in [17] and in [2].

The starting point is the formula for $K_{\lambda\mu}(t)$ in Theorem 3.4(b). To match the setup in [11] we shall work in a slightly different setting (corresponding to the Weyl group $W$ and the weight lattice of the reductive group $GL_n(\mathbb{C})$). In this case the vector space $\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}^n$ has orthonormal basis $\varepsilon_1, \ldots, \varepsilon_n$, where $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 in the $i$th coordinate, the Weyl group is the symmetric group $S_n$ acting on $\mathbb{R}^n$ by permuting the coordinates, the weight lattice $P$ is replaced by the lattice

$$\mathbb{Z}^n = \{(\gamma_1, \ldots, \gamma_n) \mid \gamma_i \in \mathbb{Z}\} \quad \text{and} \quad \delta = (n-1, n-2, \ldots, 2, 1, 0) \tag{4.1}$$

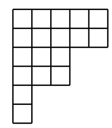
replaces the element $\rho$. The positive roots are given by $R^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i < j \leqslant n\}$ and the Schur functions (defined as in (2.6)) are viewed as (Laurent) polynomials in the variables $x_1, \ldots, x_n$, where $x_i = x^{\varepsilon_i}$ and the symmetric group $S_n$ acts by permuting the variables. If $w \in S_n$ then $(-1)^{\ell(w)} = \det(w)$ is the *sign* of the permutation $w$ and the straightening law for Schur functions (see (2.7) and [11, I paragraph after (3.1)]) is

$$s_\mu = (-1)^{\ell(w)} s_{w\circ\mu}, \quad \text{where} \quad w\circ\mu = w(\mu+\delta) - \delta, \tag{4.2}$$

for $w \in S_n$ and $\mu \in \mathbb{Z}^n$. The set of *partitions*

$$\mathcal{P} = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0\} \tag{4.3}$$

takes the role played by the set $P^+$. Conforming to the conventions in [11] so that gravity goes up and to the left, each partition $\mu = (\mu_1, \ldots, \mu_n) \in \mathcal{P}$ is identified with the collection of boxes in a corner which has $\mu_i$ boxes in row $i$, where, as for matrices, the rows and columns of $\mu$ are indexed from top to bottom and left to right, respectively. For example, with $n = 7$,

$(5, 5, 3, 3, 1, 1, 0) =$  .

For each pair $1 \leqslant i < j \leqslant n$ define the *raising operator* $R_{ij} \colon \mathbb{Z}^n \to \mathbb{Z}^n$ (see (3.13) and [11, I §1 (1.14)]) by

$$R_{ij}\mu = \mu + \varepsilon_i - \varepsilon_j \quad \text{and define} \quad (R_{i_1j_1} \cdots R_{i_lj_l})s_\mu = s_{R_{i_1j_1}\cdots R_{i_lj_l}\mu}, \tag{4.4}$$

for a sequence of pairs $i_1 < j_1, \ldots, i_l < j_l$. Using the straightening law (4.2) any Schur function $s_\mu$ indexed by $\mu \in \mathbb{Z}^n$ with $\mu_1 + \cdots + \mu_n \geqslant 0$ is either equal to 0 or to $\pm s_\lambda$ for some $\lambda \in \mathcal{P}$. Composing the action of raising operators on Schur functions $s_\lambda$ should be avoided. For example, if $n = 2$ and $s_1$ denotes the transposition in the symmetric group $S_2$ then, by the straightening law, $s_{(0,1)} = -s_{s_1((0,1)+(1,0))-(1,0)} = -s_{(1,1)-(1,0)} = -s_{(0,1)}$ giving that $s_{(0,1)} = 0$ and so

$$R_{12}(R_{12}s_{(-1,2)}) = R_{12}s_{(0,1)} = R_{12} \cdot 0 = 0, \qquad \text{whereas}$$

$$(R_{12}^2)s_{(-1,2)} = s_{(1,0)} = x_1 + x_2.$$

With notation as in (4.2) and (4.4) we may define the *Hall–Littlewood polynomials* for this type A case by (see Theorem 3.4(b) and [11, III (4.6)])

$$Q_\mu = \left( \prod_{1 \leqslant i < j \leqslant n} \frac{1}{1 - tR_{ij}} \right) s_\mu, \quad \text{for all } \mu \in \mathbb{Z}^n, \tag{4.5}$$

and the *Kostka–Foulkes polynomials* $K_{\lambda\mu}(t)$, $\lambda, \mu \in \mathcal{P}$, by

$$Q_\mu = \sum_{\lambda \in \mathcal{P}} K_{\lambda\mu}(t) s_\lambda. \tag{4.6}$$

