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Matrix Units for Centralizer Algebras

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We compute matrix units for Brauer's centralizer algebras and Hecke algebras of type A. This can be used to construct a complete system of matrix units of the centralizers of tensor products of classical Lie groups (except $SO(2n)$) and their quantum deformations. The calculation is done by induction inspired by path models for special operator algebras. It is similar to the calculation of Young's orthogonal matrix units for the symmetric group as given by Rutherford.

Classically, the finite dimensional irreducible representations of the Lie group $G = Gl(n)$ were determined by decomposing tensor products of the standard representation into irreducibles, see Weyl [Wy]. This decomposition was determined by looking at the algebra of centralizers of $\rho^{\otimes f}(G)$, the fth tensor product representation on $V^{\otimes f} = V \otimes V \otimes \cdots \otimes V$ (f times), $\dim V = n$. In the case of $G = Gl(n)$, this algebra is a quotient of the group algebra of the symmetric group, $\mathbf{C}S_f$. Using Young symmetrizers to decompose $\mathbf{C}S_f$ it became possible to decompose the $Gl(n)$ module $V^{\otimes f}$ into irreducibles.

Weyl and Brauer attempted a similar approach in the cases of the orthogonal, $O(n)$, and symplectic groups, $Sp(2n)$. In [Br], Brauer determined algebras, here denoted by D_f, $f \in \mathbf{N}$ which map surjectively onto the centralizers of tensor products of these groups. Mainly because of difficulties arising from the fact that these algebras are not semisimple in general, this approach proved to be complicated and other methods were used to find the irreducible components in the orthogonal and symplectic cases.

In this paper we use recent results about the semisimplicity of D_f (see [W2]) to compute a complete set of matrix units for D_f if it is semisimple. If the Brauer algebras are not semisimple, our formulas can be easily modified to compute matrix units for the semisimple quotient which appears as centralizer. Hence one obtains, at least in principle, a complete

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decomposition of tensor products of the standard representations of the full orthogonal and symplectic groups into their irreducibles.

We compute, by similar methods, also matrix units of Hecke algebras and of a q version of the Brauer algebras. It can be shown that the centralizers of tensor powers of the standard representation of quantum groups of types A, B, and C are quotients of those algebras, at least if the deformation parameter q is not a root of unity. So similarly as in the classical case, our matrix units can be used to describe the decomposition of $V^{\otimes f}$, V a finite dimensional vector space, into irreducible modules of a quantum group.

Our method is inspired by diagrams, usually referred to as Bratteli diagrams, which describe the inclusions $A_1 \subset A_2 \subset \cdots \subset A_f$ of finite dimensional semisimple algebras. The ascending paths in these diagrams yield a natural index set for our matrix units which is very suitable for inductive definitions. In fact it is possible to abstractly define matrix units for any sequence of algebras whose Bratteli diagram is given by a graded partially ordered set (see also [St]). For the Hecke algebras $H_f(q)$ of type A the corresponding poset is Young's lattice. Here we obtain a set of matrix units which specialize to Young's orthogonal matrix units for the symmetric groups at $q=1$. We provide inductive formulas which express these matrix units in terms of the standard generators of the Hecke algebras. This can then be used to compute matrix units of the q deformation of the Brauer algebras.

Our paper is organized as follows. In the first section, we determine explicit formulas for extending given matrix units for a pair of semisimple finite dimensional algebras $A \subset B$ to a canonical extension of B which is called the Jones "basic construction." In the second part, we compute matrix units for the Hecke algebras and the Brauer algebras and their q versions. Some of the results of this paper will also appear in [R] along with background and details.

1. General Part

Let us first fix some notation. In the following k will be a field of characteristic 0 and the algebra of all $n \times n$ matrices over k will be denoted by $M_n(k)$ or just by M_n. For convenience we will use the term *semisimple algebra* for the special case when an algebra is a finite direct sum of full matrix rings. So if A and B are semisimple k algebras, we can write them as $A = \bigoplus A_\alpha$ and $B = \bigoplus B_\beta$ with $A_\alpha \cong M_{a_\alpha}(k)$ and $B_\beta \cong M_{b_\beta}(k)$ for appropriate natural numbers a_α and b_β.

Fix for each β an index set I_β with b_β elements such that $I_\beta \cap I_{\beta'} = \varnothing$ if

$\beta \neq \beta'$. Then a set $\{e_{ij}, i, j \in I_\beta\}$ of nonzero elements in B_β is said to be a set of matrix units of B_β if

$$e_{ij} e_{kl} = \delta_{jk} e_{il},$$

where δ_{jk} is the Kronecker delta.

Let $I(B) = \bigcup I_\beta$. Then one obtains similarly a set of matrix units $\{e_{ij}, i, j \in I(B)\}$ for B, with the additional convention $e_{ij} = 0$ if i and j do not belong to the same I_β. We will usually denote a system of matrix units for B just by (e_{ij}) or, to distinguish it from systems for other algebras, by (e^B_{ij}).

Similarly, one can define an index set $I(A)$ and matrix units (e^A_{rs}) for a subalgebra $A \subset B$. The following theorem states that it is possible to "refine" a system of matrix units for A to a system for B.

THEOREM 1.1. *Let (e^A_{rs}) be a set of matrix units for $A \subset B$.*

(a) *Then there exists a system of matrix units (e^B_{ij}) for B and a matrix $G = (g_{\alpha\beta})$, labeled by the simple components of A and B such that every matrix unit e^A_{rs} can be expressed as a sum of matrix units in (e^B_{ij}). Moreover, if $e_{rs} \in A_\alpha$, exactly $g_{\alpha\beta}$ of these summands are in B_β.*

(b) *There exists a map $i \mapsto i'$ from $I(B)$ onto $I(A)$ such that*

$$e^A_{rr} = \sum_{i' = r} e^B_{ii}.$$

This equation also implies

$$e^A_{rr} e^B_{ij} = \delta_{ri'} e^B_{ij} \qquad \text{and} \qquad e^B_{ij} e^A_{rr} = \delta_{j'r} e^B_{ij}$$

and

$$e^B_{ij} e^A_{j'i'} = e^B_{ii} = e^A_{i'j'} e^B_{ji}.$$

Proof. The theorem has been proved for $k = \mathbf{C}$ in [Bt, Proposition 1.7]. It can be generalized to arbitrary k. Details also appear in [R].

