Robinson-Schensted-Knuth insertion and characters of symmetric groups and Iwahori-Hecke algebras of type A

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Dedicated to Gian-Carlo Rota

0. Introduction

The purpose of this note is to give an insertion scheme proof of the formula, [Mac] Ch. I (7.8),

$$p_\mu = \sum_{\lambda \vdash k} \chi^\lambda(\mu) s_\lambda, \tag{0.1}$$

where p_μ is the power sum symmetric function, s_λ is the Schur function and $\chi^\lambda(\mu)$ is the irreducible character of the symmetric group S_k indexed by the partition λ and evaluated at a permutation of cycle type $\mu = (\mu_1, \ldots, \mu_\ell)$. The proof of this formula will be by direct application of the Robinson-Schensted-Knuth insertion scheme and the following beautiful formula of Roichman [Ro],

$$\chi^\lambda(\mu) = \sum_Q \mathrm{rw}_1^\mu(Q), \tag{0.2}$$

where the sum is over all *standard tableaux* Q of shape λ and the μ-weight of a standard tableau is given by

$$\mathrm{rw}_q^\mu(Q) = \prod_{\substack{1\le j\le k \\ j\notin B(\mu)}} f_\mu(j,Q), \quad \text{where } B(\mu) = \{\mu_1 + \cdots + \mu_r \mid 1 \le r \le \ell\}, \text{ and} \tag{0.3}$$

$$f_\mu(j,Q) = \begin{cases} -1, & \text{if } j+1 \text{ is southwest of } j \text{ in } Q, \\ 0, & \text{if } j+1 \text{ is northeast of } j \text{ in } Q,\ j+1 \notin B(\mu), \\ & \qquad \text{and } j+2 \text{ is southwest of } j+1 \text{ in } Q, \\ q, & \text{otherwise.} \end{cases} \tag{0.4}$$

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The notation $\mathrm{rw}_1^\mu(Q)$ denotes the weight $\mathrm{rw}_q^\mu(Q)$ when $q=1$. The notations and direction for partitions and their Ferrers diagrams are as in [Mac], and "north" means in the same row or in a higher row.

Alternatively, one may assume the Frobenius formula in (0.1) and interpret this note as an elementary proof of Roichman's formula. One should note that Roichman's formula is actually more general than what I have stated above, I am considering only the Type A case, and I have stated it above only in the $q=1$ case. D. White [Wh] has also analyzed the characters of the symmetric group by an analogue of the RSK insertion scheme. His methods are different from those used in this paper.

1. Symmetric functions

Fix a positive integer n, let $x_1,\ldots,x_n$ be commuting variables and let q be an indeterminate. If λ is a partition let $s_\lambda = s_\lambda(x_1,\ldots,x_n)$ denote the Schur function associated to λ [Mac]. Define

$$q_r = q_r(x_1,x_2,\ldots,x_n;q) = \sum_{1\le i_1\le\cdots\le i_r\le n} x_{i_1}\cdots x_{i_r} q^{(\#\text{ of } i_j=i_{j+1})}(q-1)^{(\#\text{ of } i_j<i_{j+1})}, \quad (1.1)$$

and for a partition $\mu=(\mu_1,\ldots,\mu_\ell)$ define $q_\mu = q_\mu(x_1,\ldots,x_n;q) = q_{\mu_1}q_{\mu_2}\cdots q_{\mu_\ell}$. The following are well known facts:

(1.2) If $\mu\vdash k$ then

$$p_\mu = \sum_{\lambda\vdash k}\chi^\lambda(\mu)s_\lambda$$

where $\chi^\lambda(\mu)$ is the character of the symmetric group S_k associated to the partition λ evaluated at an element of cycle type μ.

(1.3) [Ra] If $\mu\vdash k$ then

$$q_\mu = \sum_{\lambda\vdash k}\chi_q^\lambda(\mu)s_\lambda$$

where $\chi_q^\lambda(\mu)$ is the character of the Iwahori-Hecke algebra of type A associated to the partition λ evaluated at the element T_{γ_μ} where $\gamma_\mu = \gamma_{\mu_1}\times\cdots\times\gamma_{\mu_\ell}\in S_{\mu_1}\times\cdots\times S_{\mu_\ell}$ and $\gamma_r=(1,2,\ldots,r)\in S_r$ (in cycle notation).

(1.4) $\chi^\lambda(\mu) = \chi_q^\lambda(\mu)\big|_{q=1}$ and $q_\mu(x_1,\ldots,x_n;1) = p_\mu(x_1,\ldots,x_n)$, where p_μ is the power sum symmetric function.