## 4.1 Insertion and Pieri rules

Let $\lambda, \mu \in \mathbb{Z}^n$ be partitions. A *column strict tableau of shape* $\lambda$ *and weight* $\mu$ is a filling of the boxes of $\lambda$ with $\mu_1$ 1s, $\mu_2$ 2s, …, $\mu_n$ $n$s, such that

(a) the rows are weakly increasing from left to right,

(b) the columns are strictly increasing from top to bottom.

If $T$ is a column strict tableau write $\mathrm{shp}(T)$ and $\mathrm{wt}(T)$ for the shape and the weight of $T$ so that

$$\begin{aligned} \mathrm{shp}(T) &= (\lambda_1, \ldots, \lambda_n), \quad \text{where } \lambda_i = \text{number of boxes in row } i \text{ of } T, \quad \text{and} \\ \mathrm{wt}(T) &= (\mu_1, \ldots, \mu_n), \quad \text{where } \mu_i = \text{number of } i\,\text{s in } T. \end{aligned}$$

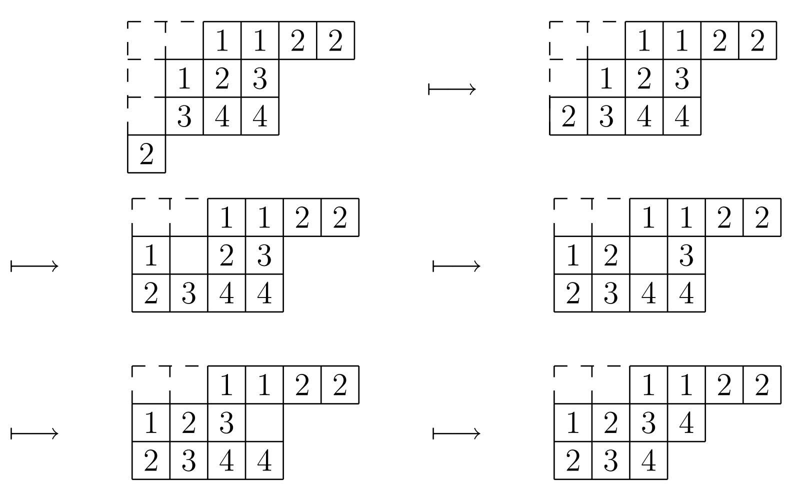
For example,

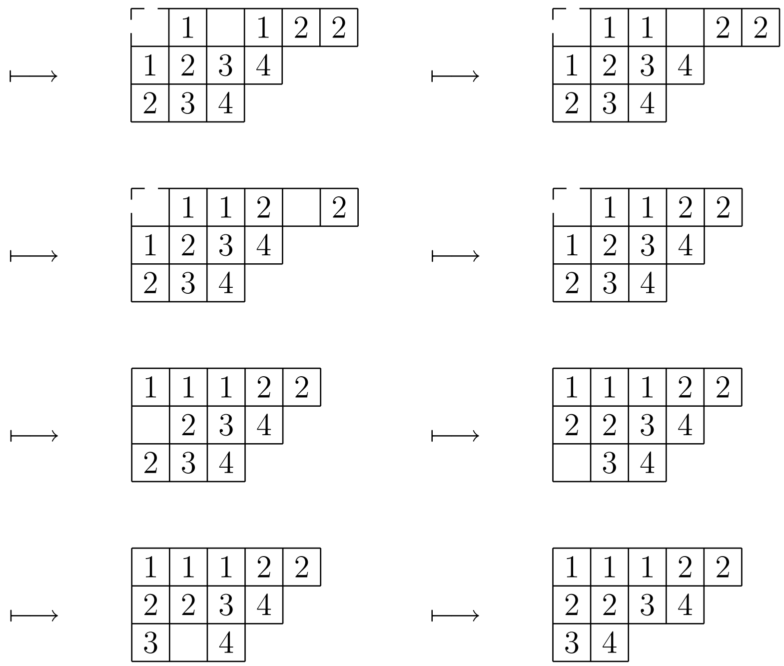
$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1&1&1&1&1&1&1&2&2 \\ \hline 2&2&2&2&3&3&4 \\ \cline{1-7} 3&3&3&4&4&4&5 \\ \cline{1-7} 4&5&5&6 \\ \cline{1-4} 6&7 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} \qquad \begin{array}{rl} \text{has} & \mathrm{shp}(T) = (9,7,7,4,2,1,0) \\ \text{and} & \mathrm{wt}(T) = (7,6,5,5,3,2,2). \end{array}$$

