UNIVERSAL VERMA MODULES AND THE MISRA-MIWA FOCK SPACE

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Abstract. The Misra-Miwa v-deformed Fock space is a representation of the quantized affine algebra $U_v(\widehat{\mathfrak{sl}}_\ell)$. It has a standard basis indexed by partitions and the non-zero matrix entries of the action of the Chevalley generators with respect to this basis are powers of v. Partitions also index the polynomial Weyl modules for $U_q(\mathfrak{gl}_N)$ as N tends to infinity. We explain how the powers of v which appear in the Misra-Miwa Fock space also appear naturally in the context of Weyl modules. The main tool we use is the Shapovalov determinant for a universal Verma module.

1. Introduction

Fock space is an infinite dimensional vector space which is a representation of several important algebras, as described in, for example, [14, Chapter 14]. Here we consider the charge zero part of Fock space, which we denote by $\mathbf{F}$, and its v-deformation $\mathbf{F}_v$. The space $\mathbf{F}$ has a standard $\mathbb{Q}$-basis $\{\,|\lambda\rangle \mid \lambda \text{ is a partition}\}$ and $\mathbf{F}_v := \mathbf{F} \otimes_{\mathbb{Q}} \mathbb{Q}(v)$. Following Hayashi [11], Misra and Miwa [23] define an action of the quantized universal enveloping algebra $U_v(\widehat{\mathfrak{sl}}_\ell)$ on $\mathbf{F}_v$. The only non-zero matrix elements $\langle \mu | F_{\bar{i}} | \lambda \rangle$ of the Chevalley generators $F_{\bar{i}}$ in terms of the standard basis occur when μ is obtained by adding a single $\bar{i}$-colored box to λ, and these are powers of v.

We show that these powers of v also appear naturally in the following context: Partitions with at most N parts index polynomial Weyl modules $\Delta(\lambda)$ for the integral quantum group $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$. Let V be the standard N dimensional representation of $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$. If the matrix element $\langle \mu | F_{\bar{i}} | \lambda \rangle$ is non-zero then, for sufficiently large N, $\left(\Delta^{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} V\right) \otimes_{\mathcal{A}} \mathbb{Q}(q)$ contains a highest weight vector of weight μ. There is a unique such highest weight vector v_μ which satisfies a certain triangularity condition with respect to an integral basis of $\Delta^{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} V$. We show that the matrix element $\langle \mu | F_{\bar{i}} | \lambda \rangle$ is equal to $v^{\mathrm{val}_{\phi_{2\ell}}(v_\mu, v_\mu)}$, where $(\cdot,\cdot)$ is the Shapovalov form and $\mathrm{val}_{\phi_{2\ell}}$ is the valuation at the cyclotomic polynomial $\phi_{2\ell}$.

Our proof is computational, making use of the Shapovalov determinant [26, 9, 20]. This is a formula for the determinant of the Shapovalov form on a weight space of a Verma module. The necessary computation is most easily done in terms of the universal Verma modules introduced in the classical case by Kashiwara [17] and studied in the quantum case by Kamita [15]. The statement for Weyl modules is then a straightforward consequence.

Before beginning, let us discuss some related work. In [19], Kleshchev carefully analyzed the $\mathfrak{gl}_{N-1}$ highest weight vectors in a Weyl module for $\mathfrak{gl}_N$, and used this information to give modular branching rules for symmetric group representations. Brundan and Kleshchev [6] have explained that highest weight vectors in the restriction of a Weyl module to $\mathfrak{gl}_{N-1}$ give information about highest weight vectors in a tensor product $\Delta(\lambda) \otimes V$ of a Weyl module with

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the standard N-dimensional representation of $\mathfrak{gl}_N$. Our computations put a new twist on the analysis of the highest weight vectors in $\Delta(\lambda) \otimes V$, as we study them in their "universal" versions and by the use of the Shapovalov determinant. Our techniques can be viewed as an application of the theory of Jantzen [12] as extended to the quantum case by Wiesner [28].

Brundan [5] generalized Kleshchev's [19] techniques and used this information to give modular branching rules for Hecke algebras. As discussed in [2, 21], these branching rules are reflected in the fundamental representation of $\widehat{\mathfrak{sl}}_p$ and its crystal graph, recovering much of the structure of the Misra-Miwa Fock space. Using Hecke algebras at a root of unity, Ryom-Hansen [25] recovered the full $U_v(\widehat{\mathfrak{sl}}_\ell)$ action on Fock space. To complete the picture one should construct a graded category, where multiplication by v in the $\widehat{\mathfrak{sl}}_\ell$ representation corresponds to a grading shift. Recent work of Brundan-Kleshchev [7] and Ariki [1] explains that one solution to this problem is through the representation theory of Khovanov-Lauda-Rouquier algebras [18, 24]. It would be interesting to explicitly describe the relationship between their category and the present work. Another related construction due to Brundan-Stroppel considers the case when the Fock space is replaced by $\wedge^m V \otimes \wedge^n V$, where V is the natural $\mathfrak{gl}_\infty$ module and m, n are fixed natural numbers.

We would also like to mention very recent work of Peng Shan [27] which independently develops a similar story to the one presented here, but using representations of a quantum Schur algebra where we use representations of $U_\varepsilon(\mathfrak{gl}_N)$. The approach taken there is somewhat different, and in particular relies on localization techniques of Beilinson and Bernstein [4].

This paper is arranged as follows. Sections 2 and 3 are background on the quantum group $U_q(\mathfrak{gl}_N)$ and the Fock space $\mathbf{F}_v$. Sections 4 and 5 explain universal Verma modules and the Shapovalov determinant. Section 6 contains the statement and proof of our main result relating Fock space and Weyl modules.

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2. The quantum group $U_q(\mathfrak{gl}_N)$ and its integral form $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$

This is a very brief review, intended mainly to fix notation. With slight modifications the construction in this section works in the generality of symmetrizable Kac-Moody algebras. See [8, Chapters 6 and 9] for details.

2.1. **The rational quantum group.** $U_q(\mathfrak{gl}_N)$ is the associative algebra over the field of rational functions $\mathbb{Q}(q)$ generated by

$$X_1, \dots, X_{N-1}, \quad Y_1, \dots, Y_{N-1}, \quad \text{and} \quad L_1^{\pm 1}, \dots, L_N^{\pm 1}, \tag{2.1}$$

with relations

$$L_iL_j = L_jL_i, \quad L_iL_i^{-1} = L_i^{-1}L_i = 1, \qquad X_iY_j - Y_jX_i = \delta_{i,j}\frac{L_iL_{i+1}^{-1} - L_{i+1}L_i^{-1}}{q - q^{-1}},$$

$$L_iX_jL_i^{-1} = \begin{cases} qX_j, & \text{if } i = j, \\ q^{-1}X_j, & \text{if } i = j+1, \\ X_j & \text{otherwise;} \end{cases} \qquad L_iY_jL_i^{-1} = \begin{cases} q^{-1}Y_j, & \text{if } i = j, \\ qY_j, & \text{if } i = j+1, \\ Y_j, & \text{otherwise;} \end{cases} \tag{2.2}$$

$$X_iX_j = X_jX_i \quad \text{and} \quad Y_iY_j = Y_jY_i, \qquad \text{if } |i-j| \geq 2,$$
$$X_i^2X_j - (q+q^{-1})X_iX_jX_i + X_jX_i^2 = Y_i^2Y_j - (q+q^{-1})Y_iY_jY_i + Y_jY_i^2 = 0, \qquad \text{if } |i-j| = 1.$$

The algebra $U_q(\mathfrak{gl}_N)$ is a Hopf algebra with coproduct and antipode given by

$$\begin{aligned} \Delta(L_i) &= L_i \otimes L_i, & & S(L_i) = L_i^{-1}, \\ \Delta(X_i) &= X_i \otimes L_iL_{i+1}^{-1} + 1 \otimes X_i, & \text{and} \quad & S(X_i) = -X_iL_i^{-1}L_{i+1}, \\ \Delta(Y_i) &= Y_i \otimes 1 + L_i^{-1}L_{i+1} \otimes Y_i, & & S(Y_i) = -L_iL_{i+1}^{-1}Y_i, \end{aligned} \tag{2.3}$$

respectively (see [8, Section 9.1]).

As a $\mathbb{Q}(q)$-vector space, $U_q(\mathfrak{gl}_N)$ has a triangular decomposition

$$U_q(\mathfrak{gl}_N) \cong U_q(\mathfrak{gl}_N)^{<0} \otimes U_q(\mathfrak{gl}_N)^0 \otimes U_q(\mathfrak{gl}_N)^{>0}, \tag{2.4}$$

where the inverse isomorphism is given by multiplication (see [8, Proposition 9.1.3]). Here $U_q(\mathfrak{gl}_N)^{<0}$ is the subalgebra generated by the Y_i for $i = 1, \ldots, N-1$, $U_q(\mathfrak{gl}_N)^{>0}$ is the subalgebra generated by the X_i for $i = 1, \ldots, N-1$, and $U_q(\mathfrak{gl}_N)^0$ is the subalgebra generated by the $L_i^{\pm 1}$ for $i = 1, \ldots, N$.

