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Affine Hecke algebras and generalized standard Young tableaux

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Abstract

This paper introduces calibrated representations for affine Hecke algebras and classifies and constructs all finite-dimensional irreducible calibrated representations. The primary technique is to provide indexing sets for controlling the weight space structure of finite-dimensional modules for the affine Hecke algebra. Using these indexing sets we show that (1) irreducible calibrated representations are indexed by skew local regions, (2) the dimension of an irreducible calibrated representation is the number of chambers in the local region, (3) each irreducible calibrated representation is constructed explicitly by formulas which describe the action of the generators of the affine Hecke algebra on a specific basis in the representation space. The indexing sets for weight spaces are generalizations of standard Young tableaux and the construction of the irreducible calibrated affine Hecke algebra modules is a generalization of A. Young's seminormal construction of the irreducible representations of the symmetric group. In this sense Young's construction has been generalized to arbitrary Lie type.

0. Introduction

The classical representation theory of the symmetric group, as developed by G. Frobenius and A. Young [47,48], has the following features:

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(a) The irreducible representations S^λ of the symmetric group S_n are indexed by partitions λ with n boxes.
(b) The dimension of S^λ is the number of standard tableaux of shape λ.
(c) The S_n-module has an elegant explicit construction: S^λ is the span of a basis $\{v_T\}$ parametrized by standard tableaux T and the action of each generator of S_n is given by a simple formula,

$$s_i v_T = \frac{1}{c(T(i)) - c(T(i+1))} v_T + \left(1 + \frac{1}{c(T(i)) - c(T(i+1))}\right) v_{s_i T}.$$

In this paper we prove analogous results for representations of affine Hecke algebras.

(A) The irreducible calibrated representations $\widetilde{H}^{(t,J)}$ of the affine Hecke algebra $\widetilde{H}$ are indexed by skew local regions (t, J).
(B) The dimension of $\widetilde{H}^{(t,J)}$ is the number of chambers in the local region (t, J).
(C) The $\widetilde{H}$-module $\widetilde{H}^{(t,J)}$ has an elegant explicit construction: $\widetilde{H}^{(t,J)}$ is the span of a basis $\{v_w \mid w \in \mathcal{F}^{(t,J)}\}$ parametrized by chambers in the local region and the action of each generator of $\widetilde{H}$ is given by a simple formula,

$$X^\lambda v_w = q^{\langle \lambda, w\gamma \rangle} v_w, \qquad T_i v_w = \frac{q - q^{-1}}{1 - t(X^{w^{-1}\alpha_i})} + \left(q^{-1} + \frac{q - q^{-1}}{1 - t(X^{w^{-1}\alpha_i})}\right) v_{s_i w}.$$

In fact, the classical theory of standard Young tableaux and partitions is a special case of our theory of chambers and local regions; this is proved in Sections 5 and 6 of this paper. Section 1 serves to fix notations and fundamental data in the form which will need it. The bulk of this material can be found in [6, Chapitres IV–VI] and Steinberg's Yale Lecture Notes [40]. Two known results which are included in Section 1 are:

(a) the determination of the center of the affine Hecke algebra, and
(b) the Pittie–Steinberg theorem, which provides a nice basis for the affine Hecke algebra over its center.

In each case we have given an elementary proof, which, hopefully, illustrates the beautiful simplicity of these powerful results. Section 2 treats the notion of weight spaces for affine Hecke algebra representations and shows how certain combinatorially defined indexing sets $\mathcal{F}^{(t,J)}$ give explicit information about the weight space structure of affine Hecke algebra modules. Section 3 classifies and constructs all irreducible calibrated affine Hecke algebra modules (for any q such that $q^2 \neq \pm 1$, *including roots of unity*). Section 4 gives the main results about the structure of the labeling sets $\mathcal{F}^{(t,J)}$ and defines a conjugation involution on them. Sections 5 and 6 show that the classical theory of standard Young tableaux is very special case of the analysis of the combinatorial structure of the sets $\mathcal{F}^{(t,J)}$. Section 7 works out the generalized standard Young tableaux in the type A, root of unity case. The resulting objects are ℓ-periodic standard Young tableaux. Section 8 describes how the generalized standard Young tableaux look in the type C, nonroot of unity case. In this case the objects are negative rotationally symmetric standard Young tableaux. It should not be difficult to work out similar explicit tableaux in terms of fillings of boxes in the other classical types.

Let us put these results into perspective.

(1) *p-adic groups and affine Hecke algebras.*

The affine Hecke algebra was introduced by Iwahori and Matsumoto [11] as a tool for studying the representations of a p-adic Lie group. In some sense, all irreducible principal series representations of the p-adic group can be determined by classifying the representations of the corresponding affine Hecke algebra. Kazhdan and Lusztig [14] (see also [8]) gave a geometric classification of all irreducible representations of the affine Hecke algebra. This classification is a q-analogue of Springer's construction of the irreducible representations of the Weyl group on the cohomology of unipotent varieties. In the q-case, K-theory takes the place of cohomology and the irreducible representations of the affine Hecke algebra are constructed as quotients of the K-theory of the Steinberg varieties. It is difficult to obtain combinatorial information from this geometric construction. So the combinatorial approach in this paper gives new information.

(2) *The theory of Young tableaux.*

The word "Young tableau" is commonly used for three very different objects in representation theory:

(1a) *partitions* with n boxes, which index representations of the symmetric group S_n,
(1b) *partitions with* $\leqslant n$ *rows*, which index the polynomial representations of $GL_n(\mathbb{C})$,
(2) *standard tableaux*, which label the basis elements of an irreducible representation of S_n,
(3) *column strict tableaux*, which label the basis elements of an irreducible polynomial representation of $GL_n(\mathbb{C})$.

The partitions in (1b) were generalized to all Lie types by H. Weyl in 1926, who showed that finite-dimensional irreducible representations of compact Lie groups are indexed by the dominant integral weights. There was much important work generalizing the column strict tableaux in (3) to other Lie types, for a survey of this work see [43]. The problem of generalizing the column strict tableaux in (3) to *all* Lie types was finally solved by the path model of Littelmann [20,21]. This paper provides a generalization of the partitions of (1a) and the standard tableaux of (2) which are valid for *all Lie types*. For important earlier work in this direction see [24, I Appendix B], Hoefsmit [10], and Ariki and Koike [1].

This paper is a revised, expanded, and updated version of the preprints [28,29]. The original preprints will not be published since the results there are contained in and expanded in this paper. Those preprints will remain available at http://www.math.wisc.edu/~ram/preprints.html.

1. The affine Hecke algebra

Though we shall never really use the data (G, B, T) it is conceptually useful to note that there is an affine Hecke algebra associated to each triple $(G \supseteq B \supseteq T)$ where

- G is a connected reductive complex algebraic group,
- B is a Borel subgroup,
- T is a maximal torus.

An example of this data is when $G = GL_n(\mathbb{C})$, B is the subgroup of upper triangular invertible matrices, and T is the subgroup of invertible diagonal matrices.

The reason that we can avoid the data $(G \supseteq B \supseteq T)$ is that it is equivalent to different data (W, C, L) where

- W is a finite real reflection group with reflection representation $\mathfrak{h}_{\mathbb{R}}^*$,
- C is a fixed fundamental chamber for the W-action,
- L is a W-invariant lattice in $\mathfrak{h}_{\mathbb{R}}^*$.

This will be our basic data. In the example where $G = GL_n(\mathbb{C})$ and B and T are the upper triangular and diagonal matrices, respectively,

$$W = S_n, \qquad \mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}^n = \sum_{i=1}^n \mathbb{R}\varepsilon_i, \qquad C = \left\{ \mu = \sum_{i=1}^n \mu_i \varepsilon_i \;\middle|\; \mu_1 \leqslant \cdots \leqslant \mu_n \right\},$$
$$L = \sum_{i=1}^n \mathbb{Z}\varepsilon_i, \tag{1.1}$$

where $W = S_n$ is the symmetric group, acting on $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}^n$ by permuting the orthonormal basis $\varepsilon_1, \ldots, \varepsilon_n$. This example will be treated in depth in Sections 5–7. We shall show that the labeling sets $\mathcal{F}^{(t,J)}$ for weight spaces of affine Hecke algebra representations that are introduced in (2.18) and Corollary 2.19 and used for the classification in Theorem 3.6 are generalizations of standard Young tableaux.

The components W and L in the data (W, C, L) are obtained from $(G \supseteq B \supseteq T)$ by

$$W = N(T)/T, \qquad X = \mathrm{Hom}(T, \mathbb{C}^*) = \left\{ X^\lambda \;\middle|\; \lambda \in L \right\},$$

where $N(T)$ is the normalizer of T in G and $\mathrm{Hom}(T, \mathbb{C}^*)$ is the set of algebraic group homomorphisms from T to $\mathbb{C}^*$. The notation is designed so that the multiplication in the group X is

$$X^\lambda X^\mu = X^{\lambda+\mu} = X^\mu X^\lambda, \quad \text{for } \mu, \lambda \in L, \tag{1.2}$$

see [7, III Section 8]. The reflection (or defining) representation of the group W is given by its action on $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R} \otimes_{\mathbb{Z}} L \cong \mathbb{R}^n$ and with respect to a W-invariant inner product $\langle\, , \rangle$ on $\mathfrak{h}_{\mathbb{R}}^*$ the group W is generated by reflections s_α in the hyperplanes

$$H_\alpha = \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle x, \alpha\rangle = 0\}, \quad \alpha \in R^+. \tag{1.3}$$

See the picture which appears just before Theorem 1.17. The *chambers* are the connected components of $\mathfrak{h}_{\mathbb{R}}^* - (\bigcup_{\alpha\in R^+} H_\alpha)$ and these are the fundamental regions for the action of W on $\mathfrak{h}_{\mathbb{R}}^*$. Fixing a choice of a fundamental chamber C corresponds to the choice of the set R^+ of positive roots, which corresponds to the choice of B in G.

In our formulation we may view the set R^+ as a labeling set for the reflecting hyperplanes H_α in $\mathfrak{h}_{\mathbb{R}}^*$ and

$$C = \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle x, \alpha\rangle > 0 \text{ for all } \alpha \in R^+\}. \tag{1.4}$$

For a root $\alpha \in R$, the *positive side* of the hyperplane H_α is the side towards C, i.e., $\{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle\lambda, \alpha\rangle > 0\}$, and the *negative side* of H_α is the side away from C.

For $w \in W$, the *inversion set* of W is

$$R(w) = \{\alpha \in R^+ \mid w\alpha \in R^-\}, \tag{1.5}$$

where $R^- = -R^+$. There is a bijection

$$\begin{aligned} W &\leftrightarrow \{\text{fundamental chambers for } W \text{ acting on } \mathfrak{h}_{\mathbb{R}}^*\}, \\ w &\mapsto w^{-1}C \end{aligned} \tag{1.6}$$

and the chamber $w^{-1}C$ is the unique chamber which is on the positive side of H_α for $\alpha \notin R(w)$ and on the negative side of H_α for $\alpha \in R(w)$.

The *simple roots* $\alpha_1, \dots, \alpha_n$ in R^+ index the walls H_{α_i} of the fundamental chamber C and the corresponding reflections $s_1, \dots, s_n$ generate W. In fact, W can be presented by generators $s_1, s_2, \dots, s_n$ and relations

$$\begin{aligned} s_i^2 &= 1, && \text{for } 1 \leqslant i \leqslant n, \\ \underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ factors}} &= \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ factors}}, && \text{for } i \neq j, \end{aligned} \tag{1.7}$$

where the (acute) angle π/m_{ij} between the hyperplanes H_{α_i} and H_{α_j} determines the value m_{ij}.

Fix $q \in \mathbb{C}^*$ with $q^2 \neq \pm 1$. The *Iwahori–Hecke algebra* H associated to (W, C) is the associative algebra over $\mathbb{C}$ defined by generators $T_1, T_2, \dots, T_n$ and relations

$$\begin{aligned} T_i^2 &= (q - q^{-1})T_i + 1, && \text{for } 1 \leqslant i \leqslant n, \\ \underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ factors}} &= \underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ factors}}, && \text{for } i \neq j, \end{aligned} \tag{1.8}$$

where m_{ij} are the same as in the presentation of W. For $w \in W$ define $T_w = T_{i_1} \cdots T_{i_p}$ where $s_{i_1} \cdots s_{i_p} = w$ is a reduced expression for w. By [6, Chapter IV, Section 2 Exercise 23], the element T_w does not depend on the choice of the reduced expression. The algebra H has dimension $|W|$ and the set $\{T_w\}_{w\in W}$ is a basis of H.

The *affine Hecke algebra* $\widetilde{H}$ associated to (W, C, L) algebra given by

$$\widetilde{H} = \mathbb{C}\text{-span}\{T_w X^\lambda \mid w \in W,\ X^\lambda \in X\} \tag{1.9}$$

where the multiplication of the T_w is as in the Iwahori–Hecke algebra H, the multiplication of the X^λ is as in (1.2) and we impose the relation

$$X^\lambda T_i = T_i X^{s_i\lambda} + (q - q^{-1}) \frac{X^\lambda - X^{s_i\lambda}}{1 - X^{-\alpha_i}}, \quad \text{for } 1 \leqslant i \leqslant n \text{ and } X^\lambda \in X. \tag{1.10}$$

This formulation of the definition of $\widetilde{H}$ is due to Lusztig [23] following work of Bernstein and Zelevinsky. The elements $T_w X^\lambda$, $w \in W$, $X^\lambda \in X$, form a basis of $\widetilde{H}$.

The group algebra of X,

$$\mathbb{C}[X] = \mathbb{C}\text{-span}\{X^\lambda \mid \lambda \in L\}, \tag{1.11}$$

is a subalgebra of $\widetilde{H}$ with a W-action obtained by linearly extending the W-action on X,

$$wX^\lambda = X^{w\lambda}, \quad \text{for } w \in W,\ X^\lambda \in X. \tag{1.12}$$

Theorem 1.13 (Bernstein, Zelevinsky, Lusztig [23, 8.1]). *The center of $\widetilde{H}$ is $\mathbb{C}[X]^W = \{f \in \mathbb{C}[X] \mid wf = f \text{ for all } w \in W\}$.*

Proof. Assume

$$z = \sum_{\lambda \in L, w \in W} c_{\lambda,w} X^\lambda T_w \in Z(\widetilde{H}).$$

Let $m \in W$ be maximal in Bruhat order subject to $c_{\gamma,m} \neq 0$ for some $\gamma \in L$. If $m \neq 1$ there exists a dominant $\mu \in L$ such that $c_{\gamma+\mu-m\mu,m} = 0$ (otherwise $c_{\gamma+\mu-m\mu,m} \neq 0$ for every dominant $\mu \in L$, which is impossible since z is a finite linear combination of $X^\lambda T_w$). Since $z \in Z(\widetilde{H})$ we have

$$z = X^{-\mu} z X^\mu = \sum_{\lambda \in L,\ w \in W} c_{\lambda,w} X^{\lambda-\mu} T_w X^\mu.$$

Repeated use of the relation (1.10) yields

$$T_w X^\mu = \sum_{\nu \in L,\ v \in W} d_{\nu,v} X^\nu T_v$$

where $d_{\nu,v}$ are constants such that $d_{w\mu,w} = 1$, $d_{\nu,w} = 0$ for $\nu \neq w\mu$, and $d_{\nu,v} = 0$ unless $v \leqslant w$. So

$$z = \sum_{\lambda \in L,\ w \in W} c_{\lambda,w} X^\lambda T_w = \sum_{\lambda \in L,\ w \in W} \sum_{\nu \in L,\ v \in W} c_{\lambda,w} d_{\nu,v} X^{\lambda-\mu+\nu} T_v$$

and comparing the coefficients of $X^\gamma T_m$ gives $c_{\gamma,m} = c_{\gamma+\mu-m\mu,m} d_{m\mu,m}$. Since $c_{\gamma+\mu-m\mu,m} = 0$ it follows that $c_{\gamma,m} = 0$, which is a contradiction. Hence $z = \sum_{\lambda\in L} c_\lambda X^\lambda \in \mathbb{C}[X]$.

The relation (1.10) gives

$$zT_i = T_i z = (s_i z)T_i + (q - q^{-1})z'$$

where $z' \in \mathbb{C}[X]$. Comparing coefficients of X^λ on both sides yields $z' = 0$. Hence $zT_i = (s_i z)T_i$, and therefore $z = s_i z$ for $1 \leqslant i \leqslant n$. So $z \in \mathbb{C}[X]^W$. □

It is often convenient to assume that W acts irreducibly on $\mathfrak{h}^*_{\mathbb{R}}$ and that the lattice L is the weight lattice

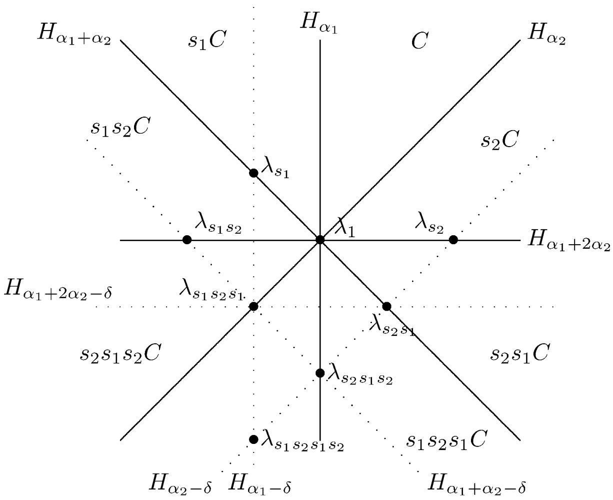
$$P = \{x \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle x, \alpha\rangle \in \mathbb{Z} \text{ for all } \alpha \in R^+\} = \sum_{i=1}^n \mathbb{Z}\omega_i, \tag{1.14}$$

where the *fundamental weights* are the elements $\omega_1, \ldots, \omega_n$ of $\mathbb{R}^n$ given by

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}, \quad \text{where } \alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i\rangle} \tag{1.15}$$

and δ_{ij} is the Kronecker delta. Many facts are easier to state in this case and the general case can always be reduced to this one. We will make some further remarks on this reduction at the end of this section.

Consider the connected regions of the negative *Shi arrangement* $\mathcal{A}^-$ [2,33–35,38,39], i.e., the arrangement of (affine) hyperplanes given by



The arrangement $\mathcal{A}^-$

Fig. 1.

$$\mathcal{A}^- = \{H_\alpha,\ H_{\alpha-\delta} \mid \alpha \in R^+\}$$
$$\text{where}\quad H_\alpha = \{x \in \mathbb{R}^n \mid \langle x, \alpha\rangle = 0\}, \quad H_{\alpha-\delta} = \{x \in \mathbb{R}^n \mid \langle x, \alpha\rangle = -1\}. \quad (1.16)$$

Each chamber $w^{-1}C$, $w \in W$, contains a unique region of $\mathcal{A}^-$ which is a cone, and the vertex of this cone is the point λ_w which appears in the following theorem.