The matrix $G = (g_{\alpha\beta})$ is called the inclusion matrix for $A \subset B$. The inclusion of A in B is conveniently described by an inclusion diagram. This is a graph with vertices arranged in 2 lines. In one line, the vertices are in 1-1 correspondence with the minimal direct summands A_α of A, in the other one with the summands B_β of B. Then a vertex corresponding to A_α is joined with a vertex corresponding to B_β by $g_{\alpha\beta}$ edges. If A and B have the same identity, one can compute the b_β's by

$$b_\beta = \sum_\alpha g_{\alpha\beta} a_\alpha. \tag{1.1}$$

If one has a sequence $k = A_0 \subset A_1 \subset A_2 \subset \cdots$ of finite dimensional semisimple algebras, the graph generated by the inclusion diagrams $A_{f-1} \subset A_f$, $f \in \mathbf{N}$, where the vertices corresponding to the simple components of A_f are arranged on the fth line of the graph, is called a *Bratteli diagram.*

EXAMPLES. (1) The standard example for this set-up is the inclusion $kS_{f-1} \subset kS_f$ of the group algebras of the symmetric groups $S_{f-1} \subset S_f$. Here one can choose as index sets, $I(kS_{f-1})$ and $I(kS_f)$, all standard tableaux with $f-1$ and f boxes, respectively. Taking the orthogonal matrix units, one sees that the map $i \mapsto i'$ is given by taking away the box containing f (see Section 2 for details).

(2) Our first example can be generalized in the following way: Let S be a graded partially ordered set (i.e., it contains a unique smallest element $\varnothing$ and for any $s \in S$ each maximal chain from $\varnothing$ to s has the same length, denoted by $\deg(s)$). Then S defines a Bratteli diagram for a sequence of algebras $A_1 \subset A_2 \subset \cdots$ in the following way (see [St]). The simple components of A_f are labeled by the elements of S of degree f. For a given $s \in S$ one defines the index set I_s to be all maximal chains from $\varnothing$ to s. For a chain $c \in I_s$ one defines c' to be the chain obtained from c by removing its last element s.

If, for each $s \in S$ of degree f, z_s is the corresponding central idempotent of A_f and if $c, d \in I_s$, one can define matrix units inductively by $e_{\varnothing\varnothing} = 1$ and

$$e_{cd} = z_s e_{c'd'}.$$

The matrix units which we are going to compute in this paper are all of this type. The difficulty consists of finding formulas for them with respect to algebraic generators of the algebras without knowing formulas for the central idempotents.

An important role will be played by traces, i.e., functionals tr: $B \to k$ such that $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for all $a, b \in B$. As there is up to scalar multiples only one trace on $M_n(k)$, any trace tr on $B = \bigoplus B_\beta$ is completely determined by a vector $\mathbf{t} = (t_\beta)$, where $t_\beta = \mathrm{tr}(p_\beta)$ and p_β is a minimal idempotent of B_β. The vector $\mathbf{t}$ is usually called the weight vector of tr. Recall that tr is nondegenerate if there exists for any $b \in B$, $b \neq 0$, a $b' \in B$ such that $\mathrm{tr}(bb') \neq 0$. In our setting this is equivalent to $t_\beta \neq 0$ for all $\beta \in I(B)$. Obviously, the matrix units form a linear basis for B. Moreover, if tr is nondegenerate with weight vectoı $\mathbf{t}$ and if $b = \sum b_{ij} e_{ij}$ with $b_{ij} \in k$ one can recover the coefficients by the formula

$$\mathrm{tr}(be_{ij}) = t_\beta b_{ji}, \qquad \text{if} \quad i, j \in I_\beta. \tag{1.2}$$

Given a trace, tr, nondegenerate on both A and B, the conditional expectation $\varepsilon_A: B \to A$ with respect to tr is defined by setting $\varepsilon_A(b)$ to be the unique element of A such that

$$\operatorname{tr}(\varepsilon_A(b)\, a) = \operatorname{tr}(ba) \qquad \text{for all} \quad a \in A, \tag{1.3}$$

for each $b \in B$. ε_A is well-defined and unique.

Let B be represented via left multiplication on itself, where we write B_ξ for the representation space B to distinguish it from the algebra B (with elements b_ξ). One obtains from ε_A an idempotent $e: B_\xi \to B_\xi$ defined by $eb_\xi = \varepsilon_A(b)_\xi$. e can be thought of as an orthogonal projection onto A with respect to the bilinear form $(b_\xi, c_\xi) \mapsto \operatorname{tr}(bc)$. The algebra $\langle B, e\rangle$ generated by B, acting via left multiplication on B_ξ, and by the idempotent e is called the Jones "basic" (or fundamental) construction for $A \subset B$. One has the following results (see [Jo, W2]).

THEOREM 1.2. *Let A, B, e, tr, and ε_A be as above. Then*

(a) *The algebra $\langle B, e\rangle$ is isomorphic to the centralizer* $\operatorname{End}_A B$ *of A acting by left multiplication on B. In particular, it is semisimple.*

(b) *There is a* 1-1 *correspondence between the simple components of A and* $\operatorname{End}_A B$ *such that if $p \in A_\alpha$ is a minimal idempotent, pe is a minimal idempotent of $\langle B, e\rangle_\alpha$. Under this correspondence, the inclusion matrix for $B \subset \langle B, e\rangle$ is the transposed G^t of the inclusion matrix for $A \subset B$.*

(c) *$ebe = \varepsilon_A(b)\, e$ for all $b \neq B$.*

(d) *Elements of the form $b_1 e b_2$, $b_1, b_2 \in B$ span $\langle B, e\rangle$ linearly.*

The following lemma computes the conditional expectations of matrix units of B.

LEMMA 1.3. *Let $e^B_{ij} \in B_\beta$ be an element of a system of matrix units obtained as a refinement from a system (e^A_{rs}). Then*

$$\varepsilon_A(e^B_{ij}) = \begin{cases} \dfrac{t_\beta}{s_\alpha} e^A_{i'j'} & \text{if both } i' \text{ and } j' \text{ are in } I_\alpha; \\[2ex] 0, & \text{if } i' \text{ and } j' \text{ belong to distinct } I_\alpha\text{'s}; \end{cases}$$

where $\mathbf{s} = (s_\alpha)$ and $\mathbf{t} = (t_\beta)$ denote the trace vectors on A and B, respectively.