2.2. **The integral quantum group.** Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. For $n, k \in \mathbb{Z}_{>0}$ and $c \in \mathbb{Z}$, let

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}}, \; x^{(k)} := \frac{x^k}{[k][k-1]\cdots[2][1]}, \text{ and } \begin{bmatrix} x; c \\ k \end{bmatrix} := \prod_{s=1}^{k} \frac{xq^{c+1-s} - x^{-1}q^{s-1-c}}{q^s - q^{-s}}, \tag{2.5}$$

in $\mathbb{Q}(q, x)$. The *restricted integral form* $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ is the $\mathcal{A}$-subalgebra of $U_q(\mathfrak{gl}_N)$ generated by $X_i^{(k)}, Y_i^{(k)}$, $L_i^{\pm 1}$ and $\begin{bmatrix} L_i; c \\ k \end{bmatrix}$ for $1 \leq i \leq N, c \in \mathbb{Z}, k \in \mathbb{Z}_{>0}$. As discussed in [22, Section 6], this is an integral form in the sense that

$$U_q^{\mathcal{A}}(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathbb{Q}(q) = U_q(\mathfrak{gl}_N). \tag{2.6}$$

As with $U_q(\mathfrak{gl}_N)$, the algebra $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ has a triangular decomposition

$$U_q^{\mathcal{A}}(\mathfrak{gl}_N) \cong U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0} \otimes U_q^{\mathcal{A}}(\mathfrak{gl}_N)^0 \otimes U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{>0}, \tag{2.7}$$

where the isomorphism is given by multiplication (see [8, Proposition 9.3.3]). In this case, $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0}$ is the subalgebra generated by the $Y_i^{(k)}$, $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{>0}$ is the subalgebra generated by the $X_i^{(k)}$, and $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^0$ is generated by $L_i^{\pm 1}$ and $\begin{bmatrix} L_i; c \\ k \end{bmatrix}$ for $1 \leq i \leq N$, $c \in \mathbb{Z}$, and $k \in \mathbb{Z}_{>0}$.

2.3. **Rational representations.** The Lie algebra $\mathfrak{gl}_N = M_N(\mathbb{C})$ of $N \times N$ matrices has standard basis $\{E_{ij} \mid 1 \leq i, j \leq N\}$, where E_{ij} is the matrix with 1 in position (i, j) and 0 everywhere else. Let $\mathfrak{h} = \text{span}\{E_{11}, E_{22}, \ldots, E_{NN}\}$. Let $\varepsilon_i \in \mathfrak{h}^*$ be the weight of $\mathfrak{gl}_N$ given by $\varepsilon_i(E_{jj}) = \delta_{i,j}$. Define

$$\begin{aligned} &\mathfrak{h}_{\mathbb{Z}}^* := \{\lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \cdots + \lambda_N\varepsilon_N \in \mathfrak{h}^* \mid \lambda_1, \ldots, \lambda_N \in \mathbb{Z}\}, \\ &(\mathfrak{h}_{\mathbb{Z}}^*)^+ := \{\lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \cdots + \lambda_N\varepsilon_N \in \mathfrak{h}_{\mathbb{Z}}^* \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N\}, \\ &P^+ := \{\lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \cdots + \lambda_N\varepsilon_N \in (\mathfrak{h}_{\mathbb{Z}}^*)^+ \mid \lambda_N \geq 0\}, \\ &R^+ := \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq N\}, \\ &Q := \text{span}_{\mathbb{Z}}(R^+), \quad Q^+ := \text{span}_{\mathbb{Z}_{\geq 0}}(R^+), \quad \text{and} \quad Q^- := \text{span}_{\mathbb{Z}_{\leq 0}}(R^+). \end{aligned} \tag{2.8}$$

to be the set of *integral weights*, the set of *dominant integral weights*, the set of *dominant polynomial weights*, the set of *positive roots*, the *root lattice*, the *positive part of the root lattice*, and the the *negative part of the root lattice*, respectively.

For an integral weight $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_N\varepsilon_N$, the *Verma module* $M(\lambda)$ for $U_q(\mathfrak{gl}_N)$ of highest weight λ is

$$M(\lambda) := U_q(\mathfrak{gl}_N) \otimes_{U_q(\mathfrak{gl}_N)^{\geq 0}} \mathbb{Q}(q)_\lambda, \tag{2.9}$$

where $\mathbb{Q}(q)_\lambda = \mathrm{span}_{\mathbb{Q}(q)}\{v_\lambda\}$ is the one dimensional vector space over $\mathbb{Q}(q)$ with $U_q(\mathfrak{gl}_N)^{\geq 0}$ action given by

$$X_i \cdot v_\lambda = 0 \quad \text{and} \quad L_j \cdot v_\lambda = q^{\lambda_j} v_\lambda, \qquad \text{for } 1 \leq i \leq N-1,\ 1 \leq j \leq N. \tag{2.10}$$

Theorem 2.1. (see [8, Chapter 10.1]) *If $\lambda \in (\mathfrak{h}^*_\mathbb{Z})^+$ then $M(\lambda)$ has a unique finite dimensional quotient $\Delta(\lambda)$ and the map $\lambda \mapsto \Delta(\lambda)$ is a bijection between $(\mathfrak{h}^*_\mathbb{Z})^+$ and the set of isomorphism classes of irreducible finite dimensional $U_q(\mathfrak{gl}_N)$-modules.*

A *singular vector* in a representation of $U_q(\mathfrak{gl}_N)$ is a vector v such that $X_i \cdot v = 0$ for all i.

2.4. **Integral representations.** The *integral Verma module* $M^\mathcal{A}(\lambda)$ is the $U_q^\mathcal{A}(\mathfrak{gl}_N)$-submodule of $M(\lambda)$ generated by v_λ. The *integral Weyl module* $\Delta^\mathcal{A}(\lambda)$ is the $U_q^\mathcal{A}(\mathfrak{gl}_N)$-submodule of $\Delta(\lambda)$ generated by v_λ. Using (2.6) and (2.4),

$$M^\mathcal{A}(\lambda) \otimes_\mathcal{A} \mathbb{Q}(q) = M(\lambda), \quad \text{and} \quad \Delta^\mathcal{A}(\lambda) \otimes_\mathcal{A} \mathbb{Q}(q) = \Delta(\lambda). \tag{2.11}$$

In general, $\Delta^\mathcal{A}(\lambda)$ is not irreducible as a $U_q^\mathcal{A}(\mathfrak{gl}_N)$ module.

3. Partitions and Fock space

We now describe the v-deformed Fock space representation of $U_v(\widehat{\mathfrak{sl}}_\ell)$ constructed by Misra and Miwa [23] following work of Hayashi [11]. Our presentation largely follows [3, Chapter 10].

3.1. **Partitions.** A partition λ is a finite length non-increasing sequence of positive integers. Associated to a partition is its Ferrers diagram. We draw these diagrams as in Figure 1 so that, if $\lambda = (\lambda_1, \ldots, \lambda_N)$, then λ_i is the number of boxes in row i (rows run southeast to northwest $\nwarrow$). Say that λ is contained in μ if the diagram for λ fits inside the diagram for μ and let μ/λ be the collection of boxes of μ that are not in λ. For each box $b \in \lambda$, the *content* $c(b)$ is the horizontal position of b and the *color* $\bar{c}(b)$ is the residue of $c(b)$ modulo ℓ. In Figure 1, the numbers $c(b)$ are listed below the diagram. The *size* $|\lambda|$ of a partition λ is the total number of boxes in its Ferrers diagram.

The set P^+ of dominant polynomial weights from Section 2.3 is naturally identified with partitions with at most N parts. If $\lambda \in P^+$ then

$$\Delta(\lambda) \otimes \Delta(\varepsilon_1) \cong \bigoplus_{\substack{1 \leq k \leq N \\ \lambda + \varepsilon_k \in P^+}} \Delta(\lambda + \varepsilon_k) \tag{3.1}$$

as $U_q(\mathfrak{gl}_N)$-modules. The diagram of $\lambda + \varepsilon_k$ is obtained from the diagram of λ by adding a box on row k, and $\Delta(\lambda + \varepsilon_k)$ appears in the sum on the right side of (3.1) if and only if $\lambda + \varepsilon_k$ is a partition. See, for example, [10, Section 6.1, Formula 6.8] for the classical statement, and [8, Proposition 10.1.16] for the quantum case.