Theorem 1.17 [42]. *Suppose that W acts irreducibly on $\mathfrak{h}_{\mathbb{R}}^*$ and that $X = \{X^\lambda \mid \lambda \in P\}$ where P is the weight lattice. The algebra $\mathbb{C}[X]$ is a free $\mathbb{C}[X]^W$-module with*

$$\textit{basis } \{X^{\lambda_w} \mid w \in W\}, \quad \textit{where } \lambda_w = w^{-1}\Bigl(\sum_{s_i w < w} \omega_i\Bigr).$$

Proof. The proof is accomplished by establishing three facts:

(a) Let f_y, $y \in W$, be a family of elements of $\mathbb{Z}[X]$. Then $\det(zf_y)$ is divisible by $\prod_{\alpha\in R^+}(X^\alpha - 1)^{|W|/2}$.
(b) $\det\bigl(zX^{\lambda_y}\bigr)_{z,y\in W} = \prod_{\alpha>0}(1 - X^\alpha)^{|W|/2}$.
(c) If $f \in \mathbb{Z}[X]$ then there is a unique solution to the equation

$$\sum_{w\in W} a_w X^{\lambda_w} = f, \quad \text{with } a_w \in \mathbb{Z}[X]^W.$$

(a) For each $\alpha \in R^+$ subtract row zf_y from row $s_\alpha z f_y$. Then this row is divisible by $(1 - X^{-\alpha})$. Since there are $|W|/2$ pairs of rows $(zf_y, s_\alpha z f_y)$ the whole determinant is divisible by $(1 - X^{-\alpha})^{|W|/2}$. For $\alpha, \beta \in R^+$ the factors $(1 - X^{-\alpha})$ and $(1 - X^{-\beta})$ are coprime, and so $\det(zf_y)$ is divisible by $\prod_{\alpha\in R^+}(1 - X^{-\alpha})^{|W|/2}$. This product and the product in the statement of (a) differ by the unit $(X^{2\rho})^{|W|/2}$ in $\mathbb{Z}[X]$.

(b) By (a), $\det(zX^{\lambda_y})$ is divisible by $\prod_{\alpha\in R^+}(X^\alpha - 1)^{|W|/2}$. The top coefficient of $\det(zX^{\lambda_y})$ is equal to

$$\prod_{z\in W} zX^{\lambda_z} = \prod_{z\in W} \prod_{\substack{i\\ s_i z < z}} X^{\omega_i} = \prod_{i=1}^n X^{(|W|/2)\omega_i} = (X^\rho)^{|W|/2},$$

and the top coefficient of $\prod_{\alpha\in R^+}(X^\alpha - 1)^{|W|/2}$ is $(X^{2\rho})^{|W|/2}$.

(c) Assume that $a_y \in \mathbb{Z}[X]^W$ are solutions of the equation $\sum_{y\in W} X^{\lambda_y} a_y = f$. Act on this equation by the elements of W to obtain the system of $|W|$ equations

$$\sum_{y\in W} \bigl(zX^{\lambda_y}\bigr) a_y = zf, \quad z \in W.$$

By (a) the matrix $(zX^{\lambda_y})_{z,y\in W}$ is invertible and so this system has a unique solution with $a_y \in \mathbb{Z}[X]^W$. In fact, the a_y can be obtained by Cramer's rule. Cramer's rule provides an expression for a_y as a quotient of two determinants. By (a) and (b) the denominator divides the numerator to give an element of $\mathbb{Z}[X]$. Since each determinant is an alternating function, the quotient is an element of $\mathbb{Z}[X]^W$. $\square$

Remark. In [42] Steinberg proves this type of result in full generality without the assumptions that W acts irreducibly on $\mathfrak{h}_{\mathbb{R}}^*$ and $L = P$. Note also that the proof given above is sketchy, particularly in the aspect that the top coefficient of the determinant is what we have claimed it is. See [42] for a proper treatment of this point.

1.18. Deducing the $\widetilde{H}_L$ representation theory from $\widetilde{H}_P$

It is often easier to work with the representation theory of $\widetilde{H}$ in the case when $L = P$. It is important to be able to convert from this case to the case of a general lattice L. If W acts irreducibly on $\mathfrak{h}_{\mathbb{R}}^*$ then the lattice L satisfies

$$Q \subseteq L \subseteq P, \quad \text{where} \quad P = \sum_{i=1} \mathbb{Z}\omega_i \quad \text{and} \quad Q = \sum_{i=1} \mathbb{Z}\alpha_i$$

are the weight lattice and the *root lattice*, respectively. The group $\Omega = P/Q$ is a finite group (either cyclic or isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$). It corresponds to the center of the corresponding complex algebraic group. Let us denote the corresponding affine Hecke algebras by

$$\widetilde{H}_Q \subseteq \widetilde{H}_L \subseteq \widetilde{H}_P,$$

according which lattice is used to make the group X.

Theorem 1.19 [30]. *Then there is an action of the finite group P/L on $\widetilde{H}_P$, by ring automorphisms, such that*

$$\widetilde{H}_L = \big(\widetilde{H}_P\big)^{P/L} = \big\{h \in \widetilde{H}_P \;\big|\; gh = h \text{ for all } g \in P/L\big\},$$

is the subalgebra of fixed points under the action of the group P/L.

This theorem is exactly what is needed to apply a (not very well known) version of Clifford theory to completely classify the representations of $\widetilde{H}_L$ in terms of the representations of $\widetilde{H}_P$, see [30].

2. $\widetilde{H}$-modules

2.1. Weights

In view of the results in Section 1.18 we shall (for the remainder of this paper, except Sections 5–7 where we use the data in (1.1)) assume that $L = P$ in the definition of the group X and $\widetilde{H}$, see (1.2), (1.9), and (1.14). The Weyl group acts on

$$T = \mathrm{Hom}(X, \mathbb{C}^*) = \{\text{group homomorphisms } t : X \to \mathbb{C}^*\} \quad \text{by} \quad (wt)\big(X^\lambda\big) = t\big(X^{w^{-1}\lambda}\big).$$

Let M be a finite dimensional $\widetilde{H}$-module and let $t \in T$. The *t-weight space* and the *generalized t-weight space* of M are

$$M_t = \big\{m \in M \;\big|\; X^\lambda m = t\big(X^\lambda\big)m \text{ for all } X^\lambda \in X\big\} \quad \text{and}$$
$$M_t^{\mathrm{gen}} = \big\{m \in M \;\big|\; \text{for each } X^\lambda \in X,\ \big(X^\lambda - t\big(X^\lambda\big)\big)^k m = 0 \text{ for some } k \in \mathbb{Z}_{>0}\big\},$$

respectively. Then

$$M = \bigoplus_{t \in T} M_t^{\mathrm{gen}} \tag{2.2}$$

is a decomposition of M into Jordan blocks for the action of $\mathbb{C}[X]$, and we say that t is a *weight* of M if $M_t^{\mathrm{gen}} \neq 0$. Note that $M_t^{\mathrm{gen}} \neq 0$ if and only if $M_t \neq 0$. A finite-dimensional $\widetilde{H}$-module

$$M \text{ is } \textit{calibrated} \text{ if } M_t^{\mathrm{gen}} = M_t, \text{ for all } t \in T.$$

Remark. The term tame is sometimes used in place of the term calibrated particularly in the context of representations of Yangians, see [26]. The word calibrated is preferable since tame also has many other meanings in different parts of mathematics.

Let M be a simple $\widetilde{H}$-module. As an $X(T)$-module, M contains a simple submodule and this submodule must be one-dimensional since all irreducible representations of a commutative algebra are one dimensional. Thus, a simple module always has $M_t \neq 0$ for some $t \in T$.

2.3. Central characters

The Pittie–Steinberg theorem, Theorem 1.17, shows that, as vector spaces,

$$\widetilde{H} = H \otimes \mathbb{C}[X] = H \otimes \mathbb{C}[X]^W \otimes \mathcal{K}, \quad \text{where } \mathcal{K} = \mathbb{C}\text{-span}\big\{X^{\lambda_w} \;\big|\; w \in W\big\},$$

and H is the Iwahori–Hecke algebra defined in (1.8). Thus $\widetilde{H}$ is a free module over $Z(\widetilde{H}) = \mathbb{C}[X]^W$ of rank $\dim(H) \cdot \dim(\mathcal{K}) = |W|^2$. By Dixmier's version of Schur's lemma (see

[44, Lemma 0.5.1]), $Z(\widetilde{H})$ acts on a simple $\widetilde{H}$-module by scalars and so it follows that every simple $\widetilde{H}$-module is finite dimensional of dimension $\leqslant |W|^2$. Theorem 2.12(d) below will show that, in fact, the dimension of a simple module is $\leqslant |W|$.

Let M be a simple $\widetilde{H}$-module. The *central character* of M is an element $t \in T$ such that

$$pm = t(p)m, \qquad \text{for all } m \in M,\ p \in \mathbb{C}[X]^W = Z(\widetilde{H}).$$

The element t is only determined up to the action of W since $t(p) = wt(p)$ for all $w \in W$. Because of this, any element of the orbit Wt is referred to as the *central character* of M.

Because $P = L$ in the construction of X, a theorem of Steinberg [41, 3.15, 4.2, 5.3] tells us that the stabilizer W_t of a point $t \in T$ under the action of W is the reflection group

$$W_t = \langle s_\alpha \mid \alpha \in Z(t) \rangle, \quad \text{where } Z(t) = \{\alpha \in R^+ \mid t(X^\alpha) = 1\}.$$

Thus the orbit Wt can be viewed in several different ways via the bijections

$$Wt \leftrightarrow W/W_t \leftrightarrow \{w \in W \mid R(w) \cap Z(t) = \emptyset\} \leftrightarrow \left\{\begin{matrix}\text{chambers on the positive}\\ \text{side of } H_\alpha \text{ for } \alpha \in Z(t)\end{matrix}\right\}, \quad (2.4)$$

where the last bijection is the restriction of the map in (1.6). If the root system $Z(t)$ is generated by the simple roots α_i that it contains then W_t is a parabolic subgroup of W and $\{w \in W \mid R(w) \cap Z(t)\}$ is the set of minimal length coset representatives of the cosets in W/W_t.

2.5. Principal series modules

For $t \in T$ let $\mathbb{C}v_t$ be the one-dimensional $\mathbb{C}[X]$-module given by

$$X^\lambda v_t = t(X^\lambda)v_t, \quad \text{for } X^\lambda \in X.$$

The *principal series representation* $M(t)$ is the $\widetilde{H}$-module defined by

$$M(t) = \widetilde{H} \otimes_{\mathbb{C}[X]} \mathbb{C}v_t = \operatorname{Ind}_{\mathbb{C}[X]}^{\widetilde{H}}(\mathbb{C}v_t). \quad (2.6)$$

The module $M(t)$ has basis $\{T_w \otimes v_t \mid w \in W\}$ with H acting by left multiplication.

If $w \in W$ and $X^\lambda \in X$ then the defining relation (1.10) for $\widetilde{H}$ implies that

$$X^\lambda(T_w \otimes v_t) = t(X^{w\lambda})(T_w \otimes v_t) + \sum_{u<w} a_u(T_u \otimes v_t), \quad (2.7)$$

where the sum is over $u < w$ in the Bruhat–Chevalley order and $a_u \in \mathbb{C}$. Let $W_t = \operatorname{Stab}(t)$ be the stabilizer of t under the W-action. It follows from (2.7) that the eigenvalues of X on $M(t)$ are of the form wt, $w \in W$, and by counting the multiplicity of each eigenvalue we have

$$M(t) = \bigoplus_{wt \in Wt} M(t)^{\text{gen}}_{wt} \quad \text{where } \dim\big(M(t)^{\text{gen}}_{wt}\big) = |W_t|, \text{ for all } w \in W. \quad (2.8)$$

In particular, if t is regular (i.e., when W_t is trivial), there is a unique basis $\{v_{wt} \mid w \in W\}$ of $M(t)$ determined by

$$X^\lambda v_{wt} = (wt)\big(X^\lambda\big)v_{wt}, \quad \text{for all } w \in W \text{ and } \lambda \in P,$$
$$v_{wt} = T_w \otimes v_t + \sum_{u<w} a_{wu}(t)(T_u \otimes v_t), \quad \text{where } a_{wu}(t) \in \mathbb{C}. \tag{2.9}$$

Let $t \in T$. The *spherical vector* in $M(t)$ is

$$\mathbf{1}_t = \sum_{w\in W} q^{\ell(w)} T_w \otimes v_t. \tag{2.10}$$

Up to multiplication by constants this is the unique vector in $M(t)$ such that $T_w\mathbf{1}_t = q^{\ell(w)}\mathbf{1}_t$ for all $w \in W$. The following is due to Kato [12, Proposition 1.20 and Lemma 2.3].

Proposition 2.11. *Let $t \in T$ and let W_t be the stabilizer of t under the W-action.*

(a) *If $W_t = \{1\}$ and v_{wt}, $w \in W$ is the basis of $M(t)$ defined in* (2.9) *then*

$$\mathbf{1}_t = \sum_{z\in W} t(c_z), \quad \text{where } c_z = \prod_{\alpha\in R(w_0 z)} \frac{q - q^{-1}X^\alpha}{1 - X^\alpha}.$$

(b) *The spherical vector $\mathbf{1}_t$ generates $M(t)$ if and only if $t(\prod_{\alpha\in R^+}(q^{-1} - qX^\alpha)) \neq 0$.*
(c) *The module $M(t)$ is irreducible if and only if $\mathbf{1}_{wt}$ generates $M(wt)$ for all $w \in W$.*

Proof. The proof is accomplished in exactly the same way as done for the graded Hecke algebra in [17, Proposition 2.8]. The only changes which need to be made to [17] are

(1) Use $T_i(\sum_{w\in W} q^{\ell(w)}T_w) = q(\sum_{w\in W} q^{\ell(w)}T_w)$ and $\mathbf{1}_t = (\sum_{w\in W} q^{\ell(w)}T_w)v_t$ and the τ-operators defined in Proposition 2.14 for the proof of (a). (We have included this result in this section since it is really a result about the structure of principal series modules. Though the proof uses the τ-operators, which we will define in the next section, there is no logical gap here.)
(2) For the proof of (b) use the Steinberg basis $\{X^{\lambda_y} \mid y \in W\}$ and the determinant $\det(X^{z^{-1}\lambda_y})$ from Theorem 1.17(b) in place of the basis $\{b_y \mid w \in W\}$ and the determinant used in [17]. □

Part (b) of the following theorem is due to Rogawski [32, Proposition 2.3] and part (c) is due to Kato [12, Theorem 2.1]. Parts (a) and (d) are classical.

Theorem 2.12. *Let $t \in T$ and $w \in W$ and define $P(t) = \{\alpha \in R^+ \mid t(X^\alpha) = q^{\pm 2}\}$.*

(a) *If $W_t = \{1\}$ then $M(t)$ is calibrated.*
(b) *$M(t)$ and $M(wt)$ have the same composition factors.*

(c) *$M(t)$ is irreducible if and only if $P(t) = \emptyset$.*
(d) *If M is a simple $\widetilde{H}$-module with $M_t \neq 0$ then M is a quotient of $M(t)$.*

Proof. (a) follows from (2.8) and the definition of calibrated. Part (b) accomplished exactly as in [17, Proposition 2.8] and (c) is a direct consequence of the definition of $P(t)$ and Proposition 2.11.

(d) Let m_t be a nonzero vector in M_t. If v_t is as in the construction of $M(t)$ in (2.6) then, as $\mathbb{C}[X]$-modules, $\mathbb{C}m_t \cong \mathbb{C}v_t$. Thus, since induction is the adjoint functor to restriction there is a unique $\widetilde{H}$-module homomorphism given by

$$\begin{aligned} \phi : M(t) &\to M, \\ v_t &\mapsto m_t. \end{aligned}$$

This map is surjective since M is irreducible and so M is a quotient of $M(t)$. □

2.13. The τ operators

The following proposition defines maps $\tau_i : M_t^{\text{gen}} \to M_{s_i t}^{\text{gen}}$ on generalized weight spaces of finite-dimensional $\widetilde{H}$-modules M. These are "local operators" and are only defined on weight spaces M_t^{gen} such that $t(X^{\alpha_i}) \neq 1$. In general, τ_i does not extend to an operator on all of M.

Proposition 2.14. *Fix i, let $t \in T$ be such that $t(X^{\alpha_i}) \neq 1$ and let M be a finite-dimensional $\widetilde{H}$-module. Define*

$$\begin{aligned} \tau_i : M_t^{\text{gen}} &\to M_{s_i t}^{\text{gen}}, \\ m &\mapsto \left(T_i - \frac{q - q^{-1}}{1 - X^{-\alpha_i}}\right)m. \end{aligned}$$

(a) *The map $\tau_i : M_t^{\text{gen}} \to M_{s_i t}^{\text{gen}}$ is well defined.*
(b) *As operators on M_t^{gen}, $X^\lambda \tau_i = \tau_i X^{s_i \lambda}$, for all $X^\lambda \in X$.*
(c) *As operators on M_t^{gen}, $\tau_i \tau_i = (q - q^{-1}X^{\alpha_i})(q - q^{-1}X^{-\alpha_i})/((1 - X^{\alpha_i})(1 - X^{-\alpha_i}))$.*
(d) *Both maps $\tau_i : M_t^{\text{gen}} \to M_{s_i t}^{\text{gen}}$ and $\tau_i : M_{s_i t}^{\text{gen}} \to M_t^{\text{gen}}$ are invertible if and only if $t(X^{\alpha_i}) \neq q^{\pm 2}$.*
(e) *Let $1 \leqslant i \neq j \leqslant n$ and let m_{ij} be as in (1.7). Then*

$$\underbrace{\tau_i \tau_j \tau_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{\tau_j \tau_i \tau_i \cdots}_{m_{ij} \text{ factors}},$$

whenever both sides are well defined operators on M_t^{gen}.

Proof. (a) The element X^{α_i} acts on M_t^{gen} by $t(X^{\alpha_i})$ times a unipotent transformation. As an operator on M_t^{gen}, $1 - X^{-\alpha_i}$ is invertible since it has determinant $(1 - t(X^{-\alpha_i}))^d$ where

$d = \dim(M_t^{\text{gen}})$. Since this determinant is nonzero $(q-q^{-1})/(1-X^{-\alpha_i}) = (q-q^{-1}) \times (1-X^{-\alpha_i})^{-1}$ is a well defined operator on M_t^{gen}. Thus the definition of τ_i makes sense.

Since $(q-q^{-1})/(1-X^{-\alpha_i})$ is not an element of $\widetilde{H}$ or $\mathbb{C}[X]$ it should be viewed only as an operator on M_t^{gen} in calculations. With this in mind it is straightforward to use the defining relation (1.10) to check that

$$X^\lambda \tau_i m = X^\lambda \left(T_i - \frac{q-q^{-1}}{1-X^{-\alpha_i}}\right) m = \left(T_i - \frac{q-q^{-1}}{1-X^{-\alpha_i}}\right) X^{s_i\lambda} m = \tau_i X^{s_i\lambda} m \quad \text{and}$$

$$\tau_i\tau_i m = \left(T_i - \frac{q-q^{-1}}{1-X^{-\alpha_i}}\right)\left(T_i - \frac{q-q^{-1}}{1-X^{-\alpha_i}}\right) m = \frac{(q-q^{-1}X^{\alpha_i})(q-q^{-1}X^{-\alpha_i})}{(1-X^{\alpha_i})(1-X^{-\alpha_i})} m,$$

for all $m \in M_t^{\text{gen}}$ and $X^\lambda \in X$. This proves (a)–(c).

(d) The operator X^{α_i} acts on M_t^{gen} as $t(X^{\alpha_i})$ times a unipotent transformation. Similarly for $X^{-\alpha_i}$. Thus, as an operator on M_t^{gen} $\det((q-q^{-1}X^{\alpha_i})(q-q^{-1}X^{-\alpha_i})) = 0$ if and only if $t(X^{\alpha_i}) = q^{\pm 2}$. Thus part (c) implies that $\tau_i\tau_i$, and each factor in this composition, is invertible if and only if $t(X^{\alpha_i}) \neq q^{\pm 2}$.