For partitions $\lambda$ and $\mu$ and, more generally, for any two sets $\mathcal{S}, \mathcal{W} \subseteq \mathcal{P}$ write

$$\begin{aligned} B(\lambda) &= \{\text{column strict tableaux } T \mid \mathrm{shp}(T) = \lambda\}, \\ B(\lambda)_\mu &= \{\text{column strict tableaux } T \mid \mathrm{shp}(T) = \lambda,\ \mathrm{wt}(T) = \mu\}, \quad (4.7) \\ B(\mathcal{S})_\mathcal{W} &= \{\text{column strict tableaux } T \mid \mathrm{shp}(T) \in \mathcal{S},\ \mathrm{wt}(T) \in \mathcal{W}\}. \end{aligned}$$

Let $\lambda$ and $\gamma$ be partitions such that $\gamma \subseteq \lambda$ (as collections of boxes in a corner, that is $\gamma_i \leqslant \lambda_i$ for $1 \leqslant i \leqslant n$). The *skew shape* $\lambda/\gamma$ is the collection of boxes of $\lambda$ which are not in $\gamma$. The *jeu de taquin* reduces a column strict filling of a skew shape $\lambda/\gamma$ to a column strict tableau of partition shape. At each step "gravity" moves one box up or to the left without violating the column strict condition (weakly increasing in rows, strictly increasing in columns). Once an empty box on the northwest side of the skew shape starts to move it must continue and exit the southeast border of the skew shape before another empty box can start its exit. The jeu de taquin is most easily illustrated by example:

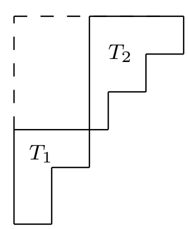





In this example $\lambda = (6, 4, 4, 1)$, $\gamma = (2, 1, 1)$ and the resulting column strict tableau is of shape $(5, 4, 2)$. The result of the jeu de taquin is independent of the choice of order of the moves ([3, §1.2, Claim 2] which is proved in [3, §2 and §3].

The *plactic monoid* is the set $B(\mathcal{P})$ of column strict tableaux with product given by

$T_1 * T_2 =$ jeu de taquin reduction of




Because the result of the jeu de taquin is independent of the choice of the order of the moves this is an associative monoid.

If $x$ is a "letter", that is, a column strict tableau of shape $(1) = \square$, then

$$\begin{array}{l} x * T \text{ is the } \textit{column insertion} \text{ of } x \text{ into } T, \quad \text{and} \\ T * x \text{ is the } \textit{row insertion} \text{ of } x \text{ into } T. \end{array} \tag{4.8}$$

The shape $\lambda$ of $P = T * x$ differs from the shape $\gamma$ of $T$ by single box and so if $\gamma$ and $P$ are given then the pair $(T, x)$ can be recovered by "uninserting" the box $\lambda/\gamma$ from $P$. The tableaux $P$ and $T$ differ by at most one entry in each row. The entries where $P$ and $T$ differ form the *bumping path of* $x$. The bumping path begins with $x$ in the first row of $P$ and ends at the entry in the box $\lambda/\gamma$. For example,

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&2&2\\\hline 2&3&3&4\\\cline{1-4} 4&4&4&5\\\cline{1-4} 6\\\cline{1-1}\end{array} \quad * \quad \boxed{1} \quad = \quad \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&\mathbf{1}&2\\\hline 2&\mathbf{2}&3&4\\\cline{1-4} \mathbf{3}&4&4&5\\\cline{1-4} \mathbf{4}\\\cline{1-1} \mathbf{6}\\\cline{1-1}\end{array},$$

where the bold face entries form the bumping path.