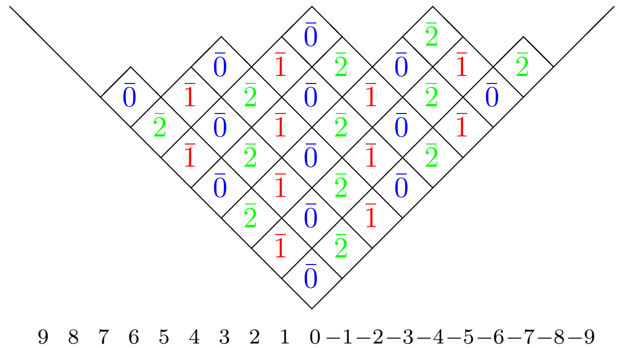
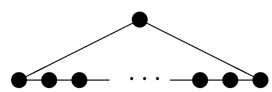


Figure 1. The partition $(7,6,6,5,5,3,3,1)$ with each box containing its color for $\ell = 3$. The content $c(b)$ of a box b is the horizontal position of b reading right to left. The contents of boxes are listed beneath the diagram so that $c(b)$ is aligned with all boxes b of that content.

3.2. **The quantum affine algebra.** Let $U_v'(\widehat{\mathfrak{sl}}_\ell)$ be the quantized universal enveloping algebra corresponding to the ℓ-node Dynkin diagram



More precisely, $U_v'(\widehat{\mathfrak{sl}}_\ell)$ is the algebra generated by $E_{\bar{i}}, F_{\bar{i}}, K_{\bar{i}}^{\pm 1}$, for $\bar{i} \in \mathbb{Z}/\ell\mathbb{Z}$, with relations

$$K_{\bar{i}}K_{\bar{j}} = K_{\bar{j}}K_{\bar{i}}, \quad K_{\bar{i}}K_{\bar{i}}^{-1} = K_{\bar{i}}^{-1}K_{\bar{i}} = 1, \qquad E_{\bar{i}}F_{\bar{j}} - F_{\bar{j}}E_{\bar{i}} = \delta_{\bar{i},\bar{j}}\frac{K_{\bar{i}} - K_{\bar{i}}^{-1}}{v - v^{-1}},$$

$$K_{\bar{i}}E_{\bar{j}}K_{\bar{i}}^{-1} = \begin{cases} v^2E_{\bar{j}}, & \text{if } \bar{i} = \bar{j}, \\ v^{-1}E_{\bar{j}}, & \text{if } \bar{i} = \bar{j} \pm 1, \\ E_{\bar{j}} & \text{otherwise;} \end{cases} \qquad K_{\bar{i}}F_{\bar{j}}K_{\bar{i}}^{-1} = \begin{cases} v^{-2}F_{\bar{j}}, & \text{if } \bar{i} = \bar{j}, \\ vF_{\bar{j}}, & \text{if } \bar{i} = \bar{j} \pm 1, \\ F_{\bar{j}}, & \text{otherwise;} \end{cases} \tag{3.2}$$

$$E_{\bar{i}}E_{\bar{j}} = E_{\bar{j}}E_{\bar{i}} \quad \text{and} \quad F_{\bar{i}}F_{\bar{j}} = F_{\bar{j}}F_{\bar{i}}, \qquad \text{if } |\bar{i} - \bar{j}| \geq 2,$$

$$E_{\bar{i}}^2E_{\bar{j}} - (v + v^{-1})E_{\bar{i}}E_{\bar{j}}E_{\bar{i}} + E_{\bar{j}}E_{\bar{i}}^2 = F_{\bar{i}}^2F_{\bar{j}} - (v + v^{-1})F_{\bar{i}}F_{\bar{j}}F_{\bar{i}} + F_{\bar{j}}F_{\bar{i}}^2 = 0, \qquad \text{if } |\bar{i} - \bar{j}| = 1.$$

See [8, Definition Proposition 9.1.1]. The algebra $U_v'(\widehat{\mathfrak{sl}}_\ell)$ is the quantum group corresponding to the non-trivial central extension $\widehat{\mathfrak{sl}}_\ell' = \mathfrak{sl}_\ell[t, t^{-1}] \oplus \mathbb{C}c$ of the algebra of polynomial loops in $\mathfrak{sl}_\ell$.

3.3. **Fock space.** Define *v-deformed Fock space* to be the $\mathbb{Q}(v)$ vector space $\mathbf{F}_v$ with basis $\{|\lambda\rangle \mid \lambda \text{ is a partition}\}$. Our $\mathbf{F}_v$ is only the charge 0 part of Fock space described in [16]. Fix $\bar{i} \in \mathbb{Z}/\ell\mathbb{Z}$ and partitions $\lambda \subseteq \mu$ such that μ/λ is a single box. Define

$$\begin{aligned} &A_{\bar{i}}(\lambda) := \{\text{boxes } b : b \notin \lambda, b \text{ has color } \bar{i} \text{ and } \lambda \cup b \text{ is a partition}\}, \\ &R_{\bar{i}}(\lambda) := \{\text{boxes } b : b \in \lambda, b \text{ has color } \bar{i} \text{ and } \lambda \backslash b \text{ is a partition}\}, \\ &N_{\bar{i}}^l(\mu/\lambda) := |\{b \in R_{\bar{i}}(\lambda) : b \text{ to the left of } \mu/\lambda\}| - |\{b \in A_{\bar{i}}(\lambda) : b \text{ to the left of } \mu/\lambda\}|, \\ &N_{\bar{i}}^r(\mu/\lambda) := |\{b \in R_{\bar{i}}(\lambda) : b \text{ to the right of } \mu/\lambda\}| - |\{b \in A_{\bar{i}}(\lambda) : b \text{ to the right of } \mu/\lambda\}| \end{aligned} \tag{3.3}$$

to be the set of *addable boxes of color* $\bar{i}$, the set of *removable boxes of color* $\bar{i}$, the *left removable-addable difference*, and the *right removable-addable difference*, respectively.

Theorem 3.1. (see [3, Theorem 10.6]) *There is an action of $U_v'(\widehat{\mathfrak{sl}}_\ell)$ on $\mathbf{F}_v$ determined by*

$$E_{\bar{i}}|\lambda\rangle := \sum_{\bar{c}(\lambda/\mu)=\bar{i}} v^{-N^r_{\bar{i}}(\lambda/\mu)}|\mu\rangle \qquad \text{and} \qquad F_{\bar{i}}|\lambda\rangle := \sum_{\bar{c}(\mu/\lambda)=\bar{i}} v^{N^l_{\bar{i}}(\mu/\lambda)}|\mu\rangle, \tag{3.4}$$

where $\bar{c}(\lambda/\mu)$ denotes the color of λ/μ and the sum is over partitions μ which differ from λ by removing (respectively adding) a single $\bar{i}$-colored box.

As a $U_v'(\widehat{\mathfrak{sl}}_\ell)$-module, $\mathbf{F}_v$ is isomorphic to an infinite direct sum of copies of the basic representation $V(\Lambda_0)$. Using the grading of $\mathbf{F}_v$ where $|\lambda\rangle$ has degree $|\lambda|$, the highest weight vectors in $\mathbf{F}_v$ occur in degrees divisible by ℓ, and the number of highest weight vectors in degree ℓk is the number of partitions of k. Then $\mathbf{F}_v \cong V(\Lambda_0)\otimes\mathbb{C}[x_1,x_2,\ldots]$, where x_k has degree ℓk, and $U_v'(\widehat{\mathfrak{sl}}_\ell)$ acts trivially on the second factor (see [16, Prop. 2.3]). Note that we are working with the 'derived' quantum group $U_v'(\widehat{\mathfrak{sl}}_\ell)$, not the 'full' quantum group $U_v(\widehat{\mathfrak{sl}}_\ell)$, which is why there are no δ-shifts in the summands of $\mathbf{F}_v$.

Comment 1. Comparing with [3, Chapter 10], our $N^l_{\bar{i}}(\mu/\lambda)$ is equal to Ariki's $-N^a_{\bar{i}}(\mu/\lambda)$ and our $N^r_{\bar{i}}(\mu/\lambda)$ is equal to Ariki's $-N^b_{\bar{i}}(\mu/\lambda)$. However, these numbers play a slightly different role in Ariki's work, which is explained by a different choice of conventions.

4. Universal Verma modules

The purpose of this section is to construct a family of representations which are universal Verma modules in the sense that each can be "evaluated" to obtain any given Verma module. This notion was defined by Kashiwara [17] in the classical case, and was studied in the quantum case by Kamita [15].

