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A 'Second Orthogonality Relation' for Characters of Brauer Algebras

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1. INTRODUCTION

In a first course in representation theory, one usually learns that there are two important relations for characters of a finite group, the first orthogonality relation and the second orthogonality relation. When one moves on to study algebras which are not necessarily group algebras it is not clear, *a priori,* that the study of characters would be fruitful, and one encounters various problems in developing a theory analogous to that for finite groups. It is true, however, that the characters of algebras, such as the Iwahori–Hecke algebras, the Brauer algebras and the Birman–Wenzl algebras, are well-defined and important. An analogue of the first orthogonality relation for characters of such algebras is understood and appears in [4]. In the Appendix of this paper, we show that the second orthogonality relation for characters makes sense for split semisimple algebras (although it is no longer an orthogonality relation).

This paper is concerned with the 'second orthogonality relation' for the Brauer algebras. We derive this relation explicitly. After a talk on the results in Sections 1 and 2 of this paper at University of Bordeaux I, R. Stanley sketched an alternate proof of the results in Section 1 using the combinatorial interpretation of the characters of the Brauer algebra and tools from his paper [11]. Here we present the proof of Stanley's as a theorem purely about the combinatorial rule for the characters of the Brauer algebras. Then we study the naturally occurring weight space representations of the Brauer algebra. Putting the three facets together, we are able to give a new derivation of the irreducible characters of the Brauer algebras.

I would like to mention, here in the Introduction so that it gets noticed, that I have been unable to compute the second relation for characters explicitly in either the case of the Iwahori–Hecke algebra of type A or the case of the Birman–Wenzl algebra. Computing these relations could be useful in the study of representations of quantum groups and/or q-differential posets and/or q-Hermite polynomials.

This paper is organized as follows.

In the Appendix we give an argument that there is an analogue of the second orthogonality relation for characters of a finite group in the case of any finite-dimensional algebra with a non-degenerate trace form (in particular, a split semisimple algebra). The purpose of this is to show that the study of the second relation for characters makes sense for the case of the Brauer algebra. We have put this material in an Appendix as it is primarily algebraic in nature and is not needed in the rest of paper.

In Section 2 we derive by an enumerative argument the explicit form of the second relation for characters of the Brauer algebra. It is interesting to note that the formulas can be expressed in terms of products of certain Hermite polynomials. In Section 3 we give an application of our result of Section 2 to Weyl group symmetric functions of types B, C and D. Diaconis and Shahshahani [5] have also applied these results in their study of the eigenvalues of random orthogonal and symplectic matrices. In Section 4 we present Stanley's proof of analogous 'second relation' formulas for certain numbers determined by up–down border strip tableaux.

In Section 5 we show that the 'second orthogonality relations' of Sections 2 and 4

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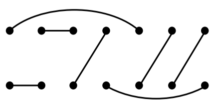
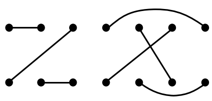
can be used to give a new proof of the fact that the numbers determined by weighted sums of up–down border strip tableaux actually are the characters of the Brauer algebras. This proof is in the same vein as Frobenius' original derivation of the characters of the symmetric groups. This approach also gives a new proof of the Frobenius formula for the Brauer algebras. This proof is quite elementary and the 'second orthogonality relations' are used in place of the Weyl character formula which was used in the original proof [9].

2. The Brauer Algebra

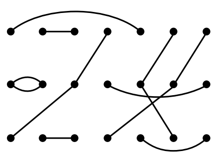
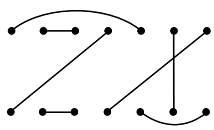
An m-diagram is a graph on two rows of m vertices each, one above the other, and $2m$ edges such that each vertex is incident to precisely one edge. The number of m diagrams is

(2.1) $$(2m-1)!! = (2m-1)(2m-3)\cdots 3\cdot 1.$$

We multiply two m-diagrams d_1 and d_2 by placing d_1 above d_2 and identifying the vertices in the bottom row of d_1 with the corresponding vertices in the top row of d_2. The resulting graph contains m paths and some number γ of closed cycles. Let d be the m-diagram the edges of which are the paths in this graph (with the cycles removed). Then the product $d_1 d_2$ is given by $d_1 d_2 = x^{\gamma} d$. For example, if

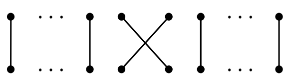
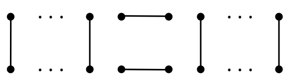
$d_1 =$  and $d_2 =$ 

then

$d_1 d_2 =$  $= x_2$ 

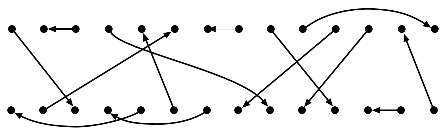
Let x be an indeterminate. The *Brauer algebra* $B_m(x)$ is the $\mathbb{C}(x)$-span of the m-diagrams. Diagram multiplication makes $B_m(x)$ an associative algebra. By convention $B_0(x) = B_1(x) = \mathbb{C}(x)$. For each complex number $\alpha \in \mathbb{C}$ one defines a Brauer algebra $B_m(\alpha)$ over $\mathbb{C}$ as the $\mathbb{C}$-linear span of the m-diagrams, where the multiplication is given as above except with x replaced by α. Although we shall state our results for the algebra $B_m(x)$, unless otherwise stated, they also hold for the algebras $B_m(\alpha)$ (replacing x by α).

Under the above multiplication the m-diagrams with only vertical edges form a symmetric group S_m inside the Brauer algebra. For $1 \leqslant i \leqslant m-1$, let

(2.2) $G_i =$  and $G_i =$ 

The elements of the set $\{G_i, E_i \mid 1 \leqslant i \leqslant f-1\}$ generate $B_m(x)$.

Cycle type. We associate to each m-diagram a partition $\tau(d) \in \hat{B}_m$ called the cycle type of d. To do this, we traverse the diagram d in the following way. Connect each vertex in the top row of the diagram d to the vertex just below it in the bottom row by a dotted line. Beginning with the first vertex (moving left to right) in the top row of d, follow the path determined by the edges and the dotted lines and assign to each edge the direction that it is traversed. Returning to the original vertex completes a *cycle* in d. If not all vertices in d have been visited, start with the first not yet visited vertex in the top row of d and traverse the cycle adjacent to it. Do this until all vertices have been visited. The diagram

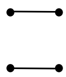
(2.3) $d =$ 

has three cycles. The first is on vertices $1, 2, 3, 6, 5$, the second on vertices $4, 7, 8, 9, 11$, and the third on vertices $10, 12, 13, 14$. To each cycle of d, let $U(c)$ denote the number of edges of c directed from bottom to top and $D(c)$ the number of edges of c directed from top to bottom. The positive integer

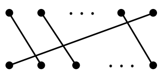
$$t(c) = |U(c) - D(c)|$$

is called the type of the cycle c. As c runs over all cycles of d, the sequence of numbers $t(c)$, arranged in decreasing order, is denoted $\tau(d)$, the *type* of the diagram d. In example (2.3) above, $\tau(d) = (3, 1, 0)$.

Characters. If d_1 is an m_1-diagram and d_2 is an m_2-diagram, then $d_1 \otimes d_2$ is the $(m_1 + m_2)$-diagram obtained by placing d_1 to the right of d_2. Let E denote the 2-diagram

(2.4) $E =$ 

and let γ_m denote the m-diagram

$\gamma_m =$ 

For a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_t)$, let $\gamma_\mu = \gamma_{\mu_1} \otimes \gamma_{\mu_2} \otimes \cdots \otimes \gamma_{\mu_t}$. We have the following theorem from [9].

(2.5) Theorem. *If d is an m-diagram and χ is a character of $B_m(x)$, then*

$$x^r \chi(d) = \chi(E^{\otimes h} \otimes \gamma_\mu),$$

where μ is the partition formed by the nonzero parts of the type $\tau(d) = (0^{m_0} 1^{m_1} 2^{m_2} \cdots)$ of the diagram d and r and h are given by $h = (m - |\mu|)/2$ and $r = h - m_0$.

The trace $\chi_{2m}^{\varnothing}$. Given a diagram d on m dots, number the dots in each row from left to right $1, 2, \ldots, m$. Let $\Omega_{2m}^{\varnothing}$ be the set of diagrams on $2m$ dots which have edges connecting $1 \to 2$, $3 \to 4, \ldots, (2m-1) \to 2m$ in the lower row. Let

$$B_{2m}^{\varnothing} = \mathbb{C}(x) - \text{span}\{d \in \Omega_{2m}^{\varnothing}\}.$$

$B_{2m}(x)$ acts on $B_{2m}^{\varnothing}$ by left multiplication. The trace of the action of $B_{2m}(x)$ on $B_{2m}^{\varnothing}$ is given by

(2.6) $$\chi_{2m}^{\varnothing}(a) = \sum_{d \in \Omega_{2m}^{\varnothing}} ad\big|_d,$$

where $a \in B_{2m}(x)$ and $ad|_d$ denotes the coefficient of the diagram d in the product ad.

