# $c$-functions and Koornwinder polynomials

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*In memory of Ian G. Macdonald*

**Abstract**

This paper develops the theory of Macdonald-Koornwinder polynomials in parallel analogy with the work done for the $GL_n$ case in [CR22]. In the context of the type $CC_n$ affine root system the Macdonald polynomials of other root systems of classical type are specializations of the Koornwinder polynomials. We derive $c$-function formulas for symmetrizers and use them to give $E$-expansions, principal specializations and norm formulas for bosonic, mesonic and fermionic Koornwinder polynomials. Finally, we explain the proof of the norm conjectures and constant term conjectures for the Koornwinder case.

***Key words*** — Macdonald polynomials, symmetric functions, Hecke algebras

## 0 Introduction

The Koornwinder polynomials are multivariate generalizations of the classical orthogonal polynomials that appear in the Askey scheme [K92]. At the top of this hierarchy we find the Askey-Wilson polynomials and the other families of orthogonal polynomials in the Askey scheme are obtained from the Askey-Wilson polynomials by specializing parameters.

Macdonald's 1987 paper [Mac87] provides a very general framework for associating orthogonal polynomials $P_\lambda$ to any affine root system. It turns out that the Koornwinder polynomials are the Macdonald polynomials for the affine root system of type $CC_n$ (in Macdonald's notation $(C_n^\vee, C_n)$) and the Askey-Wilson polynomials are the Macdonald polynomials for the affine root system of type $CC_1$. One of the key features of Macdonald's picture is that the norms $\langle P_\lambda, P_\lambda\rangle_+$ are generalizations of Macdonald's "constant terms" $\langle P_0, P_0\rangle_+$. In this way, Macdonald stated conjectures for the values of $\langle P_\lambda, P_\lambda\rangle_+$ which vastly expanded his earlier constant-term conjectures.

Cherednik introduced the double affine Hecke algebra as a tool for extending Opdam's ideas to prove the norm conjectures [C03]. This perspective pointed to a larger family of orthogonal polynomials $E_\mu$, from which the $P_\lambda$ are obtained by a process of symmetrization. All of these tools, including the proof of the norm conjectures, were wonderfully exposited in the full generality of a possibly non-reduced affine root system in Macdonald's book [Mac03]. For a wonderful history of the exciting trajectory of these amazing developments see [HKO24].

Particularly in the type $GL_n$ case, the Macdonald polynomials have been of interest to the combinatorial community because of the wealth of wonderful $q,t$-generalizations of classical combinatorial formulas in symmetric function theory. It is also stimulating that there are many fascinating connections to adjacent fields (representations of $p$-adic groups and affine Lie algebras, geometry of Hilbert schemes and affine Springer fibers, torus knot invariants, vertex models in statistical mechanics, particle process in probability, etc). For this reason it is desirable to provide expositions of the tools that

AMS Subject Classifications: Primary 05E05; Secondary 33D52.

bridge the language gaps between the generality of affine root systems and the standard conventions in classical symmetric function theory. In [CR22] we explained how, in the type $GL_n$ case, many of the combinatorial formulas can be understood from the theory of $c$-functions, which are the analogs of the Harish-Chandra $c$-functions surveyed by Helgason [Hel94] and that appear everywhere in Macdonald's monograph [Mac03].

In the type $GL_n$ case, the Macdonald polynomials depend on two parameters $q$ and $t$. In the Koornwinder case (type $CC_n$), the polynomials depend on 6 parameters, $q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}$ and $u_n^{\frac{1}{2}}$. Because of this plethora of parameters, sometimes it is not so easy to see how the combinatorial formulas familiar in the type $GL_n$ case generalize to the Koornwinder case. This paper follows the same pattern as our earlier paper [CR22], generalizing from the type $GL_n$ case to the Koornwinder (type $CC_n$) case.

There is a constantly increasing literature on Koornwinder polynomials. There are interesting technical advances and also fascinating connections to other fields (see, for example, [CGdGW16, CMW23, Ra17, RW15, Yam20, YY21]). The foundational work in [Nou95, Ra01, Sah99, Sto00, vD95, vD96], among others, continues to be extremely useful for clarifying the role and position of the double affine Hecke algebra as a tool for the Koornwinder case. There are also important and very useful surveys of the theory of Koornwinder polynomials (see, for example, [Sto04, Sto21]). There is a significant intersection between the content of this paper and the content presented in these surveys. We hope that our combinatorial and $c$-function point of view can be useful in continuing research on Koornwinder polynomials.

The plan of this paper is as follows:

At the end of this introduction we include some remarks on the different sets of parameters used in the literature and establish the ones we will use for the paper. Since there are 6 different parameters to keep track of and lots of literature to navigate, perhaps this dictionary will be useful to readers (as it was for us). Following these remarks we present a diagram of the affine root systems of classical types together with the specializations of the parameters for obtaining the Macdonald polynomials of the corresponding type from Koornwinder polynomials, which are the Macdonald polynomials for the affine root system of type $(C_n^\vee, C_n)$. A thorough study can be found in [YY21].

Section 1 is dedicated to the affine Weyl group and root system for type $CC_n$, and we include the affine coroots and affine roots. Our new contribution in Section 1 is the diagram giving the relative positions of the affine root systems of classical type and the specializations that give the Macdonald polynomials of the other classical type from Koornwinder polynomials. Although these relationships are, in principle, well-known (from Macdonald [Mac72] and Bruhat-Tits [BT72]), we have not seen this way of presenting this information, which we find very useful, and not broadly known.

In Section 2 we introduce the main tools for working with Koornwinder polynomials, including the $c$-functions and the double affine Hecke algebra. Our contribution here is to provide a framework for the DAHA in terms of $c$-functions, which makes the, sometimes daunting, formulas for the operators on the polynomial representation seem obvious and natural.

In the second half of Section 2 we introduce four families of symmetrizers together with the relations between them, their $c$-function formulas and the case when the stabilizers are nontrivial. Our contribution here is to treat the four types of symmetrizers in tandem so that their role in the theory (and the symmetry between them) becomes clearer.

Sections 3 and 4 examine the main objects of study:

(a) electronic Macdonald-Koornwinder polynomials (Section 3);

(b) bosonic Macdonald-Koornwinder polynomials;

(c) fermionic Macdonald-Koornwinder polynomials;

(d) mesonic Macdonald-Koornwinder polynomials.

We introduce the electronic Macdonald-Koornwinder polynomials $E_\mu$ as eigenfunctions of Cherednik-Dunkl operators and then give a recursive formula and a creation formula for the $E_\mu$. For the other variants, our study includes:

(a) definition of the Weyl denominators;
(b) study of the bosonic, fermionic and mesonic spaces;
(c) formulas for the Poincaré polynomial;
(d) expansions in terms of $E_\mu$, and
(e) principal specializations.

Our contribution here is to put the focus on the fermionic and mesonic Koornwinder polynomials so that the four-fold structure is clearly visible. This four-fold structure eventually leads to powerful recursions for computing norms.

Finally, Section 5 is dedicated to the study of the Macdonald-Koornwinder polynomials as a family of orthogonal polynomials. In particular, we

(a) define the Macdonald-Koornwinder inner product via multiplication by a kernel and taking the constant term and characterize the electronic and bosonic Macdonald-Koornwinder polynomials in terms of the inner product;
(b) compute adjoints of the operators from the double affine Hecke algebra;
(c) prove the going up a level and Weyl character formulas to provide recursions for norms; and
(d) use the recursions for norms to compute the norms $\langle P_\lambda, P_\lambda\rangle_+$ and the Macdonald constant term for type $CC_n$.

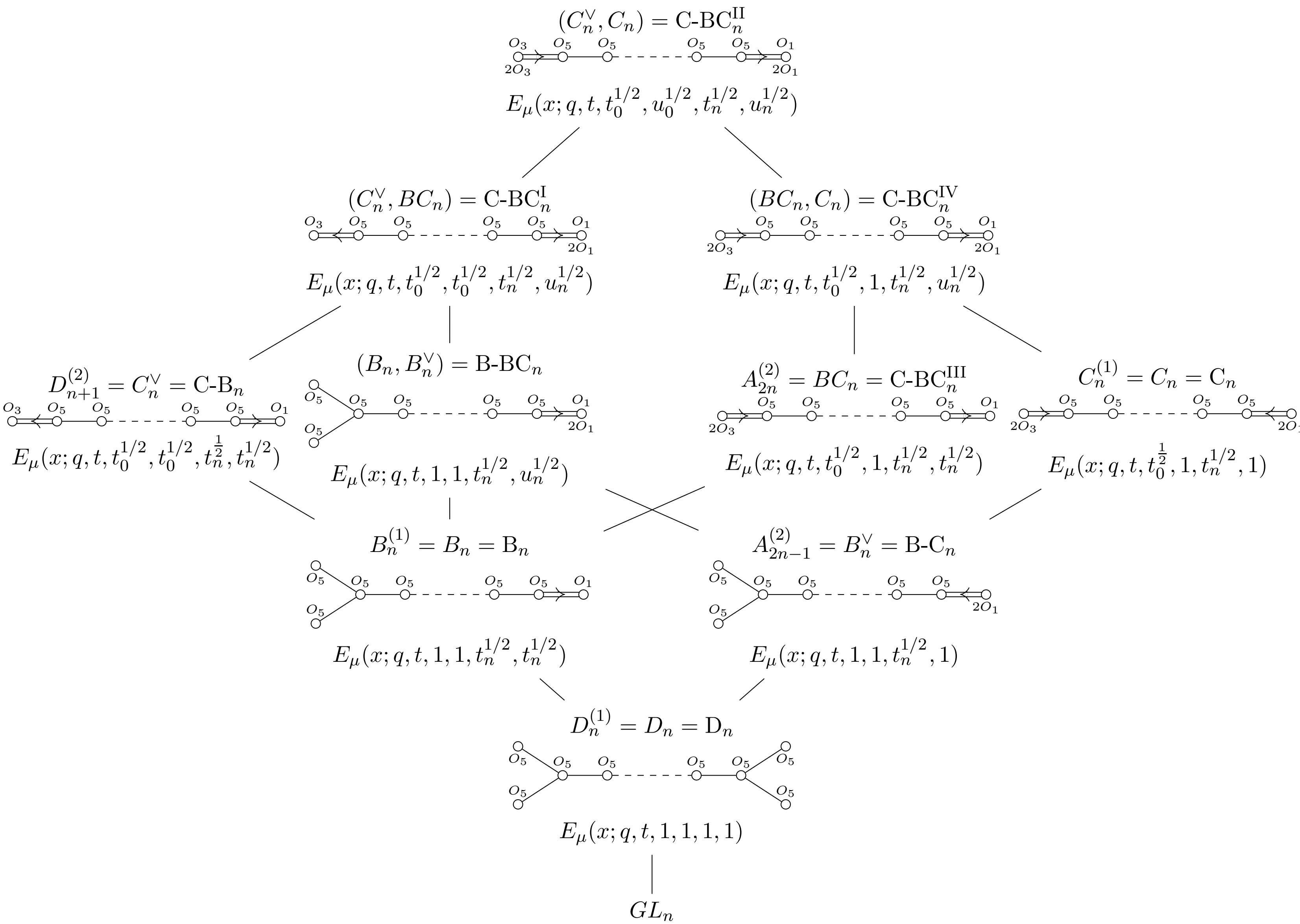
This section follows the same trajectory as that taken in [Mac03, Ch. 5]. Our contribution here is to use the fermionic and mesonic framework to organize the recursions for norms and make the proof of norm conjectures easy and natural.