4.1. **Rational universal Verma modules.** Let $\mathbb{K} := \mathbb{Q}(q, z_1, z_2, \ldots, z_N)$. This field is isomorphic to the field of fractions of $U_q(\mathfrak{gl}_N)^0$ via the map

$$\psi : U_q(\mathfrak{gl}_N)^0 \to \mathbb{K} \qquad \text{defined by} \qquad \psi(L_i^{\pm 1}) = z_i^{\pm 1}. \tag{4.1}$$

For each $\mu \in \mathfrak{h}^*_\mathbb{Z}$, define a $\mathbb{Q}(q)$-linear automorphism $\sigma_\mu \colon \mathbb{K}\to\mathbb{K}$ by

$$\sigma_\mu(z_i) := q^{(\mu,\varepsilon_i)} z_i, \qquad \text{for } 1 \le i \le N, \tag{4.2}$$

where $(\cdot,\cdot)$ is the inner product on $\mathfrak{h}^*_\mathbb{Z}$ defined by $(\varepsilon_i,\varepsilon_j)=\delta_{i,j}$. Let $\mathbb{K}_\mu = \mathrm{span}_\mathbb{K}\{v_{\mu+}\}$ be the one dimensional vector space over $\mathbb{K}$ with basis vector v_μ^+ and $U_q(\mathfrak{gl}_N)^{\ge 0}$ action given by

$$X_i \cdot v_{\mu+} = 0, \quad \text{for } 1 \le i \le N-1, \quad \text{and} \qquad a\cdot v_{\mu+} = \sigma_\mu(\psi(a))v_{\mu+}, \quad \text{for } a \in U_q(\mathfrak{gl}_N)^0. \tag{4.3}$$

The *μ-shifted rational universal Verma module* ${}^\mu\widetilde{M}$ is the $U_q(\mathfrak{gl}_N)$-module

$${}^\mu\widetilde{M} := U_q(\mathfrak{gl}_N)\otimes_{U_q(\mathfrak{gl}_N)^{\ge 0}} \mathbb{K}_\mu. \tag{4.4}$$

The universal Verma module ${}^\mu\widetilde{M}$ is actually a module over $U_q(\mathfrak{gl}_N)\otimes_{U_q(\mathfrak{gl}_N)^0}\widetilde{U}_q(\mathfrak{gl}_N)^0$, where $\widetilde{U}_q(\mathfrak{gl}_N)^0$ is the field of fractions of $U_q(\mathfrak{gl}_N)^0$. However, if we identify $\widetilde{U}_q(\mathfrak{gl}_N)^0$ with $\mathbb{K}$ using the map ψ, the action of $\widetilde{U}_q(\mathfrak{gl}_N)^0$ on ${}^\mu\widetilde{M}$ is not by multiplication, but rather is twisted by the automorphism σ_μ. It is to keep track of the difference between the action of $U_q(\mathfrak{gl}_N)^0$ and multiplication that we use different notation for the generators of $\mathbb{K}$ and $U_q(\mathfrak{gl}_N)^0$ (that is, z_i versus L_i).

4.2. **Integral universal Verma modules.** The field $\mathbb{K}$ contains an $\mathcal{A}$-subalgebra

$$\mathcal{R} \quad \text{generated by} \quad z_i^{\pm 1} \quad \text{and} \quad \begin{bmatrix} z_i; c \\ k \end{bmatrix} \qquad (1 \le i \le N, c \in \mathbb{Z}, k \in \mathbb{Z}_{>0}), \tag{4.5}$$

which is isomorphic to $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^0$ via the restriction of the map ψ in (4.1). The *integral universal Verma module* ${}^\mu\widetilde{M}^{\mathcal{R}}$ is the $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$-submodule of ${}^\mu\widetilde{M}$ generated by $v_{\mu+}$. By restricting (4.4),

$${}^\mu\widetilde{M}^{\mathcal{R}} = U_q^{\mathcal{A}}(\mathfrak{gl}_N) \otimes_{U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{\ge 0}} \mathcal{R}_\mu, \tag{4.6}$$

where $\mathcal{R}_\mu$ is the $\mathcal{R}$-submodule of $\mathbb{K}_\mu$ spanned by $v_{\mu+}$. In particular, ${}^\mu\widetilde{M}^{\mathcal{R}}$ is a free $\mathcal{R}$-module.

4.3. **Evaluation.** Let $\mathrm{ev}_\lambda^{\mathcal{R}} : \mathcal{R} \to \mathcal{A}$ be the map defined by

$$\mathrm{ev}_\lambda^{\mathcal{R}}(z_i) = q^{(\lambda,\varepsilon_i)} \qquad \text{and} \qquad \mathrm{ev}_\lambda^{\mathcal{R}} \begin{bmatrix} z_i; c \\ n \end{bmatrix} = \begin{bmatrix} q^{(\lambda,\varepsilon_i)}; c \\ n \end{bmatrix}, \tag{4.7}$$

where $(\cdot,\cdot)$ is the inner product on $\mathfrak{h}^*$ defined by $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}$.

There is a surjective $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$-module homomorphism "evaluation at λ"

$$\mathrm{ev}_\lambda : {}^\mu\widetilde{M}^{\mathcal{R}} \to M^{\mathcal{A}}(\mu+\lambda) \quad \text{defined by} \quad \mathrm{ev}_\lambda(a \cdot v_{\mu+}) := a \cdot v_{\mu+\lambda}, \quad \text{for all } a \in U_q^{\mathcal{A}}(\mathfrak{gl}_N). \tag{4.8}$$

For fixed λ, the maps $\mathrm{ev}_\lambda^{\mathcal{R}}$ and ev_λ extend to a map from the subspace of $\mathbb{K}$ and ${}^\mu\widetilde{M} = {}^\mu\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{R}} \mathbb{K}$ respectively where no denominators evaluate to 0. Where it is clear we denote both these extended maps by ev_λ.

Example 4.1. Computing the action of L_i on $v_{\mu+}$ and $v_{\mu+\lambda}$,

$$L_i \cdot v_{\mu+} = q^{(\mu,\varepsilon_i)} z_i v_{\mu+}, \qquad \text{and} \qquad \begin{aligned} L_i \cdot v_{\mu+\lambda} &= \mathrm{ev}_\lambda(q^{(\mu,\varepsilon_i)} z_i) v_{\mu+\lambda} \\ &= q^{(\mu,\varepsilon_i)} q^{(\lambda,\varepsilon_i)} v_{\mu+\lambda} = q^{(\mu+\lambda,\varepsilon_i)} v_{\mu+\lambda}. \end{aligned} \tag{4.9}$$

4.4. **Weight decompositions.** Let $\widetilde{V}$ be a $U_q(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathcal{R}$-module. For each $\nu \in \mathfrak{h}_{\mathbb{Z}}^*$, we define the *$\nu$-weight space* of $\widetilde{V}$ to be

$$\widetilde{V}_\nu := \{v \in \widetilde{V} : L_i \cdot v = q^{(\nu,\varepsilon_i)} z_i v\}. \tag{4.10}$$

The universal Verma module ${}^\mu\widetilde{M}^{\mathcal{R}}$ is a $U_q(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathcal{R}$-module, where the second factor acts as multiplication. The weight space ${}^\mu\widetilde{M}_\eta \neq 0$ if and only if $\eta = \mu - \nu$ with ν in the positive part Q^+ of the root lattice. These non-zero weight spaces and the weight decomposition of ${}^\mu\widetilde{M}$ can be described explicitly by

$${}^\mu\widetilde{M}^{\mathcal{R}}_{\mu-\nu} = U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0}_{-\nu} \cdot \mathcal{R}_\mu \qquad \text{and} \qquad {}^\mu\widetilde{M}^{\mathcal{R}} = \bigoplus_{\nu \in Q^+} {}^\mu\widetilde{M}^{\mathcal{R}}_{\mu-\nu}. \tag{4.11}$$

Here $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0}_{-\nu}$ is defined using the grading of $U_q(\mathfrak{gl}_N)^{<0}$ with $F_i \in U_q(\mathfrak{gl}_N)^{<0}_{-(\varepsilon_i - \varepsilon_{i+1})}$.

4.5. **Tensor products.** Let $\widetilde{V}$ be a $U_q^{\mathcal{A}}(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathcal{R}$-module and W a $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$-module. The tensor product $\widetilde{V} \otimes_{\mathcal{A}} W$ is a $U_q^{\mathcal{A}}(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathcal{R}$-module, where the first factor acts via the usual coproduct and the second factor acts by multiplication on $\widetilde{V}$. In the case when $\widetilde{V}$ and W both have weight space decompositions, the weight spaces of $\widetilde{V} \otimes_{\mathcal{A}} W$ are

$$(\widetilde{V} \otimes_{\mathcal{A}} W)_\nu = \bigoplus_{\gamma+\eta=\nu} \widetilde{V}_\gamma \otimes_{\mathcal{A}} W_\eta. \tag{4.12}$$

We also need the following:

Proposition 4.2. *The tensor product of a universal Verma module with a Weyl module satisfies*

$$\left({}^{\mu}\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} \Delta^{\mathcal{A}}(\nu)\right) \otimes_{\mathcal{R}} \mathbb{K} \cong \left(\bigoplus_{\gamma} ({}^{\mu+\gamma}\widetilde{M}^{\mathcal{R}})^{\oplus \dim \Delta^{\mathcal{A}}(\nu)_\gamma}\right) \otimes_{\mathcal{R}} \mathbb{K}. \tag{4.13}$$