Proof. Let $a \in A$. Then there exist scalars $a_{r,s} \in k$, $r, s \in I(A)$ such that $a = \sum a_{rs} e^A_{rs}$. Using the identities in Theorem 1.1(b) and (1.2) one obtains

$$\operatorname{tr}(e^B_{ij} a) = \operatorname{tr}(e^A_{i'i'} e^B_{ij} e^A_{j'j'} a) = \operatorname{tr}(e^B_{ij}(e^A_{j'j'} a e^A_{i'i'})). \tag{$*$}$$

If i' and j' belong to distinct I_α's, $e^A_{j'j'}$ and $e^A_{i'i'}$ belong to different simple components. Hence $e^A_{j'j'}ae^A_{i'i'}=0$, and therefore, using (1.3) and (*), $\varepsilon_A(e^B_{ij})=0$, which shows the second case of the claim. If i' and j' belong to the same I_α, it follows from Theorem 1.1 that

$$\mathrm{tr}(e^B_{ij}a)=\mathrm{tr}(e^B_{ij}a_{j'i'}e^A_{j'i'})=a_{j'i'}\,\mathrm{tr}(e^B_{ii})=a_{j'i'}t_\beta.$$

On the other hand, we have by (1.2) and Theorem 1.1(b) that

$$\mathrm{tr}((t_\beta/s_\alpha)\,e^A_{i'j'}\,a)=(t_\beta/s_\alpha)\,s_\alpha a_{j'i'}=a_{j'i'}t_\beta.$$

It follows from the last 2 equations

$$\mathrm{tr}(e^B_{ij}a)=\frac{t_\beta}{s_\alpha}\,\mathrm{tr}(e^A_{i'j'}a)\qquad\text{for all}\quad a\in A.$$

Hence $\varepsilon_A(e^B_{ij})=(t_\beta/s_\alpha)\,e^A_{i'j'}$ by (1.2).

In the following we will assume that all the entries $g_{\alpha\beta}$ of G are either 0 or 1. The following results in this section can also be shown in the general case with essentially the same proof, but messier notation (more precisely, one would have several copies of the same index set I_β in the union below which would require an additional index). If G is as above it follows from Theorem 1.2 that the index set I^α for $\langle B, e\rangle_\alpha$ can be identified with the union of all those I_β for which $g_{\alpha\beta}\neq 0$, i.e.,

$$I^\alpha=\bigcup_{\beta:\,g_{\alpha\beta}\neq 0} I_\beta.$$

THEOREM 1.4. *Let $i, j\in I^\alpha$ with $i\in I_\beta$ and $j\in I_{\beta'}$ and let (e^B_{ij}) be a system of matrix units for B which is obtained as a refinement from a system of matrix units for A as in Theorem* 1.1. *Fix $\bar{\imath}\in I_\beta$ and $\bar{\jmath}\in I_{\beta'}$ such that $\bar{\imath}'=\bar{\jmath}'\in I_\alpha$. Define*

$$e^\alpha_{ij}=\frac{s_\alpha}{\sqrt{t_\beta t_{\beta'}}}\,e^B_{i\bar{\imath}}ee^B_{\bar{\jmath}j}.$$

Then e^α_{ij} is independent of the choices of $\bar{\imath}$ and $\bar{\jmath}$ and $(e^\alpha_{ij}, i, j\in I^\alpha)$ forms a set of matrix units for $\langle B, e\rangle_\alpha$. We thus obtain a complete set of matrix units for $\langle B, e_A\rangle$ which is a refinement of (e^B_{ij}).

Proof. Let $i, j, k, l\in I^\alpha$ and choose indices $\bar{\imath}, \bar{\jmath}, \bar{k}, \bar{l}$ in I^α satisfying the conditions for the definition of e^α_{ij} and e^α_{kl}. One has that

$$e^\alpha_{ij}e^\alpha_{kl}=\gamma e^B_{i\bar{\imath}}ee^B_{\bar{\jmath}j}e^B_{k\bar{k}}ee^B_{\bar{l}l},$$

where $\gamma \in k$, $\gamma \neq 0$. By definition of matrix units, the right hand side of this product is equal to 0 if $j \neq k$. In the second case we can assume that we have β, β', and β'' such that $i \in I_\beta$, $j = k \in I_{\beta'}$, and $l \in I_{\beta''}$. In this case one has

$$\gamma = \frac{s_\alpha^2}{\sqrt{t_\beta t_{\beta'}^2 t_{\beta''}}}.$$

Recall that by Theorem 1.2 one has $ebe = \varepsilon_A(b)\, e$ for all $b \in B$. So if $j = k$ it follows from the last 2 equations and Lemma 1.3 that

$$e_{ij}^\alpha e_{kl}^\alpha = \gamma e_{i\bar{i}}^B e e_{\bar{j}\bar{k}}^B e e_{\bar{l}l}^B = \gamma \frac{t_{\beta'}}{s_\alpha} e_{i\bar{i}}^B e_{\bar{j}'\bar{k}'}^A e e_{\bar{l}l}^B = \frac{s_\alpha}{\sqrt{t_\beta t_{\beta''}}} e_{i\bar{k}}^B e e_{\bar{l}l}^B = e_{il}^\alpha.$$

To show that these matrix units are nonzero, we use the representation of $\langle B, e\rangle$ on B_ξ. It follows

$$\begin{aligned} e_{ij}^\alpha (e_{jj})_\xi &= \frac{s_\alpha}{\sqrt{t_\beta t_{\beta'}}} e_{i\bar{i}}^B \varepsilon_A(e_{\bar{j}j}^B)_\xi = \sqrt{\frac{t_{B'}}{t_\beta}} (e_{i\bar{i}}^B e_{\bar{j}'\bar{j}'}^A)_\xi \\ &= \sqrt{\frac{t_{\beta'}}{t_\beta}} (e_{i\bar{i}}^B e_{\bar{i}'\bar{i}'}^A)_\xi = \sqrt{\frac{t_{\beta'}}{t_\beta}} (e_{i\bar{i}}^B)_\xi. \end{aligned}$$

