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Combinatorics of the q-Basis of Symmetric Functions

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A basis of symmetric functions, which we denote by $q_\lambda(X; q, t)$, was introduced in the work of Ram and King and Wybourne in order to describe the irreducible characters of the Hecke algebras of type A. In this work we give combinatorial descriptions of the expansions of the functions $q_\lambda(X; q, t)$ in terms of the classical bases of symmetric functions and apply these results in determining the determinant of the character table of the Iwahori–Hecke algebras and in giving a generating function for certain permutation statistics.

INTRODUCTION

In this paper we are concerned with studying the symmetric functions $q_\lambda(x_1, x_2, ..., x_n; q, t)$ given by the generating function

$$\prod_{i=1}^{n} \frac{1 - x_i tz}{1 - x_i qz} = (q - t) \sum_{r \geqslant 0} q_r(x_1, x_2, ..., x_n; q, t)\, z^r,$$

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for nonnegative integers r, and by

$$q_\lambda = q_{\lambda_1} q_{\lambda_2} \cdots q_{\lambda_m},$$

for a partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$. Recently these functions have become important in describing the characters of the Iwahori–Hecke algebras of type A (see [R, KW]). In this paper we give combinatorial rules for expanding these functions in terms of the classical bases of symmetric functions, m_λ (the monomial symmetric functions), p_λ (the power sum symmetric functions), s_λ (the Schur functions), e_λ (the elementary symmetric functions), h_λ (the homogeneous symmetric functions), and f_λ (the forgotten symmetric functions), in the notation of [Mac].

The combinatorial rule for expanding the q_λ in terms of the Schur functions has appeared already in [R, vdJ, RR]. We give a new proof of this result that is more efficient and does not need the Littlewood–Richardson rule for computing products of Schur functions. The functions $q_\lambda(x_1, x_2, ..., x_n; q, t)$ with appropriate specialization of q and t become, up to a constant factor, multiples of $h_\lambda(x_1, x_2, ..., x_n)$, $e_\lambda(x_1, x_2, ..., x_n)$, or $p_\lambda(x_1, x_2, ..., x_n)$. Thus our combinatorial interpretations of the entries of the transition matrices between the q_λ's and the other classical bases of symmetric functions are naturally q-analogues of the combinatorial interpretations of the entries of the transition matrices between the classical bases of symmetric functions. In particular, our transition matrices will involve q-counting brick tabloids and special rim hook tabloids as developed by Eğecioğlu and Remmel in [ER] and [ER2] and q-counting bi-brick permutations as developed by Kulikauskas and Remmel in [KR]. Following the derivation of the transition matrices we give applications of these results of the character tables of the Iwahori–Hecke algebras and to permutation statistics.

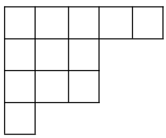
This paper is organized as follows. In Section 1 we give a brief summary of the language of λ-ring notation for working with symmetric functions. In Section 2 we derive several identities involving the symmetric functions $q_\lambda(X; q, t)$ and some identities describing the dual basis $q^*_\lambda(X; q, t)$ of the basis $q_\lambda(X; q, t)$. These identities are crucial to our derivations of the combinatorial interpretations of the entries of the transition matrices between the $q_\lambda(X; q, t)$ and the other bases of symmetric functions. In Section 3 we describe the transition matrices between the $q_\lambda(X; q, t)$ and the homogeneous, elementary, and power sum symmetric functions. In Section 4 we give the rules for expanding the q_λ in terms of monomial and forgotten symmetric functions. In Section 5 we give the derivation of the rule for expanding the q_λ in terms of the Schur functions and in Section 6 we give a rule for expanding the Schur functions in terms of the $q_\lambda(X; q, t)$. In Section 7 we use these results to calculate the determinant of the character table of the Iwahori–Hecke algebra of type A and in Section 8 we use our transition matrix results and a q-version of the homomorphism of Brenti [Br] to obtain a result on permutation statistics.

1. λ-RING NOTATION

Partitions

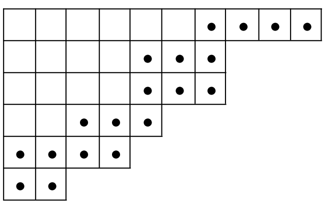
We shall adopt the notations in [Mac] for partitions. In particular, if $\lambda=(\lambda_1, \lambda_2, ...)$, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$ is a partition, then $l(\lambda)$ denotes the length (number of nonzero parts) of λ, $|\lambda|$, the weight (sum of the parts) of λ. If $|\lambda|=m$ we write $\lambda \vdash m$. Often we shall use the notation $(1^{m_1}2^{m_2}\cdots)$ for a partition, so that m_i denotes the number of parts equal to i in the partition. The *conjugate* partition is denoted λ'. A partition λ is *contained* in a partition μ, $\lambda \subseteq \mu$, if $\lambda_i \leqslant \mu_i$ for all i.

In the standard fashion (see [Mac]), to each partition $\lambda=(\lambda_1, \lambda_2, ...)$, we associate a *Ferrers diagram* of $l(\lambda)$ rows of boxes such that row i contains λ_i boxes. For example,

$(5, 3, 3, 1) =$ 

is a partition of length 4.

If λ and μ are partitions such that $\mu \subseteq \lambda$, then λ/μ shall denote the *skew diagram* determined by the set theoretic difference of the Ferrers diagrams λ and μ. In the following diagram the filled boxes form the skew diagram $(10, 7, 7, 5, 4, 2)/(6, 4, 4, 2)$.



Every partition can be expressed as a skew diagram in the form $\lambda=\lambda/\varnothing$.

Alphabets

We think of an alphabet X as a sum of commuting variables, so that, for example, $X_n = x_1 + x_2 + \cdots + x_n$ is the set of commuting variables

$\{x_1, x_2, ..., x_n\}$. From this point of view one may use the following notations:

$$\{x_1, x_2, ..., x_n\} = X_n,$$
$$\{y_1, y_2, ..., y_m\} = Y_m,$$
$$\{x_i y_j\}_{1 \leqslant i \leqslant n,\, 1 \leqslant j \leqslant m} = X_n Y_m,$$

and

$$\{x_1, ..., x_n, y_1, ..., y_m\} = X_n + Y_m.$$

Extending this idea, let $-X$ denote a *formal* (anti-)alphabet such that $X + (-X) = 0$.

Schur Functions

A column strict tableau of shape λ/μ is a filling of the boxes of the skew diagram λ/μ such that each box is filled with an element of the set $\{1, ..., n\}$ and such that the numbers are strictly increasing down the columns of λ/u and weakly increasing across the rows of λ/μ. For partitions λ, μ with $\mu \subseteq \lambda$, the skew Schur function $s_{\lambda/\mu}(X)$ in the alphabet X is defined by

$$s_{\lambda/\mu}(X) = \sum_T X^T \tag{1.1}$$

where the sum is taken over all column strict tableaux T of shape λ/μ and $X^T = x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n}$ where t_i is the number of i's in T. The set of $s_\lambda(X)$ as λ runs over all partitions forms a basis of the ring of symmetric functions in X. As in [Mac] Sect. 2 we shall let $n \to \infty$ and assume that X is an alphabet of infintely many variables $x_1, x_2, ...$.

We have the following properties of Schur functions:

$$\begin{aligned} s_{\lambda/\mu}(X+Y) &= \sum_{\mu \subseteq \gamma \subseteq \lambda} s_{\lambda/\gamma}(X)\, s_{\gamma/\mu}(Y), && \text{(sum rule)} \\ s_{\lambda/\mu}(-X) &= (-1)^{|\lambda/\mu|}\, s_{\lambda'/\mu'}(X), && \text{(duality rule)} \\ s_{\lambda/\mu}(zX) &= z^{|\lambda/\mu|} s_{\lambda/\mu}(X), && \text{(homogeneity)} \end{aligned} \tag{1.2}$$

where $|\lambda/\mu|$ denotes the number of boxes in the skew diagram λ/μ and λ' denotes the conjugate of the partition λ. For proofs of the first two, see [Mac] Chapter I, (5.10), p. 46, and (3.10), p. 26. The third property follows immediately from the definition of the Schur function.

Power Sum Symmetric Functions

Define the power sum symmetric functions by

$$p_r(x) = x^r,$$

where x is a single variable,

$$\begin{aligned} p_r(X+Y) &= p_r(X) + p_r(Y), \\ p_r(XY) &= p_r(X)\, p_r(Y), \\ p_r(-X) &= -p_r(X). \end{aligned} \tag{1.3}$$

The Cauchy Kernel

For an alphabet X, define the *Cauchy kernel*

$$\Omega(X) = \exp\left(\sum_{r \geqslant 1} \frac{p_r(X)}{r}\right).$$

We have the following properties of the Cauchy kernel:

$$\begin{aligned} \Omega(X+Y) &= \Omega(X)\, \Omega(Y), \\ \Omega(-X) &= \frac{1}{\Omega(X)}, \\ \Omega(XY) &= \sum_{\lambda} s_\lambda(X)\, s_\lambda(Y). \end{aligned} \tag{1.4}$$

Further, one shows that (see [Mac]),

$$\Omega(X_n) = \prod_{i=1}^{n} \frac{1}{1 - x_i}, \tag{1.5}$$

where X_n denotes the alphabet $X_n = x_1 + \cdots + x_n$.

Homogeneous and Elementary Symmetric Functions

Define functions $h_r(X)$ and $e_r(X)$, $r \geqslant 0$, by the generating functions

$$\Omega(Xz) = \sum_{r \geqslant 0} h_r(X)\, z^r,$$

$$\Omega(-Xz) = \sum_{r \geqslant 0} (-1)^r\, e_r(X)\, z^r,$$

and, for partitions λ, define

$$h_\lambda(X)=h_{\lambda_1}(X)\,h_{\lambda_2}(X)\cdots,$$
$$e_\lambda(X)=e_{\lambda_1}(X)\,e_{\lambda_2}(X)\cdots. \tag{1.6}$$

Monomial and Forgotten Symmetric Functions

There is a standard inner product given by making the Schur functions an orthonormal basis

$$\langle s_\lambda(X), s_\mu(X)\rangle=\delta_{\lambda\mu}$$

where $\delta_{\lambda\mu}$ is the Kronecker delta. The monomial symmetric functions $m_\lambda(X)$ and the forgotten symmetric functions $f_\lambda(X)$ are defined as the dual bases to the homogeneous and elementary symmetric functions respectively,

$$\langle h_\mu(X), m_\lambda(X)\rangle=\delta_{\lambda\mu},$$
$$\langle e_\mu(X), f_\lambda(X)\rangle=\delta_{\lambda\mu}.$$

We shall need the following properties of monomial symmetric functions:

$$m_\lambda(X+Y)=\sum_{\alpha\cup\beta=\lambda} m_\alpha(X)\,m_\beta(Y), \qquad \text{(sum rule)}$$
$$m_\lambda(-X)=(-1)^{|\lambda|}\,f_\lambda(X), \qquad \text{(duality rule)} \tag{1.7}$$

where the sum in the first identity is over all partitions α and β for which the union of α and β as multisets is λ. The first identity essentially follows from [Mac] I Sect. 4, Ex. 3 pt. (b), it is given explicitly in [G]. For the second identity see [Mac] I Sect. 2, p. 15.

The Automorphism ω

There is an involutive automorphism ω of the ring of symmetric functions that satisfies

$$\omega(p_r(X))=(-1)^{r-1}\,p_r(X),$$
$$\omega(h_r(X))=e_r(X),$$
$$\omega(e_r(X))=h_r(X), \tag{1.8}$$
$$\omega(s_\lambda(X))=s_{\lambda'}(X),$$
$$\omega(m_\lambda(X))=f_\lambda(X).$$

2. THE $q_\lambda(X; q, t)$ BASIS OF SYMMETRIC FUNCTIONS

Using the λ-ring notation of the previous section define functions $q_r(X; q, t)$, $r \geqslant 0$, by the generating function

$$\Omega((Xq - Xt)\, z) = (q - t) \sum_{r \geqslant 0} q_r(X; q, t)\, z^r,$$

and for partitions λ, define

$$q_\lambda(X; q, t) = q_{\lambda_1}(X; q, t)\, q_{\lambda_2}(X; q, t) \cdots. \tag{2.1}$$

Note that with this definition $q_0(X; q, t) = 1/(q - t)$. These functions differ only slightly from the functions $\mathbf{q}_\lambda(x; t)$ given in [Mac] III (2.9). In fact, provided $q \neq 0$,

$$q_\lambda(X; q, t) = q^{|\lambda|} q_\lambda(X; 1, t/q) = q^{|\lambda|}(1 - (t/q))^{-l(\lambda)}\, \mathbf{q}_\lambda(x; t/q).$$

This relationship between the functions $q_\lambda(X; q, t)$ and $\mathbf{q}_\lambda(x; t/q)$ shows that, in essence, the functions $q_\lambda(X; q, t)$ depend only on a single auxiliary parameter, t/q. We use the two parameter version just for convenience.

(2.2) PROPOSITION. *For each integer $r > 0$,*

(a) $q_r(X; q, t) = \sum_{m=1}^{r} (-t)^{r-m}\, q^{m-1} s_{(m, 1^{r-m})}(X)$.