(e) Let $t \in T$ be regular. By part (a), the definition of the τ_i, and the uniqueness in (2.9), the basis $\{v_{wt}\}_{w\in W}$ of $M(t)$ in (2.9) is given by

$$v_{wt} = \tau_w v_t, \tag{2.15}$$

where $\tau_w = \tau_{i_1}\cdots\tau_{i_p}$ for a reduced word $w = s_{i_1}\cdots s_{i_p}$ of w. Use the defining relation (1.10) for $\widetilde{H}$ to expand the product of τ_i and compute

$$\begin{aligned} v_{w_0t} &= \underbrace{\cdots\tau_i\tau_j\tau_i}_{m_{ij}\text{ factors}} v_t = \underbrace{\cdots T_iT_jT_i}_{m_{ij}\text{ factors}} v_t + \sum_{w<w_0} T_wP_wv_t = T_{w_0}\otimes v_t + \sum_{w<w_0} t(P_w)T_w\otimes v_t \\ &= \underbrace{\cdots\tau_j\tau_i\tau_j}_{m_{ij}\text{ factors}} v_t = \underbrace{\cdots T_jT_iT_j}_{m_{ij}\text{ factors}} v_t + \sum_{w<w_0} T_wQ_wv_t = T_{w_0}\otimes v_t + \sum_{w<w_0} t(Q_w)T_w\otimes v_t \end{aligned}$$

where P_w and Q_w are rational functions in the X^λ. By the uniqueness in (2.9), $t(P_w) = a_{w_0w}(t) = t(Q_w)$ for all $w \in W$, $w \neq w_0$. Since the values of P_w and Q_w coincide on all generic points $t \in T$ it follows that

$$P_w = Q_w \quad \text{for all } w \in W,\ w \neq w_0. \tag{2.16}$$

Thus,

$$\underbrace{\cdots\tau_i\tau_j\tau_i}_{m_{ij}\text{ factors}} = T_{w_0} + \sum_{w<w_0} T_wP_w = T_{w_0} + \sum_{w<w_0} T_wQ_w = \underbrace{\cdots\tau_j\tau_i\tau_j}_{m_{ij}\text{ factors}},$$

whenever both sides are well defined operators on M_t^{gen}. $\square$

Let $t \in T$ and recall that

$$Z(t) = \{\alpha \in R^+ \mid t(X^\alpha) = 1\} \quad \text{and} \quad P(t) = \{\alpha \in R^+ \mid t(X^\alpha) = q^{\pm 2}\}. \tag{2.17}$$

If $J \subseteq P(t)$ define

$$\mathcal{F}^{(t,J)} = \{w \in W \mid R(w) \cap Z(t) = \emptyset,\ R(w) \cap P(t) = J\}. \tag{2.18}$$

We say that the pair (t, J) is a *local region* if $\mathcal{F}^{(t,J)} \neq \emptyset$. Under the bijection (2.4) the set $\mathcal{F}^{(t,J)}$ maps to the set of chambers whose union is the set of points $x \in \mathfrak{h}_{\mathbb{R}}^*$ which are

(a) on the positive side of the hyperplanes H_α for $\alpha \in Z(t)$,
(b) on the positive side of the hyperplanes H_α for $\alpha \in P(t)\backslash J$,
(c) on the negative side of the hyperplanes H_α for $\alpha \in J$.

See the picture in Example 4.11(d). In this way the local region (t, J) really does correspond to a region in $\mathfrak{h}_{\mathbb{R}}^*$. This is a connected convex region in $\mathfrak{h}_{\mathbb{R}}^*$ since it is cut out by half spaces in $\mathfrak{h}_{\mathbb{R}}^* \cong \mathbb{R}^n$. The elements $w \in \mathcal{F}^{(t,J)}$ index the *chambers $w^{-1}C$ in the local region* and, as J runs over the subsets of $P(t)$, the sets $\mathcal{F}^{(t,J)}$ form a partition of the set $\{w \in W \mid R(w) \cap Z(t) = \emptyset\}$ (which, by (2.4), indexes the cosets in W/W_t).

Corollary 2.19. *Let M be a finite dimensional $\widetilde{H}$-module. Let $t \in T$ and let $J \subseteq P(t)$. Then*

$$\dim(M_{wt}^{\text{gen}}) = \dim(M_{w't}^{\text{gen}}), \quad \textit{for } w, w' \in \mathcal{F}^{(t,J)}.$$

Proof. Suppose $w, s_i w \in \mathcal{F}^{(t,J)}$. We may assume that $s_i w > w$. Then $\alpha = w^{-1}\alpha_i > 0$, $\alpha \notin R(w)$ and $\alpha \in R(s_i w)$. Now, $R(w) \cap Z(t) = R(s_i w) \cap Z(t)$ implies $t(X^\alpha) \neq 1$, and $R(w) \cap P(t)$ implies $t(X^\alpha) \neq q^{\pm 2}$. Since $wt(X^{\alpha_i}) = t(X^{w^{-1}\alpha_i}) = t(X^\alpha) \neq 1$ and $wt(X^{\alpha_i}) \neq q^{\pm 2}$, it follows from Proposition 2.14(d) that the map $\tau_i : M_{wt}^{\text{gen}} \to M_{s_i wt}^{\text{gen}}$ is well defined and invertible. It remains to note that if $w, w' \in \mathcal{F}^{(t,J)}$, then $w' = s_{i_1} \cdots s_{i_\ell} w$ where $s_{i_k} \cdots s_{i_\ell} w \in \mathcal{F}^{(t,J)}$ for all $1 \leqslant k \leqslant \ell$. This follows from the fact that $\mathcal{F}^{(t,J)}$ corresponds to a connected convex region in $\mathfrak{h}_{\mathbb{R}}^*$. $\square$

3. Classification of calibrated representations

For simple roots α_i and α_j in R and let R_{ij} be the rank-two root subsystem of R generated by α_i and α_j. A weight $t \in T$ is *calibratable* if, for every pair i, j, $i \neq j$, t is a weight of a calibrated representation of the rank-two affine Hecke (sub)algebra generated by T_i, T_j and $\mathbb{C}[X]$. A local region

$$(t, J) \text{ is } \textit{skew} \quad \text{if} \quad wt \text{ is calibratable for all } w \in \mathcal{F}^{(t,J)}.$$

The classification of irreducible representations of rank-two affine Hecke algebras given in [27] can be used to state this condition combinatorially. Specifically, a weight $t \in T$ is *calibratable* if

(a) for all simple roots α_i, $1 \leqslant i \leqslant n$, $t(X^{\alpha_i}) \neq 1$, and
(b) for all pairs of simple roots α_i and α_j such that $\{\alpha \in R_{ij} \mid t(X^\alpha) = 1\} \neq \emptyset$, the set $\{\alpha \in R_{ij} \mid t(X^\alpha) = q^{\pm 2}\}$ contains more than two elements.

Condition (a) says that t is regular for all rank-1 subsystems of R generated by simple roots. This condition guarantees that the weight is "calibratable" (i.e., appears as a weight of some calibrated representation) for all rank-1 affine Hecke subalgebras of $\widetilde{H}$. Condition (b) is an "almost regular" condition on t with respect to rank-2 subsystems generated by simple roots.

Remark. The conversion between the definition of calibratable weight and the combinatorial condition given in (a) and (b) is as follows. Consider a rank-two affine Hecke algebra $\widetilde{H}$.

(A) By Theorem 2.12, (a) and (d), local regions (t, J) with t regular satisfy (a) and (b) and always contribute calibrated representations of $\widetilde{H}$.
(B) Using the notation of [27], the local regions (t, J) with t nonregular and which satisfy both conditions (a) and (b) are:

type A_2: none,
type C_2: $(t_b, \{\alpha_1\})$ and $(t_b, \{\alpha_1, \alpha_1 + \alpha_2\})$ (for each of these $P(t)$ contains 3 elements),
type G_2: (t_e, J) with $J \neq \emptyset$ and $J \neq P(t_e)$ (for each of these $P(t_e)$ contains 4 elements).

From (A) and (B) it follows that the local regions which satisfy (a) and (b) do contribute calibrated weights. The following shows that the other local regions do not contribute calibratable weights.

(C) By Lemma 3.1(a) local regions (t, J) with a weight $\xi = wt$, $w \in \mathcal{F}^{(t,J)}$ such that $\xi(X^{\alpha_i}) = 1$ do not satisfy (a) and, by inspection of the tables in [27], they never contribute a calibrated representation.
(D) Using the notation of [27], the local regions which satisfy condition (a) but not condition (b) are

type A_2: $(t_c, \{\alpha_2\})$ and $(t_d, \{\alpha_1\})$,
type B_2: $(t_d, \{\alpha_2\})$,
type G_2: $(t_i, \{\alpha_2\})$, $(t_f, \{\alpha_1\})$.

(Note that to satisfy (b) $Z(t)$ must be nonempty.) From the tables in [27] we see that none of these local regions supports a calibrated representation.

Remark. The paper [27] does not treat roots of unity. However, it is interesting to note that, *provided* $q^2 \neq \pm 1$, the methods of [27] go through without change to classify all representations of rank-two affine Hecke algebras even when q^2 is a root of unity. This classification can be used (as in the previous remark) to show that (a) and (b) above still characterize calibratable weights when q^2 is a root of unity such that $q^2 \neq \pm 1$. The key point is that Lemma 1.19 of [27] still holds. If $q^2 = -1$ then Lemma 1.19 of [27] breaks down at the next to last line of the proof in the statement "… forces $\phi(wt(T_j))$, to have Jordan blocks of size 1 …." When $q^2 = -1$ it is possible that $\phi(wt(T_j))$ has a Jordan block of size 2. If $q^2 = 1$ then one can change the definition of the τ-operators and use similar methods to produce a complete analysis of simple $\widetilde{H}$-modules, but we shall not do this here, choosing instead to exclude the case $q^2 = 1$ for simplicity of exposition.

The following lemma provides fundamental results about the structure of irreducible calibrated $\widetilde{H}$-modules. We omit the proof since it is accomplished in exactly the same way as in [17, Lemmas 4.1 and 4.2].

Lemma 3.1. *Let M be an irreducible calibrated module. Then, for all $t \in T$ such that $M_t \neq 0$,*

(a) *If $t \in T$ such that $M_t \neq 0$ then $t(X^{\alpha_i}) \neq 1$ for all $1 \leqslant i \leqslant n$.*
(b) *If $t \in T$ such that $M_t \neq 0$ then $\dim(M_t) = 1$.*
(c) *If $t \in T$ such that M_t and $M_{s_i t}$ are both nonzero then the map $\tau_i : M_t \to M_{s_i t}$ is a bijection.*

This lemma together with the classification of irreducible modules for rank-two affine Hecke algebras gives the following fundamental structural result for irreducible calibrated $\widetilde{H}$-modules. The proof is essentially the same as the proof of Proposition 4.3 in [17]. We repeat the proof here for continuity.

Theorem 3.2. *If M is an irreducible calibrated $\widetilde{H}$-module with central character $t \in T$ then there is a unique skew local region (t, J) such that*

$$\dim(M_{wt}) = \begin{cases} 1 & \text{for all } w \in \mathcal{F}^{(t,J)}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Lemma 3.1(b) all nonzero generalized weight spaces of M have dimension 1 and by Lemma 3.1(c) all τ-operators between these weight spaces are bijections. This already guarantees that there is a unique local region (t, J) which satisfies the condition. It only remains to show that this local region is skew.

Let $\widetilde{H}_{ij}$ be the subalgebra generated by T_i, T_j and $\mathbb{C}[X]$. Since M is calibrated as an $\widetilde{H}$-module it is calibrated as a $\widetilde{H}_{ij}$-module and so all factors of a composition series of M as an $\widetilde{H}_{ij}$-module are calibrated. Thus the weights of M are calibratable. So (t, J) is a skew local region. □

The following proposition shows that the weight space structure of calibrated representations, as determined in Theorem 3.2, essentially forces the $\widetilde{H}$-action on a weight basis. The proof is quite similar to the proof of Proposition 4.4 in [17]. However, we include the details since there is a technicality here; to make the conclusion in (3.4) we use the fact that the group X corresponds to the weight lattice $L = P$.

Proposition 3.3. *Let M be a calibrated $\widetilde{H}$-module and assume that for all $t \in T$ such that $M_t \neq 0$,*

$$\text{(A1)} \quad t\big(X_i^{\alpha}\big) \neq 1 \quad \textit{for all } 1 \leqslant i \leqslant n, \quad \textit{and} \quad \text{(A2)} \quad \dim(M_t) = 1.$$

For each $b \in T$ such that $M_b \neq 0$ let v_b be a nonzero vector in M_b. The vectors $\{v_b\}$ form a basis of M. Let $(T_i)_{cb} \in \mathbb{C}$ and $b(X^\lambda) \in \mathbb{C}$ be given by

$$T_i v_b = \sum_c (T_i)_{cb} v_c \quad \textit{and} \quad X^\lambda v_b = b\big(X^\lambda\big) v_b.$$

Then

(a) $(T_i)_{bb} = (q - q^{-1})/(1 - b(X^{-\alpha_i}))$, *for all v_b in the basis,*
(b) *if $(T_i)_{cb} \neq 0$ then $c = s_i b$,*
(c) $(T_i)_{b,s_i b}(T_i)_{s_i b,b} = (q^{-1} + (T_i)_{bb})(q^{-1} + (T_i)_{s_i b,s_i b})$.

Proof. The defining equation for $\widetilde{H}$,

$$X^\lambda T_i - T_i X^{s_i\lambda} = \big(q - q^{-1}\big)\frac{X^\lambda - X^{s_i\lambda}}{1 - X^{-\alpha_i}},$$

forces

$$\sum_c \big(c\big(X^\lambda\big)(T_i)_{cb} - (T_i)_{cb} b\big(X^{s_i\lambda}\big)\big)v_c = \big(q - q^{-1}\big)\frac{b(X^\lambda) - b(X^{s_i\lambda})}{1 - b(X^{-\alpha_i})} v_b.$$

Comparing coefficients gives

$$\begin{aligned} c\big(X^\lambda\big)(T_i)_{cb} - (T_i)_{cb} b\big(X^{s_i\lambda}\big) &= 0, \quad \text{if } b \neq c, \quad \text{and} \\ b\big(X^\lambda\big)(T_i)_{bb} - (T_i)_{bb} b\big(X^{s_i\lambda}\big) &= \big(q - q^{-1}\big)\frac{b(X^\lambda) - b(X^{s_i\lambda})}{1 - b(X^{-\alpha_i})}. \end{aligned}$$

These relations give:

$$\text{if} \quad (T_i)_{cb} \neq 0 \quad \text{then} \quad b\big(X^{s_i\lambda}\big) = c\big(X^\lambda\big) \quad \text{for all } X^\lambda \in X, \quad \text{and}$$

$$(T_i)_{bb} = \frac{q - q^{-1}}{1 - b(X^{-\alpha_i})} \quad \text{if} \quad b(X^{-\alpha_i}) \neq 1 \text{ and } b\big(X^\lambda\big) \neq b\big(X^{s_i\lambda}\big) \text{ for some } X^\lambda \in X.$$

By assumption (A1), $b(X^{\alpha_i}) \neq 1$ for all i. For each fundamental weight ω_i, $X^{\omega_i} \in X$ and $b(X^{s_i\omega_i}) = b(X^{\omega_i - \alpha_i}) \neq b(X^{\omega_i})$ since $b(X^{\alpha_i}) \neq 1$. Thus we conclude that

$$T_i v_b = (T_i)_{bb} v_b + (T_i)_{s_i b, b} v_{s_i b} \quad \text{with } (T_i)_{bb} = \frac{q - q^{-1}}{1 - b(X^{-\alpha_i})}. \tag{3.4}$$

This completes the proof of (a) and (b). By the definition of $\widetilde{H}$, the vector

$$T_i^2 v_b = \big((T_i)_{bb}^2 + (T_i)_{b,s_i b}(T_i)_{s_i b,b}\big) v_b + \big((T_i)_{bb} + (T_i)_{s_i b, s_i b}\big)(T_i)_{s_i b,b} v_{s_i b}$$

must equal

$$\big((q - q^{-1})T_i + 1\big) v_b = \big((q - q^{-1})(T_i)_{bb} + 1\big) v_b + \big(q - q^{-1}\big)(T_i)_{s_i b,b} v_{s_i b}.$$

Using the formula for $(T_i)_{bb}$ and $(T_i)_{s_i b, s_i b}$, we find $(T_i)_{bb} + (T_i)_{s_i b, s_i b} = (q - q^{-1})$. So, by comparing coefficients of v_b, we obtain the equation

$$\begin{aligned} (T_i)_{b,s_i b}(T_i)_{s_i b,b} &= \big(q - (T_i)_{bb}\big)\big((T_i)_{bb} + q^{-1}\big) \\ &= \big(q^{-1} + (T_i)_{bb}\big)\big(q^{-1} + (T_i)_{s_i b, s_i b}\big). \qquad \square \end{aligned}$$

Theorem 3.5. *Let (t, J) be a skew local region and let $\mathcal{F}^{(t,J)}$ index the chambers in the local region (t, J). Define*

$$\widetilde{H}^{(t,J)} = \mathbb{C}\text{-span}\big\{ v_w \;\big|\; w \in \mathcal{F}^{(t,J)} \big\},$$

so that the symbols v_w are a labeled basis of the vector space $\widetilde{H}^{(t,J)}$. Then the following formulas make $\widetilde{H}^{(t,J)}$ into an irreducible $\widetilde{H}$-module: For each $w \in \mathcal{F}^{(t,J)}$,

$$\begin{aligned} X^\lambda v_w &= (wt)\big(X^\lambda\big) v_w, && \text{for } X^\lambda \in X, \quad \text{and} \\ T_i v_w &= (T_i)_{ww} v_w + \big(q^{-1} + (T_i)_{ww}\big) v_{s_i w}, && \text{for } 1 \leqslant i \leqslant n, \end{aligned}$$

where $(T_i)_{ww} = (q - q^{-1})/(1 - (wt)(X^{-\alpha_i}))$, and we set $v_{s_i w} = 0$ if $s_i w \notin \mathcal{F}^{(t,J)}$.

Proof. Since (t, J) is a skew local region $(wt)(X^{-\alpha_i}) \neq 1$ for all $w \in \mathcal{F}^{(t,J)}$ and all simple roots α_i. This implies that the coefficient $(T_i)_{ww}$ is well defined for all i and $w \in \mathcal{F}^{(t,J)}$.

By construction, the nonzero weight spaces of $\widetilde{H}^{(t,J)}$ are $(\widetilde{H}^{(t,J)})^{\text{gen}}_{wt} = (\widetilde{H}^{(t,J)})_{wt}$ where $w \in \mathcal{F}^{(t,J)}$. Since $\dim(\widetilde{H}^{(t,J)}) = 1$ for $u \in \mathcal{F}^{(t,J)}$, any proper submodule N of $\widetilde{H}^{(t,J)}$ must have $N_{wt} \neq 0$ and $N_{w't} = 0$ for some $w \neq w'$ with $w, w' \in \mathcal{F}^{(t,J)}$. This is a contradiction to Corollary 2.19. So $\widetilde{H}^{(t,J)}$ is irreducible if it is an $\widetilde{H}$-module.

It remains to show that the defining relations for $\widetilde{H}$ are satisfied. This is accomplished as in the proof of [17, Theorem 4.5]. The only relation which is tricky to check is the braid relation. This can be verified as in [17] or it can be checked by case by case arguments (as in [28]). $\square$

We summarize the results of this section with the following corollary of Theorem 3.2 and the construction in Theorem 3.5.

Theorem 3.6. *Let M be an irreducible calibrated $\widetilde{H}$-module. Let $t \in T$ be* (*a fixed choice of*) *the central character of M and let $J = R(w) \cap P(t)$ for any $w \in W$ such that $M_{wt} \neq 0$. Then (t, J) is a skew local region and $M \cong \widetilde{H}^{(t,J)}$ where $\widetilde{H}^{(t,J)}$ is the module defined in Theorem* 3.5.