Proof. Fix $\nu \in P^+$. In general, $M(\lambda+\mu)\otimes\Delta(\nu)$ has a Verma filtration (see, for example, [13, Theorem 2.2]) and if $\lambda+\mu+\gamma$ is dominant for all γ such that $\Delta(\nu)_\gamma \neq 0$ then

$$M(\lambda+\mu)\otimes\Delta(\nu) \cong \bigoplus_{\gamma} M(\lambda+\mu+\gamma)^{\oplus \dim \Delta(\nu)_\gamma}, \tag{4.14}$$

which can be seen by, for instance, taking central characters. The proposition follows since this is true for a Zariski dense set of weights λ. □

5. The Shapovalov form and the Shapovalov determinant

5.1. **The Shapovalov form.** The *Cartan involution* $\omega : U_q(\mathfrak{gl}_N) \to U_q(\mathfrak{gl}_N)$ is the $\mathbb{Q}(q)$-algebra anti-involution of $U_q(\mathfrak{gl}_N)$ defined by

$$\omega(L_i^{\pm 1}) = L_i^{\pm 1}, \qquad \omega(X_i) = Y_i L_i L_{i+1}^{-1}, \qquad \omega(Y_i) = L_i^{-1} L_{i+1} X_i. \tag{5.1}$$

The map ω is also a co-algebra involution. An ω*-contravariant* form on a $U_q(\mathfrak{gl}_N)$-module V is a symmetric bilinear form $(\cdot,\cdot)$ such that

$$(u, a\cdot v) = (\omega(a)\cdot u, v), \qquad \text{for } u,v\in V \text{ and } a \in U_q(\mathfrak{gl}_N). \tag{5.2}$$

It follows by the same argument used in the classical case [26] that there is an ω-contravariant form (the Shapovalov form) on each Verma module $M(\lambda)$ and this is unique up to rescaling. The radical of $(\cdot,\cdot)$ is the maximal proper submodule of $M(\lambda)$, so $\Delta(\lambda) = M(\lambda)/\mathrm{Rad}(\cdot,\cdot)$ for all $\lambda \in P^+$. In particular, $(\cdot,\cdot)$ descends to an ω-contravariant form on $\Delta(\lambda)$.

Since ω fixes $U_q^{\mathcal{A}}(\mathfrak{gl}_N) \subseteq U_q(\mathfrak{gl}_N)$, there is a well defined notion of an ω-contravariant form on a $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ module. In particular, the restriction of the Shapovalov form on $\Delta(\lambda)$ to $\Delta^{\mathcal{A}}(\lambda)$ is ω-contravariant.

5.2. **Universal Shapovalov forms.** There are surjective maps of $\mathcal{A}$-algebras $p_- : U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0} \to \mathbb{Q}(q)$ and $p_+ : U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{>0} \to \mathbb{Q}(q)$ defined by $p_-(F_i) = 0$ and $p_+(E_i) = 0$, for $1 \le i \le N$. Using the triangular decomposition (2.7), there is an $\mathcal{A}$-linear surjection

$$\pi_0 := p_- \otimes \mathrm{Id} \otimes p_+ : U_q^{\mathcal{A}}(\mathfrak{gl}_N) \cong U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0} \otimes_{\mathcal{A}} U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{0} \otimes_{\mathcal{A}} U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{>0} \to U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{0}. \tag{5.3}$$

The *standard universal Shapovalov form* is the $\mathcal{R}$-bilinear form $(\cdot,\cdot)_{{}^{\mu}\widetilde{M}^{\mathcal{R}}} : {}^{\mu}\widetilde{M}^{\mathcal{R}} \otimes {}^{\mu}\widetilde{M}^{\mathcal{R}} \to \mathcal{R}$ defined by

$$(a_1\cdot v_{\mu+}, a_2\cdot v_{\mu+})_{{}^{\mu}\widetilde{M}^{\mathcal{R}}} = \big(\sigma_\mu \circ \psi \circ \pi_0\big)(\omega(a_2)a_1) \tag{5.4}$$

for all $a_1, a_2 \in U_q^{\mathcal{R}}(\mathfrak{gl}_N)^{<0}$. Here ψ and σ_μ are as in (4.1) and (4.2). Since

$$(a_1a_2\cdot v_{\mu+}, a_3\cdot v_{\mu+})_{{}^{\mu}\widetilde{M}^{\mathcal{R}}} = \big(\sigma_\mu\circ\psi\circ\pi_0\big)(\omega(a_2)\omega(a_1)a_3) = (a_2\cdot v_{\mu+}, \omega(a_1)a_3\cdot v_{\mu+})_{{}^{\mu}\widetilde{M}^{\mathcal{R}}} \tag{5.5}$$

for $a_1,a_2,a_3 \in U_q(\mathfrak{gl}_N)$, the form $(\cdot,\cdot)_{{}^{\mu}\widetilde{M}^{\mathcal{R}}}$ is ω-contravariant. As with the usual Shapovalov form, distinct weight spaces are orthogonal, where weight spaces are defined as in Section 4.4.

Evaluation at λ gives an $\mathcal{A}$-valued ω-contravariant form $(\cdot,\cdot)_{M^{\mathcal{A}}(\mu+\lambda)}$ on $M^{\mathcal{A}}(\mu+\lambda)$ by

$$(\mathrm{ev}_\lambda(u_1),\mathrm{ev}_\lambda(u_2))_{M^{\mathcal{A}}(\mu+\lambda)} = \mathrm{ev}_\lambda\left((u_1,u_2)_{{}^\mu\widetilde{M}^{\mathcal{R}}}\right), \qquad \text{for } u_1,u_2 \in {}^\mu\widetilde{M}^{\mathcal{R}}. \tag{5.6}$$

The form $(\cdot,\cdot)_{{}^\mu\widetilde{M}^{\mathcal{R}}}$ can be extended by linearity to an ω-contravariant form $(\cdot,\cdot)_{{}^\mu\widetilde{M}}$ on ${}^\mu\widetilde{M}$.

5.3. **The Shapovalov determinant.** Let $\widetilde{V}$ be a $(U_q^{\mathcal{A}}(\mathfrak{gl}_N)\otimes_{\mathcal{A}}\mathcal{R})$-module with a chosen ω-contravariant form. Let B_η be an $\mathcal{R}$ basis for the η-weight space $\widetilde{V}_\eta$ of $\widetilde{V}$. Let $\det\widetilde{V}_{B_\eta}$ be the determinant of the form evaluated on the basis B_η. Changing the basis B_η changes the determinant by a unit in $\mathcal{R}$ and we sometimes write $\det\widetilde{V}_\eta$ to mean the determinant calculated on an unspecified basis ($\det\widetilde{V}_\eta$ which is only defined up to multiplication by unit in $\mathcal{R}$). The *Shapovalov determinant* is

$$\det\widetilde{M}^{\mathcal{R}}_\eta := \det((b_i,b_j)_{\widetilde{M}^{\mathcal{R}}})_{b_i,b_j\in B_\eta}. \tag{5.7}$$

Define the *partition function* $p\colon \mathfrak{h}^*\to\mathbb{Z}_{\geq 0}$ by

$$p(\gamma) := \dim M(0)_\gamma. \tag{5.8}$$

Then $p(\gamma)=\dim M(\lambda)_{\gamma+\lambda}$ for any λ, and $\eta\notin Q^-$ implies that $p(\eta)=0$ and $\det\widetilde{M}^{\mathcal{R}}_\eta = 1$.

Theorem 5.1. (see [9, Proposition 1.9A], [20, Theorem 3.4], [26]) *For any weight η,*

$$\det\widetilde{M}^{\mathcal{R}}_\eta = c_\eta \prod_{\substack{1\leq i<j\leq N\\ m>0}} \left(z_iz_j^{-1} - q^{2m+2i-2j}z_i^{-1}z_j\right)^{p(\eta+m\varepsilon_i-m\varepsilon_j)}, \tag{5.9}$$

where c_η is a unit in $\mathcal{R}\otimes_{\mathcal{A}}\mathbb{Q}(q)=\mathbb{Q}(q)[z_1^{\pm1},\dots,z_N^{\pm1}]$.