For each pair of positive integers k, m, define $f_k(m)$ as follows:

$$\text{if } k \text{ is odd,} \qquad f_k(m) = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ (m-1)!!k^{m/2}, & \text{if } m \text{ is even;} \end{cases}$$

$$\text{if } k \text{ is even,} \qquad f_k(m) = \sum_{s=0}^{\lfloor m/2 \rfloor} \binom{m}{2s}(2s-1)!!k^s,$$

where $(m-1)!!$ is defined by (2.1). If $\mu = (1^{m_1}2^{m_2}\cdots)$ is a partition, then define

$$f(\mu) = \prod_k f_k(m_k). \tag{2.7}$$

(2.8) THEOREM. *Let $\chi_{2m}^{\varnothing}$ be the trace of the action of $B_{2m}(x)$ on $B_{2m}^{\varnothing}$. Then:*
(1) *if μ is a partition of $2m-2h$, $0 \leqslant h \leqslant m$, then*

$$\chi_{2m}^{\varnothing}(E^{\otimes h} \otimes \gamma_\mu) = x^h \chi_{2m-2h}^{\varnothing}(\gamma_\mu);$$

(2) *if $\mu = (1^{m_1}2^{m_2}\cdots)$ is a partition of $2m$, then*

$$\chi_{2m}^{\varnothing}(\gamma_\mu) = f(\mu),$$

where $f(\mu)$ is given as in (2.7).

PROOF. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ be a partition of $2m-2h$ for some $0 \leqslant h \leqslant m$, and let $a = E^{\otimes h} \otimes \gamma_\mu$ on $B_{2m}^{\varnothing}$. Let $d \in \Omega_{2m}^{\varnothing}$. Since a contains horizontal edges connecting $1 \to 2$, $3 \to 4, \ldots, (2h-1) \to 2h$ in the upper row we know that $ad|_d \neq 0$ only if d also has edges connecting $1 \to 2, \ldots, (2h-1) \to 2h$ in the upper row. In this case we have that $d = E^{\otimes h} \otimes d_1$, where d_1 is a diagram on $2m-2h$ dots. We obtain that

$$\begin{aligned}\chi_{2m}^{\varnothing}(E^{\otimes h} \otimes \gamma_\mu) &= \sum_{d \in \Omega_{2m}^{\varnothing}} ad|_d = \sum_{d \in \Omega_{2m}^{\varnothing}} (E^{\otimes h} \otimes \gamma_\mu)(E^{\otimes h} \otimes d_1)|_{E^{\otimes h} \otimes d_1} \\ &= x^h \sum_{d_1 \in \Omega_{2m-2h}^{\varnothing}} \gamma_\mu d_1|_d = x^h_{\chi 2m-2h}(\gamma_\mu),\end{aligned}$$

proving (1).

Let $\mu = (\mu_1, \ldots, \mu_s)$ be a partition of $2m$. We shall refer to the factors γ_{μ_i} in $\gamma_\mu = \gamma_{\mu_1} \otimes \cdots \otimes \gamma_{\mu_s}$ as the cycles of γ_μ. If d is a diagram in $\Omega_{2m}^{\varnothing}$ then, since γ_μ is a permutation,

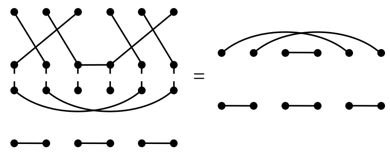
$$\gamma_\mu d|_d = \begin{cases} 1, & \text{if } \gamma_\mu d = d, \\ 0, & \text{otherwise.} \end{cases}$$

Thus

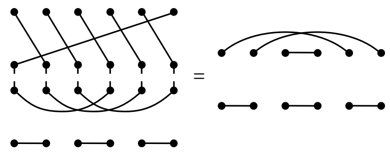
$$\chi_{2m}^{\varnothing}(\gamma_\mu) = (\text{number of } d \in \Omega_{2m}^{\varnothing} \text{ such that } \gamma_\mu d = d). \tag{2.9}$$

Let $d \in \Omega_{2m}^{\varnothing}$ and assume that $\gamma_\mu d = d$. Imagine that γ_μ is placed above d in order to compute the product $\gamma_\mu d$. We shall say that a horizontal edge in the top row of d which connects a dot which is underneath a cycle γ_{μ_i} in γ_μ to a dot which is underneath a cycle γ_{μ_j} in γ_μ to a dot which is underneath a cycle γ_{μ_j} in γ_μ is an edge connecting the cycles γ_{μ_i} and γ_{μ_j}. Then one must have that:
(1) d does not contain any horizontal edges in the top row which connect cycles of γ_μ of different length.
(2) If there is an edge in d connecting two cycles of length k, then all dots under the

first cycle must be connected to dots in the second cycle. There are exactly k different possible configurations of edges on these cycles:



(3) If d contains an edge which connects a cycle to itself, then the cycle must be even.
(4) There is exactly one possible configuration of edges under a single cycle of even length which is connected to itself:



Suppose that μ contains m_k cycles of length k.

Case 1. k is odd. Then it follows from (3) that m_k must be even and the cycles of length k must be connected in pairs. There are $(m_k-1)!!=(m_k-1)(m_k-3)\cdots 3\cdot 1$ ways to pair the cycles and k ways to connect each of these $m/2$ pairs. This gives a total of

$$(m_k-1)!!k^{m_k/2}=f_k(m_k)$$

choices for the edges in d under the cycles of length k in γ_μ.

Case 2. k is even. By (2), we may choose any even number, $2s$, of cycles to be connected in pairs and pair them in $(2s-1)!!$ ways. By (2) again, each of the s pairs may be connected in k ways. By (4), each of the remaining cycles is connected to itself in a unique way. Summing over s, this gives a total of

$$\sum_{s=0}^{\lfloor m_k \rfloor}\binom{m_k}{2s}(2s-1)!!k^s=f_k(m_k)$$

choices for the edges in d under the cycles of length k in γ_μ.

Thus, the number of $d\in\Omega_{2m}^{\varnothing}$ such that $\gamma_\mu d=d$ is $\prod_k f_k(m_k)$, where $\mu=(1^{m_1}2^{m_2}\cdots)$ and the theorem follows from (2.9). □

The trace of the regular representation. Let Ω_m denote the set of all diagrams on m dots. By definition, these diagrams form a basis of the Brauer algebra $B_m(x)$. $B_m(x)$ acts on itself by both left multiplication and right miltiplication. The bitrace of these two actions is given by

$$\operatorname{btr}_m(a_1,a_2)=\sum_{d\in\Omega_m}a_1da_2\big|_d,$$

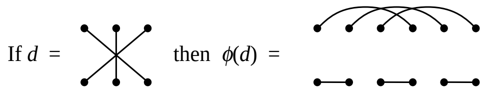
where $a_1,a_2\in B_m(x)$ and $a_1da_2\big|_d$ denotes the coefficient of the diagram d in the product a_1da_2. We shall refer to btr_m as the bitrace of the regular representation of the Brauer algebra.

(2.10) THEOREM. *Let c, b be diagrams in Ω_m. Then*

$$\mathrm{btr}_m(c, b) = \chi_{2m}^{\varnothing}(c \otimes b),$$

where $\chi_{2m}^{\varnothing}$ is given by (2.6).

PROOF. First we describe a simple bijection between Ω_m and $\Omega_{2m}^{\varnothing}$. Let $d \in \Omega_m$. Number the vertices of the top row of d, left to right, with $1, 2, \ldots, m$ and the vertices of the bottom row of d, right to left, with $m+1, m+2, \ldots, 2m$. Arranging the vertices of d into a single row of $2m$ vertices, in order, determines a diagram $\phi(d)$ of $\Omega_{2m}^{\varnothing}$. In some sense, ϕ takes a diagram, splits it down the horizontal centerline, puts a hinge on the right-hand side, and opens it up to be twice as long.



It is clear that ϕ is a bijection.

The following computation is now immediate:

$$\mathrm{btr}_m(c, b) = \sum_{d \in \Omega_m} cdb\big|_d = \sum_{\phi(d) \in \Omega_m} (c \otimes q)\phi(d)\big|_{\phi(d)} = \chi_{2m}^{\varnothing}(c \otimes q),$$

where q is the same diagram as b except *turned* over. The fact that $\chi_{2m}^{\varnothing}(c \otimes q) = \chi_{2m}^{\varnothing}(c \otimes b)$ follows from Theorem (2.5) and the fact that the cycle type of b is the same as the cycle type of q. □

(2.11) COROLLARY. *Let $\mu = (1^{m_1}2^{m_2}\cdots)$ be a partition of $m - 2k$, $0 \leqslant k \leqslant \lfloor m/2 \rfloor$, and let $\nu = (1^{n_1}2^{n_2}\cdots)$ be a partition of $m - 2\ell$, $0 \leqslant \ell \leqslant \lfloor m/2 \rfloor$. Let $\mu \cup \nu$ denote the partition given by $\mu \cup \nu = (1^{(m_1+n_1)}2^{(m_2+n_2)}\cdots)$. Then*

$$\mathrm{btr}_m(E^{\otimes k} \otimes \gamma_\mu, E^{\otimes \ell} \otimes \gamma_\nu) = x^{k+\ell} f(\mu \cup \nu),$$

where $f(\mu \cup \nu)$ is given by (2.7).

(2.12) COROLLARY. *Let $\mu = (1^{m_1}2^{m_2}\cdots)$ be a partition of $m - 2h$, $0 \leqslant h \leqslant \lfloor m/2 \rfloor$, and let $\hat{\mu} = (2^{m_2}\cdots)$. Then the trace Tr_m of the regular representation of $B_m(x)$ is given by*

$$\mathrm{Tr}_m(E^{\otimes h} \otimes \gamma_\mu) = \begin{cases} x^h f(\hat{\mu})(m + m_1 - 1)!!, & \text{if } m + m_1 \text{ is even,} \\ 0, & \text{if } m + m_1 \text{ is odd,} \end{cases}$$

where $f(\hat{\mu})$ is given by (2.7).