4. The structure of local regions

Recall that the Weyl group acts on

$$T = \mathrm{Hom}(X, \mathbb{C}^*) = \{\text{group homomorphisms } t : X \to \mathbb{C}^*\} \quad \text{by } (wt)\big(X^\lambda\big) = t\big(X^{w^{-1}\lambda}\big).$$

Any element $t \in T$ is determined by the values $t(X^{\omega_1}), t(X^{\omega_2}), \ldots, t(X^{\omega_n})$. For $t \in T$ define the *polar decomposition*

$$t = t_r t_c, \qquad t_r, t_c \in T \text{ such that } t_r\big(X^\lambda\big) \in \mathbb{R}_{>0} \text{ and } \big|t_c\big(X^\lambda\big)\big| = 1,$$

for all $X^\lambda \in X$. There is a unique $\gamma \in \mathbb{R}^n$ and a unique $\nu \in \mathbb{R}^n/P$ such that

$$t_r\big(X^\lambda\big) = \mathrm{e}^{\langle \gamma, \lambda \rangle} \quad \text{and} \quad t_c\big(X^\lambda\big) = \mathrm{e}^{2\pi \mathrm{i} \langle \nu, \lambda \rangle} \quad \text{for all } \lambda \in P. \tag{4.1}$$

In this way we identify the sets $T_r = \{t \in T \mid t = t_r\}$ and $T_c = \{t \in T \mid t = t_c\}$ with $\mathfrak{h}^*_{\mathbb{R}}$ and $\mathfrak{h}^*_{\mathbb{R}}/P$, respectively.

For this paragraph (our goal here is (4.3) below) assume that q is not a root of unity (we will treat the type A, root of unity case in detail in Section 7). The representation theory of $\widetilde{H}$ is "the same" for any q which is not a root of unity, i.e. provided q is not a root of unity, the classification and construction of simple $\widetilde{H}$-modules can be stated uniformly in terms of the parameter q. Suppose $t \in T$ is such that $t = t_r$ and $\gamma \in \mathfrak{h}^*_{\mathbb{R}}$ is such that

$$t = \mathrm{e}^\gamma, \quad \text{in the sense that} \quad t\big(X^\lambda\big) = \mathrm{e}^{\langle \gamma, \lambda \rangle} \quad \text{for all } X^\lambda \in X.$$

For the purposes of representation theory (as in Theorem 3.5) t indexes a central character and so we should assume that γ is chosen nicely in its W-orbit. When

$$q = \mathrm{e} \quad \text{and} \quad \gamma \text{ is dominant, i.e., } \langle \gamma, \alpha \rangle \geqslant 0 \quad \text{for all } \alpha \in R^+, \tag{4.2}$$

then

$$Z(t) = Z(\gamma), \qquad P(t) = P(\gamma), \quad \text{and} \quad \mathcal{F}^{(t,J)} = \mathcal{F}^{(\gamma,J)} \quad \text{for a subset } J \subseteq P(t),$$

where

$$Z(\gamma) = \big\{\alpha \in R^+ \ \big|\ \langle \gamma, \alpha \rangle = 0\big\}, \qquad P(\gamma) = \big\{\alpha \in R^+ \ \big|\ \langle \gamma, \alpha \rangle = 1\big\},$$

$$\mathcal{F}^{(\gamma,J)} = \{ w \in W \mid R(w) \cap Z(\gamma) = \emptyset,\ R(w) \cap P(\gamma) = J \}. \tag{4.3}$$

In this case the combinatorics of local regions is a new chapter in the combinatorics of the Shi arrangement defined in (1.16). Other aspects of the combinatorics of the Shi arrangement can be found in [2,33–35,37–39], and there are several additional places in the literature [35], [46, 1.11, 2.6], [15,16] which indicate that there is a deep (and not yet completely understood) connection between the structure and representation theory of the affine Hecke algebra and the combinatorics of the Shi arrangement.

4.4. Intervals in Bruhat order

Using the formulation in (4.3), Theorem 4.6 will give a complete description of the structure of $\mathcal{F}^{(\gamma,J)}$ as a subset of the Weyl group when q is not a root of unity. We will treat the type A, root of unity cases in Section 7.

The *weak Bruhat order* is the partial order on W given by

$$v \leqslant w \quad \text{if } R(v) \subseteq R(w), \tag{4.5}$$

where $R(w)$ denotes the inversion set of $w \in W$ as defined in (1.5). This definition of the weak Bruhat order is not the usual definition but is equivalent to the usual one by [4, Proposition 2]. A set of positive roots K is *closed* if $\alpha, \beta \in K$, $\alpha + \beta \in R^+$ implies that $\alpha + \beta \in K$. The *closure* $\overline{K}$ of a subset $K \subseteq R^+$ is the smallest closed subset of R^+ containing K. A set of positive roots $K \subseteq R^+$ is the inversion set of some permutation $w \in W$ if and only if K is closed and $K^c = R^+ \setminus K$ is closed (see [4, Proposition 2] or [17, Theorem 5.1]).

The following theorem is proved in [17, Section 5]. The proof of part (b) of the theorem relies crucially on a theorem of J. Losonczy [22].

Theorem 4.6. *Let $\gamma \in \mathfrak{h}^*_{\mathbb{R}}$ be dominant (i.e., $\langle \gamma, \alpha \rangle \geqslant 0$ for all $\alpha \in R^+$) and let $J \subseteq P(\gamma)$. Let $\mathcal{F}^{(\gamma,J)}$ be as given in* (4.3).

(a) *Then $\mathcal{F}^{(\gamma,J)}$ is nonempty if and only if J satisfies the condition*

$$\text{if} \quad \beta \in J, \alpha \in Z(\gamma) \text{ and } \beta - \alpha \in R^+ \quad \text{then} \quad \beta - \alpha \in J.$$

(b) *The sub-root system $R_{[\gamma]} = \{\alpha \in R \mid \langle \gamma, \alpha \rangle \in \mathbb{Z}\}$, has Weyl group*

$$W_{[\gamma]} = \langle s_\alpha \mid \alpha \in R_{[\gamma]} \rangle$$

and if $W^{[\gamma]} = \{\sigma \in W \mid R(\sigma) \cap R_{[\gamma]} = \emptyset\}$ then

$$\mathcal{F}^{(\gamma,J)} = W^{[\gamma]} \cdot [\tau_{\max}, \tau_{\min}],$$

where $\tau_{\max}, \tau_{\min} \in W_{[\gamma]}$ are determined by

$$R(\tau_{\max}) \cap R_{[\gamma]} = \overline{J} \quad \text{and} \quad R(\tau_{\min}) \cap R_{[\gamma]} = \overline{(P(\gamma) \setminus J) \cup Z(\gamma)}^c,$$

the complement is taken in the set of positive roots of $R_{[\gamma]}$, *and* $[\tau_{\min}, \tau_{\max}]$ *denotes the interval between* $w_{\min}$ *and* $w_{\max}$ *in the weak Bruhat order in* $W_{[\gamma]}$.

4.7. Conjugation

Assume that γ is dominant (i.e., $\langle\gamma,\alpha\rangle \geqslant 0$ for all $\alpha \in R^+$) and $J \subseteq P(\gamma)$. Let $\mathcal{F}^{(\gamma,J)}$ be as given in (4.3). The *conjugate* of (γ, J) and of $w \in \mathcal{F}^{(\gamma,J)}$ are defined by

$$(\gamma, J)' = \big(-u\gamma, -u\big(P(\gamma) \setminus J\big)\big) \quad \text{and} \quad \begin{array}{ccc} \mathcal{F}^{(t,J)} & \stackrel{1-1}{\longleftrightarrow} & \mathcal{F}^{(t,J)'}, \\ w & \leftrightarrow & w' = wu^{-1}, \end{array} \tag{4.8}$$

where u is the minimal length coset representative of $w_0W_\gamma \in W/W_\gamma$ and w_0 is the longest element of W. In Section 6.7 we shall show that these maps are generalizations of the classical conjugation operation on partitions.

Theorem 4.9. *The conjugation maps defined in* (4.8) *are well defined involutions.*

Proof. (a) Since γ is dominant, $-u\gamma = -w_0\gamma$ is dominant and thus $\langle -u\gamma, -u\alpha\rangle = 1$ only if $-u\alpha > 0$. Thus the equation $\langle -u\gamma, -u\alpha\rangle = 1 \Leftrightarrow \langle\gamma,\alpha\rangle = 1$ gives that $P(-u\gamma) = -uP(\gamma)$.

(b) Let $v \in W_\gamma$ such that $w_0 = uv$. (By [6, IV, Section 1 Exercise 3], v is unique.) Then $R^+ \supseteq -w_0Z(\gamma) = -uvZ(\gamma) = uZ(\gamma)$, and it follows that

$$Z(-u\gamma) = R^+ \cap \big\{\alpha \in R \;\big|\; \langle u\gamma,\alpha\rangle = 0\big\} = R^+ \cap \big(uZ(\gamma) \cup -uZ(\gamma)\big) = uZ(\gamma).$$

(c) Let $R^- = -R^+$ be the set of negative roots in R. Let $v \in W_\gamma$ such that $w_0 = uv$. Then v is the longest element of W_γ and $R(v) = Z(\gamma)$. Thus, since $w_0R^- = R^+$,

$$\begin{aligned} R(u) &= \{\alpha \in R \mid \alpha \in R^+,\ w_0v\alpha \in R^-\} = \{\alpha \in R \mid \alpha \in R^+,\ v\alpha \in R^+\}, \\ &= R^+ \backslash R(v) = R^+ \backslash Z(\gamma). \end{aligned}$$

(d) The weight $-u\gamma = -uv\gamma = -w_0\gamma$ is dominant and $-u(P(\gamma)\backslash J) \subseteq P(-u\gamma)$ since $-uP(\gamma) = P(-u\gamma)$. This shows that $(\gamma, J)'$ is well defined.

(e) Write $w_0 = uv$ where v is the longest element of W_γ. Similarly, write $w_0 = u'v'$ where u' is the minimal length coset representative of $w_0W_{w_0\gamma}$ and v' is the longest element in $W_{w_0\gamma}$. Conjugation by w_0 is an involution on W which takes simple reflections to simple reflections and $W_{w_0\gamma} = w_0W_\gamma w_0$. It follows that $v' = w_0vw_0$. This gives

$$u'u = (w_0v')(w_0v) = w_0w_0vw_0w_0v = 1,$$

and so the second map in (4.8) is an involution.

(f) Using (e) and (a),

$$\begin{aligned} -u'\big(P(-u\gamma)\big\backslash\big(-u\big(P(\gamma)\backslash J\big)\big)\big) &= -u'\big(-uP(\gamma)\big\backslash\big(-u\big(P(\gamma)\backslash J\big)\big)\big) \\ &= P(\gamma)\big\backslash\big(P(\gamma)\backslash J\big) = J, \end{aligned}$$

and so the first map in (4.8) is an involution.

(g) Let $w \in \mathcal{F}^{(\gamma,J)}$ and let $w' = wu^{-1}$. Since $R(w) \cap Z(\gamma) = \emptyset$,

$$\begin{aligned} &u^{-1}R\big(wu^{-1}\big) \cap Z(\gamma) \\ &\quad = \big\{\beta \in R \;\big|\; u\beta \in R\big(wu^{-1}\big),\ \beta \in Z(\gamma)\big\} \\ &\quad = \big\{\beta \in R \;\big|\; u\beta \in R^+, wu^{-1}u\beta \in R^-,\ \beta \in Z(\gamma)\big\} \\ &\quad = \big\{\beta \in R \;\big|\; \beta \in u^{-1}R^+,\ w\beta \in R^-,\ \beta \in Z(\gamma)\big\} \\ &\quad = \big\{\beta \in R \;\big|\; \beta \in u^{-1}R^+,\ \beta \in R(w),\ \beta \in Z(\gamma)\big\} \quad (\text{since } Z(\gamma) \subseteq R^+) \\ &\quad = \big\{\beta \in R \;\big|\; \beta \in u^{-1}R^+,\ \beta \in R(w) \cap Z(\gamma)\big\} \\ &\quad = \emptyset, \end{aligned}$$

and thus, by (b),

$$R(w') \cap Z(-u\gamma) = R\big(wu^{-1}\big) \cap uZ(\gamma) = u\big(u^{-1}R\big(wu^{-1}\big) \cap Z(\gamma)\big) = \emptyset.$$

Since $R(w) \cap P(\gamma) = J$,

$$\begin{aligned} &-u^{-1}R\big(wu^{-1}\big) \cap P(\gamma) \\ &\quad = \big\{\beta \in R \;\big|\; -u\beta \in R\big(wu^{-1}\big),\ \beta \in P(\gamma)\big\} \\ &\quad = \big\{\beta \in R \;\big|\; -u\beta \in R^+,\ -wu^{-1}u\beta \in R^-,\ \beta \in P(\gamma)\big\} \\ &\quad = \big\{\beta \in R \;\big|\; u\beta \in R^-,\ w\beta \in R^+,\ \beta \in P(\gamma)\big\} \\ &\quad = \big\{\beta \in R \;\big|\; \beta \in R(u),\ \beta \in R^+\backslash R(w),\ \beta \in P(\gamma)\big\} \quad (\text{since } P(\gamma) \subseteq R^+) \\ &\quad = \big\{\beta \in R \;\big|\; \beta \in R^+\backslash Z(\gamma),\ \beta \in R^+\backslash R(w),\ \beta \in P(\gamma)\big\} \\ &\quad = \big\{\beta \in R \;\big|\; \beta \in R^+\backslash Z(\gamma),\ \beta \in P(\gamma)\backslash J\big\} \quad (\text{since } R(w) \cap P(\gamma) = J) \\ &\quad = P(\gamma)\backslash J, \quad \text{since } Z(\gamma) \text{ and } P(\gamma) \text{ are disjoint.} \end{aligned}$$

Thus, by (a),

$$\begin{aligned} R(w') \cap P(-u\gamma) &= R\big(wu^{-1}\big) \cap -uP(\gamma) = -u\big(-u^{-1}R\big(wu^{-1}\big) \cap P(\gamma)\big) \\ &= -u\big(P(\gamma)\backslash J\big), \end{aligned}$$

and so the second map in (4.8) is well defined. □

Remark 4.10. In type A, the conjugation involution coincides with the duality operation for representations of $\mathfrak{p}$-adic $GL(n)$ defined by Zelevinsky [49]. Zelevinsky's involution has been studied further in [18,19,25] and extended to general Lie type by Kato [13] and Aubert [3]. For $\widetilde{H}$-modules in type A, this is the involution on modules induced by the

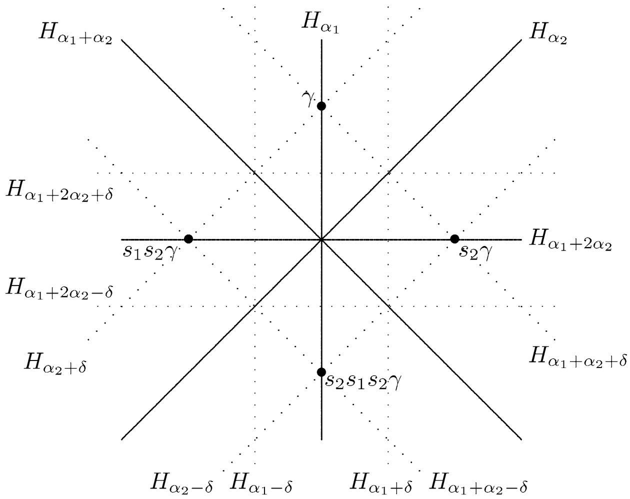


Fig. 2.

Iwahori–Matsumoto involution of $\widetilde{H}$ and is detected on the level of characters: it sends an irreducible $\widetilde{H}$-module L to the unique irreducible L^* with $\dim((L^*)_t^{\mathrm{gen}}) = \dim(L_{t^{-1}}^{\mathrm{gen}})$ for each $t \in T$. I would like to thank J. Brundan for clarifying this remark and making it precise.

Examples 4.11. (a) If γ is dominant and is generic (as an element of C) then $Z(\gamma) = P(\gamma) = \emptyset$ and $\mathcal{F}^{(\gamma,\emptyset)} = W$.

(b) Let ρ be defined by $\langle \rho, \alpha_i \rangle = 1$, for all $1 \leqslant i \leqslant n$. Then

$$Z(\rho) = \emptyset, \qquad P(\rho) = \{\alpha_1, \ldots, \alpha_n\}, \quad \text{and} \quad \mathcal{F}^{(\rho,J)} = \{w \in W \mid D(w) = J\},$$

where $D(w) = \{\alpha_i \mid ws_i < w\}$ is the *right descent set* of $w \in W$. The sets $\mathcal{F}^{(\gamma,J)}$ which arise here are fundamental to the theory of descent algebras [9,31,36].

(c) This example is a generalization of (b). Suppose that (γ, J) is a local region such that γ is regular and integral (i.e., $\langle \gamma, \alpha \rangle \in \mathbb{Z}_{>0}$ for all $\alpha \in R^+$). Then

$$Z(\gamma) = \emptyset, \qquad P(\gamma) \subseteq \{\alpha_1, \ldots, \alpha_n\}, \quad \text{and} \quad \mathcal{F}^{(\gamma,J)} = \{w \in W \mid D(w) \cap P(\gamma) = J\}.$$

(d) Let R be the root system of type C_2 with simple roots $\alpha_1 = \varepsilon_1$ $\alpha_2 = \varepsilon_2 - \varepsilon_1$, where $\{\varepsilon_1, \varepsilon_2\}$ is an orthonormal basis of $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}^2$. The positive roots are $R^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$. Let $\gamma \in \mathbb{R}^2$ be given by $\langle \gamma, \alpha_1 \rangle = 0$ and $\langle \gamma, \alpha_2 \rangle = 1$. Then γ is dominant (i.e., in $\overline{C}$) and integral and

$$Z(\gamma) = \{\alpha_1\} \quad \text{and} \quad P(\gamma) = \{\alpha_2, \alpha_1 + \alpha_2\}.$$

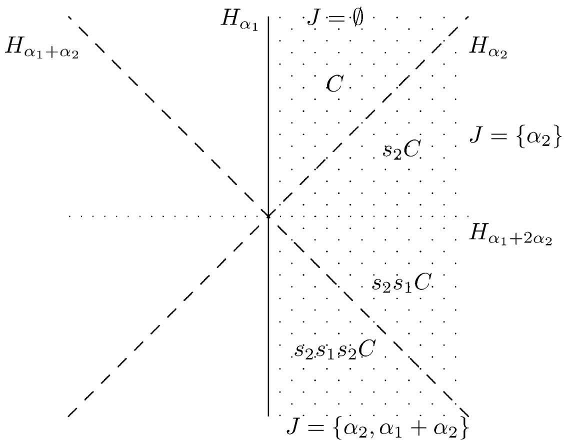


Fig. 3.

Figure 3 displays the local regions $\mathcal{F}^{(\gamma,J)}$ as regions in $\mathfrak{h}^*_{\mathbb{R}}$, see the remarks after (2.18).

The solid line is the hyperplane corresponding to the root in $Z(\gamma)$ and the dashed lines are the hyperplanes corresponding to the roots in $P(\gamma)$.

(e) Let R be the root system of type C_2 as in (d). Let $\gamma \in \mathbb{R}^2$ be defined by

$$\langle \gamma, \alpha_1 \rangle = 0, \qquad \langle \gamma, \alpha_2 \rangle = \frac{1}{2}.$$

Then

$$Z(\gamma) = \{\alpha_1\}, \qquad P(\gamma) = \{\alpha_1 + 2\alpha_2\}.$$

If $J = P(\gamma)$ then the unique minimal element $w_{\min}$ of $\mathcal{F}^{(\gamma,J)}$ has $R(w_{\min}) = \{\alpha_2, \alpha_1 + 2\alpha_2\} \neq \overline{J} = J$.