Proposition 5.2. *Fix $\mu,\eta\in\mathfrak{h}^*_{\mathbb{Z}}$ with $\eta-\mu\in Q^-$. Choose an $\mathcal{A}$-basis $B_{\eta-\mu}$ for $U_q^{\mathcal{A}}(\mathfrak{gl}_N)_{\eta-\mu}$. Consider the $\mathcal{R}$-bases $\widetilde{B}_{\eta-\mu} := \{b\cdot v_+ \mid b\in B_{\eta-\mu}\}$ for $\widetilde{M}^{\mathcal{R}}_{\eta-\mu}$ and ${}^\mu\widetilde{B}_\eta := \{b\cdot v_{\mu+}\mid b\in B_{\eta-\mu}\}$ for ${}^\mu\widetilde{M}^{\mathcal{R}}_\eta$. Then $\det{}^\mu\widetilde{M}^{\mathcal{R}}_{({}^\mu\widetilde{B}_\eta)} = \sigma_\mu\big(\det\widetilde{M}^{\mathcal{R}}_{\widetilde{B}_{\eta-\mu}}\big)$.*

Proof. For $b,b'\in B_{\eta-\mu}$,

$$(b\cdot v_{\mu+}, b'\cdot v_{\mu+})_{{}^\mu\widetilde{M}^{\mathcal{R}}} = \sigma_\mu\circ\psi\circ\pi_0(\omega(b')b) = \sigma_\mu\big((b\cdot v_{0+},b'\cdot v_{0+})_{\widetilde{M}^{\mathcal{R}}}\big). \tag{5.10}$$

The result follows by taking determinants. □

5.4. **Contravariant forms on tensor products.** If V and W are $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$-modules with ω-contravariant forms $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_W$, define an $\mathcal{A}$-bilinear form $(\cdot,\cdot)_{W\otimes V}$ by $(w_1\otimes v_1, w_2\otimes v_2)_{W\otimes V} = (w_1,w_2)_W(v_1,v_2)_V$. Similarly, for a $U_q^{\mathcal{A}}(\mathfrak{gl}_N)\otimes_{\mathcal{A}}\mathcal{R}$ module $\widetilde{W}$ with $\mathcal{R}$-bilinear ω-contravariant form $(\cdot,\cdot)_{\widetilde{W}}$, define a $\mathcal{R}$-bilinear form $(\cdot,\cdot)_{\widetilde{W}\otimes_{\mathbb{Q}(q)}V}$ on $\widetilde{W}\otimes_{\mathbb{Q}(q)}V$ by

$$(u_1\otimes v_1, u_2\otimes v_2)_{\widetilde{W}\otimes_{\mathbb{Q}(q)}V} = (u_1,u_2)_{\widetilde{W}}(v_1,v_2)_V. \tag{5.11}$$

Since ω is a coalgebra involution (i.e., $\Delta(\omega(a)) = (\omega\otimes\omega)\Delta(a)$, for $a\in U_q(\mathfrak{gl}_N)$), the forms $(\cdot,\cdot)_{V\otimes W}$ and $(\cdot,\cdot)_{{}^\mu\widetilde{M}\otimes_{\mathbb{Q}(q)}V}$ are ω-contravariant.

In the case when $\widetilde{W} = {}^\mu\widetilde{M}^{\mathcal{R}}$, evaluation of the ω-contravariant form $(\cdot,\cdot)_{{}^\mu\widetilde{M}^{\mathcal{R}}\otimes_{\mathcal{A}}V}$ at λ gives an ω-contravariant form $(\cdot,\cdot)_{M^{\mathcal{A}}(\mu+\lambda)\otimes_{\mathcal{A}}V}$:

$$\begin{aligned}(u_1\otimes v_1,u_2\otimes v_2)_{M^{\mathcal{A}}(\mu+\lambda)\otimes_{\mathcal{A}}V} &= \mathrm{ev}_\lambda\left((u_1\otimes v_1,u_2\otimes v_2)_{{}^\mu\widetilde{M}^{\mathcal{R}}\otimes_{\mathcal{A}}V}\right)\\ &= (\mathrm{ev}_\lambda(u_1)\otimes v_1, \mathrm{ev}_\lambda(u_2)\otimes v_2)_{M(\mu+\lambda)\otimes_{\mathcal{A}}V},\end{aligned} \tag{5.12}$$

for $u_1, u_2 \in {}^{\mu}\widetilde{M}$ and $v_1, v_2 \in V$. As in Section 4.3, this evaluation can be extended to the $\mathcal{A}$-submodule of the rational module where no denominators evaluate to zero.

6. The Misra-Miwa formula for $F_{\bar{i}}$ from $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ representation theory

Let us prepare the setting for our main result (Theorem 6.1). Fix $\ell \geq 2$ and a partition λ. Let N a positive integer greater than the number of parts of λ. All calculations below are in terms of representations of $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$.

• Let $V = \Delta^{\mathcal{A}}(\varepsilon_1)$ be the standard N-dimensional module. Since $\Delta^{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} \mathbb{Q}(q) = \Delta(\lambda)$, Equation (3.1) implies

$$\big(\Delta^{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} V\big) \otimes_{\mathcal{A}} \mathbb{Q}(q) \simeq \bigoplus \Delta^{\mathcal{A}}(\lambda + \varepsilon_{k_j}) \otimes_{\mathcal{A}} \mathbb{Q}(q), \tag{6.1}$$

where the sum is over those indices $1 = k_1 < k_2 < \cdots < k_{m_\lambda} \leq N$ for which $\lambda + \varepsilon_{k_j}$ is a partition. For ease of notation let $\mu^{(j)} = \lambda + \varepsilon_{k_j}$.

• Fix an $\mathcal{A}$-basis $\{v_1, \ldots, v_N\}$ of V where v_k has weight ε_k and $Y_i(v_k) = \delta_{i,k} v_{k+1}$. Recursively define singular weight vectors $v_{\mu^{(j)}}$ in $\big(\Delta^{\mathcal{A}}(\lambda) \otimes V\big) \otimes_{\mathcal{A}} \mathbb{Q}(q)$ by:

(i) $v_{\mu^{(1)}} = v_\lambda \otimes v_1$.

(ii) For each k, the submodule W_k of $(\Delta(\lambda) \otimes_{\mathcal{A}} V) \otimes_{\mathcal{A}} \mathbb{Q}(q)$ generated by $\{v_\lambda \otimes v_i \mid 1 \leq i \leq k\}$ contains all weight vectors of $(\Delta(\lambda) \otimes_{\mathcal{A}} V) \otimes_{\mathcal{A}} \mathbb{Q}(q)$ of weight greater than or equal to $\lambda + \varepsilon_k$. Thus, using (6.1), for each $1 \leq j \leq m_\lambda$ there is a one-dimensional space of singular vectors of weight $\mu^{(j)}$ in W_{k_j}, and this is not contained in $W_{k_{j-1}}$ (since $k_j > k_{j-1}$). This implies that there unique singular vector $v_{\mu^{(j)}}$ of weight $\mu^{(j)}$ in

$$v_\lambda \otimes v_{k_j} + \bigoplus_{1 \leq i < j} U_q(\mathfrak{gl}_N) v_{\mu^{(i)}} \subseteq \big(\Delta^{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} V\big) \otimes_{\mathcal{A}} \mathbb{Q}(q), \tag{6.2}$$

where we recall that $U_q(\mathfrak{gl}_N) = U_q^{\mathcal{A}}(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathbb{Q}(q)$.

• There is a unique ω-contravariant form on $\Delta^{\mathcal{A}}(\lambda)$ normalized so that $(v_\lambda, v_\lambda) = 1$ and a unique ω-contravariant form on V normalized so that $(v_1, v_1) = 1$. As in section 5.4, define a ω-contravariant form on $\big(\Delta^{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} V\big) \otimes_{\mathcal{A}} \mathbb{Q}(q)$ by $(u_1 \otimes w_1, u_2 \otimes w_2) = (u_1, u_2)(w_1, w_2)$. For each $1 \leq j \leq m_\lambda$, define an element $r_j(\lambda) \in \mathbb{Q}(q)$ by

$$r_j(\lambda) := (v_{\mu^{(j)}}, v_{\mu^{(j)}}). \tag{6.3}$$

Theorem 6.1. *The Misra-Miwa operators $F_{\bar{i}}$ from Section 3.3 satisfy*

$$F_{\bar{i}} |\lambda\rangle = \sum_{\bar{c}(b^{(j)}) = \bar{i}} v^{val_{\phi_{2\ell}} r_j(\lambda)} |\mu^{(j)}\rangle, \tag{6.4}$$

where $b^{(j)}$ is the box $\mu^{(j)}/\lambda$, $\bar{c}(b^{(j)})$ is the color of box $b^{(j)}$ as in Figure 1, $\phi_{2\ell}$ is the $2\ell^{th}$ cyclotomic polynomial in q and $val_{\phi_{2\ell}} r$ is the number of factors of $\phi_{2\ell}$ in the numerator of r minus the number of factors of $\phi_{2\ell}$ in the denominator of r.

The proof of Theorem 6.1 will occupy the rest of this section. We will first prove a similar statement, Proposition 6.6, where the role of the Weyl modules is played by the universal Verma modules from Section 4. For ease of notation, let $\widetilde{M}^{\mathcal{R}}$ denote the module ${}^{0}\widetilde{M}^{\mathcal{R}}$ from section 4.2.

Definition 6.2. *Recursively define singular weight vectors $v_{\varepsilon_k+} \in \Big(\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V\Big) \otimes_{\mathcal{R}} \mathbb{K}$ and elements $s_k \in \mathbb{K}$ for $1 \leq k \leq N$ by*

(i) $v_{\varepsilon_1+} = v_+ \otimes v_1$.