PROOF. This follows immediately from Corollary (2.11) by noting that $\mathrm{Tr}_m(a) = \mathrm{btr}(a, 1)$ for all $a \in B_m(x)$. □

The second relation for characters of the Brauer algebra. The algebra $B_m(x)$ is a split semisimple algebra over $\mathbb{C}(x)$ with irreducible representations labelled by the partitions in the set

$$\hat{B}_m = \{\lambda \vdash (f - 2k) \mid 0 \leqslant k \leqslant \lfloor m/2 \rfloor\}.$$

Except for a finite number of $\alpha \in \mathbb{Z}$, $B_m(\alpha)$ is a split semisimple algebra over $\mathbb{C}$ and has irreducible representations indexed by the elements of $\hat{B}_m$ (see [14]).

(2.13) Theorem. *Suppose that* $B_m(\alpha)$ *is semisimple and that, for each* $\lambda \in \hat{B}_m$, χ^λ *denotes the irreducible character of* $B_m(\alpha)$ *corresponding to* λ. *Let* $\mu = (1^{m_1}2^{m_2}\cdots)$ *be a partition of* $m-2k$, $0 \leqslant k \leqslant \lfloor m/2 \rfloor$, *and let* $\nu = (1^{n_1}2^{n_2}\cdots)$ *be a partition of* $m-2\ell$, $0 \leqslant \ell \leqslant \lfloor m/2 \rfloor$. *Let* $\mu \cup \nu$ *denote the partition given by* $\mu \cup \nu = (1^{(m_1+n_1)}2^{(m_2+n_2)}\cdots)$. *Then*

$$\sum_{\lambda \in B_m} \chi^\lambda(E^{\otimes k} \otimes \gamma_\mu)\chi^\lambda(E^{\otimes \ell} \otimes \gamma_\nu) = \alpha^{k+\ell} f(\mu \cup \nu),$$

where $f(\mu \cup \nu)$ *is given by* (2.7).

Proof. This follows from Corollary (2.11) in exactly the same way that Theorem (A.4) follows from Theorem (A.3) in the Appendix. □

Remark. Diaconis and Shahshahani [5] have applied this result in order to study the eigenvalues of random orthogonal and symplectic matrices.

Hermite polynomials. If k is even, then

$$\text{(2.14)} \qquad e^{(k/2)t^2} = \sum_{m \geqslant 0} \frac{k^m t_k^{2m}}{2^m m!} = \sum_{m \geqslant 0} \frac{k^m t^{2m}(2m-1)!!}{(2m)!} = \sum_{m \geqslant 0} f_k(m) \frac{t^m}{m!},$$

and, if k is odd,

$$\text{(2.15)} \qquad e^{(k/2)t^2+t} = \sum_{m,s \geqslant 0} \frac{k^s t^{2s+m}}{2^s m! s!} = \sum_{m,s \geqslant 0} \frac{k^s t^m (2s-1)!!}{(m-2s)!(2s)!}$$
$$= \sum_{m,s \geqslant 0} k^s (2s-1)!! \binom{m}{2s} \frac{t^m}{m!} = \sum_{m \geqslant 0} f_k(m) \frac{t^m}{m!},$$

where $f_k(m)$ is as defined in (2.7).

The Hermite polynomials $H_m(x)$ are given by the following generating function:

$$e^{2xt-t^2} = \sum_{m \geqslant 0} H_m(x) \frac{t^m}{m!}.$$

If we make a change of variables $t \to t\sqrt{-k/2}$ in this generating function we obtain that

$$e^{xt\sqrt{-2k}+(k/2)t^2} = \sum_{m \geqslant 0} (-k/2)^{m/2} H_m(x) \frac{t^m}{m!}.$$

Setting $x = -\sqrt{-1/2k}$,

$$\text{(2.16)} \qquad e^{(k/2)t^2+t} = \sum_{m \geqslant 0} (-k/2)^{m/2} H_m(\sqrt{-1/2k}) \frac{t^m}{m!},$$

and, setting $x = 0$,

$$\text{(2.17)} \qquad e^{(k/2k)t^2} = \sum_{m \geqslant 0} (-k/2)^{m/2} H_m(0) \frac{t^m}{m!}.$$

It follows by comparing (2.14), (2.15) and (2.16), (2.17) that:

(2.18) Proposition. *The values* $f_k(m)$ *defined in* (2.7) *satisfy*

$$f_k(m) = (-k/2)^{m/2} H_m(x), \qquad \text{where } x = \begin{cases} 0, & \text{if } k \text{ is odd,} \\ \sqrt{-1/2k}, & \text{if } k \text{ is even,} \end{cases}$$

and $H_m(x)$ *is the mth Hermite polynomial.*

This means that several important traces on the Brauer algebra are given by products of Hermite polynomials. It is not known whether this is merely coincidence or whether there is a good algebraic reason why they should be connected. Combinatorially, it is not too surprising that the Hermite polynomials appear as it is well known [13] that the Hermite polynomials can be given a combinatorial interpretation in terms of k-matchings in complete graphs. Since the m-diagrams which form a basis of the Brauer algebra can be viewed as matchings in the complete graph on $2m$ points, it is not unreasonable that they should be related to the Hermite polynomials.

3. An Application to Weyl Group Symmetric Functions

Type A. The Weyl group of type A_{n-1} is the symmetric group S_n. Define $\Lambda = \mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}]^{S_n}$, i.e. the set of S_n-invariant polynomials in $x_1, x_2, \ldots, x_n$, where S_n acts by permuting the n variables.

For each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, define

$$s_\lambda = \frac{\det(x_i^{\lambda_j+n-j})}{\det(x_i^{n-j})}.$$

The Schur functions s_λ, form a basis of Λ [7, Ch. I (3.2)]. Define an inner product on Λ by defining

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}.$$

For each integer $r > 0$, define $p_r = \sum_{i=1}^n x_i^r$, and for a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ define $p_\mu = p_{\mu_1} p_{\mu_2} \cdots p_{\mu_\ell}$. One has the following standard results.

(Frobenius formula Type A, [7, Ch. I (7.8)]) Let $\mu \vdash m$. Then

$$p_\mu = \sum_{\lambda \vdash m} \chi_S^\lambda(\mu) s_\lambda, \tag{3.1}$$

where χ_S^λ is the irreducible character of S_m corresponding to $\lambda \vdash m$.

([7, Ch. I (4.7)]) Let μ and ν be partitions. Then (assuming n large)

$$\langle p_\mu, p_\nu \rangle = \delta_{\mu\nu} \mu?, \tag{3.2}$$

where $\mu?$ is the constant given by $\mu? = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$, if $\mu = (1^{m_1} 2^{m_2} \cdots)$.

Type B. The Weyl group of type B is the hyperoctahedral group, H_n, of signed permutation matrices. H_n can be given by generators $s_1, s_2, \ldots, s_n$ and relations

$$\begin{aligned} s_i^2 &= 1, && 1 \le i \le n, \\ s_i s_j &= s_j s_i, && |i-j| > 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, && 1 \le i \le n-2, \\ s_{n-1} s_n s_{n-1} s_n &= s_n s_{n-1} s_n s_{n-1}. \end{aligned}$$

Let $x_1, x_2, \ldots, x_n$ be commuting variables. Define an action of H_n on $\mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}]$ by

$$s_i x_j = \begin{cases} x_{i+1}, & \text{if } j = i, \\ x_i, & \text{if } j = i+1, \\ x_j, & \text{otherwise,} \end{cases} \qquad \text{for } 1 \le i \le n-1,$$

and

$$s_n x_j = \begin{cases} x_n^{-1}, & \text{if } j = n, \\ x_j, & \text{otherwise.} \end{cases}$$

Define $\Lambda_b = \mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}]^{H_n}$ so that Λ_b is the set of H_n-invarinat Laurent polynomials in $x_1, x_2, \ldots, x_n$.

For each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ define

$$sb_\lambda = \frac{\det(x_i^{\lambda_j+n-j+\frac{1}{2}} - x_i^{-(\lambda_j+n-j+\frac{1}{2})})}{\det(x_i^{n-j+\frac{1}{2}} - x_i^{-(n-j+\frac{1}{2})})}. \tag{3.3}$$

The sb_λ form a basis of Λ_b. (A general proof, for any Weyl group, is given in [2, Ch. VI, §3.3 Prop. 2]. This proof is essentially the same as that given for type A in [7, Ch. I (3.2)].) Define an inner product on Λ_b by defining

$$\langle sb_\lambda, sb_\mu \rangle_b = \delta_{\lambda\mu}.$$

For each integer $r > 0$, define $pb_r = 1 + \sum_{i=1}^n x_i^r + x_i^{-r}$, and for a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ define $pb_\mu = pb_{\mu_1} pb_{\mu_2} \cdots pb_{\mu_\ell}$. The following analogue of (3.1) is proved in [9]. We shall give a new, almost completely combinatorial, proof in Section 5:

(3.4) THEOREM (Frobenius formula Type B). *Suppose that $n \gg m$. Let $\mu \in \hat{B}_m$ and suppose that $\mu \vdash m - 2k$. Then*

$$x^k_{pb_\mu} = \sum_{\lambda \in \hat{B}_m} \chi^\lambda(E^{\otimes k} \otimes \gamma_\mu) sb_\lambda,$$

where χ^λ is the irreducible character of $B_m(x)$ corresponding to $\lambda \in \hat{B}_m$ and $E^{\otimes k} \otimes \gamma_\mu$ is as in (2.4).

(3.5) THEOREM. *Let μ and ν be partitions and let $k = |(|\mu| - |\nu|)|$. Suppose that $\mu = (1^{m_1} 2^{m_2} \cdots)$ and $\nu = (1^{n_1} 2^{n_2} \cdots)$ and define $\mu \cup \nu = (1^{m_1+n_1} 2^{m_2+n_2} \cdots)$. Then* (*assuming n large*)

$$\langle pb_\mu, pb_\nu \rangle_b = \begin{cases} 0, & \textit{if } k \textit{ is odd}, \\ f(\mu \cup \nu), & \textit{otherwise}, \end{cases}$$

where $f(\mu \cup \nu)$ is given as in (2.7).