5. The connection to standard Young tableaux

In this section we shall show that the combinatorics of local regions is a generalization of the combinatorics of standard Young tableaux. Let us first make some general definitions, which we will show later provide generalizations of standard objects in the Young tableaux theory. This section is a (purely combinatorial) study of the local regions in the form which appears in (4.3), and therefore corresponds to the representation theory of affine Hecke algebras when q is not a root of unity.

5.1. *Definitions*

Let $\gamma \in \mathfrak{h}^*_{\mathbb{R}}$ be dominant and let

$$Z(\gamma) = \{\alpha \in R^+ \mid \langle \gamma, \alpha \rangle = 0\}, \qquad P(\gamma) = \{\alpha \in R^+ \mid \langle \gamma, \alpha \rangle = 1\},$$
$$\mathcal{F}^{(\gamma,J)} = \{w \in W \mid R(w) \cap Z(\gamma) = \emptyset,\ R(w) \cap P(\gamma) = J\},$$

as in (4.3).

(a) A *local region* is a pair (γ, J) such that $\mathcal{F}^{(\gamma,J)}$ is nonempty.
(b) A *ribbon* is a local region (γ, J) such that γ is regular, i.e., $\langle \gamma, \alpha \rangle \neq 0$ for all $\alpha \in R$.
(c) An element $\gamma \in \overline{C}$ is *calibratable* if
   (1) for all simple roots α_i, $1 \leqslant i \leqslant n$, $\langle \gamma, \alpha_i \rangle \neq 0$, and
   (2) for all pairs of simple roots α_i and α_j such that $\{\alpha \in R_{ij} \mid \langle \gamma, \alpha \rangle = 0\} \neq \emptyset$, the set $\{\alpha \in R_{ij} \mid \langle \gamma, \alpha \rangle = 1\}$ contains more than two elements.
(d) A *skew local region* is a local region (γ, J) such that $w\gamma$ is calibratable for all $w \in \mathcal{F}^{(\gamma,J)}$. All ribbons are skew.
(e) A *column* (*respectively row*) *reading tableau* is a minimal (respectively maximal) element of $\mathcal{F}^{(\gamma,J)}$ in the weak Bruhat order.
(f) If $\alpha \in R$ the α*-axial distance* for $w \in \mathcal{F}^{(\gamma,J)}$ is the value $d_\alpha(w) = \langle w\gamma, \alpha \rangle$.

Remarks. (1) Theorem 4.6(b) shows that, up to a shift, the set $\mathcal{F}^{(\gamma,J)}$ has a unique maximal and a unique minimal element and is an interval in the weak Bruhat order. This is the fundamental importance of the notions of the row reading and the column reading tableaux. Theorem 6.9 in Section 6 will show how Theorem 4.6(b) is a generalization of a Young tableaux result of Björner and Wachs [5, Theorem 7.2].

(2) The definition of skew local regions is forced by the representation theory of the affine Hecke algebra (see Theorem 3.6, the classification of irreducible calibrated representations). In Proposition 6.4 below we shall show that the skew local regions and the ribbons are generalizations of the skew shapes and border strips which are used in the theory of symmetric functions [24, I, Section 5 and I, Section 3, Exercise 11]

(3) The axial distances control the denominators which appear in the construction of irreducible representations of the affine Hecke algebra in Theorem 3.5. In Section 6.1 we shall see how they are analogues of the axial distances used by A. Young [48] in his constructions of the irreducible representations of the symmetric group.

To summarize, a brief dictionary between local regions combinatorics and the Young tableaux combinatorics:

skew local regions	$\leftrightarrow$	skew shapes λ/μ,
ribbons	$\leftrightarrow$	border strips,
local regions	$\leftrightarrow$	general configurations of boxes,
$\mathcal{F}^{(\gamma,J)}$	$\leftrightarrow$	the set of standard tableaux $\mathcal{F}^{\lambda/\mu}$.

The remainder of this section and the next section explain in greater detail the conversions indicated in this dictionary.

5.2. *The root system*

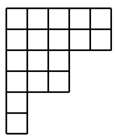
Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ be an orthonormal basis of $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}^n$ so that each sequence $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n$ is identified with the vector $\gamma = \sum_i \gamma_i \varepsilon_i$. The root system of type A_{n-1} is given by the sets

$$R = \left\{\pm(\varepsilon_j - \varepsilon_i) \,\middle|\, 1 \leqslant i, j \leqslant n\right\} \quad \text{and} \quad R^+ = \{\varepsilon_j - \varepsilon_i \mid 1 \leqslant i < j \leqslant n\}.$$

The Weyl group is $W = S_n$, the symmetric group, acting by permutations of the ε_i.

5.3. *Partitions, skew shapes, and standard tableaux*

A partition λ is a collection of n boxes in a corner. We shall conform to the conventions in [24] and assume that gravity goes up and to the left.



Any partition λ can be identified with the sequence $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots)$ where λ_i is the number of boxes in row i of λ. The rows and columns are numbered in the same way as for matrices. We shall always use the word *diagonal* to mean a major diagonal. In the example above $\lambda = (553311)$ and the diagonals of λ (from southwest to northeast) contain 1, 1, 1, 2, 3, 3, 2, 2, 2, and 1 box, respectively.

If λ and μ are partitions such that $\mu_i \leqslant \lambda_i$ for all i write $\mu \subseteq \lambda$. The *skew shape* λ/μ consists of all boxes of λ which are not in μ. Let λ/μ be a skew shape with n boxes. Number the boxes of each skew shape λ/μ along diagonals from southwest to northeast and

write box_i to indicate the box numbered i.

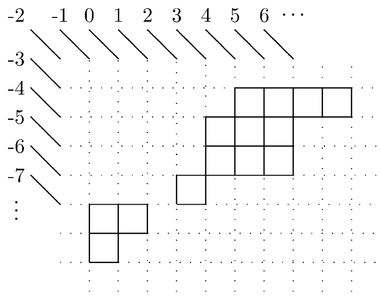
See Example 5.8 below. A *standard tableau of shape* λ/μ is a filling of the boxes in the skew shape λ/μ with the numbers $1, \ldots, n$ such that the numbers increase from left to right in each row and from top to bottom down each column. Let $\mathcal{F}^{\lambda/\mu}$ be the set of standard tableaux of shape λ/μ. Given a standard tableau p of shape λ/μ define the *word* of p to be the permutation

$$w_p = \begin{pmatrix} 1 & \cdots & n \\ p(\text{box}_1) & \ldots & p(\text{box}_n) \end{pmatrix} \tag{5.4}$$

where $p(\text{box}_i)$ is the entry in box_i of the standard tableau.

5.5. Placed skew shapes

Let λ/μ be a skew shape with n boxes. Imagine placing λ/μ on a piece of infinite graph paper where the diagonals of the graph paper are indexed consecutively (with elements of $\mathbb{Z}$) from southwest to northeast.



The *content* of a box b is

$$c(b) = \text{diagonal number of box } b.$$

Identify the sequence

$$\gamma = \big(c(\text{box}_1), c(\text{box}_2), \ldots, c(\text{box}_n)\big) \quad \text{with } \gamma = \sum_{i=1}^{n} c(\text{box}_i)\varepsilon_i \in \mathbb{R}^n. \tag{5.6}$$

The pair $(\gamma, \lambda/\mu)$ is a *placed skew shape*. It follows from the definitions in Section 5.1 that

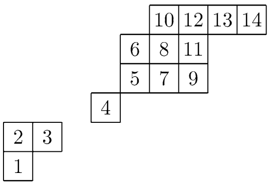
$$Z(\gamma) = \{\varepsilon_j - \varepsilon_i \mid j > i \text{ and box}_j \text{ and box}_i \text{ are in the same diagonal}\} \quad \text{and}$$
$$P(\gamma) = \{\varepsilon_j - \varepsilon_i \mid j > i \text{ and box}_j \text{ and box}_i \text{ are in adjacent diagonals}\}.$$

Define

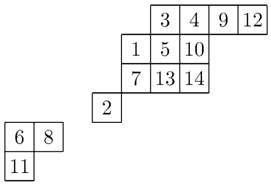
$$J = \left\{ \varepsilon_j - \varepsilon_i \;\middle|\; \begin{array}{l} j > i \\ \text{box}_j \text{ and box}_i \text{ are in adjacent diagonals} \\ \text{box}_j \text{ is northwest of box}_i \end{array} \right\}, \tag{5.7}$$

where *northwest* means strictly north and weakly west.

Examples 5.8. The following diagrams illustrate standard tableaux and the numbering of boxes in a skew shape λ/μ.



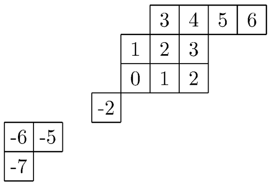
λ/μ with boxes numbered



A standard tableau p of shape λ/μ

The word of the standard tableau p is the permutation $w_p = (11, 6, 8, 2, 7, 1, 13, 5, 14, 3, 10, 4, 9, 12)$ (in one-line notation).

The following picture shows the contents of the boxes in the placed skew shape $(\gamma, \lambda/\mu)$ with $\gamma = (-7, -6, -5, -2, 0, 1, 1, 2, 2, 3, 3, 4, 5, 6)$.



Contents of the boxes of $(\gamma, \lambda/\mu)$

In this case $J = \{\varepsilon_2 - \varepsilon_1, \varepsilon_6 - \varepsilon_5, \varepsilon_8 - \varepsilon_7, \varepsilon_{10} - \varepsilon_8, \varepsilon_{10} - \varepsilon_9, \varepsilon_{11} - \varepsilon_9, \varepsilon_{12} - \varepsilon_{11}\}$.

Theorem 5.9. *Let $(\gamma, \lambda/\mu)$ be a placed skew shape and let J be as defined in* (5.7). *Let $\mathcal{F}^{\lambda/\mu}$ be the set of standard tableaux of shape λ/μ and let $\mathcal{F}^{(\gamma,J)}$ be the set defined in Section* 5.1. *Then the map*

$$\mathcal{F}^{\lambda/\mu} \overset{1-1}{\longleftrightarrow} \mathcal{F}^{(\gamma,J)}, \qquad p \leftrightarrow w_p,$$

where w_p is as defined in (5.4)*, is a bijection.*

Proof. If $w = (w(1) \cdots w(n))$ is a permutation in S_n then

$$R(w) = \{\varepsilon_j - \varepsilon_i \mid j > i \text{ such that } w(j) < w(i)\}.$$

The theorem is a consequence of the following chain of equivalences:

The filling p is a standard tableau if and only if, for all $1 \leqslant i < j \leqslant n$,

(a) $p(\mathrm{box}_i) < p(\mathrm{box}_j)$ if box_i and box_j are on the same diagonal,
(b) $p(\mathrm{box}_i) < p(\mathrm{box}_j)$ if box_j is immediately to the right of box_i, and
(c) $p(\mathrm{box}_i) > p(\mathrm{box}_j)$ if box_j is immediately above box_i.

These conditions hold if and only if

(a) $\varepsilon_j - \varepsilon_i \notin R(w_p)$ if $\varepsilon_j - \varepsilon_i \in Z(\gamma)$,
(b) $\varepsilon_j - \varepsilon_i \notin R(w_p)$ if $\varepsilon_j - \varepsilon_i \in P(\gamma) \setminus J$,
(c) $\varepsilon_j - \varepsilon_i \in R(w_p)$ if $\varepsilon_j - \varepsilon_i \in J$,

which hold if and only if

(a) $\alpha \notin R(w_p)$ if $\alpha \in Z(\gamma)$,
(b) $\alpha \notin R(w_p)$ if $\alpha \in P(\gamma) \setminus J$, and
(c) $\alpha \in R(w_p)$ if $\alpha \in J$.

Finally, these are equivalent to the conditions $R(w_p) \cap Z(\gamma) = \emptyset$ and $R(w_p) \cap P(\gamma) = J$. □

5.10. Placed configurations

We have described how one can identify placed skew shapes $(\gamma, \lambda/\mu)$ with certain pairs (γ, J). One can extend this conversion to associate placed configurations of boxes to more general pairs (γ, J). The resulting configurations are not always skew shapes.

Let (γ, J) be a pair such that $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a dominant integral weight and $J \subseteq P(\gamma)$. (The sequence γ is a dominant integral weight if $\gamma_1 \leqslant \cdots \leqslant \gamma_n$ and $\gamma_i \in \mathbb{Z}$ for all i.) If J satisfies the condition

$$\text{if} \quad \beta \in J, \alpha \in Z(\gamma), \text{and } \beta - \alpha \in R^+ \quad \text{then} \quad \beta - \alpha \in J$$

then (γ, J) will determine a placed configuration of boxes (see Theorem 4.6). As in the placed skew shape case, think of the boxes as being placed on graph paper where the boxes on a given diagonal all have the same content. (The boxes on each diagonal are allowed to slide along the diagonal as long as they do not pass through the corner of a box on an adjacent diagonal.) The sequence γ describes how many boxes are on each diagonal and the set J determines how the boxes on adjacent diagonals are placed relative to each other. We want

$$\gamma = \sum_{i=1}^{n} c(\text{box}_i)\varepsilon_i$$

and

(a) if $\varepsilon_j - \varepsilon_i \in J$ then box_j is northwest of box_i, and
(b) if $\varepsilon_j - \varepsilon_i \in P(\gamma) \setminus J$ then box_j is southeast of box_i,

where the boxes are numbered along diagonals in the same way as for skew shapes, *southeast* means weakly south and strictly east, and *northwest* means strictly north and weakly west.

If we view the pair (γ, J) as a placed configuration of boxes then the *standard tableaux* are fillings p of the n boxes in the configuration with $1, 2, \ldots, n$ such that, for all $i < j$,

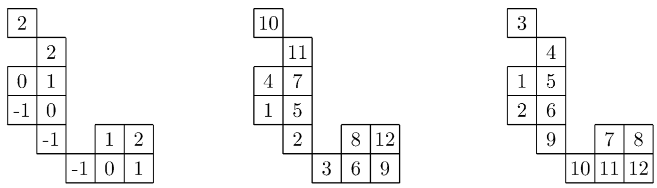
(a) $p(\mathrm{box}_i) < p(\mathrm{box}_j)$ if box_i and box_j are on the same diagonal,
(b) $p(\mathrm{box}_i) < p(\mathrm{box}_j)$ if box_i and box_j are on adjacent diagonals and box_j is southeast of box_i, and
(c) $p(\mathrm{box}_i) > p(\mathrm{box}_j)$ if box_i and box_j are on adjacent diagonals and box_j is northwest of box_i.

As in (5.6) the permutation in $\mathcal{F}^{(\gamma,J)}$ which corresponds to the standard tableau p is $w_p = (p(\mathrm{box}_1), \ldots, p(\mathrm{box}_n))$. The following example illustrates the conversion.

Example. Suppose $\gamma = (-1, -1, -1, 0, 0, 0, 1, 1, 1, 2, 2, 2)$ and

$$J = \{\varepsilon_4 - \varepsilon_1, \varepsilon_4 - \varepsilon_2, \varepsilon_4 - \varepsilon_3, \varepsilon_5 - \varepsilon_2, \varepsilon_5 - \varepsilon_3, \varepsilon_7 - \varepsilon_5, \varepsilon_7 - \varepsilon_6, \varepsilon_8 - \varepsilon_6, \varepsilon_{10} - \varepsilon_9, \\ \varepsilon_{10} - \varepsilon_8, \varepsilon_{10} - \varepsilon_7, \varepsilon_{11} - \varepsilon_9, \varepsilon_{11} - \varepsilon_8, \varepsilon_{11} - \varepsilon_7, \varepsilon_{12} - \varepsilon_9\}.$$

The placed configuration of boxes corresponding to (γ, J) is as given below.



contents of boxes   numbering of boxes   a standard tableau

5.11. *Books of placed configurations*

The general case, when $\gamma = (\gamma_1, \ldots, \gamma_n)$ is an arbitrary element of $\mathbb{R}^n$ and $J \subseteq P(\gamma)$, is handled as follows. First group the entries of γ according to their $\mathbb{Z}$-coset in $\mathbb{R}$. Each group of entries in γ can be arranged to form a sequence

$$\beta + C_\beta = \beta + (z_1, \ldots, z_k) = (\beta + z_1, \ldots, \beta + z_k), \\ \text{where } 0 \leqslant \beta < 1,\ z_i \in \mathbb{Z}, \text{ and } z_1 \leqslant \cdots \leqslant z_k.$$

Fix some ordering of these groups and let

$$\vec{\gamma} = (\beta_1 + C_{\beta_1}, \ldots, \beta_r + C_{\beta_r})$$

be the rearrangement of the sequence γ with the groups listed in order. Since $\vec{\gamma}$ and γ are in the same orbit it is sufficient to analyze $\vec{\gamma}$ (γ corresponds to the central character of the corresponding affine Hecke algebra representations and thus any convenient element of the orbit is appropriate, see Section 2.3).

The decomposition of $\vec{\gamma}$ into groups induces decompositions

$$Z(\vec{\gamma}) = \bigcup_{\beta_i} Z_{\beta_i}, \qquad P(\vec{\gamma}) = \bigcup_{\beta_i} P_{\beta_i}, \quad \text{and,} \quad \text{if } J \subseteq P(\vec{\gamma}), \quad \text{then } J = \bigcup_{\beta_i} J_{\beta_i},$$

where $J_{\beta_i} = J \cap P_{\beta_i}$. Each pair (C_β, J_β) is a placed shape of the type considered in the previous subsection and we may identify $(\vec{\gamma}, J)$ with the *book of placed shapes* $((C_{\beta_1}, J_{\beta_1}), \ldots, (C_{\beta_r}, J_{\beta_r}))$. We think of this as a *book* with *pages* numbered by the values $\beta_1, \ldots, \beta_r$ and with the placed configuration determined by $(C_{\beta_i}, J_{\beta_i})$ on page β_i. In this form the *standard tableaux* of shape $(\vec{\gamma}, J)$ are fillings of the n boxes in the book with the numbers $1, \ldots, n$ such that the filling on each page satisfies the conditions for a standard tableau in Section 5.10.