(b) $q_r(X_n; q, t) = \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_r \leqslant n} (q - t)^{\mathrm{Card}(\{j \mid i_j < i_{j+1}\})}\, q^{\mathrm{Card}(\{j \mid i_j = i_{j+1}\})} x_{i_1} x_{i_2} \cdots x_{i_r}$.

(c) $\omega(q_r(X; q, t)) = q_r(X; -t, -q)$.

(d) $q_r(X; q, 0) = q^{r-1} h_r(X)$.

(e) $q_r(X; 0, t) = (-t)^{r-1}\, e_r(X)$.

(f) $q_r(X; q, q) = q^{r-1} p_r(X)$.

Proof. (a) This result is the special case $\lambda = \varnothing$ of the result proved in Proposition (5.3).

(b) We have that

$$\Omega((X_n q - X_n t)\, z) = \frac{1 - tx_1 z}{1 - qx_1 z} \cdot \frac{1 - tx_2 z}{1 - qx_2 z} \cdots \frac{1 - tx_n z}{1 - qx_n z}.$$

Substitute

$$\frac{1 - tx_i z}{1 - qx_i z} = 1 + (q - t)\, x_i z \sum_{v_i \geqslant 0} (qx_i z)^{v_i}.$$

The result follows by multiplying out and taking the coefficient of z^r. We note that a combinatorial proof of the equivalence of (a) and (b) can be found in [RR].

(c)–(e) follow immediately from (a) and the fact that $h_r = s_{(r)}$ and $e_r = s_{(1^r)}$.

(f) follows immediately from (b).

For each positive integer k, define

$$[k]_{q,t} = (q^k - t^k)/(q-t) = q^{k-1} + tq^{k-2} + \cdots + t^{k-2}q + t^{k-1},$$

and

$$[k]_{q,t}! = [k]_{q,t}\,[k-1]_{q,t} \cdots [1]_{q,t}. \tag{2.3}$$

The following identities are generalizations of the identities [Mac] I (2.6′) and [Mac] I (2.11).

(2.4) PROPOSITION. *Let $q_r(X; q, t)$ be the symmetric functions defined by (2.1) and let $e_r(X)$, $h_r(X)$ and $p_r(X)$ denote the elementary, homogeneous and power symmetric functions respectively.*

(a) $$(t-s)\,q_r(X; t, s) + (q-t)(t-s)\left[\sum_{j=1}^{r-1} q_j(X; q, t)\,q_{r-j}(X; t, s)\right] + (q-t)\,q_r(X; q, t) = (q-s)\,q_r(X; q, s).$$

(b) $$q_r(X; q, t) + \left[\sum_{j=1}^{r-1} h_j(X)\,t^j q_{r-j}(X; q, t)\right] - h_r(X)[r]_{q,t} = 0.$$

(c) $$q_r(X; q, t) + \left[\sum_{j=1}^{r-1} e_j(X)(-q)^j\,q_{r-j}(X; q, t)\right] + (-1)^r\,e_r(X)[r]_{q,t} = 0.$$

(d) $$rq_r(X; q, t) - \left[\sum_{j=1}^{r-1} p_j(X)(q^j - t^j)\,q_{r-j}(X; q, t)\right] - p_r(X)[r]_{q,t} = 0.$$

(e) $$\sum_{j=0}^{r} (-t)^{r-j}\,[j]_{q,t}\,h_j(X)\,e_{r-j}(X) = (q-t)\,q_r(X; q, t).$$

Proof. Although an easy algebraic proof of the identities in Proposition (2.4) is possible, given that several of the goals of this paper are combinatorial, we prove them in a combinatorial fashion.

(2.4.a) The identity

$$(q-t)\, q_r(X; q, t) = \Omega(Xq - Xt)\big|_{z^r} = \prod_{i=1}^{n} (1 - tx_i z) \prod_{i=1}^{n} \frac{1}{1 - qx_i z}\Bigg|_{z^r}$$

defines the functions $q_r(X; q, t)$. By multiplying out the factors of the forms $(1 - tx_i z)$ and $1/(1 - qx_i z) = \sum_{k \geqslant 0} q^k x_i^k z^k$, one see that $(q-t)\, q_r(X; q, t)$ has the following combinatorial expression

$$(q-t)\, q_r(X; q, t) = \sum_{S = (i_1 \leqslant \cdots \leqslant i_r) \in \mathcal{Q}_r} wt(S)\, x_{i_1} \cdots x_{i_r} \tag{2.5}$$

where $\mathcal{Q}_r$ is the set of sequences $S = i_1 \leqslant \cdots \leqslant i_r$ such that each element is marked with M or U, and each maximal block of equal elements in S is marked with a word in the formal language given by $(M + \varepsilon)\{U\}^*$ where ε denotes the empty word. In other words, each sequence S is marked according to the following rules: If $i_{j_1} = i_{j_2} = \cdots = i_{j_l}$ is a maximal block of equal elements of the sequence S, then

(1) i_{j_1} is marked with M or U.

(2) $i_{j_2}, ..., i_{j_l}$ each must be marked with U.

The weight of a marked sequence S is defined to be $wt(S) = (-t)^{\# M}\, q^{\# U}$, where $\# M$ and $\# U$ denote the number of M markings and the number of U markings in S respectively.

With this in mind we have that

$$\begin{aligned}(p-s)\, q_r(X; p, s) &+ (q-t)(p-s)\left[\sum_{j=1}^{r-1} q_j(X; q, t)\, q_{r-j}(X; p, s)\right] \\ &+ (q-t)\, q_r(X; q, t) \\ &= \sum_{j=0}^{r} \sum_{\substack{S_1 = (i_1 \leqslant \cdots \leqslant i_j) \in \mathcal{Q}_j \\ S_2 = (i_{j+1} \leqslant \cdots \leqslant i_r) \in \mathcal{Q}_{r-j}}} wt_1(S_1)\, wt_2(S_2)\, x_{i_1} \cdots x_{i_j} x_{i_{j+1}} \cdots x_{i_r},\end{aligned}$$

where the sum is over all pairs of increasing sequences S_1 and S_2 such that the total length of S_1 and S_2 is r and such that each maximal block of equal elements in S_1 (resp. in S_2) is marked with a word in the formal language $(M_1 + \varepsilon)\{U_1\}^*$ (resp. $(M_2 + \varepsilon)\{U_2\}^*$). The weights of the sequences S_1 and S_2 are given by $wt_1(S_1) = (-t)^{\# M_1}\, q^{\# U_1}$ and $wt_2(S_2) = (-s)^{\# M_2}\, p^{\# U_2}$ respectively and we interpret the weight of the empty sequence, $wt_i(\varepsilon)$, to be 1 for $i = 1, 2$.

The two sequences S_1 and S_2 can be combined into a single increasing sequence S of length r such that each maximal block of equal elements in the sequence S is marked with a word in the formal language $(M_1+\varepsilon)\{U_1\}^*(M_2+\varepsilon)\{U_2\}^*$ and the weight of S is given by

$$wt(S)=wt_1(S_1)\,wt_2(S_2)=(-t)^{\# M_1}\,q^{\# U_1}(-s)^{\# M_2}\,p^{\# U_2}.$$

Let $\mathcal{M}_r$ denote the set of these sequences so that we may write

$$(p-s)\,q_r(X;p,s)+(q-t)(p-s)\left[\sum_{j=1}^{r-1} q_j(X;q,t)\,q_{r-j}(X;p,s)\right]$$
$$+(q-t)\,q_r(X;q,t)=\sum_{S=(i'_1\leqslant\cdots\leqslant i'_r)\in\mathcal{M}_r} wt(S)\,x_{i'_1}\cdots x_{i'_r}. \qquad (2.6)$$

Now set $t=p$. We define a sign reversing involution ψ on the sequences in $\mathcal{M}_r$. For a sequence $S\in\mathcal{M}_r$, let i_k be the first element in S which is marked with either M_1 or U_2. Then $\psi(S)$ is obtained from S by changing the mark of i_k from U_2 (resp. M_1) to M_1 (resp. U_2) and moving i_k to the start (resp. the end) of the maximal block of equal elements which contains i_k. If no such i_k exists, then $\psi(S)=S$. For example, the following are paired under ψ.

$$\begin{matrix} M_2 & M_1 & U_1 & U_1 & M_2 & U_2 & U_2 & M_1 & U_1 & M_2 & U_2 \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \end{matrix} \leftrightarrow \begin{matrix} M_2 & U_1 & U_1 & M_2 & U_2 & U_2 & U_2 & M_1 & U_1 & M_2 & U_2 \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \end{matrix}$$

The two sequences have weights of $(-t)^2\,q^3(-s)^3\,t^3$ and $(-t)\,q^3(-s)^3\,t^4$ respectively.

Under this involution, only increasing sequences with each maximal block of equal elements marked with a word from the formal language $\{U_1\}^*(M_2+\varepsilon)$ are fixed points. It is clear that these sequences are in bijection with the set of increasing sequences with each block of equal elements marked by a word from the formal language $(M_2+\varepsilon)\,U_1^*$. The weight generating function of these latter sequences is the function $(q-s)\,q_r(X;q,s)$ and thus

$$(t-s)\,q_r(X;t,s)+(q-t)(t-s)\left[\sum_{j=1}^{r-1} q_j(X;q,t)\,q_{r-j}(X;t,s)\right]$$
$$+(q-t)\,q_r(X;q,t)=(q-s)\,q_r(X;q,s).$$

This completes the combinatorial proof of Proposition (2.4a).

(2.4.b, c, e) Identities (2.4.b, c, e) follow from (2.4.a) by specializing $s=0$, $q=0$, and $t=0$, and using (2.2.d, e). Specializing $t=0$ or $q=0$ in $q_r(X;q,t)$ can easily be viewed combinatorially as removing all the sequences in $\mathcal{Q}_r$ that contain an M or a U respectively. We shall not expand further on this.

(2.4.d) A combinatorial proof is as follows. Rewrite (2.4.d) as

$$(q-t)\, rq_r(X; q, t) = \sum_{j=1}^{r} (q^j - t^j)\, p_j(X)\, q_{r-j}(X; q, t)(q-t). \qquad (2.7)$$

The right-hand side of (2.7) can be interpreted combinatorially as follows. Let $\mathscr{P}_j$ be the set of sequences S of j equal elements, which are either all marked with U or all marked with M, and which have weight $wt_3(S)$ equal to q^j or $-t^j$ respectively, depending on the markings of S's elements. Let $\mathscr{Q}_l$ be the set of increasing sequences marked with U or M as described in (2.5). Then (2.7) can be rewritten in the form

$$\sum_{Q=(i_1 \cdots i_r) \in \mathscr{Q}_r} r\, wt(Q)\, x_{i_1} \cdots x_{i_r} = \sum_{j=1}^{r} \sum_{P=(i, i, ..., i) \in \mathscr{P}_j} wt_3(P)\, x_i^j \times \sum_{S=(i_1 \leqslant \cdots \leqslant i_{r-j}) \in \mathscr{Q}_{r-j}} wt(S)\, x_{i_1} \cdots x_{i_{r-j}}. \qquad (2.8)$$

We now perform a sign-changing involution ρ on a pair of marked sequences (P, S) occurring in the right-hand side of (2.8). Let

$$(P, S) = \begin{pmatrix} M M M & \cdots & M \\ i \;\; i \;\; i & & i \end{pmatrix}, \begin{pmatrix} y_1\, y_2\, y_3 & \cdots & y_{r-j} \\ i_1\; i_2\; i_3 & & i_{r-j} \end{pmatrix}$$

where y_l is the mark of i_l, and hence equal to U or M.

(1) If i occurs in S with mark of M, then $\rho((P, S))$ is obtained from (P, S) by removing $\substack{M \\ i}$ from S and placing it in P.

(2) If $\substack{M \\ i}$ does not occur in S, and $j > 1$, then $\rho((P, S))$ is obtained from (P, S) by removing an $\substack{M \\ i}$ from P, and inserting it into S.

(3) In all other cases, $\rho((P, S)) = (P, S)$.

Note that, since S has at most one i which is marked with M, ρ is well-defined. For example, the following are paired under ρ.

$$\begin{pmatrix} MMMMM & UUUMUUUM \\ 2\;2\;2\;2\;2, & 1\;1\;1\;2\;2\;2\;2\;3 \end{pmatrix} \leftrightarrow \begin{pmatrix} MMMMMM & UUUUUM \\ 2\;2\;2\;2\;2\;2, & 1\;1\;1\;2\;2\;2\;3 \end{pmatrix}.$$

These have associated weights of $-t^5 q^6 (-t)^2$ and $-t^6 q^6 (-t)$ respectively.

The fixed points of this involution are of two types.

(Type 1) $j = 1$, $P = (\substack{M \\ i})$ and $\substack{M \\ i}$ does not occur in S.

(Type 2) $1 \leqslant j \leqslant r$, all elements of P are marked with U and $S \in \mathscr{Q}_{r-j}$.