PROOF. Let us suppose, for convenience, that $|\mu| \geq |\nu|$. Let $k = |\mu| - |\nu|$ and let $m = |\mu|$. Then, by (3.4),

$$pb_\mu = \sum_{\lambda \in \hat{B}_m} \chi^\lambda(\gamma_\mu) sb_\lambda \qquad \text{and} \qquad x^{k/2} pb_\nu = \sum_{\lambda \in \hat{B}_m} \chi^\lambda(E^{\otimes(k/2)} \otimes \gamma_\nu) sb_\lambda.$$

Thus,

$$\langle pb_\mu, x^{k/2} pb_\nu \rangle = \sum_{\lambda \in \hat{B}_m} \chi^\lambda(\gamma_\mu) \chi^\lambda(E^{\otimes(k/2)} \otimes \gamma_\nu).$$

The result now follows from (2.13). □

Type C. The Weyl group of type C is the hyperoctahedral group, H_n, just as in the case of type B. Define $\Lambda_c = \mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}]^{H_n}$, where the action of H_n on

$\mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}]$ is the same as in the case of type B. For each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ define

$$sc_\lambda = \frac{\det(x_i^{\lambda_j+n-j+1} - x_i^{-(\lambda_j+n-j+1)})}{\det(x_i^{n-j+1} - x_i^{-(n-j+1)})}.$$

The sc_λ form a basis of Λ_c Define an inner product on Λ_c by defining

$$\{sc_\lambda, sc_\mu\rangle_c = \delta_{\lambda\mu}.$$

For each integer $r > 0$, define $pc_r = \sum_{i=1}^m x_i^r + x_i^{-r}$ and, for a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$, define $pc_\mu = pr_{\mu_1} pc_{\mu_2} \cdots pc_{\mu_\ell}$.

The following analogue of (3.1) is proved in [9].

(3.6) Theorem (Frobenius formula type C). *Suppose that $n \gg m$. Let $\mu \in \hat{B}_m$ and suppose that $\mu \vdash m - 2k$. Then*

$$x^k(-1)^{|\mu| - \ell(\mu)} pc_\mu = \sum_{\lambda \in \hat{B}_m} \chi^{\lambda'}(E^{\otimes k} \otimes \gamma_\mu) sc_\lambda, \tag{3.7}$$

where χ^λ is the irreducible character of $B_m(x)$ corresponding to $\lambda \in \hat{B}_m$, λ' denotes the conjugate partition to λ and $E^{\otimes k} \otimes \gamma_\mu$ is as in (2.4).

(3.8) Theorem. *Let μ and ν be partitions and let $k = |(|\mu| - |\nu|)|$. Suppose that $\mu = (1^{m_1} 2^{m_2} \cdots)$ and $\nu = (1^{n_1} 2^{n_2} \cdots)$ and define $\mu \cup \nu = (1^{m_1+n_1} 2^{m_2+n_2} \cdots)$. Then* (*assuming n large*)

$$\langle pc_\mu, pc_\nu \rangle_c = \begin{cases} 0, & \text{if } k \text{ is odd,} \\ (-1)^{|\mu \cup \nu| - \ell(\mu \cup \nu)} f(\mu \cup \nu), & \text{otherwise,} \end{cases}$$

where $f(\mu \cup \nu)$ is given as in (2.7).

Proof. The proof is exactly analogous to the proof of Theorem (3.5) for type B. □

Type D. A similar result follows in exactly the same way for type D, but there are some annoying special cases which must be considered. We shall not unwind these here: the proof of an analogue of the Theorem is exactly the same, and we refer the reader to [9] for the appropriate definitions and analogue of (3.1).

4. A 'Second Orthogonality' via Tools from Differential Posets

The following approach and a sketch of the proof of Theorem (4.12) below was given by R. Stanley after a lecture by the author at the LABRI at the University of Bordeaux I in which the results of Sections 2 and 3 were presented. The combinatorial description of the characters of the Brauer algebras was given in [9]. In Section 2 we derived the 'second orthogonality' for the irreducible characters of the Brauer algebra by appealing to facts from representation theory. Stanley's approach, given below, proves this orthogonality directly from the combinatorial interpretation of the irreducible characters. In this paper we are turning the picture upside down and in Section 5 the result, Theorem (4.12), from this approach of Stanley's will be combined with the results from Sections 2 to give a new derivation of the irreducible characters of the Brauer algebras. Thus, for the moment, we do not know that the numbers $\eta^\lambda(\mu)$ in Proposition (4.11) actually are the irreducible characters of the Brauer algebras.

Let Λ denote the ring of symmetric functions. We shall view Λ as the polynomial ring

$$\Lambda = \mathbb{Q}[p_1, p_2, \ldots], \tag{4.1}$$

where p_k denotes the kth power sum symmetric function. Given a partition $\mu = (1^{m_1}2^{m_2}\cdots)$, let $\mu? = \prod_i i^{m_i} m_i!$ and let $p_\mu = p_1^{m_1} p_2^{m_2} \cdots$. For each pair of partitions λ, μ, let $\chi_S^\lambda(\mu)$ denote the irreducible character of the symmetric group corresponding to λ evaluated at a conjugacy class labelled by μ. The Schur functions are given by

$$s_\lambda = \sum_{\mu \vdash m} \frac{\chi_S^\lambda(\mu)}{\mu?} p_\mu,$$

and form a basis of Λ. There is a standard inner project on Λ which satisfies

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}, \qquad \langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu}\mu?. \tag{4.2}$$

Viewing elements of Λ as elements of the polynomial ring (4.1) [7, I §5 Ex. 3c] shows that the operator $k\,\frac{\partial}{\partial p_k}$ is the adjoint of multiplication by p_k; i.e., for any $f, g \in \Lambda$,

$$\langle p_k f, g \rangle = \left\langle f, k\frac{\partial}{\partial p_k} g \right\rangle. \tag{4.3}$$

We have that

$$k\frac{\partial}{\partial p_k} p_k - p_k k \frac{\partial}{\partial p_k} = k \cdot 1, \tag{4.4}$$

as operators on Λ. Define a generating function as follows:

$$GF = \left[\prod_{k \text{ odd}} e^{\left(p_k + k\frac{\partial}{\partial p_k}\right)t_k} \prod_{k \text{ even}} e^{\left(p_k + 1 + k\frac{\partial}{\partial p_k}\right)t_k} \right] \cdot 1.$$

We view the product in square brackets as an operator acting on $1 \in \Lambda$. GF is an element of $\mathbb{Q}[t_1, t_2, \ldots] \otimes \Lambda$. For each pair of partitions λ, μ, define values $\eta^\lambda(\mu)$ by

$$GF = \sum_\mu \frac{t^\mu}{\mu!} \sum_\lambda \eta^\lambda(\mu) s_\lambda, \tag{4.5}$$

where $t^\mu/\mu!$ is defined by

$$\frac{t^\mu}{\mu!} = \frac{t_1^{m_1}}{m_1!}\frac{t_2^{m_2}}{m_2!}\cdots \qquad \text{if } \mu = (1^{m_1}2^{m_2}\cdots).$$

We shall need the following identity from Stanley's work on differential posets, [11, Cor. 2.6a]:

$$e^{\left(p_k + k\frac{\partial}{\partial p_k}\right)t_k} = e^{\frac{1}{2}kt_k^2 + p_k t_k} e^{k\frac{\partial}{\partial p_k}t_k}, \tag{4.6}$$

$$e^{\left(p_k + 1 + k\frac{\partial}{\partial p_k}\right)t_k} = e^{\frac{1}{2}kt_k^2 + t_k + p_k t_k} e^{k\frac{\partial}{\partial p_k}t_k}.$$

(4.7) Theorem. *Let μ be a partition, let $f(\mu)$ be given by* (2.7) *and let $\eta^\varnothing(\mu)$ be as given in* (4.5). *Then*

$$\eta^\varnothing(\mu) = f(\mu).$$

Proof. It follows from Stanley's identity (4.6) that

$$GF = \left[\prod_{k \text{ odd}} e^{\frac{1}{2}kt_k^2 + p_k t_k} e^{k\frac{\partial}{\partial p_k}t_k} \prod_{k \text{ even}} e^{\frac{1}{2}kt_k^2 + t_k + p_k t_k} e^{k\frac{\partial}{\partial p_k}t_k} \right] \cdot 1.$$

Note that the operator $k\frac{\partial}{\partial p_k}$ commutes with all of the operators p_j and $j\frac{\partial}{\partial p_j}$ except p_k. Thus, we may write

$$GF = \left[\prod_{k\text{ odd}} \mathrm{e}^{\frac{1}{2}kt_k^2 + p_k t_k} \prod_{k\text{ even}} \mathrm{e}^{\frac{1}{2}kt_k^2 - t_k + p_k t_k} \prod_k \mathrm{e}^{k\frac{\partial}{\partial p_k}t_k}\right] \cdot 1.$$

Since $k\frac{\partial}{\partial p_k} \cdot 1 = 0$ for all k,

$$\mathrm{e}^{k\frac{\partial}{\partial p_k}t_k} \cdot 1 = 1$$

and we obtain that

$$\begin{aligned} GF &= \left[\prod_{k\text{ odd}} \mathrm{e}^{\frac{1}{2}kt_k^2 + p_k t_k} \prod_{k\text{ even}} \mathrm{e}^{\frac{1}{2}kt_k^2 + t_k + p_k t_k}\right] \cdot 1 \\ &= \left[\prod_{k\text{ odd}} \mathrm{e}^{\frac{1}{2}kt_k^2} \prod_{k\text{ even}} \mathrm{e}^{\frac{1}{2}kt_k^2 + t_k} \prod_k \mathrm{e}^{p_k t_k}\right] \cdot 1. \end{aligned}$$