## Acknowledgments

We thank Zajj Daugherty, Stephen Griffeth, Bogdan Ion, Alejandro H. Morales and Martha Yip for discussions which greatly helped to solidify our understanding of the topics that we treat in this paper. We thank O. Warnaar and J. Stokman for helpful and insightful comments on a preliminary version of this paper.

L.C. was partially supported by a grant from the Simons Foundation (MPS-TSM-00007362).

### 0.0.1 The poset of affine root systems of classical type




See Section 1.3 for the explanation of this diagram.

## 0.1 Remarks on parameters

Depending on the reference, the notation for the parameters varies. In this article, we follow [Nou95] and [Sah99] and mostly use the parameters

$$q, t^{\frac{1}{2}}, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}.$$

**Remark 0.1.** In an attempt to relate the parameter notations in [Mac03, §4.7], [Sah99, §3] and [C03, Def. 2.1] let

$$\tau_0' = u_0^{\frac{1}{2}}, \quad \tau_n' = u_n^{\frac{1}{2}}, \quad \tau_0 = t_0^{\frac{1}{2}}, \quad \tau_n = t_n^{\frac{1}{2}}, \quad \text{and} \quad \tau_i = \tau_i' = t^{\frac{1}{2}} \quad \text{for } i \in \{1, \dots, n-1\}.$$

The summary of (1.5.1), (4.4.1), (4.4.2), (4.4.3), and (5.1.4) in [Mac03] is that, for an affine root $a$,

$$\tau_a = (t_a t_{2a})^{\frac{1}{2}} = q^{\frac{1}{2}\kappa_a} = q^{\frac{1}{2}(k(a)+k(2a))}, \qquad \text{and} \qquad \tau_a' = t_a^{\frac{1}{2}} = q^{\frac{1}{2}\kappa_a'} = q^{\frac{1}{2}(k(a)-k(2a))}.$$

In our situation

$$\begin{aligned}
t_n^{\frac{1}{2}} &= \tau_n = q^{\frac{1}{2}\kappa_n} = t_{\varepsilon_n}^{\frac{1}{2}} t_{2\varepsilon_n}^{\frac{1}{2}} = q^{\frac{1}{2}k(\varepsilon_n)+\frac{1}{2}k(2\varepsilon_n)} = q^{\frac{1}{2}k_1+\frac{1}{2}k_2}, \\
u_n^{\frac{1}{2}} &= \tau_n' = q^{\frac{1}{2}\kappa_n'} = t_{\varepsilon_n}^{\frac{1}{2}} = q^{\frac{1}{2}k(\varepsilon_n)-\frac{1}{2}k(2\varepsilon_n)} = q^{\frac{1}{2}k_1-\frac{1}{2}k_2}, \\
t_0^{\frac{1}{2}} &= \tau_0 = q^{\frac{1}{2}\kappa_0} = t^{\frac{1}{2}}_{-\varepsilon_1+\frac{1}{2}\delta} t^{\frac{1}{2}}_{-2\varepsilon_1+\delta} = q^{\frac{1}{2}k(-\varepsilon_1+\frac{1}{2}\delta)+\frac{1}{2}k(-2\varepsilon_1+\delta)} = q^{\frac{1}{2}k_3+\frac{1}{2}k_4}, \\
u_0^{\frac{1}{2}} &= \tau_0' = q^{\frac{1}{2}\kappa_0'} = t^{\frac{1}{2}}_{-\varepsilon_1+\frac{1}{2}\delta} = q^{\frac{1}{2}k(-\varepsilon_1+\frac{1}{2}\delta)-\frac{1}{2}k(-2\varepsilon_1+\delta)} = q^{\frac{1}{2}k_3-\frac{1}{2}k_4}, \qquad \text{and} \\
t^{\frac{1}{2}} &= \tau_i = q^{\frac{1}{2}\kappa} = t^{\frac{1}{2}}_{\varepsilon_i-\varepsilon_{i+1}} = q^{\frac{1}{2}k_5}, \quad \text{for } i \in \{1, \dots, n-1\},
\end{aligned}$$

and the formulas in [Mac03, (1.5.1)] correspond to interchanging $\kappa_0$ and $\kappa_n'$. $\square$

**Remark 0.2. Askey-Wilson parameters.** In type $(C_1^\vee, C_1)$, the bosonic Macdonald polynomials $P_\lambda(q, t_1, u_1, t_0, u_0)$ are also known as the Askey-Wilson polynomials. Following [Nou95, §3], the correspondence to the original Askey-Wilson parameters is given by

$$q = q, \quad a = q^{\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{\frac{1}{2}}, \quad b = -q^{\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{-\frac{1}{2}}, \quad c = t_n^{\frac{1}{2}} u_n^{\frac{1}{2}}, \quad d = -t_n^{\frac{1}{2}} u_n^{-\frac{1}{2}}. \tag{0.1}$$

These conversions are equivalent to

$$t_0 = -q^{-1}ab, \qquad t_n = -cd, \qquad u_0 = -ab^{-1}, \qquad u_n = -cd^{-1},$$

and it is useful to note that

$$a + b = q^{\frac{1}{2}} t_0^{\frac{1}{2}} (u_0^{\frac{1}{2}} - u_0^{-\frac{1}{2}}) \qquad \text{and} \qquad c + d = t_n^{\frac{1}{2}} (u_n^{\frac{1}{2}} - u_n^{-\frac{1}{2}}).$$

Up to permutations of $a, b, c, d$, these parameters are used in [Sah00, (1)], in [CGdGW16, (17)], in [CMW23, Def. 2.2] and, with different notation, in [Mac03, (5.1.14)]. $\square$

# 1 The affine Weyl group and root system for type $CC_n$

The affine root system of type $CC_n$ (in Macdonald's notation $(C_n^\vee, C_n)$) is the structure that holds the combinatorics of Koornwinder polynomials in place as they are the Macdonald polynomials for this affine root system. The affine Weyl group $W$ plays the role of the group of symmetries of the affine root system. In this section we introduce the definitions and notations for working with the affine Weyl group $W$ and the affine root system of type $CC_n$.

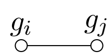
The coroots $S^\vee$ and the roots $S$ for the affine root system of type $CC_n$ play just slightly different roles in the theory, especially in the computations involving Koornwinder polynomials. One of the challenges in this work is to keep these two mirror worlds in proper focus. For this purpose, in Section 1.2, we carefully lay out two versions of the affine Weyl group $W$, one denoted $W_X$ which acts on the coroots (with 5 orbits), and one denoted $W_Y$ which acts on the roots (with 5 orbits). While the groups $W_X$, $W_Y$ and $W$ are all isomorphic, being pedantic about the notation at this early stage prevents future headaches.

To conclude this section we present a brief explanation of the reasoning for how the Macdonald polynomials of other classical types (such as $B_n$, $C_n$, $BC_n$, $A^{(2)}_{2n-1}$, $D^{(2)}_{n+1}$ etc.) are derived from the Koornwinder polynomials by specialization. This specialization process is summarized in the diagram included in Section 0.0.1.

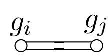
## 1.1 The affine Weyl group $W$ and the finite Weyl group $W_{\mathrm{fin}}$

Use a graphical notation for relations so that

means $g_ig_j = g_jg_i$,



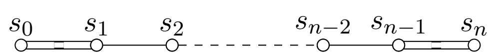
means $g_ig_jg_i = g_jg_ig_j$, and



means $g_ig_jg_ig_j = g_jg_ig_jg_i$.

The *affine Weyl group* is the group $W$ presented by generators $s_0, s_1, \dots, s_n$ and relations

$s_i^2 = 1$ and




The *finite Weyl group* is the subgroup $W_{\mathrm{fin}}$ generated by $s_1, \dots, s_n$.

Let $w \in W$. The *length of* $w$, $\ell(w)$, is the minimal $\ell \in \mathbb{Z}_{\geq 0}$ such that

$$w = s_{i_1} \dots s_{i_\ell} \quad \text{with } i_1, \dots, i_\ell \in \{0, 1, \dots, n-1, n\}.$$

The expression $w = s_{i_1} \dots s_{i_\ell}$ is a *reduced word for* $w$ and any other expression of the form $w = s_{j_1} \dots s_{j_k}$, with $j_1, \dots, j_k \in \{0, 1, \dots, n-1, n\}$, has $k \geq \ell(w)$.