(ii) *Since* $\{v_+ \otimes v_j \mid 1 \leq j \leq N\}$ *generates* $\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V$ *as a* $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{\leq 0}$ *module, Proposition 4.2 implies that, for each* $1 \leq k \leq N$, *there is a unique singular vector* v_{ε_k+} *in* $v_+ \otimes v_k + \bigoplus_{1 \leq j < k} U_q^{\mathbb{K}}(\mathfrak{gl}_N) v_{\varepsilon_j+} \subseteq \left(\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V\right) \otimes_{\mathcal{R}} \mathbb{K}$, *where* $U_q^{\mathbb{K}}(\mathfrak{gl}_N) := U_q(\mathfrak{gl}_N) \otimes_{\mathbb{Q}(q)} \mathbb{K}$ *and the factor of* $\mathbb{K}$ *acts by multiplication on* $\widetilde{M}^{\mathcal{R}}$.

Let $s_k = (v_{\varepsilon_k+}, v_{\varepsilon_k+})$.

The s_k are quantized versions of the Jantzen numbers first calculated in [12, Section 5] and quantized in [28]. It follows immediately from the definition that $s_1 = 1$.

Lemma 6.3. *For any weight* η, *up to multiplication by a power of* q,

$$\prod_{1 \leq k \leq N} s_k^{p(\eta - \varepsilon_k)} = \prod_{1 \leq k \leq N} \frac{\det \widetilde{M}^{\mathcal{R}}_{\eta - \varepsilon_k}}{\sigma_{\varepsilon_k} \det \widetilde{M}^{\mathcal{R}}_{\eta - \varepsilon_k}}, \tag{6.5}$$

where, as in Section 5.3, $\det \widetilde{M}^{\mathcal{R}}_{\eta - \varepsilon_k}$ *is the determinant of the Shapovalov form evaluated on an* $\mathcal{R}$*-basis for the* $\eta - \varepsilon_k$ *weight space of* $\widetilde{M}^{\mathcal{R}}$.

Comment 2. In order for Lemma 6.3 to hold as stated, for each $1 \leq k \leq N$, one must calculate the $\det \widetilde{M}^{\mathcal{R}}_{\eta - \varepsilon_k}$ in the numerator and denominator with respect to the same $\mathcal{R}$-basis. The power of q which appears depends on this choice of $\mathcal{R}$-bases.

Proof of Lemma 6.3. For each $\gamma \in \operatorname{span}_{\mathbb{Z}_{\leq 0}}(R^+)$ fix an $\mathcal{R}$-basis B_γ for $U_q^{\mathcal{R}}(\mathfrak{gl}_N)^{<0}_\gamma$. Consider the following three $\mathbb{K}$-bases for $\left((\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V)_\eta\right) \otimes_{\mathcal{R}} \mathbb{K}$:

$$\begin{aligned} A_\eta &:= \{(b \cdot v_+) \otimes v_k \mid b \in B_{\eta - \varepsilon_k}, 1 \leq k \leq N\}, \\ C_\eta &:= \{b \cdot (v_+ \otimes v_k) \mid b \in B_{\eta - \varepsilon_k}, 1 \leq k \leq N\}, \\ D_\eta &:= \{b \cdot v_{\varepsilon_k+} \mid b \in B_{\eta - \varepsilon_k}, 1 \leq k \leq N\}. \end{aligned} \tag{6.6}$$

Let $\det(\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V)_B$ denote the determinant of $(\cdot,\cdot)_{(\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V)_\eta}$ calculated on B, where B is one of A_η, C_η or D_η. Let $\det {}^\nu\widetilde{M}^{\mathcal{R}}_{B_{\eta-\nu}}$ denote $\det {}^\nu\widetilde{M}^{\mathcal{R}}_\eta$ calculated with respect to the basis $B_{\eta - \nu} \cdot v_{\nu+}$.

By the definition of the ω-contravariant form on $\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V$ (see Section 4.5),

$$\det(\widetilde{M}^{\mathcal{R}} \otimes V)_{A_\eta} = \prod_{k=1}^{N} (\det \widetilde{M}^{\mathcal{R}}_{B_{\eta - \varepsilon_k}})^{\dim V_{\varepsilon_k}} (\det V_{\varepsilon_k})^{\dim \widetilde{M}^{\mathcal{R}}_{\eta - \varepsilon_k}}. \tag{6.7}$$

For $1 \leq k \leq N$, V_{ε_k} is one dimensional and $\det V_{\varepsilon_k}$ is a power of q. Hence, up to multiplication by a power of q, (6.7) simplifies to

$$\det(\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V)_{A_\eta} = \prod_{k=1}^{N} \det \widetilde{M}^{\mathcal{R}}_{B_{\eta - \varepsilon_k}}. \tag{6.8}$$

Notice that $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0} \cdot v_{\varepsilon_k+}$ is isomorphic to ${}^{\varepsilon_k}\widetilde{M}$, and D_η is the union of $\mathcal{R}$-bases for each of these submodules. For each $1 \leq k \leq N$, and each $\eta \in \mathfrak{h}^*_{\mathbb{Z}}$, define an $\mathcal{R}$ basis of ${}^{\varepsilon_k}\widetilde{M}_\eta$ by

$${}^{\varepsilon_k}\widetilde{B}_\eta := \{b \cdot v_{\varepsilon_k+} \mid b \in B_{\eta - \varepsilon_k}\}. \tag{6.9}$$

Using $(v_{\varepsilon_k+}, v_{\varepsilon_k+}) = s_k$,

$$\det(\widetilde{M}^{\mathcal{R}} \otimes V)_{D_\eta} = \prod_{k=1}^{N} s_k^{\dim({}^{\varepsilon_k}\widetilde{M}^{\mathcal{R}}_\eta)} \det{}^{\varepsilon_k}\widetilde{M}^{\mathcal{R}}_{({}^{\varepsilon_k}\widetilde{B}_\eta)} = \prod_{k=1}^{N} s_k^{p(\eta-\varepsilon_k)} \sigma_{\varepsilon_k}(\det \widetilde{M}^{\mathcal{R}}_{\widetilde{B}_{\eta-\varepsilon_k}}), \tag{6.10}$$

where the last equality uses Proposition 5.2. Here, as in Section 5.3, $\det{}^{\varepsilon_k}\widetilde{M}^{\mathcal{R}}_{({}^{\varepsilon_k}\widetilde{B}_\eta)}$ is the Shapovalov determinant calculated with respect to the basis ${}^{\varepsilon_k}\widetilde{B}_\eta$.

The change of basis from A_η to C_η is unitriangular and the change of basis from C_η to D_η is unitriangular. Thus $\det(\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V)_{A_\eta} = \det(\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V)_{D_\eta}$, and so the right sides of (6.8) and (6.10) are equal. The lemma follows from this equality by rearranging. □

Lemma 6.4. *Up to multiplication by a power of q,*

$$s_k = \prod_{1 \le j < k} \left(\frac{z_j z_k^{-1} - q^{2+2j-2k} z_j^{-1} z_k}{\sigma_{\varepsilon_j}\left(z_j z_k^{-1} - q^{2+2j-2k} z_j^{-1} z_k\right)} \right). \tag{6.11}$$

Proof. Fix $1 \le k \le N$. Setting $\eta = \varepsilon_k$ in Lemma 6.3 and applying Theorem 5.1 we see that, up to multiplication by a power of q,

$$\begin{aligned}\prod_{1\le x\le N} s_x^{p(\varepsilon_k-\varepsilon_x)} &= \prod_{1\le x\le N} \frac{\det \widetilde{M}^{\mathcal{R}}_{\varepsilon_k-\varepsilon_x}}{\sigma_{\varepsilon_x}\det \widetilde{M}^{\mathcal{R}}_{\varepsilon_k-\varepsilon_x}} \\ &= \prod_{1\le x\le N} \prod_{\substack{1\le i<j\le N \\ m>0}} \left(\frac{c_{\varepsilon_k-\varepsilon_x}\left(z_i z_j^{-1} - q^{2m+2i-2j} z_i^{-1} z_j\right)}{\sigma_{\varepsilon_x}(c_{\varepsilon_k-\varepsilon_x})\sigma_{\varepsilon_x}\left(z_i z_j^{-1} - q^{2m+2i-2j} z_i^{-1} z_j\right)} \right)^{p(\varepsilon_k-\varepsilon_x+m\varepsilon_i-m\varepsilon_j)},\end{aligned} \tag{6.12}$$

where, for each $1 \le x \le N$, $c_{\varepsilon_k-\varepsilon_x}$ is a unit in $\mathbb{Q}(q)[z_1^{\pm1}, \ldots, z_N^{\pm1}]$. The value $p(\varepsilon_k - \varepsilon_x + m\varepsilon_i - m\varepsilon_j)$ is 0 unless $m = 1$ and $x \le i < j \le k$. If $i > x$, then σ_{ε_x} acts as the identity on $z_i z_j^{-1} - q^{2+2i-2j} z_i^{-1} z_j$, so the corresponding factors in the numerator and denominator cancel. Hence we need only consider factors on the right hand side where $m = 1$, $i = x$, and $x < j \le k$. If $x > k$ then $\varepsilon_k - \varepsilon_x \notin Q^-$, and hence $p(\varepsilon_k - \varepsilon_x) = 0$, so on the left hand since we only need to consider those factors where $1 \le x \le k$. Up to multiplication by a power of q, the expression reduces to

$$\begin{aligned}\prod_{1\le x\le k} s_x^{p(\varepsilon_k-\varepsilon_x)} &= \prod_{1\le x<k} \left(\frac{c_{\varepsilon_k-\varepsilon_x}}{\sigma_{\varepsilon_x}(c_{\varepsilon_k-\varepsilon_x})} \right)^{p(\varepsilon_k-\varepsilon_j)} \prod_{x<j\le k} \left(\frac{z_x z_j^{-1} - q^{2+2x-2j} z_x^{-1} z_j}{\sigma_{\varepsilon_x}\left(z_x z_j^{-1} - q^{2+2x-2j} z_x^{-1} z_j\right)} \right)^{p(\varepsilon_k-\varepsilon_j)} \\ &= \prod_{1<j\le k} \left(\prod_{1\le x<j} \frac{z_x z_j^{-1} - q^{2+2x-2j} z_x^{-1} z_j}{\sigma_{\varepsilon_x}\left(z_x z_j^{-1} - q^{2+2x-2j} z_x^{-1} z_j\right)} \right)^{p(\varepsilon_k-\varepsilon_j)}.\end{aligned} \tag{6.13}$$