It is clear from the action of p_k on Λ that the coefficient of $s_\varnothing = 1$ in $\mathrm{e}^{p_k t_k} \cdot 1$ is 1. Thus,

$$GF\Big|_{s_\varnothing} = \prod_{k\text{ odd}} \mathrm{e}^{\frac{1}{2}kt_k^2} \prod_{k\text{ even}} \mathrm{e}^{t_k}.$$

Thus, if $\mu = (1^{m_1}2^{m_2}\cdots)$,

$$\eta^\varnothing(\mu) = GF\Big|_{\frac{t^\mu}{\mu!}s_\varnothing} = \prod_{k\text{ odd}} \mathrm{e}^{\frac{1}{2}kt_k^2}\Big|_{\frac{t^{m_k}}{m_k!}} \prod_{k\text{ even}} \mathrm{e}^{\frac{1}{2}kt_k^2 + t_k}\Big|_{\frac{t^{m_k}}{m_k!}}.$$

The theorem now follows from (4.5). □

A combinatorial description of $\eta^\lambda(\mu)$. A border strip is a connected skew diagram that does not contain any 2×2 block of boxes. It is shown in [7, I §3 Ex. 11] that

$$p_k s_\lambda = \sum_{\nu\supseteq\lambda} (-1)^{r(\nu/\lambda)-1} s_\nu, \tag{4.8}$$

where the sum is over all partitions $\nu \supseteq \lambda$ such that ν/λ is a border strip of length k and $r(\nu/\lambda)$ is the number of rows in ν/λ. It follows easily from (4.2) and (4.3) that

$$k\frac{\partial}{\partial p_k} s_\lambda = \sum_{\mu\subseteq\lambda} (-1)^{r(\lambda/\mu)-1} s_\mu, \tag{4.9}$$

where the sum is over all partitions $\mu \subseteq \lambda$ such that λ/ν is a border strip of length k and $r(\lambda/\nu)$ is the number of rows in λ/ν.

Given partitions λ and μ, we shall say that λ differs from μ by a border strip if either $\lambda \subseteq \mu$ and λ/μ is a border strip or $\mu \subseteq \lambda$ and μ/λ is a border strip. We shall denote the border strip determined by λ and μ by $\mathrm{bs}(\lambda, \mu)$. The length of a border strip is the total number of boxes in the border strip. The weight of a border strip $\mathrm{bs}(\lambda, \mu)$ is

$$\mathrm{wt}(\mathrm{bs}(\lambda, \mu)) = (-1)^{k-1},$$

where k is the number of rows occupied by $\mathrm{bs}(\lambda, \mu)$. We shall make the convention that $\mathrm{wt}(\mathrm{bs}(\lambda, \lambda)) = 1$.

Let λ and $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ be partitions. A μ-up–down border strip tableau of shape λ is a sequence of partitions

$$T = (\varnothing = \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(k)} = \lambda)$$

such that for each $1 \leqslant j \leqslant k$ either:
(1) $\lambda^{(j)}$ differs from $\lambda^{(j-1)}$ by a border strip of length μ_j, or
(2) μ_j is even and $\lambda^{(j)} = \lambda^{(j-1)}$).
define the weight of the tableau T to be

$$\text{et}(T) = \prod_{j=1}^{k} \text{wt}(\text{bs}(\lambda^{(j)}, \lambda^{(j-1)})). \tag{4.10}$$

The following proposition follows from (4.8) and (4.9) and the definition of the η^λ in (4.5).

(4.11) PROPOSITION. *Let μ be a partition of m and let λ be a partition of $m - 2\ell$. Then*

$$\eta^\lambda(\mu) = \sum_T \text{wt}(T),$$

where the sum is over all μ-up–down border strip tableaux of shape λ and wt(T) *is as given in* (4.10).

A 'second orthogonality relation' for $\eta^\lambda(\mu)$

(4.12) THEOREM. *Let μ and ν be partitions of m. Suppose that $\mu = (1^{m_1}2^{m_2}\cdots)$ and $\nu = (1^{n_1}2^{n_2}\cdots)$ and define $\mu \cup \nu = (1^{m_1+n_1}2^{m_2+n_2}\cdots)$. Then*

$$\sum_{\lambda \vdash m-2k} \eta^\lambda(\mu)\eta^\lambda(\nu) = \eta^{\varnothing}(\mu \cup \nu) = f(\mu \cup \nu),$$

where the sum is over all λ partitions of $m - 2k$, $0 \leqslant k \leqslant \lfloor m/2 \rfloor$, and $f(\mu \cup \nu)$ is given by (2.7).

PROOF. The left-hand side can be written as

$$\sum_{\lambda \vdash m-2k} \eta^\lambda(\mu)\eta^\lambda(\nu) = \sum_{\lambda \vdash m-2k} \sum_{(T,S)} \text{wt}(T)\text{wt}(S),$$

where the inner sum is over all pairs (T, S) such that

$$T = (\varnothing = \rho^{(0)}, \rho^{(1)}, \ldots, \rho^{(r)} = \lambda) \qquad \text{and} \qquad S = (\varnothing = \tau^{(0)}, \tau^{(1)}, \ldots, \tau^{(s)} = \lambda),$$

are μ and ν-up–down border strip tableaux of shape λ respectively. Given a pair (T, S), the tableau

$$T * S = (\varnothing = \rho^{(0)}, \ldots, \rho^{(r)} = \lambda = \tau^{(s)}, \tau^{(s-1)}, \ldots, \tau^{(0)} = \varnothing),$$

is a $\mu \cup \nu$ up–down border strip tableau of shape $\varnothing$. It is clear from the definition of the weight of an up–down border strip tableaux that wt(T)wt(S) = wt$(T * S)$. So

$$\sum_{\lambda \vdash m-2k} \eta^\lambda(\mu)\eta^\lambda(\nu) = \sum_{\lambda \vdash m-2k} \sum_{(T,S)} \text{wt}(T)\text{wt}(S) = \sum_{T*S} \text{wt}(T*S) = \eta^{\varnothing}(\mu \cup \nu).$$

The second equality now follows from Theorem (4.7). □

5. THE IRREDUCIBLE CHARACTERS OF THE BRAUER ALGEBRA

The goal of this is section to show that one can use the 'second orthogonality relations' to give a proof of the Frobenius formula and a derivation of the characters of the Brauer algebras which is elementary in the sense that it does not use
(1) the fundamental theorem of invariant theory for the orthogonal group, or

(2) the Weyl character formula for the orthogonal group.
We will, however, make the following assumption:
(3) for a given m, the Brauer algebra $B_m(n)$ is semisimple for some $n \in \mathbb{C}$ and the number of irreducible components is the number of partitions in the set $\hat{B}_m = \{\lambda \vdash m - 2k, 0 \leqslant k \leqslant \lfloor m/2 \rfloor\}$.

REMARK. To my knowledge, it is still not known how to prove (3) without using (1). It is actually clear that (3) is true from Brauer's derivation of the Brauer algebra [3], which uses (1). Of course, much stronger results concerning the semisimplicity of the Brauer algebra are known; see [14]. A previous proof [9] of the Frobenius formula for the Brauer algebra used both (1) and (2) in a crucial way. It would be nice to be able to remove the second part of the assumption given in (3) above. For this, it would be sufficient to prove combinatorially that the center of the Brauer algebra has dimension greater than or equal to $\mathrm{Card}(\hat{B}_m)$.

We need to collect some standard facts from representation theory and symmetric functions in the context of this special case. Although the general results appear in some form in the literature [1, Ch. VIII; 2, Ch. VI; 7], we shall include the proofs for completeness, since the proofs are short and it is hard to give good references for these special cases.

(5.1) LEMMA. *If there exists $n \in \mathbb{C}$ such that $B_m(n)$ is semisimple, then, for all but a finite number of $n \in \mathbb{C}$, $B_m(n)$ is semisimple.*

PROOF. Let Tr denote the trace of the regular representation of $B_m(n)$. Let Ω_m be the basis of $B_m(n)$ given by the m-diagrams. Defilne the Gram matrix $G(n) = (\mathrm{Tr}(d_i d_j))$ with rows and columns indexed by the elements $d_i \in \Omega_m$. $B_m(n)$ is semisimple if $\det(G) \neq 0$. Let Tr_x denote the trace of the regular representation of $B_m(x)$. Let $G(x) = (\mathrm{Tr}_x(d_i d_j))$. Then, since $\mathrm{Tr}_x(d_i d_j)$ is a polynomial in x for all pairs i, j, $\det(G(x))$ is polynomial in x. By assumption, there exists an $n \in \mathbb{C}$ such that $B_m(n)$ is semisimple. So $\det(G(n)) \neq 0$ for some $n \in \mathbb{C}$. Thus $\det(G(x)) \neq 0$ and, consequently, $\det(G(n)) \neq 0$ for all but a finite number of $n \in \mathbb{C}$. □

Weight space representations. Let $I = \{-n, -(n-1), \ldots, -2, -1, 0, 1, 2, \ldots, n-1, nS\}$. Let $\{v_i \mid i \in I\}$ be a set of independent non-commuting variables. Define V to be the vector space over $\mathbb{C}$ with basis $\{v_i \mid i \in I\}$, and define

$$V^{\otimes m} = \mathbb{C} - \mathrm{span}\{v_{i_1} v_{i_2} \cdots v_{i_m} \mid i_k \in I\},$$

so that the words (simple tensors) $v_{i_1} v_{i_2} \cdots v_{i_m}$ are a basis of $V^{\otimes m}$.