### 1.1.1 Translation presentation of $W$

Define $h_{\varepsilon_1}, \dots, h_{\varepsilon_n} \in W$ by

$$h_{\varepsilon_1} = s_0s_1s_2 \cdots s_{n-1}s_ns_{n-1} \cdots s_2s_1, \qquad \text{and} \qquad h_{\varepsilon_j} = s_jh_{\varepsilon_{j-1}}s_j, \text{ for } j \in \{2, \dots, n\}.$$

For $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ define the *translation* $h_\mu$ by

$$h_\mu = (h_{\varepsilon_1})^{\mu_1} \cdots (h_{\varepsilon_n})^{\mu_n} \tag{1.1}$$

and define $u_\mu \in W$ and $v_\mu \in W_{\mathrm{fin}}$ by the equation

$$h_\mu = u_\mu v_\mu, \qquad \text{where } v_\mu \in W_{\mathrm{fin}} \text{ and } u_\mu \text{ is minimal length in the coset } h_\mu W_{\mathrm{fin}}. \tag{1.2}$$

Define an action of $W_{\mathrm{fin}}$ on $\mathbb{Z}^n$ by

$$\begin{aligned} s_i(\mu_1, \ldots, \mu_n) &= (\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \mu_i, \mu_{i+2}, \ldots, \mu_n), \qquad \text{for } i \in \{1, \ldots, n-1\}, \\ s_n(\mu_1, \ldots, \mu_n) &= (\mu_1, \ldots, \mu_{n-1}, -\mu_n). \end{aligned} \tag{1.3}$$

Then

$$W = \mathbb{Z}^n \ltimes W_{\mathrm{fin}}.$$

In other words, if $\mu, \nu \in \mathbb{Z}^n$ and $v \in W_{\mathrm{fin}}$ then

$$v h_\mu = h_{v\mu} v, \qquad h_\mu h_\nu = h_{\mu+\nu} \qquad \text{and} \qquad W = \{h_\mu v \mid \mu \in \mathbb{Z}^n, v \in W_{\mathrm{fin}}\}.$$

**Remark 1.1.** In [CR22], we use $t_\mu$ to denote the translations in the type $A$ case. We use the notation $h_\mu$ for type $CC_n$ to avoid conflict with the set of parameters for Koornwinder polynomials (specified in Section 2.1).

## 1.2 Affine coroots, affine roots and the groups $W_X$ and $W_Y$

In this subsection we set up the notation for the affine Weyl groups $W_X$ and $W_Y$. Both groups $W_X$ and $W_Y$ are isomorphic to $W$, but they serve slightly different roles and it is necessary to set up the notation to distinguish them.

### 1.2.1 The dual lattices $\mathfrak{a}_{\mathbb{Z}}$ and $\mathfrak{a}_{\mathbb{Z}}^*$

Let $\varepsilon_1, \ldots, \varepsilon_n$ and $\varepsilon_1^\vee, \ldots, \varepsilon_n^\vee$ be symbols and define dual lattices (i.e. dual free $\mathbb{Z}$-modules)

$$\mathfrak{a}_{\mathbb{Z}}^* = \{\gamma_1\varepsilon_1 + \cdots + \gamma_n\varepsilon_n \mid \gamma_1, \ldots, \gamma_n \in \mathbb{Z}\} \qquad \text{and} \qquad \mathfrak{a}_{\mathbb{Z}} = \{\mu_1\varepsilon_1^\vee + \cdots + \mu_n\varepsilon_n^\vee \mid \mu_1, \ldots, \mu_n \in \mathbb{Z}\}$$

with $\mathbb{Z}$-bilinear pairing

$$\langle\ ,\ \rangle\colon \mathfrak{a}_{\mathbb{Z}}^* \times \mathfrak{a}_{\mathbb{Z}} \to \mathbb{Z} \qquad \text{given by} \qquad \langle \varepsilon_i, \varepsilon_j^\vee \rangle = \delta_{ij}.$$

Both $\mathfrak{a}_{\mathbb{Z}}$ and $\mathfrak{a}_{\mathbb{Z}}^*$ are isomorphic to $\mathbb{Z}^n$.

### 1.2.2 The affine coroots for type $CC_n$

Let

$$Q^\vee = \mathbb{Z}\text{-span}\{\varepsilon_1^\vee, \ldots, \varepsilon_n^\vee, \tfrac{1}{2}K\}$$

be the $\mathbb{Z}$-vector space spanned by symbols $\varepsilon_1^\vee, \ldots, \varepsilon_n^\vee$ and $\frac{1}{2}K$. The affine Weyl group $W_X$ is the group of $\mathbb{Z}$-linear transformations of $Q^\vee$ generated by the transformations $s_0^\vee, s_1^\vee, \ldots, s_n^\vee$ given as follows: If $\lambda^\vee = \lambda_1\varepsilon_1^\vee + \cdots + \lambda_n\varepsilon_n^\vee + \frac{k}{2}K$ then

$$\begin{aligned} s_0^\vee\lambda^\vee &= -\lambda_1\varepsilon_1^\vee + \lambda_2\varepsilon_2^\vee + \cdots + \lambda_n\varepsilon_n^\vee + \left(\tfrac{k}{2} + \lambda_1\right)K, \\ s_n^\vee\lambda^\vee &= \lambda_1\varepsilon_1^\vee + \cdots + \lambda_{n-1}\varepsilon_{n-1}^\vee - \lambda_n\varepsilon_n^\vee + \tfrac{k}{2}K, \qquad \text{and} \\ s_i^\vee\lambda^\vee &= \lambda_1\varepsilon_1^\vee + \cdots + \lambda_{i-1}\varepsilon_{i-1}^\vee + \lambda_{i+1}\varepsilon_i^\vee + \lambda_i\varepsilon_{i+1}^\vee + \lambda_{i+2}\varepsilon_{i+2}^\vee + \cdots + \lambda_n\varepsilon_n^\vee + \tfrac{k}{2}K, \end{aligned} \tag{1.4}$$

for $i \in \{1, \dots, n-1\}$. Let

$$s_{\varepsilon_1}^\vee = s_1^\vee \cdots s_n^\vee \cdots s_1^\vee, \qquad h_{\varepsilon_1} = s_0^\vee s_{\varepsilon_1}^\vee = s_0^\vee s_1^\vee \cdots s_n^\vee \cdots s_1^\vee \qquad \text{and} \qquad h_{\varepsilon_{i+1}} = s_i^\vee h_{\varepsilon_i} s_i^\vee,$$

for $i \in \{1, \dots, n-1\}$. Then $h_{\varepsilon_1}\lambda^\vee = s_0^\vee s_{\varepsilon_1}^\vee \lambda^\vee = \lambda_1\varepsilon_1^\vee + \cdots + \lambda_n\varepsilon_n^\vee + \left(\frac{k}{2} - \lambda_1\right) K$ and

$$h_{\varepsilon_i}\lambda^\vee = \lambda_1\varepsilon_1^\vee + \cdots + \lambda_n\varepsilon_n^\vee + \left(\tfrac{k}{2} - \lambda_i\right) K, \qquad \text{for } i \in \{1, \dots, n\}.$$

If $\gamma = \gamma_1\varepsilon_1 + \cdots \gamma_n\varepsilon_n$ and $\lambda^\vee = \lambda_1\varepsilon_1^\vee + \cdots + \lambda_n\varepsilon_n^\vee + \frac{k}{2}K$ then

$$h_\gamma\lambda^\vee = h_{\varepsilon_1}^{\gamma_1} \cdots h_{\varepsilon_n}^{\gamma_n}\lambda^\vee = \lambda^\vee + (\tfrac{k}{2} - (\lambda_1\gamma_1 + \cdots + \lambda_n\gamma_n))K = \lambda^\vee + (\tfrac{k}{2} - \langle\gamma, \lambda^\vee\rangle)K,$$

and special cases of this last formula are

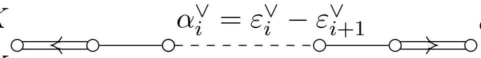
$$h_\gamma\varepsilon_i^\vee = \varepsilon_i^\vee - \gamma_i K \qquad \text{and} \qquad h_\gamma K = K. \tag{1.5}$$

The set of coroots $S^\vee$ for type $CC_n$ is the union of the five $W_X$-orbits given by

$$\begin{aligned}
O_1^\vee &= W_X \cdot \alpha_n^\vee = W_X \cdot \varepsilon_n^\vee = \{\pm\varepsilon_i^\vee + rK \mid i \in \{1, \dots, n\}, r \in \mathbb{Z}\}, \\
2O_1^\vee &= W_X \cdot 2\alpha_n^\vee = W_X \cdot 2\varepsilon_n^\vee = \{\pm 2\varepsilon_i^\vee + 2rK \mid i \in \{1, \dots, n\}, r \in \mathbb{Z}\}, \\
O_3^\vee &= W_X \cdot \alpha_0^\vee = W_X \cdot (-\varepsilon_1^\vee + \tfrac{1}{2}K) = \{\pm(\varepsilon_i^\vee + \tfrac{1}{2}(2r+1)K \mid i \in \{1, \dots, n\}, r \in \mathbb{Z}\}, \\
2O_3^\vee &= W_X \cdot 2\alpha_0^\vee = W_Y \cdot (-2\varepsilon_1^\vee + K) = \{\pm 2\varepsilon_i^\vee + (2r+1)K \mid i \in \{1, \dots, n\}, r \in \mathbb{Z}\}, \\
O_5^\vee &= W_X \cdot \alpha_1^\vee = W_X \cdot (\varepsilon_1^\vee - \varepsilon_2^\vee) = \left\{ \begin{array}{c} \pm(\varepsilon_i^\vee + \varepsilon_j^\vee) + rK, \\ \pm(\varepsilon_i^\vee - \varepsilon_j^\vee) + rK \end{array} \;\middle|\; i, j \in \{1, \dots, n\}, i < j, r \in \mathbb{Z} \right\},
\end{aligned}$$

where

$$\alpha_0^\vee = -\varepsilon_1^\vee + \tfrac{1}{2}K \qquad \alpha_i^\vee = \varepsilon_i^\vee - \varepsilon_{i+1}^\vee \qquad \alpha_n^\vee = \varepsilon_n^\vee$$



$$2\alpha_0^\vee = -2\varepsilon_1^\vee + K \qquad\qquad 2\alpha_n^\vee = 2\varepsilon_n^\vee \tag{1.6}$$

**Remark 1.2.** Throughout this paper we present several diagrams imitating the Dynkin diagram with the labeling related to the coroots (like the one above), roots, or parameters. These are merely intended for conceptual association rather than to specify relations between the objects.