Example. If $\gamma = (1/2, 1/2, 1, 1, 1, 3/2, -2, -2, -1/2, -1, -1, -1, -1/2, 1/2, 0, 0, 0)$ then one possibility for $\vec{\gamma}$ is

$$\vec{\gamma} = (-2, -2, -1, -1, -1, 0, 0, 0, 1, 1, 1, -1/2, -1/2, 1/2, 1/2, 1/2, 3/2).$$

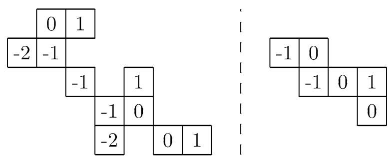
In this case $\beta_1 = 0$, $\beta_2 = 1/2$,

$$\begin{aligned} \beta_1 + C_{\beta_1} &= (-2, -2, -1, -1, -1, 0, 0, 0, 1, 1, 1), \quad \text{and} \\ \beta_2 + C_{\beta_2} &= (-1/2, -1/2, 1/2, 1/2, 1/2, 3/2). \end{aligned}$$

If $J = J_{\beta_1} \cup J_{\beta_2}$ where $J_{\beta_2} = \{\varepsilon_{14} - \varepsilon_{13}, \varepsilon_{17} - \varepsilon_{16}\}$ and

$$\begin{aligned} J_{\beta_1} = \{&\varepsilon_3 - \varepsilon_2, \varepsilon_4 - \varepsilon_2, \varepsilon_5 - \varepsilon_2, \varepsilon_6 - \varepsilon_3, \varepsilon_6 - \varepsilon_4, \varepsilon_6 - \varepsilon_5, \varepsilon_9 - \varepsilon_7, \varepsilon_9 - \varepsilon_8, \\ &\varepsilon_{10} - \varepsilon_7, \varepsilon_{10} - \varepsilon_8\} \end{aligned}$$

then the book of shapes is



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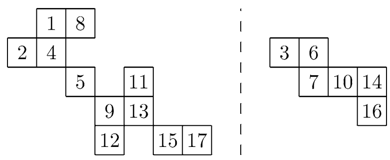
where the numbers in the boxes are the contents of the boxes. The filling



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is a standard tableau of shape $(\vec{\gamma}, J)$. This filling corresponds to the permutation

$$w = (2, 12, 4, 5, 9, 1, 13, 15, 8, 11, 17, 3, 7, 6, 10, 16, 14) \quad \text{in } \mathcal{F}^{(\vec{\gamma},J)} \subseteq S_{16}.$$

6. Skew shapes, ribbons, conjugation, etc. in type A

In this section we shall explain how the definitions in Section 5.1 correspond to classical notions in Young tableaux theory. As in the previous section let R be the root system of Type A_{n-1} as given in Section 5.2. For clarity, we shall state all of the results in this section for placed shapes (γ, J) such that γ is dominant and integral, i.e., $\gamma = (\gamma_1, \dots, \gamma_n)$ with $\gamma_1 \leqslant \cdots \leqslant \gamma_n$ and $\gamma_i \in \mathbb{Z}$. This assumption is purely for notational clarity.

6.1. Axial distance

Let (γ, J) be a local region such that γ is dominant and integral. Let $w_p \in \mathcal{F}^{(\gamma,J)}$ and let p be the corresponding standard tableau as defined by the map in Theorem 5.9. Then it follows from the definitions of γ and w_p in (5.6) and (5.4) that

$$\langle w\gamma, \varepsilon_i\rangle = \big\langle \gamma, w_p^{-1}\varepsilon_i\big\rangle = c(\mathrm{box}_{w_p^{-1}(i)}) = c\big(p(i)\big), \tag{6.2}$$

where $p(i)$ is the box of p containing the entry i.

In classical standard tableau theory the *axial distance* between two boxes in a standard tableau is defined as follows. Let λ be a partition and let p be a standard tableau of shape λ. Let $1 \leqslant i, j \leqslant n$ and let $p(i)$ and $p(j)$ be the boxes which are filled with i and j, respectively. Let (r_i, c_i) and (r_j, c_j) be the positions of these boxes, where the rows and columns of λ are numbered in the same way as for matrices. Then the *axial distance* from j to i in p is

$$d_{ji}(p) = c_j - c_i + r_i - r_j,$$

(see [45]). Rewriting this in terms of the local region (γ, J) determined by (5.7),

$$d_{ji}(p) = c\big(p(j)\big) - c\big(p(i)\big) = \langle w_p\gamma, \varepsilon_j - \varepsilon_i\rangle = d_{\varepsilon_j - \varepsilon_i}(w),$$

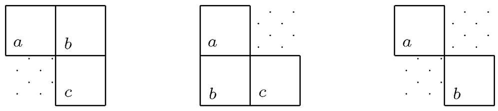
where $w_p \in \mathcal{F}^{(\gamma,J)}$ is the permutation corresponding to the standard tableau p and $d_\alpha(w_p)$ is the α-axial distance defined in (f) of Section 5.1. This shows that the axial distance defined in (f) of Section 5.1 is a generalization of the classical notion of axial distance. These numbers are crucial to the classical construction of the seminormal representations of the symmetric group given by Young (see Remark (3) of Section 5.1).

6.3. Skew shapes

The following proposition shows that, in the case of a root system of type A, the definition of skew local region coincides with the classical notion of a skew shape.

Proposition 6.4. *Let (γ, J) be a local region with γ dominant and integral. Then the configuration of boxes associated to (γ, J) is a placed skew shape if and only if (γ, J) is a skew local region.*

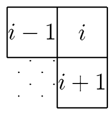
Proof. ($\Leftarrow$) We shall show that if the placed configuration corresponding to the pair (γ, J) has any 2×2 blocks of the forms



Case (1) Case (2) Case (3)

then there exists a $w \in \mathcal{F}^{(\gamma,J)}$ such that $w\gamma$ violates one of the two conditions in (c) of Section 5.1. This will show that if (γ, J) is a skew local region then the corresponding placed configuration of boxes must be a placed skew shape. In the pictures above the shaded regions indicate the absence of a box and, for reference, we have labeled the boxes with a, b, c.

Case (1). Create a standard tableau p such that the 2×2 block is filled with



by filling the region of the configuration strictly north and weakly west of box c in row reading order (sequentially left to right across the rows starting at the top), putting the next entry in box c, and filling the remainder of the configuration in column reading order (sequentially down the columns beginning at the leftmost available column). Let $w = w_p$ be the permutation in $\mathcal{F}^{(\gamma,J)}$ which corresponds to the standard tableau p. Let $p(i)$ denote the box containing i in p. Then, using the identity (6.2),

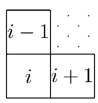
$$\langle w\gamma, \alpha_i + \alpha_{i+1}\rangle = \langle w\gamma, \varepsilon_{i+1} - \varepsilon_{i-1}\rangle = c\big(p(i+1)\big) - c\big(p(i-1)\big) = 0,$$

since the boxes $p(i+1)$ and $p(i-1)$ are on the same diagonal. However,

$$\langle w\gamma, \alpha_i\rangle = \langle w\gamma, \varepsilon_i - \varepsilon_{i-1}\rangle = c\big(p(i)\big) - c\big(p(i-1)\big) = 1, \quad \text{and}$$
$$\langle w\gamma, \alpha_{i+1}\rangle = \langle w\gamma, \varepsilon_{i+1} - \varepsilon_i\rangle = c\big(p(i+1)\big) - c\big(p(i)\big) = -1,$$

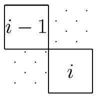
and so condition (c)(1) of Section 5.1 is violated.

Case (2). Create a standard tableau p such that the 2×2 block is filled with



by filling the region weakly north and strictly west of box c in column reading order, putting the next entry in box c, and filling the remainder of the configuration in row reading order. Using this standard tableau p, the remainder of the argument is the same as for Case (1).

Case (3). Create a standard tableau p such that the 2×2 block is filled with



by filling the region strictly north and strictly west of box b in column reading order, putting the next entry in box b, and filling the remainder of the configuration in row reading order. Let $w = w_p$ be the permutation in $\mathcal{F}^{(\gamma,J)}$ corresponding to p and let $p(i)$ denote the box containing i in p. Then

$$\langle w\gamma, \alpha_i\rangle = \langle w\gamma, \varepsilon_i - \varepsilon_{i-1}\rangle = c\big(p(i)\big) - \big(p(i-1)\big) = 0,$$

since $t(i)$ and $t(i-1)$ are on the same diagonal. Hence, condition (c)(1) of Section 5.1 is violated.

($\Rightarrow$) Let $\gamma \in \mathbb{Z}^n$ and λ/μ describe a placed skew shape (a skew shape placed on infinite graph paper). Let (γ, J) be the corresponding local region as defined in (5.7). We will show that every $w\gamma$ is calibratable for every $w \in \mathcal{F}^{(\gamma,J)}$.

Let $w \in \mathcal{F}^{(\gamma,J)}$ and let p be the corresponding standard tableau of shape λ/μ. Consider a 2×2 block of boxes of p. If these boxes are filled with

i	j
k	ℓ

then either $i < j < k < \ell$ or $i < k < j < \ell$. In both cases we have $i < \ell - 1$ and it follows that $\ell - 1$ and ℓ are not on the same diagonal. Thus

$$\langle w\gamma, \alpha_\ell\rangle = c\big(p(\ell)\big) - c\big(p(\ell-1)\big) \neq 0,$$

and so $w\gamma$ satisfies condition (a) in the definition of calibratable.

The same argument shows that one can never get a standard tableau in which ℓ and $\ell - 2$ occur in adjacent boxes of the same diagonal and thus it follows that $w\gamma$ satisfies condition (b) in the definition of calibratable. Thus (γ, J) is a skew local region. □

6.5. Ribbon shapes

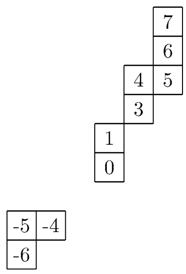
Classically, a *border strip* (or *ribbon*) is a skew shape which contains at most one box in each diagonal. Although the convention, [24, I, Section 1 p. 5], is to assume that border strips are connected skew shapes we shall *not* assume this.

Recall from (b) of Section 5.1 that a placed shape (γ, J) is a placed *ribbon* shape if γ is regular, i.e., $\langle \gamma, \alpha \rangle \neq 0$ for all $\alpha \in R$.

Proposition 6.6. *Let (γ, J) be a placed ribbon shape such that γ is dominant and integral. Then the configuration of boxes corresponding to (γ, J) is a placed border strip.*

Proof. Let (γ, J) be a placed ribbon shape with γ dominant and regular. Since $\gamma = (\gamma_1, \ldots, \gamma_n)$ is regular, $\gamma_i \neq \gamma_j$ for all $i \neq j$. In terms of the placed configuration $\gamma_i = c(\mathrm{box}_i)$ is the diagonal that box_i is on. Thus the configuration of boxes corresponding to (γ, J) contains at most one box in each diagonal. $\square$

Example. If $\gamma = (-6, -5, -4, 0, 1, 3, 4, 5, 6, 7)$ and $J = \{\varepsilon_2 - \varepsilon_1, \varepsilon_5 - \varepsilon_4, \varepsilon_7 - \varepsilon_6, \varepsilon_9 - \varepsilon_8, \varepsilon_{10} - \varepsilon_9\}$ then the placed configuration of boxes corresponding to (γ, J) is the placed border strip

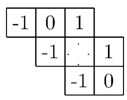


where the boxes are labeled with their contents.

6.7. *Conjugation of shapes*

Let (γ, J) be a placed shape with γ dominant and integral (i.e., $\gamma = (\gamma_1, \ldots, \gamma_n)$ with $\gamma_1 \leqslant \cdots \leqslant \gamma_n$ and $\gamma_i \in \mathbb{Z}$) and view (γ, J) as a placed configuration of boxes. In terms of placed configurations, conjugation of shapes is equivalent to transposing the placed configuration across the diagonal of boxes of content 0. The following example illustrates this.

Example. Suppose $\gamma = (-1, -1, -1, 0, 0, 1, 1)$ and $J = (\varepsilon_4 - \varepsilon_2, \varepsilon_4 - \varepsilon_3, \varepsilon_6 - \varepsilon_5, \varepsilon_7 - \varepsilon_5)$. Then the placed configuration of boxes corresponding to (γ, J) is



in which the shaded box is not a box in the configuration.

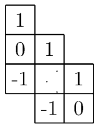
The minimal length representative of the coset $w_0 W_\gamma$ is the permutation

$$u = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 3 & 4 & 1 & 2 \end{pmatrix}.$$

We have $-u\gamma = -w_0\gamma = (-1, -1, 0, 0, 1, 1, 1)$ and

$$\begin{aligned} -u\big(P(\gamma) \setminus J\big) &= -u\{\varepsilon_4 - \varepsilon_1, \varepsilon_5 - \varepsilon_1, \varepsilon_5 - \varepsilon_2, \varepsilon_5 - \varepsilon_3, \varepsilon_6 - \varepsilon_4, \varepsilon_7 - \varepsilon_4\} \\ &= -\{\varepsilon_3 - \varepsilon_5, \varepsilon_4 - \varepsilon_5, \varepsilon_4 - \varepsilon_6, \varepsilon_4 - \varepsilon_7, \varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3\} \\ &= \{\varepsilon_5 - \varepsilon_3, \varepsilon_5 - \varepsilon_4, \varepsilon_6 - \varepsilon_4, \varepsilon_7 - \varepsilon_4, \varepsilon_3 - \varepsilon_1, \varepsilon_3 - \varepsilon_2\}. \end{aligned}$$

Thus the configuration of boxes corresponding to the placed shape $(\gamma, J)'$ is



6.8. Row reading and column reading tableaux

Let (γ, J) be a placed shape such that γ is dominant and integral and consider the placed configuration of boxes corresponding to (γ, J). The *minimal box* of the configuration is the box such that

(m_1) there is no box immediately above,
(m_2) there is no box immediately to the left,
(m_3) there is no box northwest in the same diagonal, and
(m_4) it has the minimal content of the boxes satisfying (m_1)–(m_3).

There is at most one box in each diagonal satisfying (m_1)–(m_3). Thus, (m_4) guarantees that the minimal box is unique. It is clear that the minimal box of the configuration always exists.

The *column reading* tableaux of shape (γ, J) is the filling $p_{\min}$ which is created inductively by

(a) filling the minimal box of the configuration with 1, and
(b) if $1, 2, \ldots, i$ have been filled in then fill the minimal box of the configuration formed by the unfilled boxes with $i + 1$.

The *row reading tableau* of shape (γ, J) is the standard tableau $p_{\max}$ whose conjugate $p'_{\max}$ is the column reading tableaux for the shape $(\gamma, J)'$ (the conjugate shape to (γ, J)).

Recall the definitions of the weak Bruhat order and closed subsets of roots given after Eq. (4.5).

Theorem 6.9. *Let (γ, J) be a placed shape such that γ is dominant and integral (i.e., $\gamma = (\gamma_1, \ldots, \gamma_n)$ with $\gamma_1 \leqslant \cdots \leqslant \gamma_n$ and $\gamma_i \in \mathbb{Z}$). Let $p_{\min}$ and $p_{\max}$ be the column reading* *and row reading tableaux of shape (γ, J), respectively, and let $w_{\min}$ and $w_{\max}$ be the corresponding permutations in $\mathcal{F}^{(\gamma,J)}$. Then*

$$R(w_{\min}) = \overline{J}, \qquad R(w_{\max}) = \overline{\big(P(\gamma) \setminus J\big) \cup Z(\gamma)}^c, \quad \textit{and} \quad \mathcal{F}^{(\gamma,J)} = [w_{\min}, w_{\max}],$$

where K^c denotes the complement of K in R^+ and $[w_{\min}, w_{\max}]$ denotes the interval between $w_{\min}$ and $w_{\max}$ in the weak Bruhat order.

Proof. (a) Consider the configuration of boxes corresponding to (γ, J). If $k > i$ then either $c(\text{box}_k) > c(\text{box}_i)$, or box_k is in the same diagonal and southeast of box_i. Thus when we create $p_{\min}$ we have that

$$\text{if} \quad k > i \quad \text{then} \quad \text{box}_k \text{ gets filled before box}_i \Leftrightarrow \text{box}_k \text{ is } \textit{northwest} \text{ of box}_i,$$

where the *northwest* is in a very strong sense: There is a sequence of boxes

$$\text{box}_i = \text{box}_{i_0}, \quad \text{box}_{i_1}, \quad \ldots, \quad \text{box}_{i_r} = \text{box}_k$$

such that box_{i_m} is either directly above $\text{box}_{i_{m-1}}$ or in the same diagonal and directly northwest of $\text{box}_{i_{m-1}}$. In other words,

$$\text{if} \quad k > i \quad \text{then} \quad p_{\min}(\text{box}_k) < p_{\min}(\text{box}_i) \Leftrightarrow \text{box}_k \text{ is } \textit{northwest} \text{ of box}_i.$$

So, from the formula for w_p in (5.4) we get

$$\text{if} \quad k > i \quad \text{then} \quad w_{\min}(k) < w_{\min}(i) \Leftrightarrow \varepsilon_k - \varepsilon_i \in \overline{J},$$

where $w_{\min}$ is the permutation in $\mathcal{F}^{(\gamma,J)}$ which corresponds to the filling $t_{\min}$ and $\overline{J}$ is the closure of J in R. It follows that

$$R(w_{\min}) = \overline{J}.$$

(b) There are at least two ways to prove that $R(w_{\max}) = \overline{(P(\gamma) \setminus J) \cup Z(\gamma)}^c$. One can mimic the proof of part (a) by defining the maximal box of a configuration and a corresponding filling. Alternatively one can use the definition of conjugation and the fact that $R(w_0 w) = R(w)^c$. The permutation $w_{\min}$ is the unique minimal element of $\mathcal{F}^{(\gamma,J)}$ and the conjugate of $w_{\max}$ is the unique minimal element of $\mathcal{F}^{(\gamma,J)'}$. We shall leave the details to the reader.

(c) An element $w \in W$ is an element of $\mathcal{F}^{(\gamma,J)}$ if and only if $R(w) \cap P(\gamma) = J$ and $R(w) \cap Z(\gamma) = \emptyset$. Thus $\mathcal{F}^{(\gamma,J)}$ consists of those permutations $w \in W$ such that

$$\overline{J} \subseteq R(w) \subseteq \overline{\big(P(\gamma) \setminus J\big) \cup Z(\gamma)}^c.$$

Since the weak Bruhat order is the ordering determined by inclusions of $R(w)$, it follows that $\mathcal{F}^{(\gamma,J)}$ is the interval between $w_{\min}$ and $w_{\max}$. $\square$

Example. Suppose $\gamma = (-1, -1, -1, 0, 0, 1, 1)$ and $J = \{\varepsilon_4 - \varepsilon_2, \varepsilon_4 - \varepsilon_3, \varepsilon_6 - \varepsilon_5, \varepsilon_7 - \varepsilon_5\}$. The minimal and maximal elements in $\mathcal{F}^{(\gamma,J)}$ are the permutations

$$w_{\min} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 2 & 7 & 5 & 6 \end{pmatrix} \quad \text{and} \quad w_{\max} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 5 & 6 & 2 & 7 & 3 & 4 \end{pmatrix}.$$

The permutations correspond to the standard tableaux

$$\begin{array}{|c|c|c|cc} \hline 1 & 2 & 5 & & \\ \hline & 3 & \ddots & 6 & \\ \hline & & 4 & 7 & \\ \hline \end{array} \qquad \text{and} \qquad \begin{array}{|c|c|c|cc} \hline 1 & 2 & 3 & & \\ \hline & 5 & \ddots & 4 & \\ \hline & & 6 & 7 & \\ \hline \end{array}.$$

7. The type A, root of unity case

This section describes the sets $\mathcal{F}^{(t,J)}$ in the case of the root system of Section 5.2 when $q^2 = e^{2\pi i/\ell}$, a primitive ℓth root of unity, $\ell > 2$.

Let $t \in T$. Identify t with a sequence

$$t = (t_1, \ldots, t_n) \in \mathbb{C}^n, \quad \text{where } t(X^{\varepsilon_i}) = t_i.$$

For the purposes of representation theory (see Theorem 3.6) t indexes a central character (see Section 2.3) and so t can safely be replaced by any element of its W-orbit. In this case W is the symmetric group, S_n, acting by permuting the sequence $t = (t_1, \ldots, t_n)$.

The cyclic group $\langle q^2 \rangle$ of order ℓ generated by q^2 acts on $\mathbb{C}^*$. Fix a choice of a set $\{\xi\}$ of coset representatives of the $\langle q^2 \rangle$ cosets in $\mathbb{C}^*$. Replace t with the sequence obtained by rearranging its entries to group entries in the same $\langle q^2 \rangle$-orbit, so that

$$t = \left(\xi_1 t^{(1)}, \ldots, \xi_k t^{(k)}\right),$$

where $\xi_1, \ldots, \xi_k$ are distinct representatives of the cosets in $\mathbb{C}^*/\langle q^2 \rangle$ and each $t^{(j)}$ is a sequence of the form

$$t^{(j)} = \left(q^{2\gamma_1}, \ldots, q^{2\gamma_r}\right), \quad \text{with } \gamma_1, \ldots, \gamma_r \in \{0, 1, \ldots, \ell - 1\} \text{ and } \gamma_1 \leqslant \cdots \leqslant \gamma_r.$$

As in Section 5.11 this decomposition of t into groups induces decompositions

$$Z(t) = \bigcup_{j=1}^{k} Z_{\xi_j}(t) \quad \text{and} \quad P(t) = \bigcup_{j=1}^{k} P_{\xi_j}(t),$$

and it is sufficient to analyze the case when t consists of only one group, i.e., all the entries of t are in the same $\langle q^2 \rangle$ coset.