Let $x_1, x_2, \ldots, x_n$ be commuting, independent variables. Define $x_0 = 1$ and $x_{-i} = x_i^{-1}$ for $i = 1, 2, \ldots, n$, so that x_i is defined for each $i \in I$. Define the *weight* of each word $v_{i_1} \cdots v_{i_m}$ of $V^{\otimes m}$ to be

$$\mathrm{wt}(v_{i_1} \cdots v_{i_m}) = x_{i_1} \cdots x_{i_m}.$$

Note that the weight of a word is always of the form $x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, where $a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$. For each sequence $a \in \mathbb{Z}^n$, define

$$(V^{\otimes m})_a = \mathbb{C} - \mathrm{span}\{v_{i_1} \cdots v_{i_m} \mid \mathrm{wt}(v_{i_1} \cdots v_{i_m}) = x^a\}.$$

For $0 \leqslant k \leqslant m-1$, define an action of the generators G_k and E_k of $B_m(2n+1)$, defined in (2.2), on $V^{\otimes m}$ by

$$(5.2) \qquad \begin{aligned} (v_{i_1}v_{i_2}\cdots v_{i_m})G_k &= v_{i_1}\cdots v_{i_{k-1}}v_{i_{k+1}}v_{i_k}v_{i_{k+2}}\cdots v_{i_m}, \\ (v_{i_1}v_{i_2}\cdots v_{i_m})E_k &= \delta_{i_{k_j}-i_{k+1}}\sum_{j\in I} v_{i_j}\cdots v_{i_{k-1}}v_j v_{-j}v_{i_{k+2}}\cdots v_{i_m}. \end{aligned}$$

By writing out explicitly the action of a general m-diagram one checks easily that the action defined in (5.2) extends to a well-defined action of $B_m(2n+1)$ on $V^{\otimes m}$. Since the action of the Brauer algebra on $V^{\otimes m}$ does not change the weights of the words, $(V^{\otimes m})_a$ is always a $B_m(n)$ submodule of $V^{\otimes m}$.

Let H_n denote the hyperoctahedral group of $n \times n$ signed permutation matrices defined by generators and relations in Section 3. Define an action of H_n on the variables v_i, $i \in I$, by

$$s_i v_j = \begin{cases} v_{\pm(i+1)}, & \text{if } j = \pm i, \\ v_{\pm i}, & \text{if } j = \pm(i+1), \\ v_j, & \text{otherwise,} \end{cases} \qquad \text{for } 1 \leqslant i \leqslant n-1,$$

and

$$s_n v_j = \begin{cases} v_{\mp n}, & \text{if } j = \pm n, \\ v_j, & \text{otherwise.} \end{cases}$$

and define an action of W_n on $V^{\otimes m}$ by $w(v_{i_1}\cdots v_{i_m}) = v_{w(i_1)}\cdots v_{w(i_m)}$. Define an action of H_n on monomials $x_{i_1}\cdots x_{i_m}$ and on sequences $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ by requiring that, for all words $v_{i_1}\cdots v_{i_m}$ and $w \in H_n$,

(5.3) If $\mathrm{wt}(v_{i_1}\cdots v_{i_m}) = x_1^{a_1}\cdots x_n^{a_n} = x^a$, then $\mathrm{wt}(w(v_{i_1}\cdots v_{i_m})) = w(x^a) = x^{wa}$.

(5.4) PROPOSITION. *For each $a \in \mathbb{Z}^n$, define $\lambda(a)$ to be the partition determined by rearranging the sequence $(|a_1|, |a_2|, \ldots, |a_n|)$ into decreasing order. Then*

$$(V^{\otimes m})_a \simeq (V^{\otimes m})_{\lambda(a)},$$

as $B_m(2n+1)$ modules.

PROOF. Let $w \in H_n$ and let $a \in \mathbb{Z}^n$. We first show that the action of w gives a $B_m(2n+1)$ module isomorphism

$$w\colon\ (V^{\otimes m})_a \to (V^{\otimes m})_{wa}.$$

The fact that w is a vector space isomorphism from $(V^{\otimes m})_a$ to $(V^{\otimes m})_{wa}$ follows from (5.3) and the fact that w is invertible. To show that w is a $B_m(2n+1)$-module isomorphism we must show that w commutes with the action of $B_m(2n+1)$ on $V^{\otimes m}$. One checks this for the generators G_i, E_i, $1 \leqslant i \leqslant m-1$, of $B_m(2n+1)$, with computations determined by

$$w((v_{i_1}v_{1_2})G) = v_{w(i_2)}v_{w(i_1)} = (w(v_{i_1}v_{i_2}))G,$$

and

$$w((v_{i_1}v_{i_2})E) = \delta_{i_2 i_2}\sum_{i\in I} v_{w(i)}v_{w(-i)} = \delta_{w(i_1)w(i_2)}\sum_{i\in I} v_i v_{-1} = (w(v_{i_1}v_{i_2}))E,$$

where G and E denote the generators of $B_2(2n+1)$ acting on $V^{\otimes 2}$. The proposition now follows by choosing an appropriate element $w \in H_n$ such that $wa = \lambda$. □

Symmetric functions, type B. The Weyl group of type B is the hyperoctahedral group, H_n, of signed permutation matrices defined by generators and relations in Section 3. Let $x_1, x_2, \dots, x_n$ be commuting variables. There is an action of H_n on $\mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_n, x_n^{-1}]$ given by extending linearly the action of H_n on monomials given in (5.3). Define $\Lambda_b = \mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_n, x_n^{-1}]^{H_n}$ so that Λ_b is the set of H_n-invariant Laurent polynomials in $x_1, x_2, \dots, x_n$.

For each $\alpha \in \mathbb{Z}^n$, let x^α denote $x_1^{\alpha_1} \cdots X_n^{\alpha_n}$ and define an action of H_n on $\mathbb{Z}^n$, such that $wx^\lambda = x^{w\lambda}$ for all $w \in H_n$ and all $\lambda \in \mathbb{Z}^n$. For each partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ define the monomial symmetric function mb_λ by

$$mb_\lambda = \sum_{a \in H_n\lambda} x^a,$$

where $H_n\lambda$ is the H_n orbit of λ in $\mathbb{Z}^n$. Also recall from (3.3) that, for each partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, one defines

$$sb_\lambda = \frac{\det(x_i^{\lambda_j+n-j+\frac{1}{2}} - x_i^{-(\lambda_j+n-j+\frac{1}{2})})}{\det(x_i^{n-j+\frac{1}{2}} - x_i^{-(n-j+\frac{1}{2})})}.$$

and that the sb_λ form a basis of the symmetric function ring Λ_b.

For each $w \in H_n$, let $\varepsilon(w) = \det(w)$ denote the *sign* of w. Let $\mathscr{A}$ be the vector space of alternating polynomials in $\mathbb{Z}[x_1^{\frac{1}{2}}, x_2^{-\frac{1}{2}}, x_2^{\frac{1}{2}}, x_2^{-\frac{1}{2}}, \dots, x_n^{\frac{1}{2}}, x_n^{-\frac{1}{2}}]$, i.e. polynomials f such that $wf = \varepsilon(w)f$ for all $w \in H_n$. Let $\rho = (n-\frac{1}{2}, n-\frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2})$. Then, since the elements

$$b_{\nu+\rho} = \sum_{w \in H_n} \varepsilon(w) x^{w(\nu+\rho)},$$

where λ is a partition, have no common terms, they are linearly independent elements in $\mathscr{A}$. Furthermore, it is not difficult to see that

$$sb_\nu = \frac{b_{\nu+\rho}}{b_\rho}. \tag{5.5}$$

(5.6) LEMMA. *The values $Kb^{-1}_{\lambda\mu}$ determined by*

$$mb_\mu = \sum_\lambda Kb^{-1}_{\mu\lambda} sb_\lambda$$

are all integers.

PROOF. Multiplying both sides of the defining relation for the values $Kb^{-1}_{\lambda\mu}$ by the alternating polynomial b_ρ and using (5.5) gives

$$mb_\mu b_\rho = \sum_\lambda Kb^{-1}_{\mu\lambda} b_{\lambda+\rho},$$

which is a relation in vector space $\mathscr{A}$ of alternating polynomials in $\mathbb{Z}[x_{11}^{\frac{1}{2}}, x_1^{-\frac{1}{2}}, x_2^{\frac{1}{2}}, x_2^{-\frac{1}{2}}, \dots, x_n^{\frac{1}{2}}, x_n^{-\frac{1}{2}}]$. It follows that $Kb^{-1}_{\lambda\mu} = mb_\mu b_\rho|_{x^{\lambda+\rho}}$. Thus $Kb^{-1}_{\lambda\mu}$ is an integer. □

A completely combinatorial proof of the above lemma, which actually determines a combinatorial rule for computing the coefficients $Kb^{-1}_{\lambda\mu}$, can be given in exactly the same way that Theorem (6.8) is proved in [9].

Recall the definition of the *power symmetric functions* pb_λ from Section 3. Let $\hat{B}_m$ be as defined in the beginning of this section and, for each pair of partitions $\lambda, \mu \in \hat{B}_m$ let $\eta^\lambda(\mu)$ denote the numbers defined in (4.5), given by

$$\eta^\lambda(\mu) = \sum_T \mathrm{wt}(T),$$

where the sum is over all μ-up–down border strip tableaux T of shape λ.