The *monoid of words* is the free monoid $B^*$ generated by $\{1, 2, \ldots, n\}$. The *weight* $\mathrm{wt}(w)$ of a word $w = w_1 \cdots w_n$ is

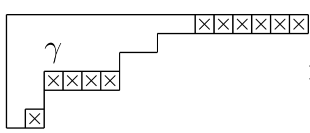
$$\mathrm{wt}(w) = \mathrm{wt}(w_1 \cdots w_n) = (\mu_1, \ldots, \mu_n) \quad \text{where} \quad \mu_i \text{ is the number of } i\text{'s in } w.$$

For example, $w = 3214566532211$ is a word of weight $\mathrm{wt}(w) = (3, 3, 2, 1, 2, 2)$. The *insertion* map

$$\begin{array}{ccc} B^* & \longrightarrow & B(\mathcal{P}) \\ w_1 \cdots w_n & \longmapsto & w_1 * \cdots * w_n \end{array} \tag{4.9}$$

is a weight preserving homomorphism of monoids.

A *horizontal strip* is a skew shape which contains at most one box in each column. The *length* of a horizontal strip $\lambda/\gamma$ is the number of boxes in $\lambda/\gamma$. The boxes containing $\times$ in the picture

$\lambda =$  form a horizontal strip $\lambda/\gamma$ of length 11.

For partitions $\mu$ and $\gamma$ and a nonnegative integer $r$ let

$$\begin{array}{rcl} \gamma \otimes (r) = (r) \otimes \gamma &=& \{\text{partitions } \lambda \mid \lambda/\gamma \text{ is a horizontal strip of length } r\}, \\ (B(r) \otimes B(\gamma))_\mu &=& \left\{ \text{pairs } v \otimes T \;\middle|\; \begin{array}{l} v \in B(r), T \in B(\gamma) \\ \text{such that } \mathrm{wt}(v) + \mathrm{wt}(T) = \mu \end{array} \right\}, \\ (B(\gamma) \otimes B(r))_\mu &=& \left\{ \text{pairs } T \otimes v \;\middle|\; \begin{array}{l} v \in B(r), T \in B(\gamma) \\ \text{such that } \mathrm{wt}(v) + \mathrm{wt}(T) = \mu \end{array} \right\}, \end{array} \tag{4.10}$$

The following lemma gives tableau versions of the Pieri rule [11, I (5.16)]. The second bijection of the lemma is proved in [3, §1.1 Proposition], and the proof of the first bijection is similar (see also [2, Propositions 2.3.4 and 2.3.11]).

**Lemma 4.1** *Let $\gamma, \mu, \tau \in \mathcal{P}$ be partitions and let $r, s \in \mathbb{Z}_{\geqslant 0}$. There are bijections*

$$\begin{array}{ccc} (B(r) \otimes B(\gamma))_\mu & \longleftrightarrow & B(\gamma \otimes (r))_\mu \\ v \otimes T & \longrightarrow & v * T \end{array} \qquad \text{and}$$

$$\begin{array}{ccc} (B(\gamma) \otimes B(s))_\tau & \longleftrightarrow & B(\gamma \otimes (s))_\tau \\ T \otimes u & \longrightarrow & T * u \end{array}.$$

### 4.2 Charge

Let $B(\mathcal{P})_{\geqslant} = \bigcup_{1\leqslant i\leqslant n} B(\mathcal{P})_{\geqslant i}$, where

$$B(\mathcal{P})_{\geqslant i} = \left\{ \text{column strict tableaux } b \;\middle|\; \begin{array}{l} \mathrm{wt}(b) = (\mu_1, \ldots, \mu_n) \text{ has} \\ \mu_1 = \cdots = \mu_{i-1} = 0 \text{ and} \\ \mu_i \geqslant \cdots \geqslant \mu_n \geqslant 0 \end{array} \right\}.$$

Let $i^k = \boxed{i}\boxed{i}\boxed{\cdots}\boxed{i}$ be the unique column strict tableau of shape $(k)$ and weight $(0, \ldots, k, 0, \ldots, 0)$, where the $k$ appears in the $i$ th entry. *Charge* is the function $\mathrm{ch}\colon B(\mathcal{P})_{\geqslant} \longrightarrow \mathbb{Z}_{\geqslant 0}$ such that

(a) $\mathrm{ch}(\emptyset) = 0$,

(b) if $T \in B(\mathcal{P})_{\geqslant(i+1)}$ and $T * i^{\mu_i} \in B(\mathcal{P})_{\geqslant i}$ then $\mathrm{ch}(T * i^{\mu_i}) = \mathrm{ch}(T)$,

(c) if $T \in B(\mathcal{P})_{\geqslant i}$ and $x$ is a letter not equal to $i$ then $\mathrm{ch}(x*T) = \mathrm{ch}(T*x)+1$.