The right hand side of (2.8) becomes

$$\sum_{\substack{P=(i)\\ S=(i_1\cdots i_{r-1})\in\mathscr{Q}1_{r-1}}} (-tx_i)\, wt(S)\, x_{i_1}\cdots x_{i_{r-1}} + \sum_{j=1}^{r} \sum_{\substack{Q=(i\cdots i)\in\mathscr{P}2_j\\ S=(i_1\cdots i_{r-j})\in\mathscr{Q}2_{r-j}}} q^j x_i^j wt(S)\, x_{i_1}\cdots x_{i_{r-j}} \tag{2.9}$$

where $\mathscr{Q}1_{r-1}$ are the type 1 fixed sequences in $\mathscr{Q}_{r-1}$ and $\mathscr{P}2_j$ and $\mathscr{Q}2_{r-j}$ are the type 2 fixed sequences in $\mathscr{P}_j$ and $\mathscr{Q}_{r-j}$ respectively.

We now show that the weight of each sequence in $\mathscr{Q}_r$ occurs exactly r times in (2.9). Let $\mathscr{Q}$ be a sequence in $\mathscr{Q}_r$ and let i_k be any element of Q.

(1) If i_k is marked with M, Q can be split into two sequences $(\binom{M}{i_k}, Q-\{\binom{M}{i_k}\})$, giving a pair of sequences appearing in the first sum in (2.9).

(2) If i_k is marked with U, let P be the subsequences of Q consisting of $\binom{U}{i_k}$ and all succeeding elements of Q which equal it. Q is then split into $(P, Q-P)$, which is a pair of sequences in the second set of sums in (2.9).

For example, we have the following splittings of Q

$$Q=\begin{pmatrix} M\,U\,U\,M\,U \\ 1\;1\;1\;2\;2 \end{pmatrix} \leftrightarrow \begin{cases} \begin{pmatrix} M & UUMU \\ 1 & 1122 \end{pmatrix} \\ \begin{pmatrix} UU & MMU \\ 11 & 122 \end{pmatrix} \\ \begin{pmatrix} U & MUMU \\ 1 & 1122 \end{pmatrix} \\ \begin{pmatrix} M & MUUU \\ 2 & 1112 \end{pmatrix} \\ \begin{pmatrix} U & MUUM \\ 2 & 1112 \end{pmatrix} \end{cases}$$

Because this splitting procedure can be performed on each of the r elements of Q, the weight of each sequence in $\mathscr{Q}_r$ appears r times in (2.9), giving the identity.

Determinantal Formulas

We can easily derive determinantal formulas by viewing each of the identities (2.4.b, c, d) as a set of linear equations. These identities are all generalizations of the analogous classical identities [Mac] I Sect. 2, Ex. 8.

(2.10) COROLLARY. *Abbreviate the symmetric functions* $q_r(X; q, t)$ *by* q_r, *the homogeneous symmetric functions* $h_r(X)$ *by* h_r, *the elementary symmetric functions* $e_r(X)$ *by* e_r, *and the power symmetric functions* $p_r(X)$ *by* p_r. *The "q-analogue" of* k, $[k]_{q,t}$ *is as in* (2.3).

(a) $$(-1)^{n-1} q_n = \det \begin{pmatrix} [1]_{q,t} h_1 & 1 & 0 & \cdot & 0 \\ [2]_{q,t} h_2 & th_1 & 1 & \cdot & 0 \\ [3]_{q,t} h_3 & t^2h_2 & th_1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ [n-1]_{q,t} h_{n-1} & t^{n-2}h_{n-2} & t^{n-3}h_{n-3} & \cdot & 1 \\ [n]_{q,t} h_n & t^{n-1}h_{n-1} & t^{n-2}h_{n-2} & \cdot & th_1 \end{pmatrix}$$

$$[n]_{q,t}!\, h_n = \det \begin{pmatrix} q_1 & -[1]_{q,t} & 0 & \cdot & 0 \\ q_2 & tq_1 & -[2]_{q,t} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ q_{n-1} & tq_{n-2} & t^2q_{n-3} & \cdot & -[n-1]_{q,t} \\ q_n & tq_{n-1} & t^2q_{n-2} & \cdot & t^{n-1}q_1 \end{pmatrix}$$

(b) $$q_n = \det \begin{pmatrix} [1]_{q,t} e_1 & 1 & 0 & \cdot & 0 \\ [2]_{q,t} e_2 & qe_1 & 1 & \cdot & 0 \\ [3]_{q,t} e_3 & q^2e_2 & qe_1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ [n-1]_{q,t} e_{n-1} & q^{n-2}e_{n-2} & q^{n-3}e_{n-3} & \cdot & 1 \\ [n]_{q,t} e_n & q^{n-1}e_{n-1} & q^{n-2}e_{n-2} & \cdot & qe_1 \end{pmatrix}$$

$$[n]_{q,t}!\, e_n = \det \begin{pmatrix} q_1 & -[1]_{q,t} & 0 & \cdot & 0 \\ q_2 & qq_1 & -[2]_{q,t} & \cdot & 0 \\ q_3 & qq_2 & q^2q_1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ q_{n-1} & qq_{n-2} & q^2q_{n-3} & \cdot & -[n-1]_{q,t} \\ q_n & qq_{n-1} & q^2q_{n-2} & \cdot & q^{n-1}q_1 \end{pmatrix}$$

(c) $$(-1)^{n-1} n!\, q_n =$$

$$\det \begin{pmatrix} p_1 & 1 & 0 & \cdot & 0 \\ p_2 & -(q-t)\,p_1 & 2 & \cdot & 0 \\ p_3 & -(q^2-t^2)\,p_2 & -(q-t)\,p_1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{n-1} & -(q^{n-2}-t^{n-2})\,p_{n-2} & -(q^{n-3}-t^{n-3})\,p_{n-3} & \cdot & n-1 \\ p_n & -(q^{n-1}-t^{n-1})\,p_{n-1} & -(q^{n-2}-t^{n-2})\,p_{n-2} & \cdot & -(q-t)\,p_1 \end{pmatrix}$$

$$(-1)^{n-1}[n]_{q,t}!\,p_n=$$
$$\det\begin{pmatrix} q_1 & [1]_{q,t} & 0 & \cdot & 0 \\ 2q_2 & (q-t)\,q_1 & [2]_{q,t} & \cdot & 0 \\ 3q_3 & (q-t)\,q_2 & (q^2-t^2)\,q_1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (n-1)\,q_{n-1} & (q-t)\,q_{n-2} & (q^2-t^2)\,q_{n-3} & \cdot & [n-1]_{q,t} \\ nq_n & (q-t)\,q_{n-1} & (q^2-t^2)\,q_{n-2} & \cdot & (q^{n-1}-t^{n-1})\,q_1 \end{pmatrix}$$

Proof. The identity (2.4.b) in terms of matrices, where the q_r are variables and the h_r are constants, is

$$\begin{pmatrix} 1 & 0 & 0 & \cdot & 0 \\ th_1 & 1 & 0 & \cdot & 0 \\ t^2h_2 & th_1 & 1 & \cdot & 0 \\ t^3h_3 & t^2h_2 & th_1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ t^{n-1}h_{n-1} & t^{n-2}h_{n-2} & \cdot & \cdot & 1 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ \cdot \\ \cdot \\ \cdot \\ q_n \end{pmatrix}=\begin{pmatrix} [1]_{q,t}\,h_1 \\ [2]_{q,t}\,h_2 \\ [3]_{q,t}\,h_3 \\ \cdot \\ \cdot \\ \cdot \\ [n]_{q,t}\,h_n \end{pmatrix}$$

The determinant of this matrix is 1. Using Cramer's rule to solve for q_n gives the first identity in (a).

The identity (2.4.b) in terms of matrices, where the h_r are variables and the q_r are constants, is

$$\begin{pmatrix} [1]_{q,t} & 0 & 0 & \cdot & 0 \\ -tq_1 & [2]_{q,t} & 0 & \cdot & 0 \\ -tq_2 & -t^2q_1 & [3]_{q,t} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -tq_{n-1} & -t^2q_{n-2} & -t^3q_{n-3} & \cdot & [n]_{q,t} \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \\ \cdot \\ \cdot \\ \cdot \\ h_n \end{pmatrix}=\begin{pmatrix} q_1 \\ q_2 \\ \cdot \\ \cdot \\ \cdot \\ q_n \end{pmatrix}$$

The determinant of this matrix is $[n]_{q,t}!$. Using Cramer's rule to solve for h_n gives the second identity in (a).

Using Cramer's rule in a similar way on identities (2.4.c, d) yields the remainder of the identities in the corollary.

The Dual Basis to the q_λ Basis

Let $q^*_\lambda(X; q, t)$ denote the dual basis to the basis $q_\lambda\,(X; q, t)$ with respect to the inner product on symmetric functions given by making the Schur functions an orthonormal basis. Using the λ-ring notation, the identity

$$\begin{aligned}\Omega(XY) &= \Omega\left((q-t)\,X\frac{Y}{q-t}\right)\\ &= \sum_\lambda h_\lambda(X_q - Xt)\, m_\lambda\left(\frac{Y}{q-t}\right)\\ &= \sum_\lambda q_\lambda(X; q, t)(q-t)^{l(\lambda)}\, m_\lambda(Y/(q-t)),\end{aligned}$$

shows that

$$q_\lambda^*(Y; q, t) = (q-t)^{l(\lambda)}\, m_\lambda(Y/(q-t)). \tag{2.11}$$

It follows from Proposition (2.2.d, e, f) that the functions $q_\lambda(X; q, t)$ are in some sense a continuous family of symmetric functions that go between the homogeneous, the elementary, and the power sum symmetric functions. By taking the dual bases, one sees that, the functions $q_\lambda^*(X; q, t)$ are a continuous family of functions that go between the monomial, the forgotten, and the power sum symmetric functions.

(2.12) THEOREM. *For each partition λ we have the following identities*

$$q_\lambda^*(Y; q, t) = \frac{1}{q^{|\lambda|} - t^{|\lambda|}} \sum_{\substack{\alpha \cup \beta = \lambda\\ \alpha \neq \varnothing}} (q-t)^{l(\alpha)}\, t^{|\beta|} m_\alpha(Y)\, q_\beta^*(Y; q, t),$$

$$q_\lambda^*(Y; q, t) = \frac{1}{q^{|\lambda|} - t^{|\lambda|}} \sum_{\substack{\alpha \cup \beta = \lambda\\ \alpha \neq \varnothing}} (-1)^{|\alpha|-1}\, (q-t)^{l(\alpha)}\, q^{|\beta|} f_\alpha(Y)\, q_\beta^*(Y; q, t),$$

where $m_\alpha(Y)$ and $f_\alpha(Y)$ denote the monomial symmetric function and the forgotten symmetric function corresponding to the partition α respectively.

Proof. Expand the function $m_\lambda(Y/(q-t))$. Using homogeneity, (1.2),

$$\begin{aligned}m_\lambda(Y/(q-t)) &= m_\lambda\left(\frac{q^{-1}Y}{1-t/q}\right)\\ &= q^{-|\lambda|} m_\lambda\left(\frac{Y}{1-t/q}\right)\\ &= q^{-|\lambda|} m_\lambda\left(Y + \frac{(t/q)\,Y}{1-t/q}\right).\end{aligned}$$

Now use the sum rule for monomial symmetric functions, (1.7), to get

$$m_\lambda(Y/(q-t)) = q^{-|\lambda|} m_\lambda\left(Y + \frac{tY}{q-t}\right)$$

$$= q^{-|\lambda|} \sum_{\alpha \cup \beta = \lambda} m_\alpha(Y)\, t^{|\beta|} m_\beta\left(\frac{Y}{q-t}\right)$$

$$= q^{-|\lambda|} \left(t^{|\lambda|} m_\lambda(Y/(q-t)) + \sum_{\substack{\alpha \cup \beta = \lambda \\ \alpha \neq \varnothing}} m_\alpha(Y)\, t^{|\beta|} m_\beta\left(\frac{Y}{q-t}\right)\right).$$

Subtracting $q^{-|\lambda|} t^{|\lambda|} m_\lambda(Y/(q-t))$ from both sides gives

$$m_\lambda(Y/(q-t))(1-(t/q)^{|\lambda|}) = q^{-|\lambda|} \sum_{\substack{\alpha \cup \beta = \lambda \\ \alpha \neq \varnothing}} m_\alpha(Y)\, t^{|\beta|} m_\beta\left(\frac{Y}{q-t}\right).$$

Multiply both sides by $q^{|\lambda|}(q-t)^{l(\lambda)}$ to get

$$(q-t)^{l(\lambda)}\, m_\lambda(Y/(q-t))(q^{|\lambda|} - t^{|\lambda|})$$

$$= \sum_{\substack{\alpha \cup \beta = \lambda \\ \alpha = \varnothing}} (q-t)^{l(\alpha)}\, t^{|\beta|} m_\alpha(Y)(q-t)^{l(\beta)}\, m_\beta\left(\frac{Y}{q-t}\right).$$

The first identity now follows by dividing both sides by $q^{|\lambda|} - t^{|\lambda|}$.

The proof of the second identity is similar. Expand

$$m_\lambda(Y/(q-t)) = m_\lambda\left(\frac{t^{-1}Y}{q/t-1}\right) = t^{-|\lambda|} m_\lambda\left(-Y + \frac{qY}{q-t}\right)$$

and use the fact that $m_\alpha(-Y) = (-1)^{|\alpha|} f_\alpha(Y)$.