To show that e_{ij}^α does not depend on the choice of $\bar{i}$ and $\bar{j}$, choose different $\bar{i}$ and/or $\bar{j}$ in the definition of e_{ij}^α. Denote these by $\bar{k}$ and $\bar{l}$ in order to distinguish them from the originals, $\bar{i}$ and $\bar{j}$. Recall that e commutes with all elements of A and that $\bar{k}' = \bar{l}'$. Then Theorem 1.1(b) gives that

$$e_{i\bar{i}}^B e e_{\bar{j}j}^B = e_{i\bar{k}}^B e_{\bar{k}'\bar{i}'}^A e e_{\bar{j}j}^B = e_{i\bar{k}}^B e e_{\bar{k}'\bar{i}'}^A e_{\bar{j}j}^B = e_{i\bar{k}}^B e e_{\bar{l}'\bar{j}'}^A e_{\bar{j}j}^B = e_{i\bar{k}}^B e e_{\bar{l}j}.$$

It follows immediately from the definition of the matrix units that the relations of Theorem 1.1(b) also hold for $B \subset \langle B, e\rangle$ with corresponding sets of matrix units (e_{ij}) and (e_{ij}^α) (with $i, j \in \bigcup I^\alpha$).

2. Matrix Units for Brauer and Hecke Algebras

Let us first fix some notation. We write a partition λ as a set of nonnegative integers $\lambda = [\lambda_1, \lambda_2, ..., \lambda_r]$, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r \geqslant 0$. The empty partition is denoted by $\varnothing$. As usual, we identify a partition with the shape given by putting λ_1 boxes in the first row, λ_2 boxes in the second, and so on. Given a partition λ, λ_i and λ'_j will denote the length of the ith row and the length of the jth column, respectively. By convention, we set $\lambda_i = 0$ if $i > r$. The weight of λ is $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_r$. We say that λ is contained in the partition μ if $\lambda_i \leqslant \mu_i$ for all i, and write $\lambda \subseteq \mu$. $\lambda \subset \mu$ denotes that

$\lambda_i < \mu_i$ for some i. A standard tableau of shape λ is a filling of λ by the numbers $1, 2, ..., f$ such that the numbers are strictly increasing down the rows and the columns. Note that there exists a 1-1 correspondence between standard tableaux of shape λ and increasing sequences of partitions $\emptyset = \lambda^0 \subset [1] = \lambda^1 \subset \lambda^2 \subset \cdots \subset \lambda^f = \lambda$ such that λ^i is the shape consisting of the boxes of t which contain the first i numbers.

We will need the following generalization of standard tableaux. Let λ be a partition with $|\lambda| \geqslant f$. An *up-down tableau* of shape λ and length f is a sequence of partitions $\emptyset = \lambda^0, \lambda^1, ..., \lambda^{f-1}, \lambda^f = \lambda$ such that λ^i differs from λ^{i-1} by exactly one block, for which we will also use the notation $|\lambda^i \Delta \lambda^{i-1}| = 1$. There exist up-down tableaux of length f for all shapes λ with $f, f-2, f-4, ...$ boxes. Observe that if λ has f boxes, every up-down tableau is necessarily a standard tableau. Let T_λ and T_λ^f denote the set of all standard tableaux of shape λ and the set of all up-down tableaux of shape λ and length f, respectively. Using the interpretation of tableaux as a sequence of partitions, one defines a map $t \mapsto t'$, where t' is the tableau corresponding to the sequence for t without the last partition. It follows almost immediately from the definitions that this map defines the bijections

$$T_\lambda \leftrightarrow \bigcup_{\mu \subset \lambda, |\mu| = |\lambda| - 1} T_\mu \tag{2.1}$$

and

$$T_\lambda^f \leftrightarrow \bigcup_{|\mu \Delta \lambda| = 1} T_\mu^{f-1}. \tag{2.2}$$

It is well known that $\mathbf{C}S_f$, the group algebra of the symmetric group S_f, is isomorphic to $\bigoplus_\lambda M_{n_\lambda}(\mathbf{C})$ where the simple components can be indexed by partitions λ with $|\lambda| = f$ and n_λ is exactly the number of standard tableaux of shape λ. This implies that matrix units, o_{mp}^λ, of $\mathbf{C}S_f$ can be indexed by allowing λ to range over partitions with $|\lambda| = f$ and m, p to range over standard tableaux of shape λ.

In the following we define orthogonal matrix units for the Hecke algebras of type A, which for $q = 1$ gives us the orthogonal matrix units for the group algebras of the symmetric groups. $H_f(q)$ is the complex algebra presented by generators $g_1, ..., g_{f-1}$ and relations

(B1) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$, $1 \leqslant i \leqslant f-2$

(B2) $g_i g_j = g_j g_i$, $|i-j| \geqslant 2$

(H) $g_i^2 = (q-1) g_i + q$, $1 \leqslant i \leqslant f-1$.

When $g = 1$, the presentation above is a standard presentation of the symmetric group algebra $\mathbf{C}S_f$. When q is neither 0 nor a root of unity, it is known that $H_f(q)$ is isomorphic to $\mathbf{C}S_f$, and the isomorphisms can be chosen to respect the imbeddings of $H_f(q)$ in $H_{f+1}(q)$ and $\mathbf{C}S_f$ in $\mathbf{C}S_{f+1}$; see, for example, [W1]. Thus for such q, the irreducible representations of $H_f(q)$ are labeled by the set Λ_f of partitions of size f, and the Bratteli diagram describing the inclusion of the sequence of Hecke algebras is the same as the one for the sequence of group algebras $\mathbf{C}S_f$.

Our definitions of idempotents is based on a set of representations π_λ, $\lambda \in \Lambda_f$, of the Hecke algebra $H_f(q)$ which yields for $q=1$ Young's orthogonal representations of the symmetric group (see [W1]). For this we also need to mention the following quantities, which figure in the definition of the representations π_λ. For $t \in T_\lambda$, define

$$d(t, i) = c(i+1) - c(i) - r(i+1) + r(i),$$

where $c(n)$ and $r(n)$ denote column and row of the box containing the number n, and for $d \in \mathbf{Z} \setminus \{0\}$ let

$$b_d(q) = \frac{q^d(1-q)}{1-q^d}.$$

Then one can define a representation π_λ on a vector space V_λ with orthonormal basis $\{v_t, t \in T_\lambda\}$ by

$$\pi_\lambda(g_i)\, v_t = b_d v_t + \frac{\sqrt{(1-q^{d+1})(1-q^{d-1})}}{1-q^d} v_{g_i(t)}, \tag{2.3}$$

where $d = d(t, i)$ and $g_i(t)$ is the tableau obtained from t by interchanging the numbers i and $i+1$. This representation follows from the ones defined in [W1] by setting $\pi_\lambda(g_i) = (q+1)\, \pi_\lambda(e_i) - id_{V_\lambda}$, where e_i is the eigenprojection corresponding to the characteristic value q of g_i. It is shown in [W1] that this defines an irreducible semisimple representation of $H_f(q)$ which reduces to Young's orthogonal representation when $q = 1$. Moreover, the representation $\pi^{(f)} = \bigoplus_{\lambda \in \Lambda_f} \pi_\lambda$ is a faithful representation of $H_f(q)$.