The proof of the existence and uniqueness of the function ch is presented beautifully in [7].

**Theorem 4.2** (Lascoux-Schützenberger [9], [17]) *For partitions $\lambda$ and $\mu$,*

$$K_{\lambda\mu}(t) = \sum_{b\in B(\lambda)_\mu} t^{\mathrm{ch}(b)},$$

*where the sum is over all column strict tableaux $b$ of shape $\lambda$ and weight $\mu$.*

**Proof** The proof is by induction on $n$. Assume that the statement of the theorem holds for all partitions $\mu = (\mu_1, \ldots, \mu_n)$. We shall prove that, for all partitions $(\mu_0, \mu) = (\mu_0, \mu_1, \ldots, \mu_n)$, $Q_{(\mu_0,\mu)}$ has an expansion

$$Q_{(\mu_0,\mu)} = \sum_{p\in B(\nu)_{(\mu_0,\mu)}} t^{\mathrm{ch}(p)} s_\nu, \tag{4.11}$$

Beginning with the expression (4.5),

$$\begin{aligned} Q_{(\mu_0,\mu)} &= \left(\prod_{0\leqslant i<j\leqslant n} \frac{1}{1-tR_{ij}}\right) s_{(\mu_0,\mu)} \\ &= \left(\prod_{j=1}^{n} \frac{1}{1-tR_{0j}}\right) \left(\prod_{1\leqslant i<j\leqslant n} \frac{1}{1-tR_{ij}}\right) s_{(\mu_0,\mu)}. \end{aligned}$$

Proposition 3.5 shows that this particular product of raising operators can be composed and so, by applying the definition of the Kostka–Foulkes polynomials (4.6),

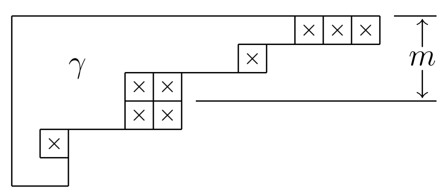
$$\begin{aligned} Q_{(\mu_0,\mu)} &= \left(\prod_{j=1}^{n} \frac{1}{1-tR_{0j}}\right) \sum_{\lambda\in\mathcal{P}} K_{\lambda\mu}(t) s_{(\mu_0,\lambda)} \\ &= \sum_{\lambda\in\mathcal{P}} K_{\lambda\mu}(t) \sum_{r\in\mathbb{Z}_{\geqslant 0}} t^r \sum_{\substack{k_1,\ldots,k_n\in\mathbb{Z}_{\geqslant 0} \\ k_1+\cdots+k_n=r}} R_{01}^{k_1}\cdots R_{0n}^{k_n} s_{(\mu_0,\lambda)} \\ &= \sum_{\lambda\in\mathcal{P}} K_{\lambda\mu}(t) \sum_{r\in\mathbb{Z}_{\geqslant 0}} t^r \sum_{\substack{k_1,\ldots,k_n\in\mathbb{Z}_{\geqslant 0} \\ k_1+\cdots+k_n=r}} s_{(\mu_0+r,\lambda-(k_1,\ldots,k_n))} \end{aligned}$$

Let $\gamma = \lambda - (k_1, \ldots, k_n)$ be such that $\lambda/\gamma$ is not a horizontal strip (usually $\gamma$ isn't even a partition). Let $m$ be the first place a violation to being a horizontal strip occurs, that is,

$$\text{let } m \text{ be minimal such that } \lambda_m - k_m < \lambda_{m+1}.$$

For example, in the followng picture, $\gamma = \lambda - (3, 1, 2, 2, 1, 0)$ and $m = 3$.