3. TRANSITION MATRICES: q_λ TO ELEMENTARY, HOMOGENEOUS, AND POWER SUM SYMMETRIC FUNCTIONS AND VICE VERSA

A *brick tabloid* of shape (r) is a row of r boxes covered with bricks such that no two bricks overlap and all boxes are covered. Let bt be a brick tabloid of shape (r) and let b_i, $1 \leqslant i \leqslant n$, be the lengths of the bricks in bt indexed from left to right. For each $1 \leqslant i \leqslant n$, let $r_i = \sum_{j \geqslant i} b_j$. Define the content $c(bt)$ of the brick tabloid to be the partition determined by the lengths b_i of the bricks in bt. Recall that for any positive integer k,

$$[k]_{q,t} = \frac{q^k - t^k}{q - t} = q^{k-1} + q^{k-2}t + q^{k-3}t^2 + \cdots + t^{k-1}.$$

Define

$$w_h(bt) = (-1)^{n-1}\, t^{r-b_n}[b_n]_{q,t}, \qquad \tilde{w}_h(bt) = \frac{1}{t^r} \prod_{i=1}^{n} \frac{t^{r_i}}{[r_i]_{q,t}},$$

$$w_e(bt) = (-1)^{r+n}\, q^{r-b_n}[b_n]_{q,t}, \qquad \tilde{w}_e(bt) = \frac{(-1)^{r+n}}{q^r} \prod_{i=1}^{n} \frac{q^{r_i}}{[r_i]_{q,t}},$$

$$w_p(bt) = (q-t)^{n-1} \prod_{i=1}^{n} \frac{[b_i]_{q,t}}{r_i}, \qquad \tilde{w}_p(bt) = b_n(t-q)^{n-1} \frac{1}{[r]_{q,t}}.$$

In these formulas, n is the number of bricks in the brick tabloid and r is the length of the brick tabloid.

(3.1) PROPOSITION. *Let B_r be the set of brick tabloids of shape (r). Let $c(bt)$ denote the content of the brick tabloid bt. Let $B_{(r)\mu}$ denote the set of brick tabloids of shape (r) and content μ. We shall abuse notation and write $q_\lambda(X)$ for $q_\lambda(X; q, t)$.*

(a) $$q_r(X) = \sum_{bt \in B_r} w_h(bt)\, h_{c(bt)}(X) = \sum_{\mu} h_\mu(X) \sum_{bt \in B_{(r)\mu}} w_h(bt).$$

(b) $$q_r(X) = \sum_{bt \in B_r} w_e(bt)\, e_{c(bt)}(X) = \sum_{\mu} e_\mu(X) \sum_{bt \in B_{(r)\mu}} w_e(bt).$$

(c) $$q_r(X) = \sum_{bt \in B_r} w_p(bt)\, p_{c(bt)}(X) = \sum_{\mu} p_\mu(X) \sum_{bt \in B_{(r)\mu}} w_p(bt).$$

(d) $$h_r(X) = \sum_{bt \in B_r} \tilde{w}_h(bt)\, q_{c(bt)}(X) = \sum_{\mu} q_\mu(X) \sum_{bt \in B_{(r)\mu}} \tilde{w}_h(bt).$$

(e) $$e_r(X) = \sum_{bt \in B_r} \tilde{w}_e(bt)\, q_{c(bt)}(X) = \sum_{\mu} q_\mu(X) \sum_{bt \in B_{(r)\mu}} \tilde{w}_e(bt).$$

(f) $$p_r(X) = \sum_{bt \in B_r} \tilde{w}_p(bt)\, q_{c(bt)}(X) = \sum_{\mu} q_\mu(X) \sum_{bt \in B_{(r)\mu}} \tilde{w}_p(bt).$$

Proof. (a) This follows immediately by recursively applying the identity

$$q_r(X; q, t) = [r]_{q,t}\, h_r(x) + \sum_{i=1}^{r-1} (-t^i)\, h_i(X)\, q_{r-i}(X; q, t) \tag{3.2}$$

from Proposition (2.4b). In particular, suppose we write

$$q_r(X; q, t) = \sum_{\mu} h_\mu(X)\, M_{(r)\,\mu}(q, t). \tag{3.3}$$

Then (3.2) and the fact that $q_1(X; q, t) = h_1(X)$ imply that the $M_{(r)\,\mu}(q, t)$'s satisfy the following recursions:

$$M_{(1)(1)}(q, t) = 1, \qquad M_{(r)(r)}(q, t) = [r]_{q,t}, \tag{3.4}$$

$$M_{(r)\,\mu}(q, t) = \sum_{k=1}^{r} (-t^k)\, M_{(r-k),\, \mu-(k)}(q, t), \tag{3.5}$$

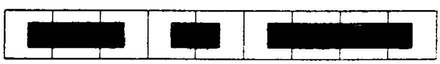
where, in (3.5), $\mu - (k)$ denotes the partition which results from removing a part of size k from μ if μ has a part of size k, and $\mu - (k)$ denotes the empty partition $\varnothing$ if μ has no part of size k. By definition, we set $M_{(s)\,\varnothing}(q, t) = 0$ for all s. It is then easy to check that if we set

$$\tilde{M}_{(r)\,\mu}(q, t) = \sum_{bt \in B_{(r)\,\mu}} w_h(bt), \tag{3.6}$$

then the $\tilde{M}_{(r)\,\mu}(q, t)$'s also satisfy the recursions (3.4–3.5). In particular, it is easy to see that the analogue of recursion (3.5) for the $\tilde{M}_{(r)\,\mu}(q, t)$ is just the result of classifying the brick tabloids in $B_{(r)\,\mu}$ by the length of their initial brick. Thus $M_{(r)\,\mu}(q, t) = \sum_{bt \in B_{(r)\,\mu}} w_h(bt)$.

The proofs of the remaining parts follow from corresponding analogues of (3.2) obtained by rewriting the identities in Proposition (2.4.b, c, d).

Example. If bt is the brick tabloid of length 9 given by $b_1 = 3$, $b_2 = 2$, $b_3 = 4$,



then $c(bt) = (2\ \ 3\ \ 4)$, and

$$w_h(bt) = (-t^3)(-t^2)[4]_{q,t} = t^5[4]_{q,t},$$

$$\tilde{w}_h(bt) = \frac{t^9}{t^3[9]_{q,t}} \cdot \frac{t^6}{t^2[6]_{q,t}} \cdot \frac{t^4}{t^4[4]_{q,t}},$$

$$w_e(bt) = (-(-q)^3)(-(-q)^2)(-(-1)^4\,[4]_{q,t}) = q^5[4]_{q,t},$$

$$\tilde{w}_e(bt) = \frac{(-1)^2\, q^9}{q^3[9]_{q,t}} \cdot \frac{(-1)^1\, q^6}{q^2[6]_{q,t}} \cdot \frac{(-1)^3\, q^4}{q^4[4]_{q,t}},$$

$$w_p(bt) = \frac{(q-t)[3]_{q,t}}{9} \cdot \frac{(q-t)[2]_{q,t}}{6} \cdot \frac{[4]_{q,t}}{4},$$

$$\tilde{w}_p(bt) = \frac{-(q^6-t^6)}{[9]_{q,t}} \cdot \frac{-(q^4-t^4)}{[6]_{q,t}} \cdot \frac{4}{[4]_{q,t}}.$$

Define a brick tabloid B of shape λ to be a Ferrers diagram of λ covered with bricks such that each row $1 \leqslant i \leqslant l(\lambda)$ is a brick tabloid bt_i of shape (λ_i). i.e.,

(1) No brick covers boxes in different rows,

(2) No two bricks overlap,

(3) Every box of λ is covered.

The content of a brick tabloid is the partition determined by the lengths of the bricks. Define weights w_h, w_e, w_p, $\tilde{w}_h$, $\tilde{w}_e$, $\tilde{w}_p$ of a brick tableau B by defining the weight to be the product of the weights of the brick strips bt_i in B; for example

$$\tilde{w}_e(B) = \prod_{i=1}^{l(\lambda)} \tilde{w}_e(bt_i).$$

The following theorem is an immediate corollary of Proposition (3.1).

(3.7) Theorem. *Let $B_{\lambda\mu}$ be the set of brick tabloids of shape λ and content μ and let the weights of a brick tabloid B be given as in the previous paragraph. Define*

$$H_{\lambda\mu}(q,t) = \sum_{B \in B_{\lambda\mu}} w_h(B), \qquad H^{-1}_{\lambda\mu}(q,t) = \sum_{B \in B_{\lambda\mu}} \tilde{w}_h(B),$$

$$E_{\lambda\mu}(q,t) = \sum_{B \in B_{\lambda\mu}} w_e(B), \qquad E^{-1}_{\lambda\mu}(q,t) = \sum_{B \in B_{\lambda\mu}} \tilde{w}_e(B),$$

$$P_{\lambda\mu}(q,t) = \sum_{B \in B_{\lambda\mu}} w_p(B), \qquad P^{-1}_{\lambda\mu}(q,t) = \sum_{B \in B_{\lambda\mu}} \tilde{w}_p(B).$$

Then

(a) $q_\lambda(X; q, t) = \sum_\mu H_{\lambda\mu}(q,t)\, h_\mu(X)$,

(b) $q_\lambda(X; q, t) = \sum_\mu E_{\lambda\mu}(q,t)\, e_\mu(X)$,

$$\text{(c)}\quad q_\lambda(X;q,t)=\sum_\mu P_{\lambda\mu}(q,t)\,p_\mu(X),$$

$$\text{(d)}\quad h_\lambda(X)=\sum_\mu H^{-1}_{\lambda\mu}(q,t)\,q_\mu(X;q,t),$$

$$\text{(e)}\quad e_\lambda(X)=\sum_\mu E^{-1}_{\lambda\mu}(q,t)\,q_\mu(X;q,t),$$

$$\text{(f)}\quad p_\lambda(X)=\sum_\mu P^{-1}_{\lambda\mu}(q,t)\,q_\mu(X;q,t).$$

In each case the sum is over all partitions μ *of* $|\lambda|$.

4. TRANSITION MATRICES: q_λ TO MONOMIAL AND FORGOTTEN SYMMETRIC FUNCTIONS AND VICE VERSA

Given partitions λ and μ and a positive integer n such that $n \geqslant l(\lambda)$ and $n \geqslant l(\mu)$, let $A_{\lambda\mu}$ be the set of $n \times n$ matrices with nonnegative integer entries and row sums λ_i and columns sumns μ_j. Define weights of a matrix $A \in A_{\lambda\mu}$ by

$$w_m(A)=q^{|\lambda|-N}(q-t)^{N-l(\lambda)}$$

and

$$w_f(A)=(-t)^{|\lambda|-N}\,(q-t)^{N-l(\lambda)},$$

where N is the number of nonzero entries in A. Define polynomials

$$M_{\lambda\mu}(q,t)=\sum_{A\in A_{\lambda\mu}} w_m(A) \qquad \text{and} \qquad F_{\lambda\mu}(q,t)=\sum_{A\in A_{\lambda\mu}} w_f(A).$$

(4.1) THEOREM.

(a) $q_\lambda(X;q,t)=\sum_\mu M_{\lambda\mu}(q,t)\,m_\mu(X)$.

(b) $q_\lambda(X;q,t)=\sum_\mu F_{\lambda\mu}(q,t)\,f_\mu(X)$.

Proof. (a) Consider the coefficient of a monomial $x^\mu=x_1^{\mu_1}x_2^{\mu_2}\cdots x_n^{\mu_n}$ in $q_\lambda(X;q,t)=q_{\lambda_1}q_{\lambda_2}\cdots$. Each monomial in q_{λ_i} is of the form $\prod_{j=1}^n x_j^{a_{ij}}$ where the a_{ij} are nonnegative integers. For each i, $\sum_j a_{ij}=\lambda_i$. It follows from Proposition (2.2.b) that the monomial $\prod_j x_j^{a_{ij}}$ appears in q_{λ_j} with coefficient

$$q^{a_{ij_1}-1}\prod_{k>1} q^{a_{ij_k}-1}(q-t)=q^{\lambda_i-l_i}(q-t)^{l_i-1}, \tag{4.2}$$

where $1 \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant n$ is the set of integers such that a_{ij_k} is nonzero and l_i is the number of nonzero a_{ij}. Using (4.2), we have that a given monomial $x^\mu = \prod_{i,j} x_j^{a_{ij}}$ appearing in q_λ appears with coefficient

$$w_m(A) = q^{|\lambda| - N}(q-t)^{N - l(\lambda)},$$

where N is the number of nonzero entries in the matrix $A = (a_{ij})_{1 \leqslant i, j \leqslant n}$.

In order for x^μ to appear in q_λ we must have that $\prod_{i,j} x_j^{a_{ij}} = \prod_j x_j^{\mu_j}$. Thus, for each j, $\sum_i a_{ij} = \mu_j$. Thus, the matrix A determined by the a_{ij} is a matrix with nonnegative entries, row sums λ_i and column sums μ_j.

(b) Applying the involution ω to part (a) and using Proposition (2.2c) gives that

$$q_\lambda(X; -t, -q) = \sum_\mu M_{\lambda\mu}(q, t) f_\mu(X).$$

Making a substitution of variables $q \to -t$ and $t \to -q$ gives the result.