It follows from the definitions (see also (2.1)) that the map $v_t \mapsto v_{t'}$ defines an isomorphism; i.e., we have

$$V_\lambda \cong \bigoplus_{\mu \subset \lambda, |\mu| = |\lambda| - 1} V_\mu. \tag{2.4}$$

It follows now immediately from the definition of π_λ in (2.3) that (2.4) describes the decomposition of V_λ into simple H_{f-1} modules. In particular, we have under the identification in (2.4)

$$\pi_\lambda(a) = \bigoplus_{\mu \subset \lambda, |\mu| = |\lambda| - 1} \pi_\mu(a) \qquad \text{for} \quad a \in H_{f-1}(q). \tag{2.5}$$

Using (2.2)–(2.5) one can define a family of idempotents $\{o_t : t \in T_f\}$ in $H_f(q)$ inductively as follows. First, $o_{[1]} = 1$. Fix $t \in T_{f+1}$ and let r be the tableau obtained by removing from t the box containing the number $f+1$. Then define

$$o_t = \prod_s \frac{o_r g_f o_r - b_{d(s,f)} o_r}{b_{d(t,f)} - b_{d(s,f)}},$$

where the product is over all $s \in T_{f+1}$ such that $s \neq t$ but removing from s the box containing the number $f+1$ also yields the tableau r. The family $\{o_t : t \in T_f\}$ is a partition of unity consisting of minimal idempotents in $H_f(q)$ such that $\pi_\lambda(o_t)$ is the orthogonal projection onto v_t for each $t \in T_\lambda$.

We generalize these results to obtain matrix units o_{mp}, labeled by standard tableaux m and p in the following way: Let $o_{pp} = o_p$. If m and p are in T_λ such that also m' and p' are of the same shape, we define

$$o_{mp} = o_{m'p'} o_p. \tag{2.6}$$

Otherwise, choose $\bar{m}$ and $\bar{p}$ in T_λ such that $\bar{m}'' = \bar{p}''$ and $\bar{m}'$ (resp. $\bar{p}'$) belongs to the same shape as m' (resp. p'). Observe that one obtains such tableaux by taking any tableau $\bar{m}$ (resp. $\bar{p}$) of shape λ such that f is in the same box as in m (resp. p) and $f-1$ is in the same box that f is in p (resp. m).

Then we define

$$o_{mp} = \frac{1-q^d}{\sqrt{(1-q^{d+1})(1-q^{d-1})}} o_{m'\bar{m}'} g_f o_{\bar{p}'p'} o_p, \tag{2.7}$$

where $d = d(\bar{m}, f)$. The following proposition shows that the elements o_{mp} are the q analogous of Young's orthogonal matrix units. A different set of minimal idempotents for the Hecke algebras, which are the q analogues of the Young symmetrizers were found by A. Gyoja [Gy].

PROPOSITION 2.1. *The elements o_{mp} as defined in* (2.6) *and* (2.7) *form a complete set of matrix units for the Hecke algebra $H_{f+1}(q)$ if q is not a root of unity. Moreover, if m, p, r are tableaux of the same shape, one also has*

$$o_{mp} o_{p'r'} = o_{mr},$$

$$o_{m'p'} o_{pr} = o_{mr}.$$

Proof. The proof goes by induction on f. As $\pi^{(f+1)}$ is faithful, it suffices to show that $\pi^{(f+1)}(o_{mp})\, v_t = \delta_{pt} v_m$. If m' and p' have the same shape, this follows from (2.5) and the properties of the o_t's. In the second case, the claim follows similarly from (2.5) and (2.3). This proves that the o_{mp}'s are matrix units. The second statement is checked similarly.

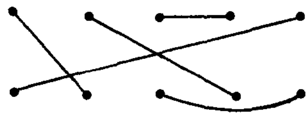
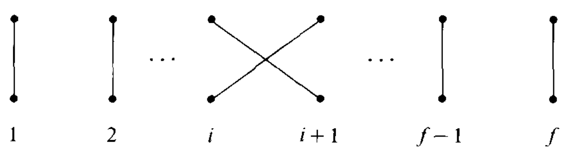


FIGURE 1

We now describe the Brauer algebras, $D_f(x)$, $f = 0, 1, 2, \ldots$. Let $\mathbf{C}[x]$ be the ring of polynomials in x over the complex numbers. We denote its quotient field by $\mathbf{C}(x)$. A diagram d on f vertices is a collection of $2f$ dots connected by f edges such that any dot belongs to only one edge. As an example, Fig. 1 is a diagram on 5 vertices. The product of two diagrams d_1 and d_2 on f vertices is determined by putting the diagram of d_1 above the diagram of d_2 and connecting the lower dots of d_1 to the upper dots of d_2. Let s be the number of cycles in that picture. The product $d_1 d_2$ is given by x^s times the resulting diagram without cycles. Figure 2 should make this clear. The $\mathbf{C}(x)$ linear span of all diagrams on f vertices with the linear extension of the multiplication of diagrams forms the Brauer algebra $D_f(x)$. It has dimension $(2f-1)(2f-3)\cdots 3\cdot 1$.