$\lambda =$




Let $s_m$ be the simple transposition in the symmetric group which switches $m$ and $m + 1$ and define

$$\tilde{\gamma} = s_m \circ \gamma, \quad \text{so that} \quad s_{(\mu_0+r,\gamma)} = -s_{(\mu_0+r,\tilde{\gamma})}.$$

Then $\tilde{\gamma} = \lambda - (\tilde{k}_1, \ldots, \tilde{k}_n)$ with $\lambda_i - \tilde{k}_i = \lambda_i - k_i$, for $i \neq m, m+1$, and

$$\lambda_m - \tilde{k}_m = \lambda_{m+1} - k_{m+1} - 1, \quad \text{and} \quad \lambda_{m+1} - \tilde{k}_{m+1} = \lambda_m - k_m + 1.$$

Thus $\tilde{\gamma}_m = \lambda_{m+1} - k_{m+1} - 1 < \lambda_{m+1}$ and so $\lambda/\tilde{\gamma}$ is not a horizontal strip. This pairing $\gamma \leftrightarrow \tilde{\gamma}$ provides a cancellation in the expression for $Q_{(\mu_0,\mu)}$ and thus

$$\begin{aligned} Q_{(\mu_0,\mu)} &= \sum_{\lambda\in\mathcal{P}} \sum_{r\in\mathbb{Z}_{\geqslant 0}} t^r K_{\lambda\mu}(t) \sum_{\substack{\gamma\in\mathcal{P} \\ \lambda\in\gamma\otimes(r)}} s_{(\mu_0+r,\gamma)} \\ &= \sum_{\gamma,r} \sum_{\substack{\lambda\in\mathcal{P} \\ \lambda\in\gamma\otimes(r)}} t^r K_{\lambda\mu}(t) s_{(\mu_0+r,\gamma)}\,, \end{aligned}$$

where $\gamma \otimes (r)$ is as defined in (4.10). Then, by the induction assumption,

$$\begin{aligned} Q_{(\mu_0,\mu)} &= \sum_{\gamma,r} \sum_{\substack{\lambda\in\mathcal{P} \\ \lambda\in\gamma\otimes(r)}} \sum_{b\in B(\lambda)_\mu} t^r t^{\mathrm{ch}(b)} s_{(\mu_0+r,\gamma)} \\ &= \sum_{\gamma,r} \sum_{b\in B(\gamma\otimes(r))_\mu} t^{r+\mathrm{ch}(b)} s_{(\mu_0+r,\gamma)}, \end{aligned}$$

with $B(\gamma\otimes(r))_\mu$ as in (4.10). By the first bijection in Lemma 4.1 this can be rewritten as

$$\begin{aligned} Q_{(\mu_0,\mu)} &= \sum_{\gamma,r} \sum_{v\otimes T\in(B(r)\otimes B(\gamma))_\mu} t^{r+\mathrm{ch}(v*T)} s_{(\mu_0+r,\gamma)} \\ &= \sum_{\gamma,r} \sum_{v\otimes T\in(B(r)\otimes B(\gamma))_\mu} t^{r+\mathrm{ch}(v*T*0^{\mu_0}))} s_{(\mu_0+r,\gamma)} \qquad (4.12) \\ &= \sum_{\gamma,r} \sum_{v\otimes T\in(B(r)\otimes B(\gamma))_\mu} t^{\mathrm{ch}(T*0^{\mu_0}*v))} s_{(\mu_0+r,\gamma)}, \end{aligned}$$

where the last two equalities come from the defining properties of the charge function ch.

Let $v\otimes T\in(B(r)\otimes B(\gamma))_\mu$ and let

$$p = T*0^{\mu_0}*v \quad \text{and} \quad \nu = \mathrm{shp}(T*0^{\mu_0}*v).$$

Let $d$ be such that

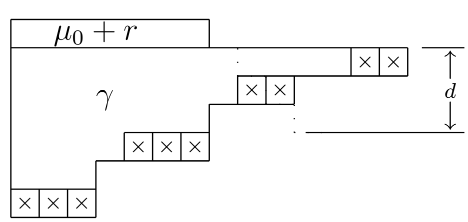
$$\mu_0+r+d > \nu_d \quad \text{and} \quad \mu_0+r+d-1 \leqslant \nu_{d-1},$$