The last two expressions are equal because they are each a product over pairs (x, j) with $1 \le x < j \le k$, and the factors of $\frac{c_{\varepsilon_k-\varepsilon_x}}{\sigma_{\varepsilon_x}(c_{\varepsilon_k-\varepsilon_x})}$ have been dropped because they are powers of q. Using the fact that $s_1 = 1$ and making the change of variables $j \to x$ and $x \to j$ on the

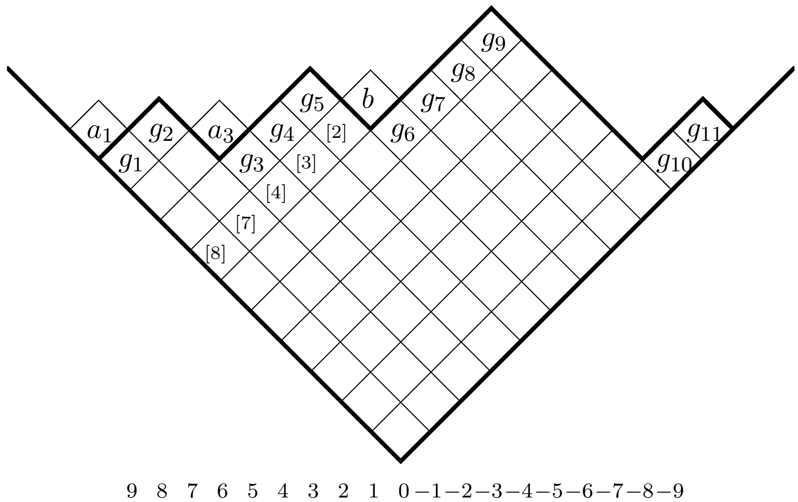


FIGURE 2. The partition enclosed by the thick lines is $\lambda = (10, 10, 8, 8, 8, 6, 6, 6, 6, 6,\ 1, 1)$. If $k = 6$ then $A(\lambda, < 6) = \{a_1, a_3\}$, $R(\lambda, < 6) = \{g_2, g_5\}$, and

$$\mathrm{ev}_\lambda(s_6) = \frac{[2]}{[3]}\frac{[3]}{[4]}\frac{[4]}{[5]}\frac{[7]}{[8]}\frac{[8]}{[9]} = \frac{[2][7]}{[5][9]} = \frac{[c(g_5) - c(b)][c(g_2) - c(b)]}{[c(a_3) - c(b)][c(a_1) - c(b)]}.$$

The factors in the numerator of the first expression are displayed. These are the q-integers corresponding to the hook lengths of the boxes in the same column as the addable box b in row 6.

right side, (6.13) becomes

$$\prod_{1<x\leq k} s_x^{p(\varepsilon_k-\varepsilon_x)} = \prod_{1<x\leq k}\left(\prod_{1\leq j<x}\frac{z_j z_x^{-1} - q^{2+2j-2x} z_j^{-1} z_x}{\sigma_{\varepsilon_j}\left(z_j z_x^{-1} - q^{2+2j-2x} z_j^{-1} z_x\right)}\right)^{p(\varepsilon_k-\varepsilon_x)}. \tag{6.14}$$

For $k \geq 2$, the lemma now follows by induction. For $k = 1$ the result simply says that $s_1 = 1$, which we already know. □

Proposition 6.5. *Let λ be a partition. Let $A(\lambda, < k)$ (resp. $R(\lambda, < k)$) be the set of boxes which can be added to (resp. removed from) λ on rows λ_j with $j < k$ such that the result is still a partition. Let $b = (\lambda + \varepsilon_k)/\lambda$ and let $c(\cdot)$ be as in Figure 1. Then, up to multiplication by a power of q,*

$$ev_\lambda(s_k) = \begin{cases} \dfrac{\prod_{r\in R(\lambda,<k)}[c(r) - c(b)]}{\prod_{a\in A(\lambda,<k)}[c(a) - c(b)]}, & \text{if } \lambda + \varepsilon_k \text{ is a partition,} \\ 0, & \text{if } \lambda + \varepsilon_k \text{ is not a partition.}\end{cases} \tag{6.15}$$

Proof. For $1 \leq j \leq N$, let g_j be the last box in row j of λ. By Lemma 6.4, up to multiplication by a power of q,

$$\mathrm{ev}_\lambda(s_k) = \mathrm{ev}_\lambda\left(\prod_{1\leq j<k}\frac{z_j z_k^{-1} - q^{2+2j-2k} z_j^{-1} z_k}{\sigma_{\varepsilon_j}(z_j z_k^{-1} - q^{2+2j-2k} z_j^{-1} z_k)}\right) = \prod_{1\leq j<k}\frac{[c(g_j) - c(b)]}{[c(g_j) - c(b) + 1]}, \tag{6.16}$$

where the last equality is a simple calculation from definitions. The denominator on the right side is never zero, and the numerator is zero exactly when $\lambda_k = \lambda_{k-1}$, so that $\lambda + \varepsilon_k$ is no longer a partition. If $\lambda_j = \lambda_{j+1}$ for any $j < k$, then there is cancellation, giving (6.15). See Figure 2. □

Proposition 6.6. *Let $N^l_{\bar{j}}(\mu/\lambda)$ be as in Section 3.3. For any partition λ,*

$$\text{(6.17)}\qquad \begin{cases} val_{\phi_{2\ell}} ev_\lambda(s_k) = N^l_{\bar{i}}(\mu/\lambda), & \text{if } \mu = \lambda + \varepsilon_k \text{ is a partition, and } \mu/\lambda \text{ is an } \bar{i} \text{ colored box,} \\ ev_\lambda(s_k) = 0, & \text{otherwise.} \end{cases}$$

Proof. By Proposition 6.5, $\mathrm{ev}_\lambda(s_k) = 0$ if $\lambda + \varepsilon_k$ is not a partition. If $\lambda + \varepsilon_k$ is a partition then

$$\text{(6.18)}\qquad \begin{array}{l} \{b \in A(\lambda, < k) : \bar{c}(b) = \bar{c}(\mu/\lambda)\} = \{b \in A_{\bar{i}}(\lambda) \mid b \text{ is to the left of } \mu/\lambda\}, \quad \text{and} \\ \{b \in R(\lambda, < k) : \bar{c}(b) = \bar{c}(\mu/\lambda)\} = \{b \in R_{\bar{i}}(\lambda) \mid b \text{ is to the left of } \mu/\lambda\}, \end{array}$$

where the notation is as in Section 3.3. Since

$$\text{(6.19)}\qquad [x] = \frac{q^x - q^{-x}}{q - q^{-1}} = q^{-x}(q - q^{-1})^{-1} \prod_{d|2x} \phi_d,$$

$[x]$ is divisible by $\phi_{2\ell}$ if and only if x is divisible by ℓ, and $[x]$ is never divisible by $\phi^2_{2\ell}$. The result now follows from Proposition 6.5. □

Proof of Theorem 6.1. Fix λ and $1 \le k \le m_\lambda$. From definitions, $(\mathrm{ev}_\lambda \otimes 1)v_{\varepsilon_{k_j}+} = v_{\mu^{(j)}}$. Thus, using (5.12),

$$\text{(6.20)}\qquad r_j(\lambda) = (v_{\mu^{(j)}}, v_{\mu^{(j)}}) = ((\mathrm{ev}_\lambda \otimes 1)v_{\varepsilon_{k_j}+}, (\mathrm{ev}_\lambda \otimes 1)v_{\varepsilon_{k_j}+}) = \mathrm{ev}_\lambda(v_{\varepsilon_{k_j}+}, v_{\varepsilon_{k_j}+}) = \mathrm{ev}_\lambda(s_{k_j}).$$

The result now follows from Proposition 6.6. □

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