### 1.2.3 The affine roots for type $CC_n$

Let

$$Q = \mathbb{Z}\text{-span}\{\varepsilon_1, \dots, \varepsilon_n, \tfrac{1}{2}\delta\}$$

be the $\mathbb{Z}$-vector space spanned by symbols $\varepsilon_1, \dots, \varepsilon_n$ and $\frac{1}{2}\delta$. The affine Weyl group $W_Y$ is the group of $\mathbb{Z}$-linear transformations of $Q$ generated by the transformations $s_0, s_1, \dots, s_n$ given as follows: If $\mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n + \frac{k}{2}\delta$ then

$$\begin{aligned}
s_0\mu &= -\mu_1\varepsilon_1 + \mu_2\varepsilon_2 + \cdots + \mu_n\varepsilon_n + \left(\tfrac{k}{2} + \lambda_1\right)\delta, \\
s_n\mu &= \mu_1\varepsilon_1 + \cdots + \mu_{n-1}\varepsilon_{n-1} - \mu_n\varepsilon_n + \tfrac{k}{2}\delta, \qquad \text{and} \\
s_i\mu &= \mu\varepsilon_1 + \cdots + \mu_{i-1}\varepsilon_{i-1} + \mu_{i+1}\varepsilon_i + \mu_i\varepsilon_{i+1} + \mu_{i+2}\varepsilon_{i+2} + \cdots + \mu_n\varepsilon_n + \tfrac{k}{2}\delta,
\end{aligned} \tag{1.7}$$

for $i \in \{1, \dots, n-1\}$. Let

$$s_{\varepsilon_1} = s_1 \cdots s_n \cdots s_1, \qquad h_{\varepsilon_1^\vee} = s_0 s_{\varepsilon_1} = s_0 s_1 \cdots s_n \cdots s_1 \qquad \text{and} \qquad h_{\varepsilon_{i+1}^\vee} = s_i t_{\varepsilon_i^\vee} s_i,$$

for $i \in \{1, \dots, n-1\}$. Then $h_{\varepsilon_1^\vee}\mu = s_0 s_{\varepsilon_1}\mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n + \left(\frac{k}{2} - \mu_1\right)\delta$ and

$$h_{\varepsilon_i^\vee}\mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n + \left(\tfrac{k}{2} - \mu_i\right)\delta, \qquad \text{for } i \in \{1, \dots, n\}.$$

If $\nu^\vee = \nu_1\varepsilon_1^\vee + \cdots \nu_n\varepsilon_n^\vee$ and $\mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n + \frac{k}{2}\delta$ then

$$h_{\nu^\vee}\mu = h_{\varepsilon_1^\vee}^{\nu_1} \cdots h_{\varepsilon_n^\vee}^{\nu_n}\mu = \mu - (\tfrac{k}{2} + (\mu_1\nu_1 + \cdots + \mu_n\nu_n))\delta = \mu + (\tfrac{k}{2} - \langle \mu, \nu^\vee \rangle)\delta,$$

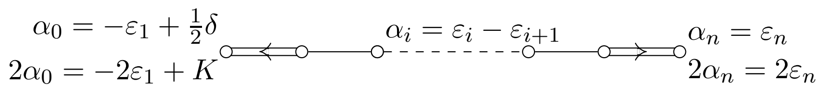
and special cases of this last formula are

$$h_{\nu^\vee}\varepsilon_i = \varepsilon_i - \nu_i\delta \qquad \text{and} \qquad h_{\nu^\vee}\delta = \delta.$$

The set of roots $S$ for type $CC_n$ is the union of the five $W_Y$-orbits given by

$$\begin{aligned}
O_1 &= W_Y \cdot \alpha_n = W_Y \cdot \varepsilon_n = \{\pm\varepsilon_i + r\delta \mid i \in \{1, \dots, n\}, r \in \mathbb{Z}\}, \\
2O_1 = O_2 &= W_Y \cdot 2\alpha_n = W_Y \cdot 2\varepsilon_n = \{\pm 2\varepsilon_i + 2r\delta \mid i \in \{1, \dots, n\}, r \in \mathbb{Z}\}, \\
O_3 &= W_Y \cdot \alpha_n = W_Y \cdot (-\varepsilon_1 + \tfrac{1}{2}\delta) = \{\pm(\varepsilon_i + \tfrac{1}{2}(2r+1)\delta \mid i \in \{1, \dots, n\}, r \in \mathbb{Z}\}, \\
2O_3 = O_4 &= W_Y \cdot 2\alpha_n = W_Y \cdot (-2\varepsilon_1 + \delta) = \{\pm 2\varepsilon_i + (2r+1)\delta \mid i \in \{1, \dots, n\}, r \in \mathbb{Z}\}, \\
O_5 &= W_Y \cdot \alpha_1 = W_Y \cdot (\varepsilon_1 - \varepsilon_2) = \left\{ \begin{matrix} \pm(\varepsilon_i + \varepsilon_j) + r\delta \\ \pm(\varepsilon_i - \varepsilon_j) + r\delta \end{matrix} \;\middle|\; i, j \in \{1, \dots, n\}, i < j, r \in \mathbb{Z} \right\},
\end{aligned}$$

where




(1.8)

## 1.3 Other classical types

With the notations as in Section 1.2.3, each affine root system of classical type is a subset of $Q$. The irreducible affine root systems of classical type (and the appropriate specializations for obtaining the Macdonald polynomials of each type from the Macdonald polynomials of type $(C_n^\vee, C_n)$) are given by the diagram in Section 0.0.1. The middle notation for each root system is the notation in Macdonald [Mac03, §1.3], the right notation is that of Bruhat-Tits [BT72] and the left notation is that of Kac [Kac, Ch. 6].

To determine the specializations, we look at the kernel for the inner product (see Section 5.1). Following [Mac03, (5.1.3)] the orthogonal polynomials are determined by the inner product which, in turn, is determined by factors of the form in (1.9) corresponding to orbits of roots.

For instance, when both $O_1$ and $2O_1$ are present then the factor corresponding to the root $\varepsilon_n$ is

$$\frac{1}{\Delta_{\varepsilon_n}\Delta_{2\varepsilon_n}} = \frac{(1 - t_n^{\frac{1}{2}} u_n^{\frac{1}{2}} x_n)(1 + t_n^{\frac{1}{2}} u_n^{-\frac{1}{2}} x_n)}{1 - x_n^2}. \tag{1.9}$$

(The notation $\frac{1}{\Delta_{\varepsilon_n}\Delta_{2\varepsilon_n}}$ for this factor is as in [Mac03, (5.1.3)]; the notation that we use for this factor in Section 5.1 is $\kappa^X_{\varepsilon_n}$.) If only the orbit $O_1$ is present then the factor is

$$\frac{1}{\Delta_{\varepsilon_n}} = \frac{1 - t_n x_n}{1 - x_n} \qquad \text{which is obtained by specializing } t_n^{\frac{1}{2}} = u_n^{\frac{1}{2}} \text{ in (1.9).}$$

If only the orbit $2O_1$ is present then the factor is

$$\frac{1}{\Delta_{2\varepsilon_n}} = \frac{1 - t_n x_n^2}{1 - x_n^2} \qquad \text{which is obtained by specializing } u_n^{\frac{1}{2}} = 1 \text{ in (1.9).}$$

In this way, the parameter specializations in the diagram in Section 0.0.1 are determined by which orbits of roots are present in the root system.

# 2 $c$-functions and DAHA relations

This section collects the tools for working with Koornwinder polynomials as polynomials in $x_1^{\pm 1}, \ldots, x_n^{\pm 1}$ that depend on 6 parameters. The root system of type $CC_n$ provides the structure for organizing the many symmetries between the variables and the various parameters and this section specifies carefully the links to the root system. The $c$-functions, introduced in Section 2.3, are a core structure to providing explicit formulas for Koornwinder expansions, specializations and norm formulas. With the notation for the $c$-functions in hand, Section 2.4 describes briefly the relations of the double affine Hecke algebra. These relations provide a convenient summary of the calculus of the operators on polynomials which are used in the rest of the paper.

## 2.1 Parameters

Let

$$q, t^{\frac{1}{2}}, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}} \quad \text{be independent parameters,}$$

and let

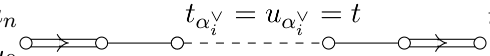
$$\mathbb{K} = \mathbb{C}(q, t^{\frac{1}{2}}, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}) \quad \text{be the field of fractions of } \mathbb{C}[q, t^{\frac{1}{2}}, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}].$$

The field $\mathbb{K}$ will be the base field for most algebras in this paper.

Recalling the simple coroots and simple roots from (1.6) and (1.8), set

$$t_{\alpha_0^\vee} = u_n \qquad t_{\alpha_i^\vee} = u_{\alpha_i^\vee} = t \qquad t_{\alpha_n^\vee} = t_n$$

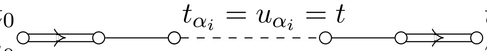
$$u_{\alpha_0^\vee} = u_0 \qquad\qquad\qquad\qquad u_{\alpha_n^\vee} = t_0 \tag{2.1}$$



and

$$t_{\alpha_0} = t_0 \qquad t_{\alpha_i} = u_{\alpha_i} = t \qquad t_{\alpha_n} = t_n$$

$$u_{\alpha_0} = u_0 \qquad\qquad\qquad\qquad u_{\alpha_n} = u_n \tag{2.2}$$



Let $S^\vee$ be the set of affine coroots and let $S$ be the set of affine roots. Define $t_{\alpha^\vee}$, $u_{\alpha^\vee}$ and $t_\alpha$, $u_\alpha$ for arbitrary coroots $\alpha^\vee$ and arbitrary roots $\alpha$ by requiring

$$t_{w\alpha} = t_\alpha \qquad \text{and} \qquad t_{w\alpha^\vee} = t_{\alpha^\vee}, \qquad \text{for } w \in W,\ \alpha^\vee \in S^\vee \text{ and } \alpha \in S. \tag{2.3}$$

The difference between the parameters for the coroots in (2.1) and the parameters for the roots in (2.2) corresponds exactly to introduction of the "dual labels" in [Mac03, (1.5.1)].