Monomial and Forgotten Symmetric Functions in Terms of the $q_\lambda(X; q, t)$

To describe the transition matrices given by

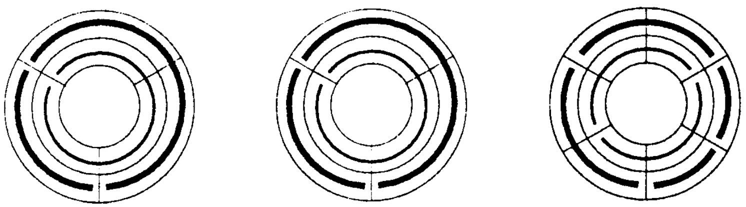
$$m_\lambda(X) = \sum_\mu M^{-1}_{\lambda\mu}(q, t)\, q_\mu(X; q, t)$$

and

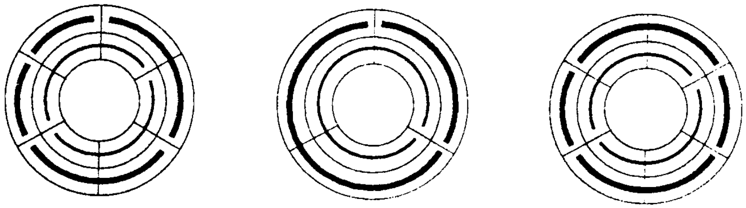
$$f_\lambda(X) = \sum_\mu F^{-1}_{\lambda\mu}(q, t)\, q_\mu(X; q, t),$$

we first need some notation and definitions.

With λ and μ partitions of n, define a (λ, μ) bi-brick permutation as follows: let $C_1, C_2, \ldots, C_k$ be a collection of cycles whose lengths sum to n. On each cycle, place an outer tier of λ bricks and an inner tier of μ bricks so that each tier contains bricks whose lengths sum to the length of the cycle. If the bricks are placed in such a way that each cycle has no rotational symmetry, the bi-brick permutation is called *primitive*. For example, if $\lambda = (1^4, 2^4)$ and $\mu = (3^4)$, the following is a primitive (λ, μ) bi-brick permutation:



The diagram below is not primitive because the third cycle displays rotational symmetry.



We have the following result from the work of Kulikauskas and Remmel.

(4.3) THEOREM. [KR] *Let $PB(\lambda, \mu)$ be the set of primitive (λ, μ) bi-brick permutations. Define constants $M^{-1}_{\lambda\mu}(1, 1)$ by*

$$M^{-1}_{\lambda\mu}(1, 1) = (-1)^{l(\lambda) + l(\mu)} \, |PB(\lambda, \mu)|,$$

where $|PB(\lambda, \mu)|$ denotes the number of primitive (λ, μ) bi-brick permutations. Then, for all λ,

$$m_\lambda(X) = \sum_\mu M^{-1}_{\lambda\mu}(1, 1) \, h_\mu(X),$$

where the sum is over all partitions μ such that $|\mu| = |\lambda|$.

The following lemma is an easy consequence of the above theorem.

(4.4) LEMMA. *The product of a monomial symmetric function and a homogeneous symmetric function is given by*

$$m_\alpha(X) \cdot h_\mu(X) = \sum_{\gamma \sqsupseteq \mu} (-1)^{l(\alpha) + l(\gamma - \mu)} \, |PB(\alpha, \gamma - \mu)| \, h_\gamma(X),$$

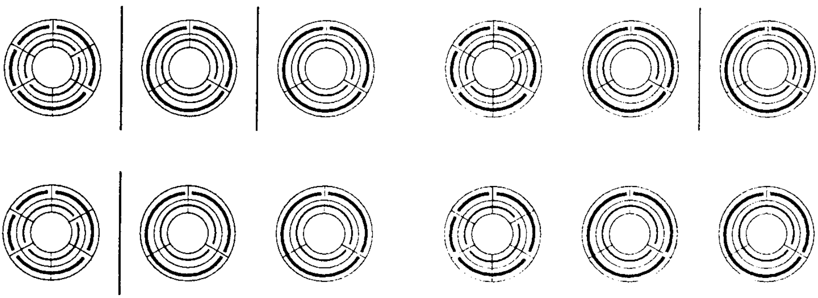
where the sum is over all partitions γ such that the parts of μ form a subset of the parts of γ and $|PB(\alpha, \gamma - \mu)|$ denotes the number of primitive $(\alpha, \gamma - \mu)$ bi-brick permutations.

Proof. Using Theorem (4.3),

$$m_\alpha(X) \cdot h_\mu(X) = \sum_\delta (-1)^{l(\alpha) + l(\delta)} \, |PB(\alpha, \delta)| \, h_\delta(X) \, h_\mu(X)$$

$$= \sum_{\gamma \sqsupseteq \mu} (-1)^{l(\alpha) + l(\gamma - \mu)} \, |PB(\alpha, \gamma - \mu)| \, h_\gamma(X).$$

Define a layered primitive (λ, μ) bi-brick permutation as a sequence $(D_1, D_2, ..., D_k)$ with $D_1 D_2 \cdots D_k$ a primitive (λ, μ) bi-brick permutation, and with each D_i being a primitive bi-brick permutation. Let $LPB(\lambda, \mu)$ be the collection of such layered primitive (λ, μ) bi-brick permutations. For example, below we give the elements in $LPB((1^4, 2^4), (3^4))$ corresponding to the primitive bi-brick permutation in the diagram above. The vertical bars separate the pieces D_i:



(4.5) THEOREM. *For each pair of partitions λ, μ such that $|\lambda| = |\mu|$ define rational functions $M^{-1}_{\lambda\mu}(q, t)$ and $F^{-1}_{\lambda\mu}(q, t)$ in q and t by*

$$M^{-1}_{\lambda\mu}(q, t) = \sum_{(D_1, ..., D_k) \in LPB(\lambda, \mu)} (t-q)^{l(\lambda)} (-1)^{l(\mu)} \times \prod_{i=2}^{k} \frac{t^{|D_i| + \cdots |D_k|}}{q^{|D_{i-1}| + \cdots |D_k|} - t^{|D_i - 1| + \cdots + |D_k|}},$$

and

$$F^{-1}_{\lambda\mu}(q, t) = \sum_{(D_1, ..., D_k) \in LPB(\lambda, \mu)} (t-q)^{l(\lambda)} (-1)^{l(\mu)} \times \prod_{i=2}^{k} \frac{(-1)^{|D_{i-1}| - 1} q^{|D_i| + \cdots + |D_k|}}{q^{|D_{i-1}| + \cdots + |D_k|} - t^{|D_{i-1}| + \cdots + |D_k|}}.$$

where $|D_j|$ is the sum of the lengths of the cycles in D_j. Then, for each partition λ,

$$m_\lambda(X) = \sum_{\mu} M^{-1}_{\lambda\mu}(q, t)\, q_\mu(X; q, t),$$

where the sum is over all partitions μ such that $|\mu| = |\lambda|$.

Proof. It is easy to show from the definition that the $M^{-1}_{\lambda\mu}(q,t)$'s satisfy the following recursions:

$$M^{-1}_{(1)(1)}(q,t)=1,$$
$$M^{-1}_{\lambda\mu}(q,t)=\sum_{\substack{\alpha\sqsubseteq\lambda\\ \beta\sqsubseteq\mu\\ |\alpha|=|\beta|>0}}(t-q)^{l(\alpha)}(-1)^{l(\beta)}\frac{t^{|\mu|-|\beta|}}{q^{|\lambda|}-t^{|\lambda|}}\times|PB(\alpha,\beta)|\,M^{-1}_{(\lambda-\alpha)(\mu-\beta)}(q,t), \tag{4.6}$$

where the notation $\alpha\sqsubseteq\lambda$ means that the parts of α form a subset of the parts of λ and we have $|\alpha|=|\beta|$. Note that these recursions completely determine $M^{-1}_{\lambda\mu}(q,t)$.

Define rational functions $\mathscr{M}_{\lambda\mu}(q,t)$ by the equation

$$m_\lambda(X)=\sum_\mu \mathscr{M}_{\mu\lambda}(q,t)\,q_\mu(X;q,t).$$

We will show that the functions $\mathscr{M}_{\lambda\mu}(q,t)$ also satisfy the recursions in (4.6). Clearly, from the definition, $\mathscr{M}_{(1)(1)}(q,t)=1$.

Because the pairs $q_\mu(X;q,t)$ and $q^*_\mu(X;q,t)$ and $h_\mu(X)$ and $m_\mu(X)$ are dual bases with respect to the inner product defined on symmetric functions that makes the Schur functions orthonormal, it follows that

$$q^*_\lambda(X;q,t)=\sum_\mu \mathscr{M}_{\lambda\mu}(q,t)\,h_\mu(X). \tag{4.7}$$

So, using Theorem 2.12, and Lemma 4.4, we have

$$\begin{aligned}
q^*_\lambda(X;q,t)&=\sum_{\substack{\alpha\cup\beta=\lambda\\ \alpha\neq\varnothing}}(q-t)^{l(\alpha)}\frac{t^{|\beta|}}{(q^{|\lambda|}-t^{|\lambda|})}m_\alpha(X)\,q^*_\beta(X;q,t)\\
&=\sum_{\substack{\alpha\cup\beta=\lambda\\ \alpha\neq\varnothing}}\sum_{\nu\vdash|\beta|}(q-t)^{l(\alpha)}\frac{t^{|\beta|}}{(q^{|\lambda|}-t^{|\lambda|})}\mathscr{M}_{\beta\nu}(q,t)\,m_\alpha(X)\,h_\nu(X)\\
&=\sum_{\substack{\alpha\cup\beta=\lambda\\ \alpha\neq\varnothing}}\sum_{\nu\vdash|\beta|}(q-t)^{l(\alpha)}\frac{t^{|\beta|}}{(q^{|\lambda|}-t^{|\lambda|})}\mathscr{M}_{\beta\nu}(q,t)\sum_{\mu\sqsupseteq\nu}(-1)^{l(\alpha)+l(\mu-\nu)}\\
&\quad\times|PB(\alpha,\mu-\nu)|\,h_\mu(X)\\
&=\sum_\mu h_\mu(X)\sum_{\substack{\alpha\cup\beta=\lambda\\ \alpha\neq\varnothing}}\sum_{\substack{\nu\vdash|\beta|\\ \nu\sqsubseteq\mu}}(t-q)^{l(\alpha)}(-1)^{l(\mu-\nu)}\frac{t^{|\beta|}}{(q^{|\lambda|}-t^{|\lambda|})}\\
&\quad\times|PB(\alpha,\mu-\nu)|\,\mathscr{M}_{\beta\nu}(q,t).
\end{aligned}$$

Since $|\mu - \nu| = |\alpha| \neq 0$ and $|\mu| - |\mu - \nu| = |\nu| = |\beta|$, letting $\gamma = \mu - \nu$ gives

$$\mathscr{M}_{\lambda\mu}(q, t) = \sum_{\substack{\alpha \sqsubseteq \lambda \\ \gamma \sqsubseteq \mu \\ \alpha, \gamma \neq \varnothing}} (t-q)^{l(\alpha)} (-1)^{l(\gamma)} \frac{t^{|\mu| - |\gamma|}}{q^{|\lambda|} - t^{|\lambda|}} |PB(\alpha, \gamma)| \mathscr{M}_{(\lambda-\alpha)(\mu-\gamma)}(q, t).$$

Thus, the functions $\mathscr{M}_{\lambda\mu}(q, t)$ satisfy the same recursions that the functions $M^{-1}_{\lambda\mu}(q, t)$ do. Since these recursions completely determine these functions it follows that $\mathscr{M}_{\lambda\mu}(q, t) = M^{-1}_{\lambda\mu}(q, t)$ for all partitions λ, μ such that $|\lambda| = |\mu|$. This completes the proof of part (a).

(b) By applying the ω transformation to both sides of the equation $m_\lambda(X) = \sum_\mu M^{-1}_{\lambda\mu}(q, t)\, q_\mu(X; q, t)$ from part (a), we get

$$f_\lambda(X) = \sum_\mu M^{-1}_{\lambda\mu}(q, t)\, q_\mu(X; -t, -q). \tag{4.8}$$

Thus, $F^{-1}_{\lambda\mu}(q, t) = M^{-1}_{\lambda\mu}(-t, -q)$, and it follows that

$$F^{-1}_{\lambda\mu}(q, t) = \sum_{(D_1, ..., D_k) \in LPB(\lambda, \mu)} (t-q)^{l(\lambda)} (-1)^{l(\mu)} \times \prod_{i=2}^{k} \frac{(-1)^{|D_{i-1}|-1} q^{|D_i| + \cdots + |D_k|}}{q^{|D_{i-1}| \cdots + |D_k|} - t^{|D_{i-1}| + \cdots + |D_k|}}.$$

5. TRANSITION MATRICES: q_λ TO SCHUR FUNCTIONS

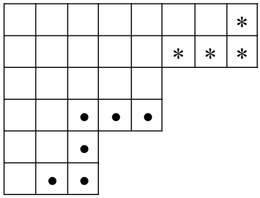
A skew diagram λ/μ is a *border strip* or *rim hook* if it is connected and contains no 2×2 block of boxes (see [Mac] I Sect. 3 Ex. 11). A skew diagram is a *broken border strip* if it contains no 2×2 block of boxes. Any broken border strip is a union of its connected components, each of which is a border strip. Define the *weight* of a border strip λ/μ by

$$\overline{wt}(\lambda/\mu; q, t) = q^{c-1}(-t)^{r-1},$$

where c is the number of columns and r is the number of rows in the border strip λ/μ. Define the *weight* of a broken border strip λ/μ by

$$\overline{wt}(\lambda/\mu; q, t) = (q-t)^{cc-1} \prod_{bs \in \lambda/\mu} \overline{wt}(bs; q, t), \tag{5.1}$$

where cc is the number of connected components (border strips) in λ/μ and the product is over the border strips bs in the broken border strip λ/μ. For convenience let us define $\overline{wt}(\lambda/\mu; q, t) = 0$ if λ/μ is not a broken border strip, and $\overline{wt}(\lambda/\lambda; q, t) = \overline{wt}(\emptyset; q, t) = 1$. The following is a broken border strip of weight $(q-t)\, q^2(-t)\, q^3(-t)^2$.