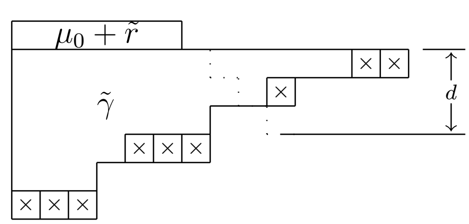
where, by convention, $\nu_0 = \mu_0+r$. If $d>1$ define $\tilde\gamma$ and $\tilde r$ by

$$\tilde\gamma = (\gamma_1,\ldots,\gamma_{d-2},\mu_0+r+d-1,\gamma_d,\ldots,\gamma_n) \quad \text{and} \quad \mu_0+\tilde r+d-1 = \gamma_{d-1},$$

so that, if $s_i$ denotes the transposition $(i,i+1)$ in the symmetric group, then $(\mu_0+\tilde r,\tilde\gamma) = (s_0\cdots s_{d-3}s_{d-2}s_{d-3}\cdots s_0)\circ(\mu_0+r,\gamma)$, and

$$s_{(\mu_0+r,\gamma)} = (-1)^{2(d-3)+1} s_{(\mu_0+\tilde r,\tilde\gamma)} = -s_{(\mu_0+\tilde r,\tilde\gamma)}. \qquad (4.13)$$

$\nu =$ 


$=$ 

Note that $\tilde{\tilde{\gamma}} = \gamma$ and $\tilde{\tilde{r}} = r$.

*Case 1:* $d > 1$ and $(\mu_0 + r, \gamma) = (\mu_0 + \tilde{r}, \tilde{\gamma})$. In this case (4.13) implies $s_{(\mu_0+r,\gamma)} = 0$.

*Case 2:* $d > 1$ and $(\mu_0 + r, \gamma) \neq (\mu_0 + \tilde{r}, \tilde{\gamma})$. Then

$$\nu \in \gamma \otimes (\mu_0 + r) \qquad \text{and} \qquad \nu \in \tilde{\gamma} \otimes (\mu_0 + \tilde{r}).$$

Row uninserting the horizontal strips $\nu/\gamma$ and $\nu/\tilde{\gamma}$ from $p$, by using the second bijection in Lemma 4.1, produces pairs

$$T \otimes u = T \otimes (0^{\mu_0} * v) \in (B(\gamma) \otimes B(\mu_0 + r))_{(\mu_0,\mu)}$$

and

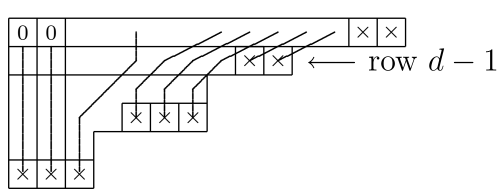
$$\tilde{T} \otimes \tilde{u} \in (B(\tilde{\gamma}) \otimes B(\mu_0 + \tilde{r}))_{(\mu_0,\mu)},$$

respectively. Consider the $\ell = \mu_0 + r$ bumping paths in the tableau $p$ which arise from $T * u$. These begin with the letters $u_1 \leqslant \ldots \leqslant u_\ell$ of $u$ and end at the boxes of the horizontal strip $\nu/\gamma$. Similarly, there are $\tilde{\ell} = \mu_0 + \tilde{r}$ bumping paths in $p$ arising from $\tilde{T} * \tilde{u}$. Note that

(a) since $u = 0^{\mu_0} * v$ begins with $\mu_0$ 0s the leftmost $\mu_0$ bumping paths in $T * u$ travel vertically, directly down the first $\mu_0$ columns of $p$, and

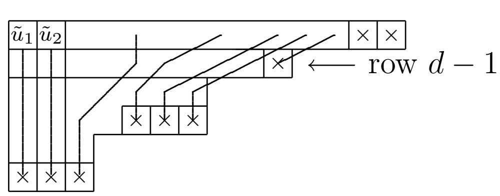
(b) in rows numbered $\geqslant d$ the bumping paths for $\tilde{T} * \tilde{u}$ coincide exactly with the bumping paths for $T * u$, since the horizontal strips $\nu/\gamma$ and $\nu/\tilde{\gamma}$ coincide exactly in rows $\geqslant d$ and these paths are obtained by uninserting the boxes in this portion of the horizontal strip.

$p =$




bumping paths in $T * u$

$p =$




bumping paths in $\tilde{T} * \tilde{u}$

Suppose there are $k$ bumping paths which end in rows $\geqslant d$. The picture above has $k = 6$ and corresponds to Case 2b below.