(5.7) LEMMA. *For each* $\mu \in \hat{B}_m$

$$pb_\mu = \sum_{\lambda \in \hat{B}_m} \eta^\lambda(\mu) sb_\lambda.$$

PROOF. This follows by induction from the identity

$$pb_r sb_\lambda = \sum_\mu \mathrm{wt}(bs(\lambda, \mu)) sb_\mu,$$

where the sum is over all partitions μ such that μ differs from λ by a border strip; we also include the case $\mu = \lambda$ when r is even. This identity is proved in exactly the same fashion as [7, I §3 Ex. 11]. A complete derivation of this formula appears in [9, Theorem (6.8)]. □

(5.8) PROPOSITION. *For each* $a \in \mathbb{Z}^n$, *let* β_a: $B_m(2n+1) \to \mathbb{C}$ *denote the character of* $B_m(2n+1)$ *acting on the weight space representation* $(V^{\otimes m})_a$. *Let* pb_μ *denote the power symmetric function as defined in Section* 3 *and let* mb_μ *denote the monomial symmetric function defined above. Then, for each* $\mu \vdash m - 2k \in \hat{B}_m$,

$$(2n+1)^k pb_\mu = \sum_\lambda \beta_\lambda(E^{\otimes k} \otimes \gamma_\mu) mb_\lambda,$$

where $E^{\otimes k} \otimes \gamma_\mu$ *is as defined as in* (2.4).

PROOF. Define a weighted trace of $B_m(2n+1)$ acting on $V^{\otimes m}$ by

$$\mathrm{wtr}(b) = \sum_{i_1,\dots,i_m} v_{i_1} \cdots v_{i_m} b\big|_{v_{i_1}\cdots v_{i_m}} \mathrm{wt}(v_{i_1} \cdots v_{i_m}), \tag{5.9}$$

for each $b \in B_m(2n+1)$, where the sum is over all sequences $i_1, i_2, \dots, i_m$ with $i_j \in I$, and where $v_{i_1} \cdots v_{i_m} b\big|_{v_{i_1}\cdots v_{i_m}}$ is the coefficient of $v_{i_1} \cdots v_{i_m}$ in $v_{i_1} \cdots v_{i_m} b$. Using Proposition (5.4), we have that $\beta_a = \beta_{\lambda(a)}$ for all $a \in \mathbb{Z}^n$. Thus, for all $b \in B_m(2n+1)$,

$$\mathrm{wtr}(b) = \sum_a \beta^a(b) x^a = \sum_\lambda \beta_\lambda(b) mb_\lambda. \tag{5.10}$$

where $mb_\lambda = \sum_{a \in H_n\lambda} x^a$. Let γ_r and E be as defined in (2.4). Then an easy computation shows that $\mathrm{wtr}(\gamma_r) = 1 + \sum_{i=1}^n x_i^r + x_i^{-r}$, and $\mathrm{wtr}(E) = 2n+1$. The proposition now follows from (5.10) and the easy fact that

$$\mathrm{wtr}(E^{\otimes k} \otimes \gamma_\mu) = \mathrm{wtr}(E)^k\, \mathrm{wtr}(\gamma_{\mu_1}) \mathrm{wtr}(\gamma_{\mu_2}) \cdots = (2n+1)^k pb_{\mu_1} pb_{\mu_2} \cdots. \qquad \square$$

Irreducible characters of Brauer algebras

(5.11) THEOREM. *Let $n \gg m$. Let the values $\eta^\lambda(\mu)$,* for *$\lambda, \mu \in \hat{B}_m$ be given as in* (4.5). *Define the characters χ^λ, $\lambda \in \hat{B}_m$ on $B_m(2n+1)$ by defining*

$$\chi^\lambda(E^{\otimes k} \otimes \gamma_\mu) = (2n+1)^k \eta^\lambda(\mu),$$

for each partition $\mu \in \hat{B}_m$. Then, up to a permutation of the elements of $\hat{B}_m$, the characters χ^λ are the irreducible characters of $B_m(2n+1)$.

PROOF. Since the functions β_λ are the characters of the weight spaces $(V^{\otimes m})_\lambda$ it follows that, for each partition $\lambda \in \hat{B}_m$,

$$\beta_\lambda = \sum_{\nu \in \hat{B}_m} C_{\lambda\nu} \chi^\nu$$

where χ^ν, $\nu \in \hat{B}_m$ denote the irreducible characters of $B_m(n)$ and where the $C_{\lambda\mu}$ are all non-negative integers. Then, for each partition $\mu \in \hat{B}_m$,

$$\begin{aligned}\sum_\nu (2n+1)^k \eta^\lambda(\mu) sb_\lambda = (2n+1)^k pb_\mu, &\qquad \text{by Lemma (5.7)},\\ = \sum_{\lambda,\nu} \beta_\lambda(\mu) Kb^{-1}_{\lambda\nu} sb_\nu, &\qquad \text{by Propositions (5.8) and (5.6)},\\ = \sum_{\gamma,\lambda,\nu} Kb^{-1}_{\lambda\nu} C_{\lambda\gamma} \chi^\gamma(\mu) sb_\nu. &\end{aligned}$$

Thus, if we define matrices

$$Kb^{-1} = (Kb^{-1}_{\lambda\mu})_{\lambda,\mu \in \hat{B}_m}, \qquad C = (C_{\lambda\nu})_{\lambda,\mu \in \hat{\beta}_m},$$
$$\eta = ((2n+1)^k \eta^\lambda(\mu))_{\lambda,\mu \in \hat{B}_m}, \qquad \chi = (\chi^\lambda(\mu))_{\mu,\mu \in \hat{B}_m},$$

then $\eta = D\chi$, where the matrix $D = (Kb^{-1})^t C$ is such that all entries are integers. It is because of Lemma (5.1) that we know that for n sufficiently large the matrix χ is square and has rows and columns indexed by the elements of $\hat{B}_m$.

It follows from the 'second orthogonality relations' (2.13) and (4.12) that

$$\lambda^t \chi = \eta^t \eta = \chi^t D^t D \chi.$$

Thus $\chi^t\chi = \chi^t D^t D\chi$, and since the character table of a semisimple algebra is always invertible, this implies that $D^t D = I$, where I is the identity matrix. Since D is a matrix with all integer entries and $D^t D = I$, one can show easily that D is a signed permutation matrix, i.e. a matrix with exactly one non-zero entry, equal to ± 1, in each row and each column.

We know that $\chi^\lambda(1^m)$ is the dimension of the irreducible representation of $B_m(n)$ indexed by λ and therefore $\chi^\lambda(1^m) > 0$. Since, by Proposition (4.12), $\eta^\lambda(1^m)$ is the number of up–down tableaux of shape λ and length m, we have that $\eta^\lambda(1^m) > 0$ for all $\lambda \in \hat{B}_m$. Since $\eta = D_\chi$, it follows that D cannot have negative entries. Therefore D is a permutation matrix. The theorem follows. □

The following Frobenius formula given in Theorem (3.4) is now an immediate corollary of Theorem (5.11) and Lemma (5.7).

(5.12) THEOREM (Frobenius formula Type B). *Suppose that $n \gg m$. Let $\mu \in \hat{B}_m$ and suppose that $\mu \vdash m - 2k$. Then*

$$(2n+1)^k pb_\mu = \sum_{\lambda \in \hat{B}_m} \chi^\lambda(E^{\otimes k} \otimes \gamma_\mu) sb_\lambda,$$

where χ^λ *is the irreducible character of* $B_m(2n+1)$ *corresponding to* $\lambda \in \hat{B}_m$ *and* $E^{\otimes k} \otimes \gamma_\mu$ *is as in* (2.4).

A determinantal formula. Given a Frobenius formula of the form given in Theorem (5.12), there is a 'standard way' of deriving a determinantal-type formula for the irreducible character. Although this procedure is well known to the experts, it is periodically rediscovered. The general method is probably originally due independently to D.-N. Verma and A. Zelevinsky. A survey appears in [8]. We derive, explicitly, the determinantal formula for the special case of the Brauer algebra in the following theorem. To our knowledge, this formula has not been explicitly worked out for the Brauer algebra, except for the case in which the characters are evaluated at the identity, which appears in [6]. The result in [6] is interesting as it describes a generating function for the dimensions of the weight space representations in terms of Bessel functions. A similar Bessel function description can also be given for the general character of the weight space, but it is not particularly nice in the general case; we shall not present it here.

For each $\lambda \in \hat{B}_m$, let χ^λ denote the irreducible character of $B_m(2n+1)$ corresponding to λ. For each $a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$, let $\lambda(a)$ be the partition determined by rearranging the sequence $(|a_1|, |a_2|, \ldots, |a_n|)$ into decreasing order. Let $\beta_{a_1} \cdot \beta_{a_2} \cdots \beta_{a_n}$ denote the character $\beta_{\lambda(a)}$ of the weight space $(V^{\otimes m})_{\lambda(a)}$.