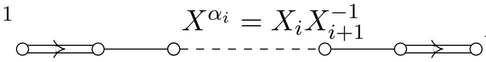
## 2.2 The polynomial rings $\mathbb{K}[X]$ and $\mathbb{K}[Y]$

Let $\mathbb{K}[X] = \mathbb{K}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ be the Laurent polynomial ring in the variables $X_1, \ldots, X_n$. Identify $\mathbb{K}[X]$ with the group algebra of $Q = \mathbb{Z}\text{-span}\{\varepsilon_1, \ldots, \varepsilon_n, \frac{1}{2}\delta\}$ via the notations

$$q^{\frac{1}{2}} = X^{\frac{1}{2}\delta} \quad \text{and} \quad X_i = X^{\varepsilon_i}, \quad \text{and} \quad q^{\frac{k}{2}} X_1^{\mu_1} \cdots X_n^{\mu_n} = X^{\frac{k}{2}\delta + \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n} = X^\mu,$$

for $i \in \{1, \ldots, n\}$ and $\mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n + \frac{k}{2}\delta \in Q$. The image of the simple roots in $\mathbb{K}[X]$ is given by

$$X^{\alpha_0} = q^{\frac{1}{2}} X_1^{-1} \qquad X^{\alpha_i} = X_i X_{i+1}^{-1} \qquad X^{\alpha_n} = X_n \tag{2.4}$$

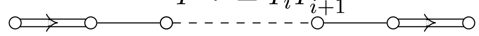


Let $\mathbb{K}[Y] = \mathbb{K}[Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}]$ be the Laurent polynomial ring in the variables $Y_1, \ldots, Y_n$. Identify $\mathbb{K}[Y]$ with the group algebra of $Q^\vee = \mathbb{Z}\text{-span}\{\varepsilon_1^\vee, \ldots, \varepsilon_n^\vee, \frac{1}{2}K\}$ via the notations

$$q^{\frac{1}{2}} = Y^{-\frac{1}{2}K} \quad \text{and} \quad Y_i = Y^{\varepsilon_i^\vee}, \quad \text{and} \quad q^{-\frac{k}{2}} Y_1^{\lambda_1} \cdots Y_n^{\lambda_n} = Y^{\frac{k}{2}K + \lambda_1\varepsilon_1^\vee + \cdots + \lambda_n \varepsilon_n^\vee} = Y^{\lambda^\vee},$$

for $i \in \{1, \ldots, n\}$ and $\lambda^\vee = \lambda_1 \varepsilon_1^\vee + \cdots + \lambda_n \varepsilon_n^\vee + \frac{k}{2}K \in Q^\vee$. The image of the simple coroots in $\mathbb{K}[Y]$ is given by

$$Y^{\alpha_0^\vee} = q^{-\frac{1}{2}} Y_1^{-1} \qquad Y^{\alpha_i^\vee} = Y_i Y_{i+1}^{-1} \qquad Y^{\alpha_n^\vee} = Y_n \tag{2.5}$$



## 2.3 $c$-functions

Let $\mathbb{K}(Y)$ be the field of fractions of the Laurent polynomial ring $\mathbb{K}[Y]$. For a coroot $\alpha^\vee$ let (see [Mac03, (4.2.2) and (4.3.9)]),

$$c^Y_{\alpha^\vee} = t_{\alpha^\vee}^{-\frac{1}{2}} \frac{(1 - t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{\frac{1}{2}} Y^{\alpha^\vee})(1 + t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{-\frac{1}{2}} Y^{\alpha^\vee})}{(1 - Y^{2\alpha^\vee})} \qquad \text{and} \qquad \kappa^Y_{\alpha^\vee} = t_{\alpha^\vee}^{\frac{1}{2}} c^Y_{\alpha^\vee}. \tag{2.6}$$

$$\text{If } t_{\alpha^\vee}^{\frac{1}{2}} = u_{\alpha^\vee}^{\frac{1}{2}} \quad \text{then} \quad c^Y_{\alpha^\vee} = \frac{t_{\alpha^\vee}^{-\frac{1}{2}} - t_{\alpha^\vee}^{\frac{1}{2}} Y^{\alpha^\vee}}{1 - Y^{\alpha^\vee}} \qquad \text{and} \qquad \kappa^Y_{\alpha^\vee} = \frac{1 - t_{\alpha^\vee} Y^{\alpha^\vee}}{1 - Y^{\alpha^\vee}}.$$

(More accurately, the function $c^Y_{\alpha^\vee}$ should be considered as a local factor of a $c$-function, see [Sto11].) The expression $\kappa^Y_{\alpha^\vee}$ is a slightly renormalized version of the $c$-function $c^Y_{\alpha^\vee}$ which, although not technically necessary, is immensely helpful for making the formulas more palatable.

In general, for arbitrary $t_{\alpha^\vee}^{\frac{1}{2}}$ and $u_{\alpha^\vee}^{\frac{1}{2}}$,

$$\begin{aligned}
c^Y_{\alpha^\vee} + c^Y_{-\alpha^\vee} &= \frac{(1 - t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{\frac{1}{2}} Y^{\alpha^\vee})(1 + t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{-\frac{1}{2}} Y^{\alpha^\vee})}{t_{\alpha^\vee}^{\frac{1}{2}}(1 - Y^{2\alpha^\vee})} + \frac{(1 - t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{\frac{1}{2}} Y^{-\alpha^\vee})(1 + t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{-\frac{1}{2}} Y^{-\alpha^\vee})}{t_{\alpha^\vee}^{\frac{1}{2}}(1 - Y^{-2\alpha^\vee})} \\
&= \frac{(1 - t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{\frac{1}{2}} Y^{\alpha^\vee})(1 + t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{-\frac{1}{2}} Y^{\alpha^\vee})}{t_{\alpha^\vee}^{\frac{1}{2}}(1 - Y^{2\alpha^\vee})} + \frac{t_{\alpha^\vee}(1 - t_{\alpha^\vee}^{-\frac{1}{2}} u_{\alpha^\vee}^{-\frac{1}{2}} Y^{\alpha^\vee})(1 + t_{\alpha^\vee}^{-\frac{1}{2}} u_{\alpha^\vee}^{\frac{1}{2}} Y^{\alpha^\vee})}{t_{\alpha^\vee}^{\frac{1}{2}}(1 - Y^{2\alpha^\vee})} \\
&= \frac{1 - t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{\frac{1}{2}} Y^{\alpha^\vee} + t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{-\frac{1}{2}} Y^{\alpha^\vee} - t_{\alpha^\vee} Y^{2\alpha^\vee} + t_{\alpha^\vee} - t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{-\frac{1}{2}} Y^{\alpha^\vee} + t_{\alpha^\vee}^{\frac{1}{2}} u_{\alpha^\vee}^{\frac{1}{2}} Y^{\alpha^\vee} - Y^{2\alpha^\vee})}{t_{\alpha^\vee}^{\frac{1}{2}}(1 - Y^{2\alpha^\vee})} \\
&= \frac{(1 + t_{\alpha^\vee})(1 - Y^{2\alpha^\vee})}{t_{\alpha^\vee}^{\frac{1}{2}}(1 - Y^{2\alpha^\vee})} = t_{\alpha^\vee}^{\frac{1}{2}} + t_{\alpha^\vee}^{-\frac{1}{2}}.
\end{aligned} \tag{2.7}$$

Let $w \in W$ and let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced word for $w$. The *coroot sequence* of the reduced word $w = s_{i_1} \cdots s_{i_\ell}$ is

$$\text{the sequence } (\beta_k^\vee \mid k \in \{1, \ldots, \ell\} \text{ and } i_k \neq \pi\}) \text{ given by} \qquad \beta_k^\vee = s_{i_\ell}^{-1} \cdots s_{i_{k+1}}^{-1} \alpha_{i_k}^\vee. \tag{2.8}$$

Then define

$$\kappa^Y_w = \prod_{\beta^\vee \in \mathrm{Inv}(w)} \kappa^Y_{\beta^\vee}, \qquad \text{where} \qquad \mathrm{Inv}(w) = \{\beta_1^\vee, \ldots, \beta_\ell^\vee\} \tag{2.9}$$

is the set of elements in a coroot sequence for a reduced word for $w$. If $w = uv$ with $\ell(u) + \ell(v) = \ell(w)$ then the coroot sequence of $w$ is $v^{-1}$ times the coroot sequence of $u$ followed by the coroot sequence of $v$ so that

$$\mathrm{Inv}(uv) = v^{-1}\mathrm{Inv}(u) \cup \mathrm{Inv}(v).$$

The inversions of elements of $W_{\text{fin}}$ come in two types: 'droite' and 'standard'. Using the indicators $d$ for 'droite' and $s$ for 'standard', let

$$(S^\vee_{0,d})^+ = \{\varepsilon^\vee_1, \dots, \varepsilon^\vee_n\} \qquad \text{and} \qquad (S^\vee_{0,s})^+ = \left\{ \begin{matrix} \varepsilon^\vee_i - \varepsilon^\vee_j, \\ \varepsilon^\vee_i + \varepsilon^\vee_j \end{matrix} \;\middle|\; i,j \in \{1,\dots,n\} \text{ with } i<j \right\},$$

and for $v \in W_{\text{fin}}$, define

$$\mathrm{Inv}_d(v) = \mathrm{Inv}(v) \cap (S^\vee_{0,d})^+ \quad \text{and} \quad \mathrm{Inv}_s(v) = \mathrm{Inv}(v) \cap (S^\vee_{0,s})^+.$$