There is a natural imbedding of $D_{f-1}(x)$ into $D_f(x)$ given by extending a diagram on $f-1$ vertices to a diagram on f vertices by adding a pair of dots connected by a vertical edge on the end. There is also a natural embedding of $\mathbf{C}(x)\, S_f$, the group algebra of the symmetric group on f letters into $D_f(x)$, given by associating a permutation $\pi \in S_f$ to the diagram on f vertices which has the ith dot of the lower row connected to the $\pi(i)$th dot in the upper row for each i. We denote the diagram



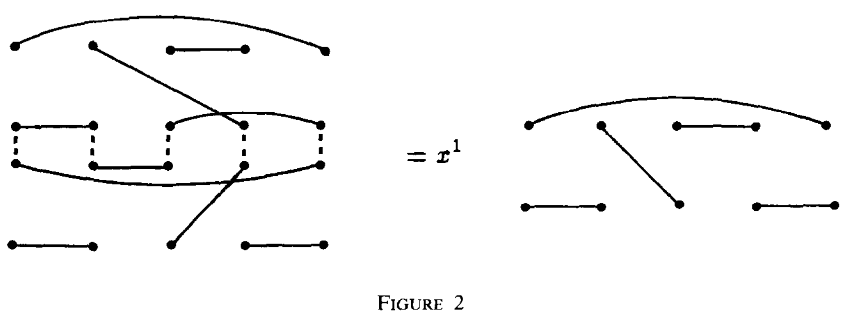
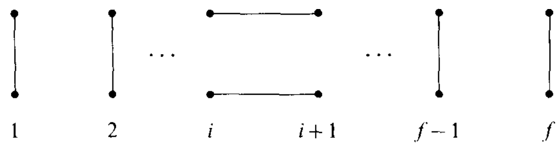


FIGURE 2

by g_i and the diagram



by e_i. Observe that the elements g_i satisfy the relations of simple reflections of the symmetric group, which are the same as the defining relations of the Hecke algebra for $q=1$. So using the same symbols for generators of the Hecke algebras and Brauer algebras should not create too much confusion. The elements $g_1, g_2, ..., g_{f-1}$ and $e_1, e_2, ..., e_{f-1}$ generate D_f. We will write $\bar{e}_i$ for $(1/x)\, e_i$. Notice that e_{f-1} and g_{f-1} commute with every element of D_{f-2} and that $\bar{e}_i$ is idempotent for each i. Denote the identity element of $D_f(x)$ by 1. Let I_f be the ideal of D_f consisting of the span of all diagrams that have at least one horizontal edge. We need the following results from [W2]:

THEOREM 2.2. (a) *There exists a nondegenerate trace* tr *on* D_f, *defined inductively by* $\mathrm{tr}(1)=1$ *and* $\mathrm{tr}(ag_fb)=\mathrm{tr}(ae_fb)=(1/x)\,\mathrm{tr}(ab)$ *for all* $a, b \in D_f$. *This definition determines* tr *completely.*

(b) I_{f+1} *is isomorphic to the basic construction for* $D_{f-1} \subset D_f$ *with the conditional expectation derived from the trace defined in* (a). *The isomorphism is given by the map* $b_1 \bar{e}_f b_2 \mapsto b_1 e b_2$, *where* e *is the projection coming from the basic construction.*

In order to apply Theorem 1.4 for computing matrix units for I_{f+1}, we need to know the weight vector of tr. Its components can be expressed by polynomials P_λ labeled by partitions λ which were derived from Weyl's dimension formulas for orthogonal and symplectic groups by El Samra and King (see [EK]). To define them, let $h(i,j)=\lambda_i+\lambda'_j-i-j+1$ be the length of the hook through the box $(i,j)\in\lambda$, i.e., the box in the ith row and jth column of λ and let

$$l(i,j)=\begin{cases}\lambda_i+\lambda_j-i-j+1 & \text{if } i\leqslant j,\\ -\lambda'_i-\lambda'_j+i+j-1 & \text{if } i>j.\end{cases} \tag{2.8}$$

Then the polynomials P_λ are given by

$$P_\lambda(x)=\prod_{(i,j)\in\lambda}\frac{x-1+l(i,j)}{h(i,j)}, \tag{2.9}$$

and the weight vector for D_f is given by $(P_\lambda/x^f)_{\lambda\in\Gamma_f}$.

We now define elements $e_{mp} \in D_f$, labeled by up-down tableaux m and p of length f and same shape λ. Let $\varnothing$ and $[1]$ also denote the unique tableau of shape $\varnothing$ and of shape $[1]$, respectively. Then we define

$$e_{\varnothing\varnothing} = e_{[1][1]} = 1. \tag{2.10}$$

If m, p are up-down tableaux of length $f > 1$ and of shape λ with $|\lambda| < f$ such that m' and p' are of shape μ and $\tilde{\mu}$, respectively, we define

$$e_{mp} = \frac{P_\lambda}{x\sqrt{P_\mu P_{\tilde{\mu}}}} e_{m's} \bar{e}_{f-1} e_{tp'} = \frac{P_\lambda}{\sqrt{P_\mu P_{\tilde{\mu}}}} e_{m's} e_{f-1} e_{tp'}, \tag{2.11}$$

where s and t are up-down tableaux of shape μ and $\tilde{\mu}$, respectively, such that $s' = t'$ is of shape λ. Observe that this is always possible as we have assumed λ to be of weight less than f.

Let $z_f = \sum_p e_{pp}$, where the summation goes over all up-down tableaux of length f and shape λ with $|\lambda| < f$. If m and p are standard tableaux of length f, one defines

$$e_{mp} = (1 - z_f)\, o_{mp}, \tag{2.12}$$

where o_{mp} is defined as in (2.6) and (2.7) with $q = 1$ and with g_i the corresponding generator of the Brauer algebra. It follows from the remark above that the o_{mp}'s form a set of matrix units of the subalgebra of D_f which is generated by the g_i's (which is isomorphic to $\mathbf{C}(x)S_f$).

THEOREM 2.3. *The elements e_{mp} defined above for each pair m and p of up-down tableaux of length f and the same shape form a complete set of matrix units for D_f. In particular, the element z_f is the central idempotent of the ideal I_f.*

Proof. The proof goes by induction on f with $f = 0$ and $f = 1$ being trivially true. By induction assumption we have that:

(1) Our system of matrix units for D_f is a refinement of the one for D_{f-1}.

(2) The simple components $D_{f,\mu}$ of D_f can be indexed by partitions μ of $f, f-2, f-4, \ldots$ boxes, since there exist up-down tableaux of length f for these shapes. Similarly, the simple components of $D_{f-1,\lambda}$ of D_{f-1} can be indexed by partitions λ of $f-1, f-3, f-5, \ldots$.

(3) The sets T^f_μ can be used for labeling a system of matrix units of $D_{f,\mu}$.