*Case 2a:* If $\mu_0 + \tilde{r} > \mu_0 + r$ then the $k$ bumping paths which end in rows $\geqslant d$ are the same or slightly "more left" in $\tilde{T} * \tilde{u}$ than in $T * u$. Since the first $\mu_0$ bumping paths cannot be any "more left" than vertical, this forces the first $\mu_0$ entries of $\tilde{u}$ to be 0 so that $\tilde{u} = 0^{\mu_0} * \tilde{v}$ for some $v \in B(\tilde{r})$.

*Case 2b:* If $\mu_0 + \tilde{r} < \mu_0 + r$ then the $k$ bumping paths which end in rows $\geqslant d$ are the same or slightly "more right" in $\tilde{T} * \tilde{u}$ than in $T * u$. We shall analyze how these $k$ paths pass through row $d-1$ in $T * u$ and in $\tilde{T} * \tilde{u}$. Divide row $d-1$ into four disjoint regions, left to right:

Region 1: the leftmost $\mu_0$ boxes of row d-1,

Region 2: the boxes which do not have a cross in them in $T * u$ (and are not in Region 1),

Region 3: the boxes which have a cross in them in $T * u$ but not in $\tilde{T} * \tilde{u}$,

Region 4: the boxes which have a cross in them in both $T * u$ and $\tilde{T} * \tilde{u}$.

Of the $k$ bumping paths of $T * u$ which end in rows $\geqslant d$ the first $\mu_0$ of these pass through Region 1 in $T * u$, and the others ($k - \mu_0$ of them) pass through Region 2. Since the total number of bumping paths (the number of crosses) in $\tilde{T} * \tilde{u}$ is $\mu_0 + \tilde{r}$ and there are some bumping paths of $\tilde{T} * \tilde{u}$ which end in row $d-1$ ($r - \tilde{r}$ of these), $k < \mu_0 + \tilde{r}$. Thus

$$k - \mu_0 < \mu_0 + \tilde{r} - \mu_0 < \mu_0 + \tilde{r} + (d-1) - \mu_0 = \mathrm{Card}(\text{Region } 2),$$

since $\mathrm{Card}(\text{Region } 1) = \mu_0$ and $\mathrm{Card}(\text{Region } 1) + \mathrm{Card}(\text{Region } 2) = \mu_0 + \tilde{r} + d - 1$. Thus there must be a box in Region 2 of $T * u$ that does not have a bumping path passing through it. All the bumping paths of $T * u$ which pass through row $d-1$ to the left of this box remain the same as bumping paths for $\tilde{T} * \tilde{u}$ and the first $\mu_0$ of these begin at an entry 0 in the first row of $p$. Thus, as in Case 2a, the first $\mu_0$ entries of $\tilde{u}$ are 0 so that $\tilde{u} = 0^{\mu_0} * \tilde{v}$ for some $v \in B(\tilde{r})$.

So,

$$\tilde{T} \otimes \tilde{u} = \tilde{T} \otimes (0^{\mu_0} * \tilde{v}), \quad \text{with } \tilde{v} \otimes \tilde{T} \in (B(\tilde{r}) \otimes B(\tilde{\gamma}))_\mu,$$

and the terms in the last line of (4.12) corresponding to the pairs $v \otimes T$ and $\tilde{v} \otimes \tilde{T}$ cancel each other because

$$T * 0^{\mu_0} * v = \tilde{T} * 0^{\mu_0} * \tilde{v} \quad \text{and } s_{(\mu_0 + r, \gamma)} = -s_{(\mu_0 + \tilde{r}, \tilde{\gamma})}.$$

*Case 3:* $d = 1$. Since $\mu_0 + r + 1 > \nu_1$ and $\nu \in \gamma \otimes (\mu_0 + r)$ the horizontal strip $\nu/\gamma$ has its boxes in each of the first $\mu_0 + r$ columns and

$$\nu = (\nu_0, \nu_1, \ldots, \nu_n) = (\mu_0 + r, \gamma_1, \ldots, \gamma_n) = (\mu_0 + r, \gamma).$$

Row uninsertion of the horizontal strip $\nu/\gamma$ from the column strict tableau $p$, i.e. using the second bijection in Lemma 4.1, recovers the pair $T \otimes (0^{\mu_0} * v)$ and shows that $0^{\mu_0} * v$ is the first row of $p$.

In conclusion, in the last line of (4.12) the terms corresponding to Case 1 vanish, the terms corresponding to Case 2 cancel, and the remaining Case 3 terms give formula (4.11), as desired. □

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