(5.2) LEMMA. *Let q and t be variables. Then, λ-ring notation,*

(a) $$s_{\mu/\nu}(q-t) = \begin{cases} (q-t)\, \overline{wt}(\mu/\nu; q, t), & \text{if } \mu/\nu \text{ is a broken border strip;} \\ 0, & \text{if } \mu/\nu \text{ is not a broken border strip.} \end{cases}$$

(b) $$s_\mu((q-t)\, z + Y) = \sum_{\nu \subseteq \mu} z^{|\mu/\nu|} (q-t)\, \overline{wt}(\mu/\nu; q, t)\, s_\nu(Y).$$

Proof. (a) By the sum rule,

$$s_{\mu/\nu}(q-t) = \sum_{\nu \subseteq \gamma \subseteq \mu} s_{\mu/\gamma}(q)\, s_{\gamma/\nu}(-t).$$

By the definition of the Schur function,

$$s_{\mu/\gamma}(q) = \begin{cases} q^k, & \text{if } \mu/\gamma \text{ is a horizontal strip of length } k; \\ 0, & \text{otherwise.} \end{cases}$$

By duality, and the definition of the Schur function,

$$\begin{aligned} s_{\gamma/\nu}(-t) &= (-1)^{|\gamma/\nu|}\, s_{\gamma'/\nu'}(t) \\ &= \begin{cases} (-t)^m, & \text{if } \gamma/\nu \text{ is a vertical strip of length } m; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, we have that μ is "gotten" from ν by adding a vertical strip (to get γ) and then adding a horizontal strip (to γ). Then μ/ν is a broken border strip (see the picture below). We have

$$s_{\mu/\nu}(q-t) = \sum_{\substack{hs,\ vs \\ \nu + vs + hs = \mu}} q^{hs}(-t)^{vs},$$

where the sum is over all horizontal strips hs and all vertical strips vs such that μ is obtained from ν by first placing vs and then placing hs.

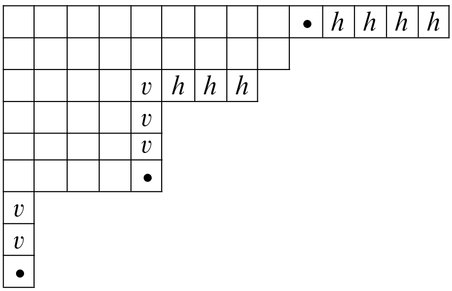
Suppose that *bs* is a border strip appearing in μ/ν. Each box in *bs* satisfies one of the following:

(1) There is a box of *bs* immediately below it

(2) There is a box of *bs* immediately to its left,

(3) Neither (1) or (2) holds.

In the picture below the boxes satisfying (1), (2), (3) are labeled with v, h, and $\bullet$ respectively. In case (1) the box must have come from the application of the vertical strip to μ and thus this box has weight $-t$. In case (2) the box must have come from the application of the horizontal strip to μ and thus this box hs weight q. In case (3) the box could have come from *either* the application of the horizontal strip *or* the vertical strip and thus this box has weight $q-t$. Each *bs* in μ/ν contains exactly one box of this type.



The result follows by noting that the product of these weights is exactly $(q-t)\,\overline{wt}(\mu/\nu;\,q,\,t)$ where $\overline{wt}(\mu/\nu;\,q,\,t)$ is as defined in (5.1).

(b) By using the sum rule, and homogeneity,

$$\begin{aligned} s_\mu((q-t)\,z+Y) &= \sum_{\nu\subseteq\mu} s_{\mu/\nu}((q-t)\,z)\,s_\nu(Y) \\ &= \sum_{\nu\subseteq\mu} z^{|\mu/\nu|} s_{\mu/\nu}(q-t)\,s_\nu(Y) \end{aligned}$$

Part b now follows by application of part a.

(5.3) Proposition. *For each* $r>0$ *and each partition* λ,

$$q_r(X;\,q,\,t)\,s_\lambda(X)=\sum_{\mu\supseteq\lambda} s_\mu(X)\,\overline{wt}(\mu/\lambda;\,q,\,t),$$

where the sum is over partitions μ *such that* μ/λ *is a broken border strip and* $|\mu/\lambda|=r$.

Proof. Let GF be a short notation for the following generating function

$$GF=(q-t)\sum_{r\geqslant 0} q_r(X;q,t)\,z^r\sum_{\lambda} s_\lambda(X)\,s_\lambda(Y).$$

It follows from the product rule for the Cauchy kernel and the definition of the $q_r(X;q,t)$ that

$$GF=\Omega((Xq-Xt)\,z)\,\Omega(XY).$$

By addition rule for the Cauchy kernel,

$$GF=\Omega(X((q-t)\,z+Y)).$$

Using the product rule for the Cauchy kernel to reexpand, we have

$$GF=\sum_{\mu} s_\mu(X)\,s_\mu((q-t)\,z+Y).$$

Now use Lemma (5.2b) to rewrite the Schur function $s_\mu((q-t)\,z+Y)$ and get

$$GF=\sum_{\mu} s_\mu(X)\sum_{\lambda\subseteq\mu} z^{|\mu/\lambda|}(q-t)\,\overline{wt}(\mu/\lambda;q,t)\,s_\lambda(Y).$$

Summarizing, we have obtained

$$\sum_{r\geqslant 0} q_r(X;q,t)\,z^r\sum_{\lambda} s_\lambda(X)\,s_\lambda(Y)=\sum_{\lambda} s_\lambda(Y)\sum_{\mu\supseteq\lambda}\overline{wt}(\mu/\lambda;q,t)\,s_\mu(X)\,z^{|\mu/\lambda|}.$$

The result now follows by taking the coefficient of $z^r s_\lambda(Y)$ on each side of this equation.

Define a μ-broken border strip tableau of shape λ to be a sequence of partitions

$$T=(\emptyset=\lambda^{(0)}\subseteq\lambda^{(1)}\subseteq\lambda^{(2)}\subseteq\cdots\subseteq\lambda^{(m)}=\lambda)$$

such that for each $1\leqslant j\leqslant m$, $\lambda^{(j)}/\lambda^{(j-1)}$ is a broken border strip of length μ_j. Define the weight of a μ-broken border strip tableau to be

$$\overline{wt}(T;q,t)=\prod_{j=1}^{m}\overline{wt}(\lambda^{(j)}/\lambda^{(j-1)};q,t), \tag{5.4}$$

where the weights $\overline{wt}(\mu/\nu;q,t)$ of broken border strips are given by (5.1).

(5.5) THEOREM. *For each pair of partitions λ, μ such that $|\lambda| = |\mu|$ let polynomials $\chi^\lambda_\mu(q, t)$ be given by*

$$\chi^\lambda_\mu(q, t) = \sum_T \overline{wt}(T; q, t),$$

where the sum is over all μ-broken border strip tableaux T of shape λ. Then

$$q_\mu(X; q, t) = \sum_{\lambda \vdash |\mu|} \chi^\lambda_\mu(q, t)\, s_\lambda.$$

Proof. This follows from Proposition (5.3) and the definition of the $q_\lambda(X; q, t)$, by induction.

6. TRANSITION MATRICES: SCHUR FUNCTIONS TO q_λ

Let us define $\eta^\lambda_\mu(q, t) \in \mathbb{C}(q, t)$ by

$$s_\lambda(X) = \sum_\mu \eta^\lambda_\mu(q, t)\, q_\mu(X; q, t). \tag{6.1}$$

The following proposition is an easy consequence of the definitions.

(6.2) PROPOSITION. (a) $q^*_\mu(X; q, t) = \sum_\lambda \eta^\lambda_\mu(q, t)\, s_\lambda(X)$, *where $q^*_\mu(X; q, t)$ is the dual basis to the basis $q_\lambda(X; q, t)$.*

(b) $\eta^{\lambda'}_\mu(q, t) = \eta^\lambda_\mu(-t, -q)$, *where λ' is the conjugate partition to the partition λ.*

Proof. (a) follows immediately since both (a) and (6.1) are equivalent to $\eta^\lambda_\mu(q, t) = \langle q^*_\mu(X; q, t)\, s_\lambda(X)\rangle$ where $\langle\ ,\ \rangle$ is the inner product on symmetric functions that makes the Schur functions an orthonormal basis.

(b) follows by applying the involution ω to (6.1) to get $s_{\lambda'}(X) = \sum_\mu \eta^\lambda_\mu(q, t)\, q_\mu(X; -t, -q)$.

There are a few special cases which are easy to compute.

(6.3) PROPOSITION. *For a partition $\lambda = (\lambda_1, \lambda_2, ...)$, let $n(\lambda) = \sum_i (i-1)\lambda_i$, and let $H_\lambda(z) = \prod_{x \in \lambda}(1 - z^{h(x)})$ be the hook length polynomial corresponding to the partition λ (see [Mac] I Sect. 1 Ex. 1 and Sect. 3 Ex. 2). Let $[r]_{q,t}$ be the "q-analogue" of r as defined in (2.3). Then, in the notation of (6.1),*

$$\eta^\lambda_{(1^r)}(q, t) = (1 - (t/q))^r \frac{(t/q)^{n(\lambda)}}{H_\lambda(t/q)} = \frac{1}{[r]_{q,t}!} \sum_T t^{r(T)} q^{\binom{r}{2} - r(T)},$$

where the sum in the last equality is over all standard tableaux T of shape λ and $r(T)$ is defined by

$$r(T)=\sum \{k\colon k+1 \text{ lies in a lower row than } k \text{ in } T\}.$$

Proof. From [Mac] Sect. 4 Ex. 2 it follows that

$$\Omega(X/1-q))=\prod_{i,j\geqslant 1}(1-x_i q^{i-1})^{-1}=\sum_\lambda \frac{q^{n(\lambda)}}{H_\lambda(q)}s_\lambda(X),$$

where $H_\lambda(q)=\prod_{x\in\lambda}(1-q^{h(x)})$ is the hook length polynomial corresponding to the partition λ. On the other hand one has that $\Omega(X/1-q))=\sum_{r\geqslant 0} e_r(X/(1-q))$, where $e_r(X)$ is the elementary symmetric function. It follows that

$$e_r(X/(1-q))=\sum_{\lambda\vdash r}\frac{q^{n(\lambda)}}{H_\lambda(q)}s_\lambda(X).$$

Then one has that

$$\begin{aligned}
q^*_{(1^r)}(X;q,t)&=(q-t)^r\, m_{1^r}(X/(q-t))\\
&=(q-t)^r\, e_r(X/(q-t))\\
&=(q-t)^r\, e_r\left(\frac{q^{-1}X}{1-(t/q)}\right)\\
&=(q-t)^r\, q^{-r}\sum_{\lambda\vdash r}\frac{(t/q)^{n(\lambda)}}{H_\lambda(t/q)}s_\lambda(X).
\end{aligned}$$

This proves the first equality in Proposition (6.3). Now if one applies [Mac] I Sect. 5 Ex. 14 formula (3) one gets that

$$\begin{aligned}
(q-t)^r\, q^{-r}\frac{(t/q)^{n(\lambda)}}{H_\lambda(t/q)}\frac{\phi_r(t/q)}{\phi_r(t/q)}&=\frac{(q-t)^r}{\phi_r(t/q)}q^{-r}\sum_T (t/q)^{r(T)}\\
&=\frac{(1-(t/q))^r}{\phi_r(t/q)}\sum_T (t/q)^{r(T)}\\
&=\frac{1}{[r]_{q,t}!}\frac{1}{q^{\binom{r}{2}}}\sum_T (t/q)^{r(T)}\\
&=\frac{1}{[r]_{q,t}!}\sum_T t^{r(T)}q^{\binom{r}{2}-r(T)},
\end{aligned}$$

where $\phi_r(q)=(1-q)\cdots(1-q^r)$. The result follows.

(6.4) PROPOSITION. *In the notation of* (6.1),

$$\eta^{\lambda}_{(r)}(q, t) = \begin{cases} (-1)^{r-k}/[r]_{q,t}, & \text{if } \lambda = (1^{r-k}k), \\ 0, & \text{otherwise}, \end{cases}$$

where $[r]_{q,t} = (q^r - t^r)/(q - t)$.

Proof.

$$\begin{aligned} q^*_{(r)}(X; q, t) &= (q-t)\, m_{(r)}(X/(q-t)) = (q-t)\; p_r(X/(q-t)) \\ &= (q-t)\, \frac{p_r(X)}{p_r(q-t)} = \frac{q-t}{q^r - t^r}\, p_r(X). \end{aligned}$$

The formula follows by expanding the power sum symmetric functions in terms of Schur functions in the standard way [Mac] Sect. 3 Ex. 11 formula (2).