Using (1), we can apply Theorem 1.4 to define matrix units for the basic construction for $D_{f-1} \subset D_f$. By Theorem 2.2(b) this will give a system of matrix units for I_{f+1}. Part (2) above gives that the simple components of I_{f+1} can be indexed by partitions of $f-1, f-3, \ldots$. By (3) and the identity (2.2), we see that T_λ^{f+1} is exactly the set we need for labeling the system of matrix units for $I_{f+1,\lambda}$ (the λth simple component of the basic construction) that are defined by Theorem 1.4. Using this labeling we see that the elements e_{mp} defined in (2.11) for up-down tableaux m and p of shape λ, $|\lambda| < f$ are exactly the system of matrix units given by Theorem 1.4. It follows that z_{f+1} is the central idempotent corresponding to I_{f+1}.

Given the natural imbedding of the symmetric group S_{f+1} into D_{f+1} we see that $D_{f+1} \cong I_{f+1} \oplus \mathbf{C}(x)\, S_{f+1}$ as vector spaces. We also have a natural isomorphism (i.e., respecting multiplication) between the group of diagrams that do not have horizontal edges and permutations on $f+1$ symbols, giving that $D_{f+1}/I_{f+1} \cong \mathbf{C}(x)\, S_f$. In particular since $(1-z_{f+1})\,\pi \neq 0$ for any permutation π in S_{f+1} a dimension argument gives that

$$\begin{aligned} D_{f+1} &= z_{f+1} D_{f+1} \oplus (1-z_{f+1})\, D_{f+1} \\ &= I_{f+1} \oplus (1-z_{f+1})\, \mathbf{C}(x)\, S_{f+1}. \end{aligned}$$

Equation (2.12) gives a set of matrix units of $(1-z_{f+1})\,\mathbf{C}(x)\, S_{f+1}$. This combined with the fact that $T_\lambda^{f+1} = T_\lambda$ for all partitions λ such that $|\lambda| = f+1$ shows that we have a complete set of matrix units of D_{f+1}.

That these matrix units satisfy the formulas in Theorem 1.1(b) follows from Theorem 1.4 and Proposition 2.1.

COROLLARY [W2]. *$D_f \cong \bigoplus_\lambda M_{n_\lambda}$, where n_λ is the number of up-down tableaux of shape λ and λ runs over Γ_f, the union of all shapes with $f, f-2, f-4, \ldots$ boxes.*

The importance of the Brauer algebra lies in the fact that for $n \in \mathbf{Z}$ the complex algebra $D_f(n)$ maps surjectively onto the centralizer of the fth tensor power of the standard representation of the orthogonal group $O(n)$ for $n > 0$ and of the symplectic group $Sp(-n)$ for $n < -1$ and n even.

To describe this connection, let V be an n dimensional vector space and let $(E_{ij})_{1 \leqslant i,j \leqslant n}$ be a set of matrix units for End V. Then one defines the matrices $G, E \in \operatorname{End}(V \otimes V)$ by

$$G(v \otimes w) = w \otimes v$$

$$E = \sum_{i,j} E_{ij} \otimes E_{ij}.$$

We embed these matrices into the linear maps on the f-fold tensor power $V^{\otimes f}$ as G_i and E_i for $i = 1, 2, \ldots, f-1$, where E_i acts as matrix E on the ith and $(i+1)$st factor of $V^{\otimes f}$ and as identity on the other ones. It was shown by Brauer that the map $\Phi: D_f(n) \to \operatorname{End} V^{\otimes f}$, which maps g_i to G_i and e_i to E_i for $i = 1, 2, ..., f-1$ induces a homomorphism from $D_f(n)$ onto $B_f(O(n))$, the centralizer of the fth tensor power of the standard representation of $O(n)$. A similar statement also holds for the centralizer of $Sp(-n)$ (see [Br] or [W2] for details).

To describe the structure of the image of that homomorphism, we need the following definitions. We say that a shape λ is n permissible for $n \in \mathbf{Z}$ if the first 2 columns of λ contain at most n boxes for $n > 0$ and if the first row contains at most $-n/2$ boxes for $n < 0$ and n even. Moreover let $B_f(n)$ be the centralizer of $O(n)$ for $n > 0$ and let it be the centralizer for $Sp(-n)$ for $n < 0$ and n even.

Call an up-down tableau n-permissible if it contains only n-permissible diagrams. If m, p are n-permissible up-down tableaux of length $f > 1$ and of shape λ with $|\lambda| < f$ such that m' and p' are of shape μ and $\tilde{\mu}$, respectively, we define

$$e_{mp}(n) = \frac{P_\lambda(n)}{\sqrt{P_\mu(n)\, P_{\tilde{\mu}}(n)}}\, e_{m's}(n)\, E_{f-1}\, e_{tp'}(n), \tag{2.13}$$

where s and t are up-down tableaux of shape μ and $\tilde{\mu}$, respectively, such that $s' = t'$ is of shape λ. Let $z_f(n) = \sum_p e_{pp}(n)$, where the summation goes over all n-permissible up-down tableaux of length f and shape λ with $|\lambda| < f$. If m and p are n-permissible up-down tableaux of length f, one defines

$$e_{mp}(n) = (1 - z_f(n))\, o_{mp}, \tag{2.14}$$

where o_{mp} is defined as in (2.7) with $q = 1$ and with G_i in place of g_i.

THEOREM 2.4. (a) *There exists a nondegenerate trace on $B_f(n)$ and $I_{f+1}(n)$ is isomorphic to the basic construction for $B_{f-1}(n) \subset B_f(n)$ with the conditional expectation derived from this trace. Then*

$$B_{f+1}(n) \cong I_{f+1}(n) \oplus \left(\bigoplus_\lambda \mathbf{C}S_{f+1,\lambda} \right),$$

where the $\mathbf{C}S_{f+1,\lambda}$ are the simple components of $\mathbf{C}S_{f+1}$ given by the shapes λ where λ is n-permissible.

(b) *The elements e_{mp} given by* (2.10)–(2.12) *where m and p are up-down tableaux which only contain n-permissible partitions, are well defined and form a complete set of matrix units for $B_f(n)$.*

Proof. Part (a) follows from [W-2, Theorem 3.4]. For part (b), [W-2, Corollary 3.5] gives that the n-permissible shapes are exactly the shapes λ

such that $P_\lambda(n) \neq 0$. Thus the e_{mp} given by (2.13) are well defined. Then one applies exactly the same proof as for Theorem 2.3.