Lemma 1.5. *If $(i_1,\ldots,i_r)$ is a sequence $1\le i_1,\ldots,i_r\le n$, define*

$$\mathrm{wt}_q(i_1,\ldots,i_r) = \begin{cases} (-1)^s q^{r-1-s}, & \text{if there exists an } s,\ 0\le s<r, \\ & \quad \text{such that } i_1<\cdots<i_s<i_{s+1}\ge\cdots\ge i_r, \\ 0, & \text{otherwise.}\end{cases}$$

Then $$q_r(x_1,\ldots,x_n;q) = \sum_{1\le i_1,\ldots,i_r\le n} x_{i_1}\cdots x_{i_r}\mathrm{wt}_q(i_1,\ldots,i_r).$$

Proof. Let $1 \le j_1 \le j_2 \le \cdots \le j_r \le n$ be an increasing sequence and let us show that

$$\sum_{1\le i_1,\ldots,i_r\le n} x_{i_1}\cdots x_{i_r}\mathrm{wt}_q(i_1,\ldots,i_r)\big|_{x_{j_1}\cdots x_{j_r}} = q^{r-\ell}(q-1)^{\ell-1},$$

where ℓ is the number of $j_k < j_{k+1}$ in the sequence $j_1 \le j_2 \le \ldots \le j_r$.

Let D be the set of distinct elements in the sequence $j_1 \le \cdots \le j_r$, let m be the maximal element of D and let $D' = D\backslash\{m\}$. For each subset S of D' let

$s = \mathrm{Card}(S)$,

$i_1 < i_2 < \cdots < i_s$ be the elements of S in increasing order,

$i_{s+1} = m$, and let

$i_{s+2} \ge i_{s+3} \ge \cdots \ge i_r$ be the remainder of the elements of the sequence $j_1,\ldots j_r$ arranged in decreasing order.

Then $x_{i_1}\cdots x_{i_r} = x_{j_1}\cdots x_{j_r}$ and $\mathrm{wt}(i_1,\ldots,i_r) = (-1)^s q^{r-1-s}$.

We have $x_{i_1}\cdots x_{i_r} = x_{j_1}\cdots x_{j_r}$ and $\mathrm{wt}_q(i_1,\ldots,i_r) \ne 0$ if and only if

(1) there exists a permutation $\sigma \in S_r$ such that $(i_1,\ldots,i_r) = (j_{\sigma(1)},\ldots,j_{\sigma(r)})$ and

(2) there exists an s such that $0 \le s < r$ and

$$j_{\sigma(1)} < \cdots < j_{\sigma(2)} < \cdots < j_{\sigma(s)} < j_{\sigma(s+1)} \ge j_{\sigma(s+2)} \ge \cdots \ge j_{\sigma(r)}.$$

It follows from this that every sequence $1 \le i_1,\ldots,i_r \le n$ such that $x_{i_1}\cdots x_{i_r} = x_{j_1}\cdots x_{j_r}$ and $\mathrm{wt}_q(i_1,\ldots,i_r) \ne 0$ is of the form given in the previous paragraph for a unique subset S of D'. Thus

$$\begin{aligned}\sum_{1\le i_1,\ldots,i_r\le n} x_{i_1}\cdots x_{i_r}\mathrm{wt}_q(i_1,\ldots,i_r)\big|_{x_{j_1}\cdots x_{j_r}} &= \sum_{S\subseteq D'} q^{r-|S|-1}(-1)^{|S|}\\ &= \sum_{s=0}^{\ell-1}\binom{\ell-1}{s}(-1)^s q^{r-s-1} = q^{r-\ell}(q-1)^{\ell-1},\end{aligned}$$

where $\ell = \mathrm{Card}(D) = (\#\text{ of } j_k < j_{k+1} \text{ in } j_1 \le \cdots \le j_r)$. ∎

Corollary 1.6. *Let $\mu = (\mu_1,\ldots,\mu_\ell)$ be a partition of k and let $B(\mu) = \{\mu_1+\cdots+\mu_r \mid 1 \le r \le \ell\}$. Then*

$$q_\mu = \sum_{i_1,\ldots,i_k} x_{i_1}\cdots x_{i_k}\mathrm{wt}_q^\mu(i_1,\ldots,i_k)$$

where the sum is over all $1 \le i_1,\ldots i_k \le n$,

$$\mathrm{wt}_q^\mu(i_1,\ldots,i_k) = \prod_{\substack{1\le j\le k\\ j\notin B(\mu)}} \phi_\mu(j;i_1,\ldots,i_k), \qquad \textit{and}$$

$$\phi_\mu(j;i_1,\ldots,i_k) = \begin{cases} -1, & \textit{if } i_j < i_{j+1},\\ 0, & \textit{if } i_j \ge i_{j+1} < i_{j+2} \textit{ and } j+1 \notin B(\mu),\\ q, & \textit{otherwise.}\end{cases}$$

Proof. The result follows immediately from the previous lemma once we note that with $\mathrm{wt}_q(i_1,\ldots,i_r)$ defined as in Lemma 1.5 we have

$$\mathrm{wt}_q(i_1,\ldots,i_r)=\prod_{j=1}^{r}\phi(j;i_1,\ldots,i_r),\quad \text{where}$$

$$\phi(j;i_1,\ldots,i_r)=\begin{cases}-1, & \text{if } i_j<i_{j+1},\\ 0, & \text{if } i_j\ge i_{j+1}<i_{j+2} \text{ and } j+1\neq r.\\ q, & \text{otherwise.}\end{cases}\qquad\blacksquare$$

The following corollary follows from the last one by setting $q=1$.