For $v \in W_{\text{fin}}$ define

$$\begin{aligned} \kappa^+_v &= \left(\prod_{\beta^\vee \in \mathrm{Inv}_s(v)} \kappa^Y_{\beta^\vee}\right)\left(\prod_{\beta^\vee \in \mathrm{Inv}_d(v)} \kappa^Y_{\beta^\vee}\right), \qquad & \kappa^\pm_v &= \left(\prod_{\beta^\vee \in \mathrm{Inv}_s(v)} \kappa^Y_{\beta^\vee}\right)\left(\prod_{\beta^\vee \in \mathrm{Inv}_d(v)} \kappa^{Y^{-1}}_{\beta^\vee}\right), \\ \kappa^-_v &= \left(\prod_{\beta^\vee \in \mathrm{Inv}_s(v)} \kappa^{Y^{-1}}_{\beta^\vee}\right)\left(\prod_{\beta^\vee \in \mathrm{Inv}_d(v)} \kappa^{Y^{-1}}_{\beta^\vee}\right), \qquad & \kappa^\mp_v &= \left(\prod_{\beta^\vee \in \mathrm{Inv}_s(w)} \kappa^{Y^{-1}}_{\beta^\vee}\right)\left(\prod_{\beta^\vee \in \mathrm{Inv}_d(w)} \kappa^Y_{\beta^\vee}\right). \end{aligned} \tag{2.10}$$

Finally, define also

$$\kappa^Y_{\mathrm{dr}} = \prod_{i=1}^n \kappa^Y_{\varepsilon^\vee_i} = \prod_{i=1}^n \frac{(1 - t_n^{\frac12} t_0^{\frac12} Y_i)(1 + t_n^{\frac12} t_0^{-\frac12} Y_i)}{(1 - Y_i^2)}$$

and

$$\kappa^Y_{\mathrm{st}} = \prod_{1 \le i<j \le n} \kappa^Y_{\varepsilon^\vee_i - \varepsilon^\vee_j} \kappa^Y_{\varepsilon^\vee_i + \varepsilon^\vee_j} = \prod_{1 \le i<j \le n} \frac{(1 - tY_iY_j^{-1})}{(1 - Y_iY_j^{-1})} \frac{(1 - tY_iY_j)}{(1 - Y_iY_j)} \tag{2.11}$$

so that

$$\kappa^Y_{w_0} = \kappa^Y_{\mathrm{dr}} \kappa^Y_{\mathrm{st}}. \tag{2.12}$$

This subsection has presented the $c$-functions and related functions $\kappa^Y_{\beta^\vee}$ in terms of the $\{Y_1, \cdots, Y_n\}$ variables. We will also consider these notions in other sets of variables, like $\{Y_1^{-1}, \cdots, Y_n^{-1}\}$, $\{X_1, \cdots, X_n\}$, and $\{X_1^{-1}, \cdots, X_n^{-1}\}$, and use notations like $\kappa^{Y^{-1}}_{\beta^\vee}$, $\kappa^X_\beta$, $\kappa^{X^{-1}}_\beta$, respectively. For example

$$\kappa^{X^{-1}}_{w_0} = \left(\prod_{i=1}^n \frac{(1 - t_n^{\frac12} u_n^{\frac12} X_i^{-1})(1 + t_n^{\frac12} u_n^{-\frac12} X_i^{-1})}{(1 - X_i^{-2})}\right)\left(\prod_{1 \le i<j \le n} \frac{(1 - tX_i^{-1}X_j)(1 - tX_i^{-1}X_j^{-1})}{(1 - X_i^{-1}X_j)(1 - X_i^{-1}X_j^{-1})}\right). \tag{2.13}$$

## 2.4 The algebras $\widetilde{H}_{\mathrm{loc}}$ and $\widetilde{H}_{\mathrm{int}}$

The Koornwinder polynomials are elements of the polynomial ring $\mathbb{K}[x_1^{\pm1}, \dots, x_n^{\pm1}]$ which are characterized, up to normalization, by the fact that they are eigenvectors for the Cherednik-Dunkl operators $Y_1, \dots, Y_n$ (see (3.5)). However the $c$-functions, which form the core calculus for working with Koornwinder polynomials, are elements of $\mathbb{K}(X)$, the field of fractions of the polynomial ring. Thus extending from $\mathbb{K}[X]$ to $\mathbb{K}(X)$ is necessary for handling the tools.

The Cherednik-Dunkl operators are elements of the double affine Hecke algebra $\widetilde{H}_{\mathrm{int}}$, which is formed by pasting the two polynomial rings $\mathbb{K}[X]$ and $\mathbb{K}[Y]$ together with a finite Hecke algebra $H_{\mathrm{fin}}$. However, the right home for the $c$-functions and the operators for creating Koornwinder polynomials is a larger algebra $\widetilde{H}_{\mathrm{loc}}$ which extends the algebra $\widetilde{H}_{\mathrm{int}}$ by extending $\mathbb{K}[X]$ and $\mathbb{K}[Y]$ to the fraction fields $\mathbb{K}(X)$ and $\mathbb{K}(Y)$. In this subsection we introduce the algebras $\widetilde{H}_{\mathrm{loc}}$ and $\widetilde{H}_{\mathrm{int}}$ by generators and relations.

Let $\mathbb{K}(X)$ and $\mathbb{K}(Y)$ be the fraction fields of $\mathbb{K}[X]$ and $\mathbb{K}[Y]$ respectively. Recall that $\mathbb{K}[X]$ is the group algebra of $Q$ and $\mathbb{K}[Y]$ is the group algebra of $Q^\vee$, that

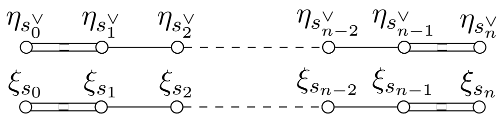
$$X^{\frac{1}{2}\delta} = q \qquad \text{and} \qquad Y^{-\frac{1}{2}K} = q^{\frac{1}{2}}$$

and that $W_X$ acts on $Q^\vee$ and $W_Y$ acts on $Q$ by the formulas given in (1.7) and (1.4) so that

$$wX^\mu = X^{w\mu} \qquad \text{and} \qquad zY^{\lambda^\vee} = Y^{z\lambda^\vee}, \tag{2.14}$$

for $w \in W_Y$, $z \in W_X$, $\lambda^\vee \in Q^\vee$ and $\mu \in Q$.

Let $\widetilde{H}_{\mathrm{loc}}$ be the $\mathbb{K}$-algebra generated by $\eta_{s_0^\vee}, \ldots, \eta_{s_n^\vee}$, $\xi_{s_0}, \ldots, \xi_{s_n}$, $T_1, \ldots, T_n$, $T_{\alpha_0}, T_{\alpha_1}, \ldots, T_{\alpha_n}$ and $T_{\alpha_0^\vee}, T_{\alpha_1^\vee}, \ldots, T_{\alpha_n^\vee}$ and $T_{s_{\varepsilon_1}}$ and $\mathbb{K}(X)$ and $\mathbb{K}(Y)$ with relations




$$\begin{aligned} \eta_{s_i}^2 &= 1, & \eta_{s_i} Y^{\lambda^\vee} &= Y^{s_i\lambda^\vee}\eta_{s_i} \\ \xi_{s_i}^2 &= 1, & \xi_{s_i} X^{\mu} &= X^{s_i\mu}\xi_{s_i}, \end{aligned} \tag{2.15}$$

for $i \in \{0, 1, \ldots, n\}$, $\lambda^\vee \in Q^\vee$ and $\mu \in Q$, and

$$T_{\alpha_i^\vee} + t_{\alpha_i^\vee}^{-\frac{1}{2}} = (1 + \eta_{s_i^\vee}) c^Y_{\alpha_i^\vee} \quad \text{and} \quad T_{\alpha_i} + t_{\alpha_i}^{-\frac{1}{2}} = (1 + \xi_{s_i}) c^{X^{-1}}_{\alpha_i}, \qquad \text{for } i \in \{0, 1, \ldots, n\}, \tag{2.16}$$

$$T_j = T_{\alpha_j} = T_{\alpha_j^\vee} \quad \text{for } j \in \{1, \ldots, n\} \qquad \text{and} \qquad T_{s_{\varepsilon_1}} = T_1 \cdots T_{n-1} T_n T_{n-1} \cdots T_1, \tag{2.17}$$

$$Y^{\varepsilon_1^\vee} = T_{\alpha_0} T_{s_{\varepsilon_1}} \qquad X^{\varepsilon_1} = (T_{\alpha_0^\vee})^{-1} T_{s_{\varepsilon_1}}^{-1} \tag{2.18}$$

$$Y^{\varepsilon_{j+1}^\vee} = T_{\alpha_j}^{-1} Y^{\varepsilon_j^\vee} T_{\alpha_j}^{-1}, \qquad \text{and} \qquad X^{\varepsilon_{j+1}} = T_{\alpha_j} X^{\varepsilon_j} T_{\alpha_j}, \qquad \text{for } j \in \{1, \ldots, n-1\}, \tag{2.19}$$

$$Y^{-\varepsilon_1^\vee} X^{\varepsilon_1} = q^{\frac{1}{2}}(u_0^{\frac{1}{2}} - u_0^{-\frac{1}{2}}) T_{s_{\varepsilon_1}}^{-1} + q T_{s_{\varepsilon_1}}^{-1} X^{-\varepsilon_1} Y^{\varepsilon_1} T_{s_{\varepsilon_1}}^{-1}. \tag{2.20}$$

This presentation of $\widetilde{H}_{\mathrm{loc}}$ is not minimal as there are many redundant generators and many redundant relations. It is designed to specify notations and list the relations that we will need, and to motivate the operators on polynomials which are the main tools for working with Macdonald polynomials in general. To be precise, since

$$(1 + \xi_{s_i}) c^{X^{-1}}_{\alpha_i} = T_{\alpha_i} + t_{\alpha_i}^{-\frac{1}{2}} = (T_{\alpha_i} - t_{\alpha_i}^{\frac{1}{2}}) + t_{\alpha_i}^{-\frac{1}{2}} + t_{\alpha_i}^{\frac{1}{2}} = (T_{\alpha_i} - t_{\alpha_i}^{\frac{1}{2}}) + c^{X^{-1}}_{\alpha_i} + c^{X^{-1}}_{-\alpha_i}$$

then

$$T_{\alpha_i} - t^{\frac{1}{2}} = -c^{X^{-1}}_{-\alpha_i} + \xi_{s_i} c^{X^{-1}}_{\alpha_i} = -c^{X^{-1}}_{-\alpha_i}(1 - \xi_{s_i}) = -c^X_{\alpha_i}(1 - \xi_{s_i}), \tag{2.21}$$

and this is the formula used to define the action of the double affine Hecke algebra on $\mathbb{K}[X]$ in (3.3).