For the sake of completeness, we also mention that the same method works for constructing matrix units for the q version $C_f(r, q)$ of Brauer's centralizer algebra (see [W3]). $C_f(r, q)$ is given by generators $g_1, g_2, ..., g_{f-1}$. We use the same symbols for the generators here as we obtain the Brauer algebra as a special case of $C_f(r, q)$ (see below). We assume that the generators are invertible and satisfy the following relations

(B1) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$,

(B2) $g_i g_j = g_j g_i$ if $|i-j| \geqslant 2$,

(R1) $(g_i - r^{-1})(g_i + q^{-1})(g_i - q) = 0$,

(R2) $e_i g_{i-1}^{\pm 1} e_i = r^{\pm 1} e_i$, where e_i is defined by the equation

(D) $$(q - q^{-1})(1 - e_i) = g_i - g_i^{-1}.$$

It can be shown that for $r = q^{n-1}$ one obtains in the limiting case $q \to 1$ the Brauer algebra $D_f(n)$. It is shown in [W3] that $C_f(r, q)$ has the same decomposition into full matrix rings as D_f except possibly if q is a root of unity or if $r = q^n$ for some $n \in \mathbf{Z}$. In particular, in this case there exists a nondegenerate trace on $C_{f-1}(r, q)$ such that $C_f(r, q)$ is isomorphic to a direct sum of Jones' basic construction for $C_{f-2}(r, q) \subset C_{f-1}(r, q)$ and the Hecke algebra $H_f(q^2)$.

For computing the weight vector of $C_f(r, q)$, the rational functions P_λ are replaced by their q-versions $Q_\lambda(r, q)$. Using the notation $[n]_q = q^n - q^{-n}$ and $[y + n]_q = rq^n - r^{-1}q^{-n}$, they are defined by

$$Q_\lambda(r, q) = \prod_{(j,j) \in \lambda} \frac{[y + \lambda_j - \lambda'_j]_q + [h(j,j)]_q}{[h(j,j)]_q} \prod_{(i,j) \in \lambda, i \neq j} \frac{[y + l(i,j)]_q}{[h(i,j)]_q}.$$

Having this information at hand, one defines matrix units $e_{mp}(r, q)$ for $C_f(r, q)$ similarly as for D_f in the following way:

(a) Define elements $o_{mp}(q^2)$, using exactly the same formulas as in (2.6) and (2.7) for the o_{mp} except that now the g_i's are the generators of $C_f(r, q)$ and q is replaced by q^2. It should be noted that these elements are only matrix elements in the quotient modulo the ideal generated by the e_i's.

(b) Assume the matrix units are known for $C_{f-1}(r, q)$. Then the matrix units for up-down tableaux m and p of length f, belonging to a shape λ with less than f boxes is given by

$$e_{mp}(r, q) = \frac{Q_\lambda}{\sqrt{Q_\mu Q_{\tilde{\mu}}}} e_{m's}(r, q)\, e_{f-1}\, e_{tp'}(r, q), \tag{2.15}$$

where s, t, μ, and $\tilde{\mu}$ are as in (2.11). If m and p belong to a shape with f boxes, we define

$$e_{mp}(r, q) = (1 - z_f(r, q))\, o_{mp}(q^2), \tag{2.16}$$

where $z_f(r, q) = \sum_p e_{pp}(r, q)$ with the summation going over all up-down tableaux p of length f belonging to shapes with less than f boxes. We have now

COROLLARY 2.5. *The elements* $e_{mp}(r, q)$, *defined in* (2.15) *and* (2.16), *form a complete set of matrix units for* $C_f(r, q)$.

Proof. The proof that the elements e_{mp} form a set of matrix units is exactly the same as the one in Theorem 2.3.

Similarly as quotients of the algebra of the symmetric groups and of the Brauer algebras appear as centralizers of classical Lie groups, their q versions appear as centralizers of tensor powers of the standard representation of quantum groups, which are q versions of the universal enveloping algebras of the corresponding Lie algebras (see [D, Ji1, 2]). The homomorphisms are given in the following way.

For type A let V be an n-dimensional vector space and let (E_{ij}) be a set of matrix units for End V. Let $R \in \mathrm{End}(V \otimes V)$ be the special solution of the quantum Yang–Baxter equation by

$$\begin{aligned} R = \sum_i qE_{ii} \otimes E_{ii} + \sum_{i<j} &(q-1)\, E_{ii} \otimes E_{jj} \\ &+ q^{1/2}(E_{ij} \otimes E_{ji} + E_{ji} \otimes E_{ij}). \end{aligned}$$

Similarly as before for the Brauer algebras, one defines the elements $R_i \in \mathrm{End}\, V^{\otimes f}$, which act as matrix R on the ith and $(i+1)$st copy of $V^{\otimes f}$ and as identity on the other copies. Then the homomorphism, mentioned before, is given by $g_i \mapsto R_i$, where g_i is a generator of H_f.

For our second example, let V be an $n+1$ dimensional vector space with $n+1$ odd. It will be convenient to label a basis $\{v_i, i \in I\}$ of V by the set

$$I = \{-n+1, -n+3, ..., -1, 0, 1, 3, ..., n-1\}.$$

Let $\check{R}(x)$ be Jimbo's solution of the $QYBE$ for type $B^{(1)}_{n/2}$ (see [Ji1]). Following Turaev [T] one extracts from this the matrix $R = (k\xi)^{-1}\, \check{R}(0)$ which has the form

$$\begin{aligned} R = \sum_{i \neq 0} &(qE_{i,i} \otimes E_{i,i} + q^{-1}E_{i,-i} \otimes E_{-i,i}) \\ &+ E_{0,0} \otimes E_{0,0} + \sum_{i \neq j, -j} E_{i,j} \otimes E_{j,i} \\ &+ (q - q^{-1}) \left(\sum_{i<j} E_{i,i} \otimes E_{j,j} - \sum_{j<-i} q^{(i+j)/2} E_{i,j} \otimes E_{-i,-j} \right). \end{aligned}$$

Defining the matrices $R_i \in \mathrm{End}\, V^{\otimes f}$ as before, one obtains a homomorphism by $g_i \mapsto R_i$, where now the g_i's are the generators of $C_f(q^n, q)$.

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