(5.13) THEOREM. *Let the notation be as in the previous paragraph. Then*

$$\chi^\lambda = \det(\beta_{\lambda_i - i + j} - \beta_{\lambda_i + 2n - i - j + 1})_{1 \leqslant i, j \leqslant n}.$$

PROOF. Comparing the 'Frobenius formula' with the equality in Proposition (5.8), we have

$$\sum_{\lambda \in \hat{B}_m} \beta_\lambda m b_\lambda = \sum_{\lambda \in \hat{B}_m} \sum_{a \in H_n \lambda} \beta_a x^a = \sum_{\nu \in \hat{B}_m} \chi^\nu s b_\nu.$$

Multiplying both sides of the relation above by b_ρ and using (5.5) gives

$$\sum_{\nu \in \hat{B}_m} \chi^\nu \sum_{w \in H_n} \varepsilon(w) x^{w(\nu+\rho)} = \sum_{\lambda \in \hat{B}_m} \sum_{a \in H_n \lambda} \sum_{v \in H_n} \beta_a \varepsilon(v) x^{a + v\rho}.$$

For simplicity, substitute $\gamma = a + v\rho - \rho$, to obtain

$$\sum_{\nu \in \hat{B}_m} \chi^\nu \sum_{u \in H_n} \varepsilon(w) x^{w(\nu+\rho)} = \sum_{\lambda \in \hat{B}_m} \sum_{a \in H_n \lambda} \left(\sum_{v \in H_n} \varepsilon(v) \beta_{\gamma + \rho - v\rho} \right) x^{\gamma+\rho}.$$

Both sides of this equation are alternating polynomials and can be written as linear combinations of the elements $b_{\lambda+\rho}$, where $\lambda \in \hat{B}_m$. Note that if λ is a partition, then $\lambda + \rho = (\lambda_1 + n - \frac{1}{2}, \ldots, \lambda_n + \frac{1}{2})$ is always such that $\lambda_1 + n - \frac{1}{2} > \ldots > \lambda_n + \frac{1}{2} > 0$. Thus, one can compare coefficients of $x^{\gamma+\rho}$, where $\gamma + \rho$ is a strictly decreasing sequence, i.e. $\gamma_1 + n - \frac{1}{2} > \cdots > \gamma_n + \frac{1}{2} > 0$. If λ is a partition, then $w(\lambda + \rho)$ is a strictly decreasing sequence iff w is the identity. Comparing coefficients gives that $\lambda = \gamma$. It follows that

$$\chi^\lambda = \sum_{v \in H_n} \varepsilon(v) \beta_{\lambda + \rho - v\rho}.$$

It remains to express the right-hand side in a determinantal form. Let $\mu = \lambda + \rho - v\rho = (\mu_1, \ldots, \mu_n)$. If $v \in H_n$ such that $v(j) = \pm i$, then the ith entry of $v\rho$ is

$\pm(n-j+\frac{1}{2})$. Thus the ith entry, μ_i, of $\mu=\lambda+\rho-v\rho$ is $\lambda_i+n-i+\frac{1}{2}\mp(n-j+\frac{1}{2})$, so that

$$\mu_i\begin{cases}\lambda_i-i+j, & \text{if } v(j)=i,\\ \lambda_i+2n-i-j+1, & \text{if } v(j)=-i.\end{cases} \tag{5.14}$$

The elements of H_n can be viewed as signed permutations $v=(v(1),v(2),\ldots,v(n))$. Each element $v\in H_n$ is of the form $v=\varepsilon\pi$, where $\varepsilon=(\pm1,\pm2,\ldots,\pm n)\in H_n$ and $\pi=(|v(1)|,\ldots,|v(n)|)\in S_n$. Thus,

$$\chi^\lambda=\sum_{v\in H_n}\varepsilon(v)\beta_{\lambda+\rho-v\rho}=\sum_{\varepsilon=(\pm1,\pm2,\ldots,\pm n)}(-1)^{\varepsilon(\varepsilon)}\sum_{\pi\in S_n}\varepsilon(\pi)\beta_{\mu_1}\cdots\beta_{\mu_n},$$

where in the last expression $\mu=\lambda+\rho-\epsilon\pi\rho$, so that μ_i is given by (5.14) and $\varepsilon(\epsilon)$ is the number of negative entries of the sequence ϵ. It is now easy to rewrite this expression in the determinantal form given in the statement of the theorem. □

REMARK. By evaluating both sides of the equation in (5.13) at the identity element of the Brauer algebra, one obtains a 'determinantal' formula for the dimension of the irreducible representation of the Brauer algebra indexed by λ. This is analogous to the determinantal formula for the dimension of the irreducible representations of the symmetric group (see [10 §3.2]). It would be interesting to derive the 'hook formula' [12, Lemma 8.7]

$$d_\lambda=\binom{k}{|\lambda|}(k-|\lambda|-1)!!\,\frac{|\lambda|!}{\prod_{(i,j)\in\lambda}h_{i,j}}$$

for the dimensions of the irreducible representations of the Brauer algebra from the determinantal formula. This would be analogous to the derivation of the hook formula for the symmetric group case originally given by Frame, Robinson and Thrall (see [10, §3.2]).

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Appendix: The Second Relation for Characters of a Split Semisimple Algebra

Let A be a finite-dimensional algebra over a field F. Let $\vec{t}: A \to F$ be a non-degenerate trace on A, i.e. a linear functional such that $\vec{t}(a_1a_2) = \vec{t}(a_2a_1)$ for all $a_1, a_2 \in A$ and such that if $a \in A$, $a \neq 0$, then there exists $b \in A$ such that $\vec{t}(ba) \neq 0$. Define a bilinear form on A by defining

$$\langle a_1, a_2 \rangle = \vec{t}(a_1, a_2)$$

for all $a_1, a_2 \in A$. Let $G = \{g_i\}$ be a basis of A and let $G^* = \{g_i^*\}$ be the dual basis to the basis G with respect to the form $\langle\,,\rangle$, i.e. the basis defined by $\langle g_i, g_j^* \rangle = \delta_{ij}$.

Given an element $a \in A$, define

(A.1) $$[a] = \sum_{g_i \in G} g_i a g_i^*.$$

One can show that the element $[a]$ is independent of the choice of the basis G.

The algebra A acts on itself by left multiplication and by right multiplication. The bitrace of these two actions can be given by

(A.2) $$\mathrm{btr}(a_1, a_2) = \sum_{g_i \in G} a_1 g_i a_2 \big|_{g_i},$$

where $a_1 g_i a_2|_{g_i}$ denotes the coefficient of g_i in the product $a_1 g_i a_2$, expanded in terms of the basis G.

(A.3) Theorem. *For any two elements $a_1, a_2 \in A$,*

$$btr(a_1, a_2) = \langle a_1, [a_2] \rangle = \langle [a_1], a_2 \rangle.$$

Proof.

$$\begin{aligned}\mathrm{btr}(a_1, a_2) &= \sum_{g_i \in G} a_1 g_i a_2 \big|_{g_i} = \sum_{g_i \in G} \langle a_1 g_i a_2, g_i^* \rangle \\ &= \sum_{g_i \in G} \vec{t}(a_1 g_i a_2 g_i^*) = \vec{t}(a_1[a_2]) = \langle a_1, [a_2] \rangle.\end{aligned}$$

The second equality follows by noting that

$$\langle a_1, [a_2]\rangle = \sum_{g_i\in G} \vec{t}(a_1 g_i a_2 g_i^*) = \sum_{g_i\in G} \vec{t}(g_i^* a_1 g_i a_2) = \langle [a_1], a_2\rangle. \qquad \square$$

(A.4) THEOREM. *If A is a split semisimple algebra over F and $\hat{A}$ is an index set for the irreducible representations of A, then*

$$\sum_{\lambda\in\hat{A}} \chi^\lambda(a_1)\chi^\lambda(a_2) = \langle a_1, [a_2]\rangle,$$

where χ^λ denotes the irreducible character of A corresponding to the irreducible representation labelled by λ.

PROOF. A acts on itself by both left and right multiplication. These two actions commute and each generates the full centralizer of the other in End(A). This fact, combined with the double centralizer theory, implies that

$$A \simeq \sum_{\lambda\in\hat{A}} A_\lambda \otimes A^\lambda, \tag{A.5}$$

as $A\otimes A^{op}$ bimodules. Here A_λ denotes the irreducible A-module indexed by $\lambda\in\hat{A}$ and A^λ the corresponding A^{op}-module. Taking traces on each side of (A.5) gives that

$$\mathrm{btr}(a_1, a_2) = \sum_{\lambda\in\hat{A}} \chi_\lambda(a_1)\chi^\lambda(a_2),$$

where χ^λ denotes the character of A corresponding to A^λ and λ_λ denotes the character of A corresponding to A_λ. It is clear that $\chi^\lambda(a) = \chi_\lambda(a)$ for all $a\in A$. The theorem now follows from Theorem (6.3). $\square$

The group algebra of a finite group. Let G be a finite group and let $A = FG$ be its group algebra over F. Let $\vec{t}$: $FG\to F$ given by

$$\vec{t}(a) = a|_1\,,$$

where $a|_1$ denotes the coefficient of the identity in the element a. $\vec{t}$ is a non-degenerate trace on FG and the dual basis to the basis of group elements is the set of g^{-1}, $g\in G$. If $g\in G$, then the element $[g]$ defined by (A.1) is given by

$$[g] = (g?) \sum_{k\in\mathscr{C}_g} k, \qquad \text{where } g? = |G|/|\mathscr{C}_g|\,,$$

and $\mathscr{C}_g$ is the conjugacy class of the element g. In this special case, Theorem (A.4) gives the classical result that if FG is split semisimple then

$$\sum_{\lambda\in G} \chi^\lambda(g)\chi^\lambda(h) = \left\langle g, (h?) \sum_{k\in\mathscr{C}_h} k \right\rangle = \begin{cases} h? & \text{if } g\in\mathscr{C}_{h^{-1}}, \\ 0, & \text{otherwise,}\end{cases} \tag{A.6}$$

where $\hat{G}$ is an index set for the irreducible representations of G and χ^λ denotes the irreducible character of G associated to $\lambda\in\hat{G}$.

The symmetric group. In the case of the symmetric group S_m the conjugacy classes are indexed by partitions $\mu\vdash m$. If $\mu = (1^{m_1}2^{m_2}\cdots)$ is a partition of m, then $\mu? = 1^{m_1}m_1!2^{m_2}m_2!\cdots$, and it follows from (A.6) that

$$\sum_{\lambda\vdash m} \chi^\lambda(\mu)\chi^\lambda(\nu) = \delta_{\mu\nu}\mu?,$$

where $\chi^\lambda(\mu)$ denotes the irreducible character of S_m indexed by λ evaluated at the conjugacy class indexed by μ.