The *double affine Hecke algebra (DAHA)* is the subalgebra $\widetilde{H}_{\mathrm{int}}$ inside $\widetilde{H}_{\mathrm{loc}}$

$$\text{generated by } X_1^{\pm 1}, \ldots, X_n^{\pm 1},\ Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}, \text{ and } T_1, \ldots, T_n.$$

The algebra $\widetilde{H}_{\mathrm{int}}$ is an integral form of $\widetilde{H}_{\mathrm{loc}}$ (alternatively, the algebra $\widetilde{H}_{\mathrm{loc}}$ is a completion, or localized form, of $\widetilde{H}_{\mathrm{int}}$). A common definition of the DAHA uses the relations listed in the following proposition, which follow without difficulty from the defining relations of $\widetilde{H}_{\mathrm{loc}}$.

**Proposition 2.1.** *Let $i \in \{0, 1, \dots, n\}$, $\lambda^\vee \in Q^\vee$ and $\mu \in Q$. Let $T_0^{\#} = q^{-\frac{1}{2}} T_{\alpha_0^\vee}^{-1} T_{s_{\varepsilon_1}}^{-1} T_{\alpha_0}^{-1}$. Then*


(2.22)

$$T_{\alpha_0^\vee} T_{\alpha_1}^{-1} T_{\alpha_0} T_{\alpha_1} = T_{\alpha_1}^{-1} T_{\alpha_0} T_{\alpha_1} T_{\alpha_0^\vee}, \qquad (T_0^{\#} - u_0^{\frac{1}{2}})(T_0^{\#} + u_0^{\frac{1}{2}}) = 0, \tag{2.23}$$

$$(T_{\alpha_i^\vee} - t_{\alpha_i^\vee}^{\frac{1}{2}})(T_{\alpha_i^\vee} + t_{\alpha_i^\vee}^{-\frac{1}{2}}) = 0 \quad \text{and} \quad T_{\alpha_i^\vee} Y^{\lambda^\vee} = Y^{s_i^\vee \lambda^\vee} T_{\alpha_i^\vee} + (c_{\alpha_i^\vee}^{Y} - t_{\alpha_i^\vee}^{-\frac{1}{2}})(Y^{\lambda^\vee} - Y^{s_i^\vee \lambda^\vee}), \tag{2.24}$$

$$(T_{\alpha_i} - t_{\alpha_i}^{\frac{1}{2}})(T_{\alpha_i} + t_{\alpha_i}^{-\frac{1}{2}}) = 0 \quad \text{and} \quad T_{\alpha_i} X^{-\mu} = X^{-s_i\mu} T_{\alpha_i} + (c_{\alpha_i}^{X^{-1}} - t_{\alpha_i}^{-\frac{1}{2}})(X^{-\mu} - X^{-s_i\mu}). \tag{2.25}$$

**Remark 2.2.** In personal communication, J. Stokman insightfully points out that the algebra $\widetilde{H}_{\mathrm{loc}}$ defined in (2.15)–(2.20) is fishy. One can Ore-localize DAHA in either the $X$-elements (so that the normalized intertwiners $\xi_{s_j}$ are in the localized algebra), or in the $Y$-elements (so that the $\eta_{s_j}$ are in the ones in the localized algebra) but not simultaneously. This difficulty is alluded to in a different form in [CR22, Remark 3.5]. However, trying to set up the accurate formal framework for handling an $X$-$Y$-localized algebra would be distracting from the combinatorial perspective of this paper. We feel that, even if the right localization formalism is not in place, the concept of an algebra $\widetilde{H}_{\mathrm{loc}}$ that contains all the useful relations for computations with these operators is a healthy point of view and so we have chosen to include it. The algebra $\widetilde{H}_{\mathrm{loc}}$ is not absolutely necessary for the results in this paper as the operators on the polynomial representation that we use are well-defined and the proofs that we give are valid.

## 2.5 Symmetrizers

There are four ways of symmetrizing/antisymmetrizing in the Koornwinder polynomial context, corresponding to the four 1-dimensional representations of $W_{\mathrm{fin}}$. These 1-dimensional representations are the analogues of the usual sign of a permutation. The four symmetrizers, and useful formulas for them, are presented in Sections 2.5, 2.6 and 2.7. The symmetrizers will be used in Section 4 to construct and manage the bosonic (symmetric), fermionic (antisymmetric) and the two mesonic (half symmetric-half antisymmetric) versions of the Koornwinder polynomials. These four symmetrized/antisymmetrized versions of Koornwinder polynomials, and the relations between them, turn out to be fundamental in the proof of the norm formulas and constant term formulas that are established in Section 5.

The finite Hecke algebra $H_{\mathrm{fin}}$ is the $\mathbb{K}$-subalgebra of $\widetilde{H}_{\mathrm{int}}$ generated by $T_1, \dots, T_{n-1}$ and $T_n$. The finite Hecke algebra

$$H_{\mathrm{fin}} \quad \text{has } \mathbb{K}\text{-basis} \quad \{T_v \mid v \in W_{\mathrm{fin}}\}, \tag{2.26}$$

where $T_v = T_{i_1} \dots T_{i_k}$ if $v = s_{i_1} \cdots s_{i_k}$ is a reduced word for $v$ in $W_{\mathrm{fin}}$. The four one dimensional representations of $H_{\mathrm{fin}}$ are

$$\chi^+ \colon H_{\mathrm{fin}} \to \mathbb{K}, \qquad \chi^\pm \colon H_{\mathrm{fin}} \to \mathbb{K}, \qquad \chi^\mp \colon H_{\mathrm{fin}} \to \mathbb{K}, \qquad \chi^- \colon H_{\mathrm{fin}} \to \mathbb{K},$$

given by

$$\chi^+(T_i) = \begin{cases} t^{\frac{1}{2}}, & \text{if } i \in \{1, \dots, n-1\}, \\ t_n^{\frac{1}{2}}, & \text{if } i = n, \end{cases} \qquad \chi^\pm(T_i) = \begin{cases} t^{\frac{1}{2}}, & \text{if } i \in \{1, \dots, n-1\}, \\ (-t_n)^{-\frac{1}{2}}, & \text{if } i = n, \end{cases}$$

$$\chi^\mp(T_i) = \begin{cases} (-t)^{-\frac{1}{2}}, & \text{if } i \in \{1, \dots, n-1\}, \\ t_n^{\frac{1}{2}}, & \text{if } i = n, \end{cases} \qquad \chi^-(T_i) = \begin{cases} (-t)^{-\frac{1}{2}}, & \text{if } i \in \{1, \dots, n-1\}, \\ (-t_n)^{-\frac{1}{2}}, & \text{if } i = n. \end{cases}$$

For $v \in W_{\mathrm{fin}}$ define $\ell_s(v)$ and $\ell_d(v)$ by

$$\chi^+(T_v) = (t^{\frac{1}{2}})^{\ell_s(v)}(t_n^{\frac{1}{2}})^{\ell_d(v)}.$$

The *Hecke symmetrizers* are

$$\text{the elements} \quad \varepsilon_+,\ \varepsilon_\pm,\ \varepsilon_\mp \text{ and } \varepsilon_- \quad \text{of } H_{\mathrm{fin}}$$

which are defined such that, in terms of the basis in (2.26), the coefficient of $T_{w_0}$ is 1 and for $w \in W_{\mathrm{fin}}$,

$$T_w\varepsilon_+ = \chi^+(T_w)\varepsilon_+, \qquad T_w\varepsilon_\pm = \chi^\pm(T_w)\varepsilon_\pm, \qquad T_w\varepsilon_\mp = \chi^\mp(T_w)\varepsilon_\mp, \qquad T_w\varepsilon_- = \chi^-(T_w)\varepsilon_-.$$

In other words, if $\Xi \in \{+,\pm,\mp,-\}$ then $T_w\varepsilon_\Xi = \chi^\Xi(T_w)\varepsilon_\Xi$.

A reduced word for the longest element of $W_{\mathrm{fin}}$ is

$$w_0 = (s_1\cdots s_n\cdots s_1)(s_2\cdots s_n\cdots s_2)\cdots(s_{n-1}s_ns_{n-1})s_n \qquad \text{and} \qquad t^{\frac{1}{2}\ell_s(w_0)}t_n^{\frac{1}{2}\ell_d(w_0)} = t^{\frac{1}{2}n(n-1)}t_n^{\frac{1}{2}n}.$$

In terms of the basis in (2.26) the symmetrizers are given explicitly by

$$\varepsilon_\Xi = \frac{1}{\chi^\Xi(T_{w_0})}\sum_{v\in W_{\mathrm{fin}}}\chi^\Xi(T_v)T_v, \qquad \text{for } \Xi \in \{+,\pm,\mp,-\}. \tag{2.27}$$

The *Poincaré polynomial for* $W_{\mathrm{fin}}$ is

$$W_0(t,t_n) = \sum_{w\in W_{\mathrm{fin}}} t^{\ell_s(w)}t_n^{\ell_d(w)} = \sum_{w\in W_{\mathrm{fin}}}\chi^+(T_w)^2. \tag{2.28}$$

Three alternate formulas for $W_0(t,t_n)$ are given in Proposition 4.3. Then

$$\begin{aligned}
\varepsilon_+^2 &= \frac{1}{\chi^+(T_{w_0})}W_0(t,t_n)\varepsilon_+, & \varepsilon_\pm^2 &= \frac{1}{\chi^\pm(T_{w_0})}W_0(t,t_n^{-1})\varepsilon_\pm, \\
\varepsilon_-^2 &= \frac{1}{\chi^-(T_{w_0})}W_0(t^{-1},t_n^{-1})\varepsilon_-, & \varepsilon_\mp^2 &= \frac{1}{\chi^\mp(T_{w_0})}W_0(t^{-1},t_n)\varepsilon_\mp.
\end{aligned} \tag{2.29}$$

**Example 2.1.** For $\Xi = \pm$,

$$\varepsilon_\pm = \frac{1}{\chi^\pm(T_{w_0})}\sum_{w\in W_{\mathrm{fin}}}\chi^\pm(T_w)T_w = t^{-\frac{1}{2}\ell_s(w_0)}(-t_n^{\frac{1}{2}})^{\ell_d(w_0)}\sum_{w\in W_{\mathrm{fin}}} t^{\frac{1}{2}\ell_s(w)}(-t_n^{-\frac{1}{2}})^{\ell_d(w)}T_w,$$

and since $T_w\varepsilon_\pm = \chi^\pm(T_w)\varepsilon_\pm$ then

$$\varepsilon_\pm^2 = \frac{1}{\chi^\pm(T_{w_0})}\sum_{w\in W_{\mathrm{fin}}}\chi^\pm(T_w)^2\varepsilon_\pm = \frac{1}{\chi^\pm(T_{w_0})}W_0(t,t_n^{-1})\varepsilon_\pm.$$

## 2.6 $c$-function formulas for symmetrizers

The definition of the Hecke symmetrizers and the formulas for them given in Section 2.5, are purely in terms of the $T_w$ in the double affine Hecke algebra $\widetilde{H}_{\mathrm{int}}$. However the Koornwinder polynomials are more naturally constructed and managed with the $\eta_w$ and $\xi_w$ that are in $\widetilde{H}_{\mathrm{loc}}$ and so it becomes desirable to have expressions for the symmetrizers that are in terms of the $\eta_w$ and $\xi_w$ (and $c$-functions). Perhaps surprisingly, these conversion formulas, presented in Proposition 2.3, are compact and elegant (and useful!). We will use them in Section 4 to provide formulas for the symmetric (bosonic), antisymmetric (fermionic) and half symmetric-half antisymmetric (mesonic) Koornwinder polynomials.