One can also give relatively simple formulas for $\eta^{\lambda}_{\mu}(q, t)$ when μ has two parts, $\mu = (s, r-s)$, or is a hook shape, $\mu = (1^{r-s}, s)$; the computations are similar to those given above and we shall not give them here.

A General Combinatorial Rule

A *rim hook* ([Mac]) is a skew diagram that is connected and does not contain any 2×2 block of boxes. We shall define the beginning of the rim hook to be the lowest and leftmost box in the rim hook (where we draw the diagrams with orientation as in Sect. 1). The length of a rim hook is the total number boxes in the rim hook. The *sign* of a rim hook h is defined to be

$$\varepsilon(h) = (-1)^{r-1}, \tag{6.5}$$

where r is the number of rows occupied by the rim hook h.

Let λ/μ be a skew shape and let $\alpha = (\alpha_1 \geqslant \cdots \geqslant \alpha_k > 0)$ be a partition such that $|\alpha| = |\lambda/\mu|$. We say that a rim hook h is a special rim hook of λ/μ if h is contained in λ/μ, h contains the leftmost square of the bottom row of λ/μ, and the removal of the squares of h from λ/μ results in a skew diagram ν/μ where $\mu \subseteq \nu \subseteq \lambda$. A *special rim hook tableau* of shape λ/μ and content $\alpha = (\alpha_1, \alpha_2, ...)$ is a sequence $(\mu = \nu_0 \subseteq \nu_1 \subseteq \cdots \subseteq \nu_k = \lambda)$ such that

(1) For $1 \leqslant i \leqslant k$, ν_i/ν_{i-1} is a special rim hook of ν_i/μ and

(2) $(|\nu_1/\nu_0|, |\nu_2/\nu_1|, ..., |\nu_k/\nu_{k-1}|)$ is a rearrangement of $(\alpha_1 \geqslant \cdots \geqslant \alpha_k)$.

The sign $\varepsilon(T)$ of a special rim hook tableau T is defined to be the product of the signs of the individual rim hooks in the tableau T,

$$\varepsilon(T)=\prod_{i=1}^{k} \varepsilon(v_i/v_{i-1}). \tag{6.6}$$

As an example, the following is a diagram of a special rim hook tableaux $T=(v_0 \subseteq v_1 \subseteq \cdots \subseteq v_5)$ of shape $(14, 9, 8, 5^3, 1^3)/(4^2, 2, 1)$ and content $(11, 10, 9, 4^2)$. In the diagram below, the i's in the diagram fill the squares of the rim hook v_i/v_{i-1}. The sign of T is $(-1)^{1-1}(-1)^{2-1}(-1)^{2-1}(-1)^{3-1}(-1)^{5-1}$.

				1	1	1	1	2	2	2	2	2	2
				2	2	2	2	2					
		3	3	4	4	4	4						
	3	3	4	4									
4	4	4	4	5									
5	5	5	5	5									
5													
5													
5													

The following result is proved in [ER2].

(6.7) PROPOSITION. *If λ/μ is a skew shape and α is a partition such that $|\lambda/\mu|=|\alpha|$ define*

$$K^{-1}_{\lambda/\mu,\,\alpha}=\sum_{T} \varepsilon(T),$$

where the sum is over all special rim hook tableaux of shape λ/μ and content α and $\varepsilon(T)$ is given by (6.6). Then the product of the monomial symmetric function $m_\alpha(X)$ and the Schur function $s_\mu(X)$ is given by

$$m_\alpha(X)\, s_\mu(X)=\sum_{\substack{\lambda \\ |\lambda/\mu|=|\alpha|}} K^{-1}_{\lambda/\mu,\,\alpha} K^{-1}_{\lambda/\mu,\,\alpha} s_\lambda(X),$$

where the sum is over all partitions λ such that λ/μ is a skew shape and such that $|\lambda/\mu|=|\alpha|$.

Define a *layered special rim hook tableau* T of shape λ and content μ to be a sequence of special rim hook tableaux

$$T=(T^{(0)}, T^{(1)}, ..., T^{(k)}),$$

such that if $T^{(i)}$ has shape $\lambda^{(i)}/\lambda^{(i-1)}$ and content $\alpha^{(i)}$ then

(1) $\varnothing = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(k)} = \lambda$, and

(2) $\alpha^{(1)} \cup \alpha^{(2)} \cup \cdots \cup \alpha^{(k)} = \mu$.

Define the weight of a layered special rim hook tableau $T = (T^{(0)}, T^{(1)}, ..., T^{(k)})$, to be

$$wt(T) = \prod_{i=1}^{k} \frac{\varepsilon(T^{(i)})(q-t)^{l(\alpha^{(i)})}\, t^{|\lambda^{(i-1)}|}}{q^{|\lambda^{(i)}|} - t^{|\lambda^{(i)}|}} \tag{6.8}$$

where the sign $\varepsilon(T^{(i)})$ is as given in (6.6).

The following theorem then follows by inductively using Proposition (6.7) to expand the right hand side of the first identity in Theorem (2.12) in terms of Schur functions.

(6.9) THEOREM. *The expansion of the Schur functions $s_\lambda(X)$ in terms of the functions $q_\lambda(X; q, t)$,*

$$s_\lambda(X) = \sum_{\lambda} \eta^\lambda_\mu(q, t)\, q_\mu(X; q, t), \qquad \textit{is given by} \qquad \eta^\lambda_\mu(q, t) = \sum_T wt(T),$$

where the sum is over all layered special rim hook tableaux T of shape λ and content μ, and the weight $wt(T)$ is given by (6.8).

Proof. Define

$$\tilde{\eta}^\lambda_\mu(q, t) = \sum_T wt(T),$$

where the sum is over all layered special rim hook tableaux T of shape λ and content μ and the weight $wt(T)$ is given by (6.8). Then it is easy to see that the $\tilde{\eta}^\lambda_\mu(q, t)$ satisfy the following recursions.

$$\begin{aligned} &\tilde{\eta}_{(1)(1)}(q, t) = 1, \\ &\tilde{\eta}^\lambda_\mu(q, t) = \sum_{\substack{\alpha \cup \beta = \mu \\ \alpha \neq \varnothing}} \sum_{\delta \vdash |\beta|} \frac{(q-t)^{l(\alpha)}\, (t)^{|\beta|}}{q^{|\mu|} - t^{|\mu|}} K^{-1}_{\lambda/\delta, \alpha} \tilde{\eta}^\delta_\beta(q, t). \end{aligned} \tag{6.10}$$

Clearly, from the definition (6.1), $\eta_{(1)(1)}(q, t) = 1$. Then, using Theorem (2.12) and Proposition (6.7),

$$\sum_{\lambda} \eta_\mu^\lambda(q,t)\, s_\lambda(X) = q_\mu^*(X;q,t)$$

$$= \sum_{\substack{\alpha\cup\beta=\mu\\ \alpha\neq\varnothing}} (q-t)^{l(\alpha)} \frac{t^{|\beta|}}{(q^{|\mu|}-t^{|\mu|})}\, m_\alpha(X)\, q_\beta^*(X;q,t)$$

$$= \sum_{\substack{\alpha\cup\beta=\mu\\ \alpha\neq\varnothing}} \sum_{\delta\vdash|\beta|} (q-t)^{l(\alpha)} \frac{t^{|\beta|}}{(q^{|\mu|}-t^{|\mu|})}\, \eta_\beta^\delta(q,t)\, m_\alpha(X)\, s_\delta(X)$$

$$= \sum_{\substack{\alpha\cup\beta=\mu\\ \alpha\neq\varnothing}} \sum_{\delta\vdash|\beta|} (q-t)^{l(\alpha)} \frac{t^{|\beta|}}{(q^{|\mu|}-t^{|\mu|})}\, \eta_\beta^\delta(q,t)$$

$$\times \sum_{\lambda\sqsupseteq\nu} K^{-1}_{\lambda/\delta,\alpha}\, s_\lambda(X)$$

$$= \sum_{\lambda} s_\lambda(X) \sum_{\substack{\alpha\cup\beta=\lambda\\ \alpha\neq\varnothing}} \sum_{\substack{\delta\vdash|\beta|\\ \delta\sqsubseteq\lambda}} (q-t)^{l(\alpha)} \frac{t^{|\beta|}}{(q^{|\mu|}-t^{|\mu|})}\, K^{-1}_{\lambda/\delta,\alpha}\, \eta_\beta^\delta(q,t).$$

Thus the η_μ^λ's satisfy the same recursions as the $\tilde{\eta}_\mu^\lambda$'s so that $\eta_\mu^\lambda(q,t) = \tilde{\eta}_\mu^\lambda(q,t)$ as claimed.

7. THE DETERMINANT OF THE CHARACTER TABLE OF THE IWAHORI–HECKE ALGEBRA

The *Iwahori–Hecke algebra of type A*, denoted $H_n(q)$, is the algebra with 1 generated over (q) by $T_{s_1}, ..., T_{s_{n-1}}$ subject to the relations

(B1) $T_{s_i} T_{s_{i+1}} T_{s_i} = T_{s_{i+1}} T_{s_i} T_{s_{i+1}}$,

(B2) $T_{s_i} T_{s_j} = T_{s_j} T_{s_i}$ if $|i-j|>1$,

(IH) $T_{s_i}^2 = (q-1)\, T_{s_i} + q$.

Let

$$\Xi(q,t) = (\chi_\mu^\lambda(q,t))$$

be the matrix with rows and columns indexed by partitions of n and entries $\chi_\mu^\lambda(q,t)$ as given in Theorem (5.5). It is known, see [R], that the matrix $\Xi(q,1)$ is the character table of the Iwahori–Hecke algebra and that the matrix $\Xi(1,1)$ is the character table of symmetric group S_n.

(7.1) THEOREM. (a) *The determinant*

$$\det(\Xi(q, t)) = \prod_{\lambda \vdash n} [\lambda]_{q, t},$$

where, for a positive integer k, $[k]_{q, t} = (q^k - t^k)/(q - t)$, *and if* $\lambda = (\lambda_1, \lambda_2, ...)$, *then* $[\lambda]_{q, t} = [\lambda_1]_{q, t} [\lambda_2]_{q, t} \cdots$.

(b) *For each partition* $\lambda = (1^{m_1} 2^{m_2} \cdots)$ *define* $i(\lambda) = 1^{m_1} 2^{m_2} \cdots$ *and* $m(\lambda) = m_1! \, m_2! \cdots$. *The determinant of the character table of the symmetric group,* S_n, *is given by*

$$\det(\Xi(1, 1)) = \prod_{\lambda \vdash n} i(\lambda) = \prod_{\lambda \vdash n} m(\lambda).$$

Proof. Note that the matrix $H(q, t) = (H_{\lambda\mu}(q, t))$ determined by Theorem (3.7) is upper triangular with respect to the lexicographic ordering on partitions. The diagonal entries are given by $H_{\lambda\lambda}(q, t) = [\lambda_1]_{q, t} [\lambda_2]_{q, t} \cdots$. Let $K = (K_{\lambda\mu})$ be the Kostka matrix determined by $h_\mu(X) = \sum_\lambda s_\lambda(X) K_{\lambda\mu}$. It is well known (see [Mac] Chapt. 1 (6.5)) that, with respect to the lexicographic ordering on partitions, K is upper triangular with diagonal entries 1. Then

$$q_\mu(X; q, t) = \sum_\rho H_{\mu\rho}(q, t) \, h_\rho = \sum_\rho \sum_\lambda H_{\mu\rho}(q, t) \, K_{\lambda\rho} s_\lambda = \sum_\lambda \chi^\lambda_\mu(q, t) \, s_\lambda(X).$$

This gives that $H(q, t) \, K^t = \Xi(q, t)$ and thus that

$$\det(\Xi(q, t)) = \det(H) \det(K^t) = \prod_{\lambda \vdash n} [\lambda]_{q, t}.$$

(b) The first equality follows by setting $q = t = 1$ in part (a). It follows from basic representation theory that inverse of $\Xi(1, 1)$ is the matrix

$$\Xi^{-1}(1, 1) = \left(\frac{\chi^\lambda_\mu(1, 1)}{i(\mu) \, m(\mu)} \right),$$

since $n!/i(\mu) \, m(\mu)$ is the number of elements in the conjugacy class of permutations of cycle type μ in S_n. Thus

$$\det(\Xi(1, 1))^{-1} = \det(\Xi^{-1}(1, 1)) = \det(\Xi(1, 1)) \prod_{\lambda \vdash n} \frac{1}{i(\lambda) \, m(\lambda)}.$$

The second equality now follows.

Remark. One might be tempted to guess that $\det(\Xi(q,t)) = \prod_{\lambda \vdash n} [m(\lambda)]_{q,t}$ where $[m(\lambda)]_{q,t} = [m_1]_{q,t}! \, [m_2]_{q,t}! \cdots [m_k]_{q,t}!$ for $\lambda = (1^{m_1} 2^{m_2} \cdots k^{m_k})$. This is, however, *not* true, as one can easily check for $n = 5$.