Now assume that

$$t = \left(q^{2\gamma_1}, \ldots, q^{2\gamma_n}\right), \quad \text{with } \gamma_1 \leqslant \cdots \leqslant \gamma_n,\ \gamma_i \in \{0, \ldots, \ell - 1\}.$$

Consider a page of graph paper with diagonals labeled by $\ldots, 0, 1, \ldots, \ell-1, 0, 1, \ldots, \ell-1, 0, 1, \ldots$ from southwest to northeast. For each local region (t, J), $J \subseteq P(t)$, we will construct an ℓ-periodic configuration of boxes for which the ℓ-periodic standard tableaux defined below will be in bijection with the elements of $\mathcal{F}^{(t,J)}$. For each $1 \leqslant i \leqslant n$, the configuration will have a box numbered i, box_i, on each diagonal which is labeled γ_i. There are an infinite number of such diagonals containing a box numbered i, since the diagonals are labeled in an ℓ-periodic fashion, but each strip of consecutive diagonals labeled $0, 1, \ldots, \ell-1$ will contain n boxes. The *content* of a box b (see [24, I, Section 1, Exercise 3]) is

$$c(b) = (\text{the diagonal number of the box } b).$$

Then

$$\begin{aligned} Z(t) &= \{\varepsilon_j - \varepsilon_i \mid i < j,\ \gamma_i = \gamma_j\} \\ &= \{\varepsilon_j - \varepsilon_i \mid i < j,\ \mathrm{box}_i \text{ and } \mathrm{box}_j \text{ are in the same diagonal}\} \end{aligned}$$

and

$$\begin{aligned} P(t) &= \left\{ \varepsilon_j - \varepsilon_i \,\middle|\, \begin{array}{l} i < j \text{ and } \gamma_j = \gamma_i + 1, \text{ or} \\ i < j, \gamma_j = \ell - 1, \text{ and } \gamma_i = 0 \end{array} \right\} \\ &= \{\varepsilon_j - \varepsilon_i \mid i < j \text{ and } \mathrm{box}_i \text{ and } \mathrm{box}_j \text{ are in adjacent diagonals}\}. \end{aligned}$$

We will use $J \subseteq P(t)$ to organize the relative positions of the boxes in adjacent diagonals:

- if $\varepsilon_j - \varepsilon_i \in J$ and if $c(\mathrm{box}_j) \neq \ell - 1$ or $c(\mathrm{box}_i) \neq 0$, place box_j northwest of box_i;
- if $\varepsilon_j - \varepsilon_i \notin J$ and if $c(\mathrm{box}_j) \neq \ell - 1$ or $c(\mathrm{box}_i) \neq 0$, place box_j southeast of box_i;
- if $\varepsilon_j - \varepsilon_i \in J$ and $c(\mathrm{box}_j) = \ell - 1$ and $c(\mathrm{box}_i) = 0$, place box_j southeast of box_i;
- if $\varepsilon_j - \varepsilon_i \notin J$ and $c(\mathrm{box}_j) = \ell - 1$ and $c(\mathrm{box}_i) = 0$, place box_j northwest of box_i.

Thus, t determines the number of boxes in each diagonal and J determines the relative positions of the boxes in adjacent diagonals. This information completely determines the ℓ-periodic configuration of boxes associated to the pair (t, J).

A *ℓ-periodic standard tableau* is an ℓ-periodic filling p of the boxes with $1, 2, \ldots, n$ such that

(a) if $i < j$ and box_i and box_j are in the same diagonal then $p(i) < p(j)$,
(b) if $i < j$ and box_i and box_j are in adjacent diagonals with box_j southwest of box_i then $p(i) < p(j)$,
(c) if $i < j$ and box_i and box_j are in adjacent diagonals with box_j northeast of box_i then $p(i) > p(j)$,

where $p(i)$ denotes the entry in box_i. An ℓ-periodic standard tableau p corresponds to a permutation in S_n via the correspondence

$$\{\text{standard tableaux}\} \leftrightarrow \mathcal{F}^{(t,J)},$$
$$p \mapsto \begin{pmatrix} 1 & 2 & \cdots & n \\ p(1) & p(2) & \cdots & p(n) \end{pmatrix}.$$

Example. Suppose that $q^2 = e^{2\pi i/4}$ and

$$t = (q^0, q^0, q^0, q^0, q^2, q^2, q^2, q^4, q^4, q^6, q^6, q^6, q^6, q^6).$$

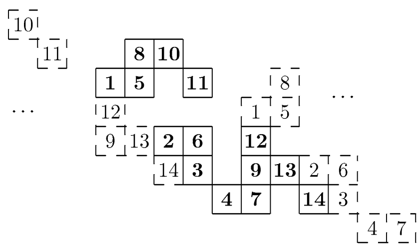
Then

$$Z(t) = \{\varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_1, \varepsilon_4 - \varepsilon_1, \varepsilon_3 - \varepsilon_2, \varepsilon_4 - \varepsilon_2, \varepsilon_4 - \varepsilon_3, \varepsilon_6 - \varepsilon_5, \varepsilon_7 - \varepsilon_5, \ldots\} \quad \text{and}$$
$$P(t) = \{\varepsilon_5 - \varepsilon_1, \varepsilon_5 - \varepsilon_2, \varepsilon_5 - \varepsilon_3, \varepsilon_5 - \varepsilon_4, \varepsilon_6 - \varepsilon_1, \ldots, \varepsilon_{14} - \varepsilon_9, \varepsilon_{10} - \varepsilon_1, \varepsilon_{10} - \varepsilon_2, \ldots, \varepsilon_{14} - \varepsilon_4\}.$$

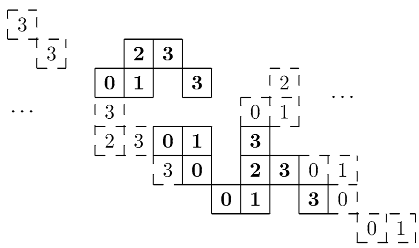
If

$$J = \{\varepsilon_5 - \varepsilon_2, \varepsilon_5 - \varepsilon_3, \varepsilon_5 - \varepsilon_4, \varepsilon_6 - \varepsilon_3, \varepsilon_6 - \varepsilon_4, \varepsilon_8 - \varepsilon_5, \varepsilon_8 - \varepsilon_6, \varepsilon_8 - \varepsilon_7, \varepsilon_9 - \varepsilon_7, \varepsilon_{10} - \varepsilon_9, \varepsilon_{11} - \varepsilon_9, \varepsilon_{12} - \varepsilon_9, \varepsilon_{12} - \varepsilon_2, \varepsilon_{12} - \varepsilon_3, \varepsilon_{12} - \varepsilon_4, \varepsilon_{13} - \varepsilon_2, \varepsilon_{13} - \varepsilon_3, \varepsilon_{13} - \varepsilon_4, \varepsilon_{14} - \varepsilon_3, \varepsilon_{14} - \varepsilon_4\}$$

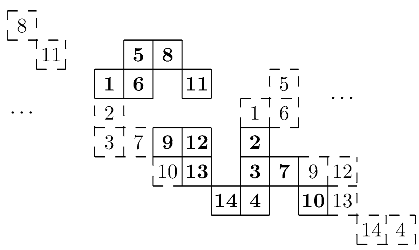
then the corresponding ℓ-periodic configuration of boxes and a sample ℓ-periodic standard tableau are



numbering of boxes



contents of boxes



a standard tableau p

8. Standard tableaux for type C in terms of boxes

8.1. The root system

Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ be an orthonormal basis of $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}^n$ and view elements $\gamma = \sum_i \gamma_i \varepsilon_i$ of $\mathbb{R}^n$ as sequences

$$\gamma = (\gamma_{-n}, \ldots, \gamma_{-1}; \gamma_1, \ldots, \gamma_n), \quad \text{such that } \gamma_{-i} = -\gamma_i. \tag{8.2}$$

The root system of type C_n is given by the sets

$$\begin{aligned} R &= \big\{\pm 2\varepsilon_i, \pm(\varepsilon_j \pm \varepsilon_i) \,\big|\, 1 \leqslant i, j \leqslant n\big\} \quad \text{and} \\ R^+ &= \{2\varepsilon_i,\ \varepsilon_j \pm \varepsilon_i \mid 1 \leqslant i < j \leqslant n\}. \end{aligned} \tag{8.3}$$

The simple roots are given by $\alpha_1 = 2\varepsilon_1$, $\alpha_i = \varepsilon_i - \varepsilon_{i-1}$, $2 \leqslant i \leqslant n$. The Weyl group $W = WC_n$ is the *hyperoctahedral group* of permutations of $-n, \ldots, -1, 1, \ldots, n$ such that $w(-i) = -w(i)$. This groups acts on the ε_i by the rule $w\varepsilon_i = \varepsilon_{w(i)}$, with the convention that $\varepsilon_{-i} = -\varepsilon_i$.

For this type C case there is a nice trick. View the root system as

$$\begin{aligned} R &= \big\{\pm(\varepsilon_j \pm \varepsilon_i) \,\big|\, i < j,\ i, j \in \{\pm 1, \ldots, \pm n\}\big\} \quad \text{and} \\ R^+ &= \big\{\varepsilon_j - \varepsilon_i \,\big|\, i < j,\ i, j \in \{\pm 1, \ldots, \pm n\}\big\}, \end{aligned} \tag{8.4}$$

with the convention that $\varepsilon_{-i} = -\varepsilon_i$. In this notation $\varepsilon_i - \varepsilon_{-i} = 2\varepsilon_i$ and $\varepsilon_{-i} - \varepsilon_{-j} = \varepsilon_j - \varepsilon_i$. This way the type C root system "looks like" a type A root system and many computations can be done in the same way as in type A.

8.5. Rearranging γ

We analyze the structure of the sets $\mathcal{F}^{(\gamma,J)}$ as considered in (4.3). This corresponds to when the q in the affine Hecke algebra is not a root of unity. The analysis in this case is analogous to the method that was used in Section 5.11 to create books of placed configurations in the type A case.

Let $\gamma \in \mathbb{R}^n$. Apply an element of the Weyl group to γ to "arrange" the entries of γ so that, for each $i \in \{1, \ldots, n\}$,

$$\gamma_i \in \left[z + \frac{1}{2}, z\right], \quad \text{for some } z \in \mathbb{Z}.$$

Then

$$\gamma_{-i} = -\gamma_i \in \left[z', z' + \frac{1}{2}\right], \quad \text{for some } z' \in \mathbb{Z}.$$

As in the type A case, the sets $Z(\gamma)$ and $P(\gamma)$ can be partitioned according to the $\mathbb{Z}$ cosets of the elements of γ and it is sufficient to consider each $\mathbb{Z}$-coset separately and then assemble the results in "books of pages." There are three cases to consider:

Case $\boldsymbol{\beta}$. The $\mathbb{Z}$-coset $\beta + \mathbb{Z}$, $\beta \in (1/2, 1)$. Then

$$\gamma = (-\beta - z_n \leqslant \cdots \leqslant -\beta - z_2 \leqslant -\beta - z_1;\ \beta + z_1 \leqslant \beta + z_2 \leqslant \cdots \leqslant \beta + z_n), \quad z_i \in \mathbb{Z},$$

Case 1/2. The $\mathbb{Z}$-coset $1/2 + \mathbb{Z}$. Then

$$\begin{aligned}\gamma = {} & (-1/2 - z_n \leqslant \cdots \leqslant -1/2 - z_2 \leqslant -1/2 - z_1; \\ & 1/2 + z_1 \leqslant 1/2 + z_2 \leqslant \cdots \leqslant 1/2 + z_n), \quad z_i \in \mathbb{Z}_{\geqslant 0},\end{aligned}$$

Case 0. The $\mathbb{Z}$-coset $\mathbb{Z}$. Then

$$\gamma = (-z_n \leqslant \cdots \leqslant -z_2 \leqslant -z_1;\ z_1 \leqslant z_2 \leqslant \cdots \leqslant z_n), \quad z_i \in \mathbb{Z}_{\geqslant 0}.$$

It is notationally convenient to let $z_{-i} = -z_i$.

8.6. Boxes and standard tableaux

Let us assume that the entries of γ all lie in a single $\mathbb{Z}$-coset and describe the resulting standard tableaux. The general case is obtained by creating books of pages of standard tableaux where the pages correspond to the different $\mathbb{Z}$-cosets of entries in γ.

The placed configuration of boxes is determined as follows.

8.7. Case β, $\beta \in (1/2, 1)$

Assume that $\gamma \in \mathfrak{h}^*_{\mathbb{R}}$ is of the form

$$\gamma = (-\beta - z_n \leqslant \cdots \leqslant -\beta - z_2 \leqslant -\beta - z_1;\ \beta + z_1 \leqslant \beta + z_2 \leqslant \cdots \leqslant \beta + z_n), \quad z_i \in \mathbb{Z}.$$

Place boxes on two pages of infinite graph paper. These pages are numbered β and $-\beta$ and each page has the diagonals numbered consecutively with the elements of $\mathbb{Z}$, from bottom left to top right. View these two pages, page β and page $-\beta$, as "linked." For each $1 \leqslant i \leqslant n$ place box_i on diagonal z_i of page β and box_{-i} on diagonal $-z_i$ of page $-\beta$. The boxes on each diagonal are arranged in increasing order from top left to bottom right. The placement of boxes on page $-\beta$ is a $180°$ rotation of the placement of the boxes on page β.

Using the notation for the root system of type C_n in (8.4)

$$\begin{aligned}P(\gamma) &= \{\varepsilon_j - \varepsilon_i \mid j > i \text{ and box}_i \text{ and box}_j \text{ are in adjacent diagonals}\} \quad \text{and} \\ Z(\gamma) &= \{\varepsilon_j - \varepsilon_i \mid j > i \text{ and box}_i \text{ and box}_j \text{ are in the same diagonal}\}.\end{aligned}$$

Note that $\varepsilon_{-i} - \varepsilon_{-j} \in Z(\gamma)$ if and only if $\varepsilon_j - \varepsilon_i \in Z_\beta(\gamma)$, and similarly $\varepsilon_{-i} - \varepsilon_{-j} \in P_\beta(\gamma)$ if and only if $\varepsilon_j - \varepsilon_i \in P_\beta(\gamma)$. If $J \subseteq P(\gamma)$ arrange the boxes on adjacent diagonals according to the rules

(1) if $\varepsilon_j - \varepsilon_i \in J$ place box_j northwest of box_i, and
(2) if $\varepsilon_j - \varepsilon_i \in P(\gamma)\backslash J$ place box_j southeast of box_i.

A *standard tableau* is a negative rotationally symmetric filling p of the $2n$ boxes with $-n, \ldots, -1, 1, \ldots, n$ such that

(a) $p(\text{box}_i) < p(\text{box}_j)$ if $j > i$ and box_j and box_i are in the same diagonal,
(b) $p(\text{box}_i) > p(\text{box}_j)$ if $j > i$, box_i and box_j are in adjacent diagonals and box_j is northwest of box_i,
(c) $p(\text{box}_i) < p(\text{box}_j)$ if $j > i$, box_i and box_j are in adjacent diagonals and box_j is southeast of box_i.

The negative rotational symmetry means that the filling of the boxes on page $-\beta$ is the same as the filling on page β except rotated by 180° and with all entries in the boxes multiplied by -1.

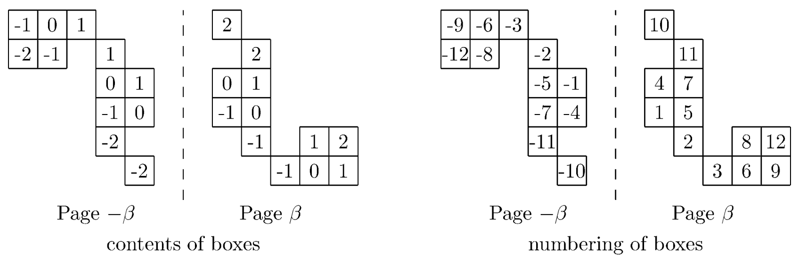
Example. Suppose $\beta \in (1/2, 1)$, and

$$\begin{aligned}\gamma &= (-\beta; \beta) \\ &\quad + (-2,-2,-2,-1,-1,-1,0,0,0,1,1,1;-1,-1,-1,0,0,0,1,1,1,2,2,2) \\ &= (-\beta-2,-\beta-2,-\beta-2,-\beta-1,-\beta-1,-\beta-1,-\beta,-\beta,-\beta,-\beta+1, \\ &\quad -\beta+1,-\beta+1; \\ &\quad \beta-1,\beta-1,\beta-1,\beta,\beta,\beta,\beta+1,\beta+1,\beta+1,\beta+2,\beta+2,\beta+2)\end{aligned}$$

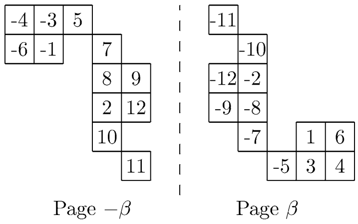
and

$$\begin{aligned}J = \{&\varepsilon_4-\varepsilon_1, \varepsilon_{-1}-\varepsilon_{-4}, \varepsilon_4-\varepsilon_2, \varepsilon_{-2}-\varepsilon_{-1}, \varepsilon_4-\varepsilon_3, \varepsilon_{-3}-\varepsilon_{-4}, \varepsilon_5-\varepsilon_2, \varepsilon_{-2}-\varepsilon_{-5}, \\ &\varepsilon_5-\varepsilon_3, \varepsilon_{-3}-\varepsilon_{-5}, \varepsilon_7-\varepsilon_5, \varepsilon_{-5}-\varepsilon_{-7}, \varepsilon_7-\varepsilon_6, \varepsilon_{-6}-\varepsilon_{-7}, \varepsilon_8-\varepsilon_6, \varepsilon_{-6}-\varepsilon_{-8}, \\ &\varepsilon_{10}-\varepsilon_9, \varepsilon_{-9}-\varepsilon_{-10}, \varepsilon_{10}-\varepsilon_8, \varepsilon_{-8}-\varepsilon_{-10}, \varepsilon_{10}-\varepsilon_7, \varepsilon_{-7}-\varepsilon_{-10}, \varepsilon_{11}-\varepsilon_9, \\ &\varepsilon_{-9}-\varepsilon_{-11}, \varepsilon_{11}-\varepsilon_8, \varepsilon_{-8}-\varepsilon_{-11}, \varepsilon_{11}-\varepsilon_7, \varepsilon_{-7}-\varepsilon_{-11}, \varepsilon_{12}-\varepsilon_9, \varepsilon_{-9}-\varepsilon_{-12}\}.\end{aligned}$$

The placed configuration of boxes corresponding to (γ, J) is



and a sample negative rotationally symmetric standard tableau is



a standard tableau

8.8. Case 1/2

Assume that $\gamma \in \mathfrak{h}_{\mathbb{R}}^*$ is of the form

$$\gamma = (-1/2 - z_n \leqslant \cdots \leqslant -1/2 - z_2 \leqslant -1/2 - z_1;$$
$$1/2 + z_1 \leqslant 1/2 + z_2 \leqslant \cdots \leqslant 1/2 + z_n), \quad z_i \in \mathbb{Z}_{\geqslant 0}.$$

Place boxes on a page of infinite graph paper which has its diagonals numbered consecutively with the elements of $1/2 + \mathbb{Z}$, from bottom left to top right. This page has page number 1/2. For each $i \in \{\pm 1, \ldots, \pm n\}$ place box_i on diagonal $1/2 + z_i$ and box_{-i} on diagonal $-1/2 - z_i$. The boxes on each diagonal are arranged in increasing order from top left to bottom right and the placement of boxes is negative rotationally symmetric in the sense that a 180° rotation takes box_i to box_{-i}.