Corollary 1.7.

$$p_\mu=\sum_{i_1,\ldots,i_k} x_{i_1}\cdots x_{i_k}\mathrm{wt}_1^\mu(i_1,\ldots,i_k)$$

where the sum is over all $1\le i_1,\ldots,i_k\le n$*, and* wt_1^μ *is as given in Corollary 1.6 except with* $q=1$.

2. Insertion

Let us assume that the variables $x_1,\ldots,x_n$ are ordered so that $x_1<x_2<\cdots<x_n$. A *column strict tableau of shape* λ is a filling of the boxes of the Ferrers diagram of λ with entries from $\{x_1,\ldots,x_n\}$ such that

(1) The entries in the rows are weakly increasing, left to right, and
(2) The entries in the columns are increasing, top to bottom.

If P is a column strict tableau let x^P be the product of the entries in P. One has $s_\lambda=\sum_P x^P$, where the sum is over all column strict tableaux P of shape λ.

The Robinson-Schensted-Knuth (RSK) insertion scheme gives a bijection between sequences $(x_{i_1},\ldots,x_{i_k})$ and pairs (P,Q) where P is a column strict tableau and Q is a standard tableau and P and Q are the same shape. (The original references for the RSK insertion scheme are [Sz], [Sch] and [Kn], for an expository treatment see [Sag]). The pair of tableaux obtained by RSK insertion of the sequence $x_{i_1},\ldots,x_{i_k}$ is the pair $(P,Q)=(P_k,Q)$ determined recursively by setting

$P_0=\emptyset$, $P_j=(P_{j-1}\leftarrow x_{i_j})$, and

the box of Q containing k is the box created upon the insertion of x_{i_j} into P_{j-1}.

Here $P_{j-1}\leftarrow x_{i_j}$ denotes the column strict tableau obtained by column insertion of the letter x_{i_j} into the column strict tableau P_{j-1}. The following proposition is a well known fact about RSK-insertion.

Proposition 2.1. *Let* P_{j-1} *be a column strict tableau and consider the insertions* $(P_{j-1}\leftarrow x_{i_j})\leftarrow x_{i_{j+1}}$.

(1) If $x_{i_j} < x_{i_{j+1}}$ then the box created upon insertion of $x_{i_{j+1}}$ into $P_j = (P_{j-1} \leftarrow x_{i_j})$ appears southwest of the box created upon insertion of x_{i_j} into P_{j-1}.

(2) If $x_{i_j} \geq x_{i_{j+1}}$ then the box created upon insertion of $x_{i_{j+1}}$ into $P_j = (P_{j-1} \leftarrow x_{i_j})$ appears northeast of the box created upon insertion of x_{i_j} into P_{j-1}.

Our goal now is to prove Roichman's formula. For each pair of partitions $\lambda, \mu \vdash k$ define a polynomial $\eta_q^\lambda(\mu) \in \mathbb{Z}[q]$ by (the q-version of) the right hand side of Roichman's formula (0.2), i.e. define

$$\eta_q^\lambda(\mu) = \sum_Q \mathrm{rw}_q^\mu(Q), \tag{2.2}$$

where the sum is over all standard tableaux Q of shape λ and the μ-weight of a standard tableau is as given in (0.3).

Theorem 2.3. *If $\mu \vdash k$, then $q_\mu = \sum_{\lambda \vdash k} \eta_q^\lambda(\mu) s_\lambda$.*

Proof. RSK-insertion of a sequence $x_{i_1}, \ldots, x_{i_k}$ produces a pair (P, Q) consisting of a column-strict tableau P and a standard tableau Q where the shape λ of the tableau P is a partition with k boxes. Thus RSK-insertion combined with Proposition 2.1 implies the following identity

$$q_\mu = \sum_{i_1,\ldots,i_k} x_{i_1} \cdots x_{i_k} \mathrm{wt}_q^\mu(i_1, \ldots, i_k) = \sum_{(P,Q)} x^P \mathrm{rw}_q^\mu(Q)$$

$$= \sum_{\lambda \vdash k} \left(\sum_Q \mathrm{rw}_q^\mu(Q) \right) \left(\sum_P x^P \right) = \sum_{\lambda \vdash k} \eta_q^\lambda(\mu) s_\lambda. \qquad \blacksquare$$

In view of the previous theorem we have that (1.3) is equivalent to Roichman's theorem where Roichman's theorem is the statement that

$$\chi_q^\lambda(\mu) = \eta_q^\lambda(\mu).$$

We get the following corollary of Theorem 2.3 by setting $q = 1$.

Corollary 2.4.

(a) If $\mu \vdash k$ then $p_\mu = \sum_{\lambda \vdash k} \eta_1^\lambda(\mu) s_\lambda$.

(b) For $\lambda, \mu \vdash k$ the character of the symmetric group S_k associated to the partition λ evaluated at an element of cycle type μ is given by

$$\chi^\lambda(\mu) = \sum_Q \mathrm{rw}_1^\mu(Q),$$

where the sum is over all standard tableaux Q of shape λ and $\mathrm{rw}_1^\mu(Q)$ is as given in (0.3) except with $q = 1$.

3. References

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