8. A COMBINATORIAL INTERPRETATION OF $\xi(q_n(X; q, 1))$

In [Br], F. Brenti defines a homomorphism ξ on the ring of symmetric functions and shows that the resulting polynomials arise naturally by enumerating sets of permutations of S_n with respect to the number of excedances. Recently, D. Beck and J. Remmel [BR] have extended this work by applying it to the various bases of the ring of symmetric functions and giving a q-analogue of Brenti's work. We now apply their homomorphism to $q_n(X; q, 1)$ and obtain a combinatorial interpretation for it.

Define the homomorphism ξ from the ring of symmetric functions to the ring $\mathbb{Q}(q)[x]$ by

$$\xi(e_k(X)) = \frac{(1-x)^{k-1}}{[k]!} \, q^{\binom{k}{2}},$$

where $e_k(X)$ denotes the elementary symmetric function and $[k]! = [k][k-1] \cdots [2][1]$, and $[j] = (q^j - 1)/(q-1)$. We want to determine $\xi(q_n(X; q, 1))$. We begin with some preliminary definitions.

Let $(\sigma_1, ..., \sigma_k)$ be a composition of n. Let $R(1^{\sigma_1} \cdots k^{\sigma_k})$ be the set of rearrangements of σ_1 1's, σ_2 2's, etc. For a sequence $r = r_1 r_2 \cdots r_n$ in this set, let

$$\overline{inv}(r) = \sum_{i<j} \chi(r_i < r_j),$$

where for a statement A, $\chi(A)$ is 1 if A is true and 0 otherwise. For a permutation τ written in cycle notation, define $\overline{inv}(\tau)$ analogously on each cycle. In other words, if $\tau = C_1, C_2, ..., C_k$, where each cycle is written so that it ends with its largest element, for a given cycle $C_i = (c_1, ..., c_l)$, let $\overline{inv}(C_i) = \sum_{i<j} \chi(c_i < c_j)$ and $\overline{inv}(\tau) = \sum_{i=1}^{k} \overline{inv}(C_i)$. For example, $\overline{inv}((1,3)(2,5,4,6)) = 6$. We shall use the following identity of Carlitz. A proof is given in [St] Sect. 1.3.17. For $\sigma_1 \cdots \sigma_k$ a composition of n,

$$\sum_{r \in R(1^{\sigma_1} \cdots k^{\sigma_k})} q^{\overline{inv}(r)} = \frac{[n]!}{[\sigma_1]! \cdots [\sigma_k]!} \tag{8.1}$$

For τ a permutation define the *excedances* of τ by

$$exc(\tau) = \{i : i < \tau(i)\}.$$

(8.2) THEOREM. *Let $\mathscr{C}(n)$ denote the set of permutations of cycle type (n). Then, for $\sigma_1, ..., \sigma_k$ a composition of n,*

$$\sum_{\sigma_1, ..., \sigma_k} \frac{[n-1]!}{[\sigma_1]! \cdots [\sigma_{k-1}]! \, [\sigma_k - 1]!} (x-1)^{n-k} \, q^{\binom{\sigma_1}{2} + \cdots + \binom{\sigma_{k-1}}{2} + \binom{\sigma_k - 1}{2} + n - 1}$$

$$= \sum_{\pi \in \mathscr{C}(n)} x^{|exc(\pi)|} q^{\overline{inv}(\pi)}.$$

Proof. Using (8.1), we have

$$\sum_{\sigma_1, ..., \sigma_k} \frac{[n-1]!}{[\sigma_1]! \cdots [\sigma_{k-1}]! \, [\sigma_k - 1]!} (x-1)^{n-k} \times q^{\binom{\sigma_1}{2} + \cdots + \binom{\sigma_{k-1}}{2} + \binom{\sigma_k - 1}{2} + n - 1}$$

$$= \sum_{\sigma_1, ..., \sigma_k} \sum_{r \in R(1^{\sigma_1} \cdots (k-1)^{\sigma_{k-1}} k^{\sigma_k - 1})} (x-1)^{n-k} \times q^{\overline{inv}(r)} q^{\binom{\sigma_1}{2} + \cdots + \binom{\sigma_{k-1}}{2} + \binom{\sigma_k - 1}{2} + n - 1}. \tag{8.3}$$

Each factor of $(x-1)$ can be used to mark the sequence as follows. Mark the elements in $r = (r_1, ..., r_{n-1})$ so that each occurrence of i in r except the first one is marked with either x or -1. End the sequence with an extra mark of x or -1. Let $MR_{n-1}(1^{\sigma_1} \cdots (k-1)^{\sigma_{k-1}} k^{\sigma_k - 1})$ represent the set of such marked sequences. (The subscript of $n-1$ is to remind us that, for example,

$$\begin{matrix} & & x & & -1 & -1 & & & x & x & -1 \\ 1 & 1 & 2 & & 2 & & 1 & & 3 & 4 & 3 & 3 \end{matrix}$$

is an element of $MR_9(1^3 2^2 3^3 4^1)$). For any marked sequence r, let $v(r)$ (resp. $w(r)$) be the number of x's (resp. the number of -1's) in the marked sequence r. With this interpretation, (8.3) becomes

$$\sum_{\sigma_1, ..., \sigma_k} \sum_{r \in MR_{n-1}(1^{\sigma_1} \cdots (k-1)^{\sigma_{k-1}} k^{\sigma_k - 1})} x^{v(r)} (-1)^{w(r)} \times q^{\overline{inv}(r)} q^{\binom{\sigma_1}{2} + \cdots + \binom{\sigma_{k-1}}{2} + \binom{\sigma_k - 1}{2} + n - 1}$$

$$= \sum_{r \in MS_{n-1}} x^{v(r)} (-1)^{w(r)} \, q^{\overline{inv}(r)} q^{\binom{\sigma_1}{2} + \cdots + \binom{\sigma_{k-1}}{2} + \binom{\sigma_k - 1}{2} + n - 1}, \tag{8.4}$$

where MS_n represents the set of marked sequences r of length n with elements from $1, 2, ..., n$ that have the following properties:

1. If i is an element of r then $i-1$ is also an element of r.

2. For each i in r, the first i is not marked and all subsequent occurrences of i are marked with either x or -1.

3. There is an extra mark of either x or -1 at the end of r.

For example, the following is a sequence in MS_7:

$$\begin{matrix} & x & & -1 & & -1 & x & x \\ 2 & 2 & 1 & 1 & 3 & 1 & 1 & \end{matrix}.$$

Let $M\mathscr{C}(n)$ be the set of permutations in $\mathscr{C}(n)$ written so that n occurs at the end of the cycle, and the cycle is marked according to the following rules:

1. n is marked with either x or -1.

2. The first element in the cycle is not marked.

3. All other elements in the cycle are either not marked, or marked with x or -1.

We now give a bijection ρ between the sets MS_{n-1} and $M\mathscr{C}(n)$. Given a sequence r in MS_{n-1}, the cycle $\rho(r) \in M\mathscr{C}(n)$ is constructed as follows.

1. The cycle begins with the positions of the 1's in r, in order, followed by the positions of the 2's in r, in order, etc.

2. The element i in the cycle is marked with the mark in position i of r.

3. n is the last element in the cycle and is marked with the extra mark of r.

For example, if

$$r = \begin{matrix} & & x & & -1 & -1 & x & x & x & -1 & -1 & x \\ 2 & 3 & 3 & 1 & 2 & 1 & 3 & 1 & 1 & 2 & 1 & \end{matrix}$$

then

$$\rho(r) = \begin{matrix} & -1 & x & x & -1 & & -1 & -1 & & x & x & x \\ 4 & 6 & 8 & 9 & 11 & 1 & 5 & 10 & 2 & 3 & 7 & 12 \end{matrix}.$$

Note that if the element in position i of $\rho(r)$ is the jth unmarked element of $\rho(r)$, then the element as well as all the consecutively marked elements that follow (not including n, the final element in the cycle) give the positions of the j's in r. For example, in $\rho(r)$ above, 1 is the second unmarked element, and there are two marked elements that immediately follow it, namely 5 and 10. This shows that the 2's in r are in positions 1, 5, and 10. In this way, we can see that ρ is a bijection. Given some cycle f in $\mathscr{C}(n)$, we can construct $r = \rho^{-1}(f)$ by using the mark on n, the last element in f, as the extra mark of r, and using the ith unmarked element of f as well as the consecutive marked elements that follow it (not including n, the last element in the cycle) to give the positions and marks of the i's in r.

We have that $v(r) = v(\rho(r))$ and $w(r) = w(\rho(r))$. Also, if $f = \rho(r)$, then

$$\overline{inv}(f) = \overline{inv}(r) + \binom{\sigma_1}{2} + \cdots + \binom{\sigma_{k-1}}{2} + \binom{\sigma_k - 1}{2} + n - 1,$$

because the factor of $\binom{\sigma_i}{2}$ counts the contribution to $\overline{inv}(f)$ created by replacing the sequence of i's in r with increasing integers, and the factor of q^{n-1} counts the contribution of the final n in the cycle to $\overline{inv}(f)$.

Thus (8.4) equals

$$\sum_{f \in M\mathscr{C}(n)} q^{\overline{inv}(f)} x^{v(f)} (-1)^{w(f)}. \tag{8.5}$$

Next we want to cancel terms with negative signs. To this end, we define an involution $\psi: M\mathscr{C}(n) \to M\mathscr{C}(n)$ as follows. In a cycle f, starting at the left, find the first element in the cycle, f_i, with $i > 1$, such that

1. f_i is marked with -1, or
2. f_i is unmarked and $f_i > f_{i-1}$.

In the first case, $\psi(f)$ is the cycle obtained by removing the -1 marking from f_i. In the second case, $\psi(f)$ is the cycle obtained by adding a -1 marking to f_i. The fixed points to this involution are cycles in $M\mathscr{C}(n)$ such that all marked elements are marked with x, each block of marked elements increases from left to right, and if an element is unmarked, it is smaller than the one preceeding it. For example, the following cycle is fixed under ψ.

$$\begin{matrix} x & x & x & & & x & x & & x & x & x \\ 4 & 6 & 9 & 11 & 1 & 5 & 10 & 2 & 3 & 7 & 12 \end{matrix}.$$

The number of x's in the marked cycle is exactly equal to the number of excedences of the cycle. Hence, (8.5) equals

$$\sum_{f \in \mathscr{C}(n)} x^{|exc(f)|} q^{\overline{inv}(f)}.$$

This completes the proof of Theorem 8.2.

We now give the combinatorial interpretation of $\xi(q_n(X; q, 1))$.

(8.6) THEOREM. *The image of $q_n(X; q, 1)$ under the q-analogue of the Brenti homomorphism has a combinatorial interpretation given by the equation*

$$[n-1]!\, \xi(q_n(X; q, 1)) = \sum_{f \in \mathscr{C}(n)} x^{|exc(f)|} q^{\overline{inv}(f)},$$

where the sum is over all permutations f of cycle type (n).

Proof. Using Theorem (3.7b),

$$\begin{aligned}
[n-1]!\, \xi(q_n(X: q, 1)) &= \sum_{\mu \vdash n} [n-1]!\, E_{(n)\,\mu}(q, 1)\, \xi(e_\mu) \\
&= \sum_{\mu \vdash n} \sum_{B = (d_1, ..., d_l) \in B_{(n)\,\mu}} (1-x)^{n-l(\mu)} \\
&\quad \times \frac{[n-1]!}{[\mu_1]!\, [\mu_2]! \cdots [\mu_l]!} (-1)^{n-l(\mu)} \\
&\quad \times q^{n-d_l} [d_l]\, q^{\binom{d_1}{2} + \cdots + \binom{d_{l-1}}{2} + \binom{d_l}{2}} \\
&= \sum_{\mu \vdash n} \sum_{B = (d_1, ..., d_l) \in B_{(n)\,\mu}} (x-1)^{n-l(\mu)} \\
&\quad \times \frac{[n-1]!}{[d_1]! \cdots [d_{l-1}]!\, [d_l]!} [d_l] \\
&\quad \times q^{\binom{d_1}{2} + \cdots + \binom{d_{l-1}}{2} + \binom{d_l}{2}} q^{n-d_l} \\
&= \sum_{d_1, ..., d_l} (x-1)^{n-l} \frac{[n-1]!}{[d_1]! \cdots [d_{l-1}]!\, [d_l - 1]!} \\
&\quad \times q^{\binom{d_1}{2} + \cdots + \binom{d_{l-1}}{2} + \binom{d_l - 1}{2} + n - 1},
\end{aligned}$$

where if B is a brick tabloid of shape (n) and content μ, we write $B = (d_1, ..., d_l)$ if $(d_1, ..., d_l)$ are the lengths of the μ-bricks from left to right that occur in B. The sum in the third equality is over all compositions of n. The third equality follows from the fact that $\binom{d_l}{2} + (n - d_l) = \binom{d_l - 1}{2} + n - 1$. The theorem now follows from Theorem (8.2).

(8.7) COROLLARY. *If $\lambda = (\lambda_1, ..., \lambda_k) = 1^{l_1} 2^{l_2} \cdots n^{l_n}$ is a partition of n, and $\mathscr{C}_\lambda$ represents the set of permutations of cycle type λ, then we have*

$$\binom{n}{k} \left(\prod_{i=1}^{k} [\lambda_i - 1]! \right) \frac{1}{l_1! \cdots l_n!} \xi(q_\lambda(X; q, 1)) = \sum_{\sigma \in \mathscr{C}_\lambda} x^{|exc(\sigma)|} q^{\overline{inv}(\sigma)}.$$

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