Using the root system notation in (8.4),

$$P(\gamma) = \{\varepsilon_j - \varepsilon_i \mid j > i \text{ and } \mathrm{box}_i \text{ and } \mathrm{box}_j \text{ are in adjacent diagonals}\} \quad \text{and}$$
$$Z(\gamma) = \{\varepsilon_j - \varepsilon_i \mid j > i \text{ and } \mathrm{box}_i \text{ and } \mathrm{box}_j \text{ are in the same diagonal}\}.$$

Note that it is the formulation of the root system of type C_n in (8.4) which makes the description of $P(\gamma)$ and $Z(\gamma)$ nice in this case. If $J \subseteq P(\gamma)$ arrange the boxes on adjacent diagonals according to the rules

(1) if $\varepsilon_j - \varepsilon_i \in J$ place box_j northwest of box_i, and
(2) if $\varepsilon_j - \varepsilon_i \in P(\gamma)\backslash J$ place box_j southeast of box_i.

A *standard tableau* is a negative rotationally symmetric filling p of the $2n$ boxes with $-n, \ldots, -1, 1, \ldots, n$ such that

(a) $p(\mathrm{box}_i) < p(\mathrm{box}_j)$ if $j > i$ and box_j and box_i are in the same diagonal,
(b) $p(\mathrm{box}_i) > p(\mathrm{box}_j)$ if $j > i$, box_i and box_j are in adjacent diagonals, and box_j is northwest of box_i,
(c) $p(\mathrm{box}_i) < p(\mathrm{box}_j)$ if $j > i$, box_i and box_j are in adjacent diagonals, and box_j is southeast of box_i.

The negative rotational symmetry means that the filling of the boxes is the same if each entry is multiplied by -1 and the configuration is rotated by $180°$.

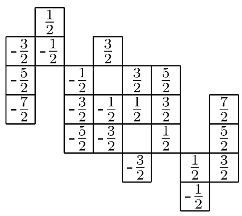
Example. Suppose

$$\gamma = \left(-\tfrac{7}{2}, -\tfrac{5}{2}, -\tfrac{5}{2}, -\tfrac{3}{2}, -\tfrac{3}{2}, -\tfrac{3}{2}, -\tfrac{3}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \tfrac{5}{2}, \tfrac{7}{2}\right)$$

and

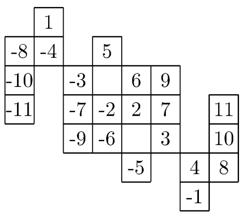
$$\begin{aligned} J = \{&\varepsilon_{11} - \varepsilon_{10}, \varepsilon_{-10} - \varepsilon_{-11}, \varepsilon_{10} - \varepsilon_8, \varepsilon_{-8} - \varepsilon_{-10}, \varepsilon_9 - \varepsilon_7, \varepsilon_{-7} - \varepsilon_{-9}, \varepsilon_9 - \varepsilon_8, \\ &\varepsilon_{-8} - \varepsilon_{-9}, \varepsilon_7 - \varepsilon_3, \varepsilon_{-3} - \varepsilon_{-7}, \varepsilon_7 - \varepsilon_4, \varepsilon_{-4} - \varepsilon_{-7}, \varepsilon_6 - \varepsilon_2, \varepsilon_{-2} - \varepsilon_{-6}, \varepsilon_6 - \varepsilon_3, \\ &\varepsilon_{-3} - \varepsilon_{-6}, \varepsilon_6 - \varepsilon_4, \varepsilon_{-4} - \varepsilon_{-6}, \varepsilon_5 - \varepsilon_4, \varepsilon_{-4} - \varepsilon_{-5}, \varepsilon_5 - \varepsilon_3, \varepsilon_{-3} - \varepsilon_{-5}, \varepsilon_5 - \varepsilon_2, \\ &\varepsilon_{-2} - \varepsilon_{-5}, \varepsilon_2 - \varepsilon_{-1}, \varepsilon_3 - \varepsilon_{-1}, \varepsilon_4 - \varepsilon_{-1}, \varepsilon_1 - \varepsilon_{-1}\} \\ = \{&\varepsilon_{11} - \varepsilon_{10}, \varepsilon_{10} - \varepsilon_8, \varepsilon_9 - \varepsilon_7, \varepsilon_9 - \varepsilon_8, \varepsilon_7 - \varepsilon_3, \varepsilon_7 - \varepsilon_4, \varepsilon_6 - \varepsilon_2, \varepsilon_6 - \varepsilon_3, \varepsilon_6 - \varepsilon_4, \\ &\varepsilon_5 - \varepsilon_4, \varepsilon_5 - \varepsilon_3, \varepsilon_5 - \varepsilon_2, \varepsilon_2 + \varepsilon_1, \varepsilon_3 + \varepsilon_1, \varepsilon_4 + \varepsilon_1, 2\varepsilon_1\}. \end{aligned}$$

The placed configuration of boxes corresponding to (γ, J) is as given below:



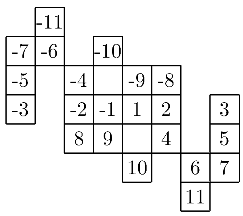
Page $\frac{1}{2}$

contents of boxes



Page $\frac{1}{2}$

numbering of boxes



Page $\frac{1}{2}$

a standard tableau

8.9. Case 0

Assume that $\gamma \in \mathfrak{h}_{\mathbb{R}}^*$ is of the form

$$\gamma = (-z_n \leqslant \cdots \leqslant -z_2 \leqslant -z_1;\ z_1 \leqslant z_2 \leqslant \cdots \leqslant z_n), \quad z_i \in \mathbb{Z}_{\geqslant 0}.$$

Place boxes on a page of infinite graph paper which has its diagonals numbered consecutively with the elements of $\mathbb{Z}$, from bottom left to top right. This page has page number 0. For each $i \in \{\pm 1, \ldots, \pm n\}$ place box_i on diagonal z_i and box_{-i} on diagonal $-z_i$. The boxes on each diagonal are arranged in increasing order from top left to bottom right and the placement of boxes is negative rotationally symmetric in the sense that a $180°$ rotation takes box_i to box_{-i}.

Using the root system notation in (8.4),

$$P(\gamma) = \{\varepsilon_j - \varepsilon_i \mid j > i \text{ and box}_i \text{ and box}_j \text{ are in adjacent diagonals}\} \quad \text{and}$$
$$Z(\gamma) = \{\varepsilon_j - \varepsilon_i \mid j > i \text{ and box}_i \text{ and box}_j \text{ are in the same diagonal}\}.$$

If $J \subseteq P(\gamma)$ arrange the boxes on adjacent diagonals according to the rules

(1) if $\varepsilon_j - \varepsilon_i \in J$ place box_j northwest of box_i, and
(2) if $\varepsilon_j - \varepsilon_i \in P(\gamma) \backslash J$ place box_j southeast of box_i.

A *standard tableau* is a negative rotationally symmetric filling p of the $2n$ boxes with $-n, \ldots, -1, 1, \ldots, n$ such that

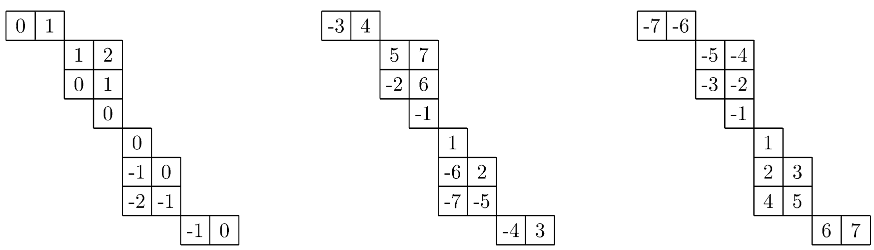
(a) $p(\text{box}_i) < p(\text{box}_j)$ if $j > i$ and box_j and box_i are in the same diagonal,
(b) $p(\text{box}_i) > p(\text{box}_j)$ if $j > i$, box_i and box_j are in adjacent diagonals, and box_j is northwest of box_i,
(c) $p(\text{box}_i) < p(\text{box}_j)$ if $j > i$, box_i and box_j are in adjacent diagonals, and box_j is southeast of box_i.

The negative rotational symmetry means that the filling of the boxes is the same if each entry is multiplied by -1 and the configuration is rotated by 180°.

Example. Suppose $\gamma = (-2, -1, -1, -1, 0, 0, 0; 0, 0, 0, 1, 1, 1, 2)$ and

$$\begin{aligned} J = \{&\varepsilon_4 - \varepsilon_1, \varepsilon_{-1} - \varepsilon_{-4}, \varepsilon_4 - \varepsilon_2, \varepsilon_{-2} - \varepsilon_{-4}, \varepsilon_4 - \varepsilon_3, \varepsilon_{-3} - \varepsilon_{-2}, \varepsilon_5 - \varepsilon_1, \varepsilon_{-1} - \varepsilon_{-5}, \\ &\varepsilon_5 - \varepsilon_2, \varepsilon_{-2} - \varepsilon_{-5}, \varepsilon_5 - \varepsilon_3, \varepsilon_{-3} - \varepsilon_{-5}, \varepsilon_6 - \varepsilon_1, \varepsilon_{-1} - \varepsilon_{-6}, \varepsilon_6 - \varepsilon_2, \varepsilon_{-2} - \varepsilon_{-6}, \\ &\varepsilon_6 - \varepsilon_3, \varepsilon_{-3} - \varepsilon_{-6}, \varepsilon_7 - \varepsilon_6, \varepsilon_{-6} - \varepsilon_{-7}, \varepsilon_6 - \varepsilon_{-1}, \varepsilon_1 - \varepsilon_{-6}, \varepsilon_5 - \varepsilon_{-1}, \varepsilon_1 - \varepsilon_{-5}, \\ &\varepsilon_4 - \varepsilon_{-1}, \varepsilon_1 - \varepsilon_{-4}, \varepsilon_5 - \varepsilon_{-2}, \varepsilon_2 - \varepsilon_{-5}, \varepsilon_4 - \varepsilon_{-2}, \varepsilon_2 - \varepsilon_{-4}\} \\ = \{&\varepsilon_4 - \varepsilon_1, \varepsilon_4 - \varepsilon_2, \varepsilon_4 - \varepsilon_3, \varepsilon_5 - \varepsilon_1, \varepsilon_5 - \varepsilon_2, \varepsilon_5 - \varepsilon_3, \varepsilon_6 - \varepsilon_1, \varepsilon_6 - \varepsilon_2, \varepsilon_6 - \varepsilon_3, \\ &\varepsilon_7 - \varepsilon_6, \varepsilon_6 + \varepsilon_1, \varepsilon_5 + \varepsilon_1, \varepsilon_5 + \varepsilon_2, \varepsilon_4 + \varepsilon_1, \varepsilon_4 + \varepsilon_2\}. \end{aligned}$$

The placed configuration of boxes corresponding to (γ, J) is as given below:



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contents of boxes | number of boxes | a standard tableau

8.10. A posteriori the analysis of the three cases β, $1/2$, and 0, it becomes evident that the trick of using the formulation of the root system of type C_n in (8.4) provides a completely uniform description of the configurations of boxes and standard tableaux corresponding to type C_n local regions. All three cases give negative rotationally invariant tableaux. We could not ask for nature to work out more perfectly.

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References

[1] S. Ariki, K. Koike, A Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr S_n$ and construction of its irreducible representations, Adv. Math. 106 (1994) 216–243.
[2] C. Athanasiadis, S. Linusson, A simple bijection for the regions of the Shi arrangement of hyperplanes, Discrete Math. 204 (1–3) (1999) 27–39.
[3] M. Aubert, Dualité dans le groupe de Grothendieck de la categorie des représentations lisses de longueur finie d'un groupe réductif p-adique, Trans. Amer. Soc. 347 (1995) 2179–2189.
[4] A. Björner, Orderings of Coxeter groups, in: Combinatorics and Algebra, Boulder, CO, 1983, Contemp. Math., Vol. 34, Amer. Math. Society, Providence, 1984, pp. 175–195.
[5] A. Björner, M. Wachs, Generalized quotients in Coxeter groups, Trans. Amer. Math. Soc. 308 (1) (1988) 1–37.
[6] N. Bourbaki, Groupes et algèbres de Lie, Chapitres 4, 5 et 6, Elements de Mathématique, Hermann, Paris, 1968.
[7] R. Carter, G. Segal, I.G. Macdonald, Lectures on Lie Groups and Lie Algebras, London Math. Soc. Stud. Texts, Vol. 32, Cambridge Univ. Press, Cambridge, 1995.
[8] N. Chriss, V. Ginzburg, Representation Theory and Complex Geometry, Birkhäuser, 1997.
[9] A. Garsia, C. Reutenauer, A decomposition of Solomon's descent algebra, Adv. Math. 77 (2) (1989) 189–262.
[10] P.N. Hoefsmit, Representations of Hecke algebras of finite groups with BN-pairs of classical type, PhD thesis, University of British Columbia, 1974.
[11] N. Iwahori, H. Matsumoto, On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups, Publ. Math. Inst. Hautes Études Sci. 40 (1972) 81–116.
[12] S. Kato, Irreducibility of principal series representations for Hecke algebras of affine type, J. Fac. Sci. Univ. Tokyo Sec. IA 28 (1981) 929–943.
[13] S. Kato, Duality for representations of a Hecke algebra, Proc. Amer. Math. Soc. 119 (1993) 941–946.
[14] D. Kazhdan, G. Lusztig, Proof of the Deligne–Langlands conjecture for Hecke algebras, Invent. Math. 87 (1987) 153–215.
[15] B. Kostant, A generalization of the Bott–Borel–Weil theorem and Euler number multiplets of representations, Conference Moshé Flato 1999 (Dijon), Lett. Math. Phys. 52 (1) (2000) 61–78.
[16] C. Krattenthaler, L. Orsina, P. Papi, Enumeration of ad-nilpotent $\mathfrak{b}$-ideals for simple Lie algebras, Adv. Appl. Math, Special issue in memory of Rodica Simion, to appear.

[17] C. Kriloff, A. Ram, Representations of graded Hecke algebras, Represent. Theory 6 (2002) 31–69; http://www.ams.org/ert/home-2002.html.
[18] H. Knight, A. Zelevinsky, Representations of quivers of type A and the multisegment duality, Adv. Math. 117 (2) (1996) 273–293.
[19] B. Leclerc, J.-Y. Thibon, E. Vasserot, Zelevinsky's involution at roots of unity, J. Reine Angew. Math. 513 (1999) 33–51.
[20] P. Littelmann, Paths and root operators in representation theory, Ann. of Math. (2) 142 (1995) 499–525.
[21] P. Littelmann, The path model for representations of symmetrizable Kac–Moody algebras, in: Proceedings of the International Congress of Mathematicians, Vol. 1, Zürich, 1994, Birkhäuser, Basel, 1995, pp. 298–308.
[22] J. Losonczy, Standard Young tableaux in the Weyl group setting, J. Algebra 220 (1999) 255–260.
[23] G. Lusztig, Singularities, character formulas, and a q-analog of weight multiplicities, in: Analysis and Topology on Singular Spaces, II, III, Luminy, 1981, Astérisque 101–102 (1983) 208–229.
[24] I.G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd edition, Oxford Math. Monographs, Oxford Univ. Press, New York, 1995.
[25] C. Moeglin, J.-L. Waldspurger, L'involution de Zelevinski, J. Reine Angew. Math. 372 (1986) 136–177.
[26] M. Nazarov, V. Tarasov, Representations of Yangians with Gelfand–Zetlin bases, J. Reine Angew. Math. 496 (1998) 181–212.
[27] A. Ram, Representations of rank two affine Hecke algebras, in: Advances in Algebra and Geometry (University of Hyderabad Conference 2001), Hidustan Book Agency, New Delhi, India, 2002, pp. 57–91.
[28] A. Ram, Calibrated representations of affine Hecke algebras, Preprint, 1998, http://www.math.wisc.edu/~ram/preprints.html.
[29] A. Ram, Standard Young tableaux for finite root systems, Preprint, 1998, http://www.math.wisc.edu/~ram/preprints.html.
[30] A. Ram, J. Ramagge, Affine Hecke algebras, cyclotomic Hecke algebras and Clifford theory, to appear in: V. Lakshmibai, R. Sridharan (Eds.), A volume in honor of the 70th birthday of C.S. Seshadri, 2003.
[31] C. Reutenauer, Free Lie Algebras, in: London Math. Soc. Monogr. (N.S.), Vol. 7, Oxford Univ. Press, New York, 1993.
[32] J. Rogawski, On modules over the Hecke algebra of a p-adic group, Invent. Math. 79 (1985) 443–465.
[33] J.-Y. Shi, The number of $\oplus$-sign types, Quart. J. Math. Oxford Ser. (2) 48 (1997) 93–105.
[34] J.-Y. Shi, Left cells in affine Weyl groups, Tôhoku Math. J. (2) 46 (1) (1994) 105–124.
[35] J.-Y. Shi, Sign types corresponding to an affine Weyl group, J. London Math. Soc. (2) 35 (1987) 56–74.
[36] L. Solomon, A Mackey formula in the group ring of a Coxeter group, J. Algebra 41 (2) (1976) 255–264.
[37] L. Solomon, H. Terao, The double Coxeter arrangement, Preprint, 1997.
[38] R. Stanley, Hyperplane arrangements, interval orders, and trees, Proc. Nat. Acad. Sci. USA 93 (1996) 2620–2625.
[39] R. Stanley, Hyperplane arrangements, parking functions and tree inversions, in: Mathematical Essays in Honor of Gian-Carlo Rota, Birkhäuser, Boston, 1998.
[40] R. Steinberg, Lectures on Chevalley groups, Notes prepared by John Faulkner and Robert Wilson, Yale University, New Haven, CT, 1968.
[41] R. Steinberg, Endomorphisms of linear algebraic groups, Mem. Amer. Math. Soc. 80 (1968) 1–108.
[42] R. Steinberg, On a theorem of Pittie, Topology 14 (1975) 173–177.
[43] S. Sundaram, Tableaux in the representation theory of the classical Lie groups, in: Invariant Theory and Tableaux, Minneapolis, MN, 1988, IMA Vol. Math. Appl., Vol. 19, Springer-Verlag, New York, 1990, pp. 191–225.
[44] N. Wallach, Real reductive groups I, Pure Appl. Math., Vol. 132, Academic Press, Boston, MA, 1988.
[45] H. Wenzl, Hecke algebras of type A_n and subfactors, Invent. Math. 92 (1988) 349–383.
[46] N. Xi, Representations of Affine Hecke Algebras, Lecture Notes in Math., Vol. 1587, Springer-Verlag, Berlin, 1994.
[47] A. Young, The Collected Papers of Alfred Young 1873–1940, Math. Expositions, Vol. 21, University of Toronto Press, 1977.
[48] A. Young, On quantitative substitutional analysis (sixth paper), Proc. London. Math. Soc. (2) 34 (1931) 196–230.
[49] A. Zelevinsky, Induced representations of $\mathfrak{p}$-adic groups II: On irreducible representations of $GL(n)$, Ann. Sci. École Norm. Sup. (4) 13 (1980) 165–210.