For $w \in W_{\text{fin}}$ let

$$\xi_w = \xi_{s_{i_1}} \cdots \xi_{s_{i_\ell}} \qquad \text{and} \qquad \eta_w = \eta_{s_{i_1}^\vee} \cdots \eta_{s_{i_\ell}^\vee},$$

if $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced word for $w$. There are four $X$-symmetrizers

$$e_+^X = \sum_{w\in W_{\text{fin}}} \xi_w, \qquad e_-^X = \sum_{w\in W_{\text{fin}}} (-1)^{\ell_s(w)+\ell_d(w)}\xi_w,$$
$$e_\pm^X = \sum_{w\in W_{\text{fin}}} (-1)^{\ell_d(w)}\xi_w, \qquad e_\mp^X = \sum_{w\in W_{\text{fin}}} (-1)^{\ell_s(w)}\xi_w,$$

and four $Y$-symmetrizers

$$e_+^Y = \sum_{w\in W_{\text{fin}}} \eta_w, \qquad e_-^Y = \sum_{w\in W_{\text{fin}}} (-1)^{\ell_s(w)+\ell_d(w)}\eta_w,$$
$$e_\pm^Y = \sum_{w\in W_{\text{fin}}} (-1)^{\ell_d(w)}\eta_w, \qquad e_\mp^Y = \sum_{w\in W_{\text{fin}}} (-1)^{\ell_s(w)}\eta_w.$$

The following proposition writes the Hecke symmetrizers in terms of the $X$-symmetrizers and the $Y$-symmetrizers.

**Proposition 2.3.** *Let $\kappa_{\text{st}}^X$, $\kappa_{\text{dr}}^X$ be as defined in* (2.11)*.*

$$\chi^+(T_{w_0})\varepsilon_+ = e_+^X \kappa_{\text{st}}^{X^{-1}} \kappa_{\text{dr}}^{X^{-1}} = e_+^Y \kappa_{\text{st}}^Y \kappa_{\text{dr}}^Y \qquad \textit{and} \qquad \chi^+(T_{w_0})\varepsilon_- = \kappa_{\text{st}}^X \kappa_{\text{dr}}^X e_-^X = \kappa_{\text{st}}^{Y^{-1}} \kappa_{\text{dr}}^{Y^{-1}} e_-^Y,$$
$$\chi^+(T_{w_0})\varepsilon_\pm = \kappa_{\text{dr}}^X e_\pm^X \kappa_{\text{st}}^{X^{-1}} = \kappa_{\text{dr}}^{Y^{-1}} e_\pm^Y \kappa_{\text{st}}^Y \qquad \textit{and} \qquad \chi^+(T_{w_0})\varepsilon_\mp = \kappa_{\text{st}}^X e_\mp^X \kappa_{\text{dr}}^{X^{-1}} = \kappa_{\text{st}}^{Y^{-1}} e_\mp^Y \kappa_{\text{dr}}^Y.$$

*Proof.* Let us prove the formula $\chi^+(T_{w_0})\varepsilon_\pm = \kappa_{\text{dr}}^X e_\pm^X \kappa_{\text{st}}^{X^{-1}}$. The proof for the other cases is similar.

Let

$$R_\pm = \kappa_{\text{dr}}^X e_\pm^X \kappa_{\text{st}}^{X^{-1}}.$$

For $i \in \{1, \ldots, n-1\}$, and using (2.21),

$$\begin{aligned}(T_i - t^{\frac12})R_\pm &= (T_i - t^{\frac12})\kappa_{\text{dr}}^X e_\pm^X \kappa_{\text{st}}^{X^{-1}} = -c_{\alpha_i}^X(1-\xi_{s_i})\kappa_{\text{dr}}^X e_\pm^X \kappa_{\text{st}}^{X^{-1}} \\ &= -\kappa_{\text{dr}}^X c_{\alpha_i}^X (1-\xi_{s_i}) e_\pm^X \kappa_{\text{st}}^{X^{-1}} = -\kappa_{\text{dr}}^X c_{\alpha_i}^X \cdot 0 \cdot \kappa_{\text{st}}^{X^{-1}} = 0, \qquad \text{so that} \quad T_i R_\pm = t^{\frac12} R_\pm.\end{aligned}$$

Using (2.16),

$$\begin{aligned}(T_n + t_n^{-\frac12})R_\pm &= (T_n + t_n^{-\frac12})\kappa_{\text{dr}}^X e_\pm^X \kappa_{\text{st}}^{X^{-1}} = (1+\xi_{s_n})c_{\alpha_n}^{X^{-1}} \kappa_{\text{dr}}^X e_\pm^X \kappa_{\text{st}}^{X^{-1}} = (1+\xi_{s_n})\kappa_{\text{dr}}^X c_{\alpha_n}^{X^{-1}} e_\pm^X \kappa_{\text{st}}^{X^{-1}} \\ &= \kappa_{\text{dr}}^X (1 + \frac{c_{\alpha_n}^{X^{-1}}}{c_{\alpha_n}^X}\xi_{s_n}) c_{\alpha_n}^{X^{-1}} e_\pm^X \kappa_{\text{st}}^{X^{-1}} = \kappa_{\text{dr}}^X \left( c_{\alpha_n}^{X^{-1}} + \frac{c_{\alpha_n}^{X^{-1}}}{c_{\alpha_n}^X} c_{\alpha_n}^X \xi_{s_n} \right) e_\pm^X \kappa_{\text{st}}^{X^{-1}} \\ &= \kappa_{\text{dr}}^X \left( c_{\alpha_n}^{X^{-1}} + c_{\alpha_n}^{X^{-1}} \xi_{s_n} \right) e_\pm^X \kappa_{\text{st}}^{X^{-1}} = \kappa_{\text{dr}}^X \left( c_{\alpha_n}^{X^{-1}} - c_{\alpha_n}^{X^{-1}} \right) e_\pm^X \kappa_{\text{st}}^{X^{-1}} = 0,\end{aligned}$$

so that $T_n R_\pm = -t_n^{-\frac12} R_\pm$.

Since $T_i = \xi_{s_i} c_{\alpha_i}^{X^{-1}} + (c_{\alpha_i}^{X^{-1}} - t^{-\frac12})$ and the coefficient of $T_{w_0}$ is 1 then there are rational functions $a_w^{X^{-1}}$ such that

$$\begin{aligned}\chi^+(T_{w_0})\varepsilon_\pm = \chi^+(T_{w_0}) \sum_{w\in W_{\text{fin}}} \xi_w a_w^{X^{-1}} &= \chi^+(T_{w_0})\xi_{w_0} c_{w_0}^{X^{-1}} + \chi^+(T_{w_0}) \sum_{w<w_0} \xi_w a_w^{X^{-1}} \\ &= \kappa_{\text{dr}}^X \xi_{w_0} \kappa_{\text{st}}^{X^{-1}} + \chi^+(T_{w_0}) \sum_{w<w_0} \xi_w a_w^{X^{-1}}.\end{aligned}$$

The element $\varepsilon_\pm \in H_{\mathrm{fin}}$ is determined by the conditions that the coefficient of $T_{w_0}$ is 1, and $T_n\varepsilon_\pm = -t_n^{-\frac{1}{2}}\varepsilon_\pm$ and $T_i\varepsilon_\pm = t^{\frac{1}{2}}\varepsilon_\pm$ for $i \in \{1,\dots,n-1\}$. So $\chi^+(T_{w_0})\varepsilon_\pm = R_\pm$. □

### 2.7 Symmetrizers and stabilizers

The finite Weyl group $W_{\mathrm{fin}}$ acts on $\mathbb{Z}^n$ by the formulas in (1.3). Since the action of $W_{\mathrm{fin}}$ on $\mathbb{Z}^n$ is not free, there are elements of $\mathbb{Z}^n$ that have nontrivial stabilizer, and one is forced to confront these stabilizers. This subsection computes formulas for the symmetrizers which take into account, and allow us to manage, the cases when the stabilizer is nontrivial.

Let

$$(\mathbb{Z}^n_{\geq 0})^+ = \{\lambda = (\lambda_1,\dots,\lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n \geq 0\}.$$

The set $(\mathbb{Z}^n_{\geq 0})^+$ is a set of representatives of the $W_{\mathrm{fin}}$-orbits on $\mathbb{Z}^n$. For $\lambda \in (\mathbb{Z}^n_{\geq 0})^+$, let

$$W_\lambda = \{w \in W_{\mathrm{fin}} \mid w\lambda = \lambda\} \quad \text{and} \quad W^\lambda = \left\{ \begin{array}{c} \text{minimal length representatives} \\ \text{of cosets in } W_{\mathrm{fin}}/W_\lambda \end{array} \right\}.$$

Let $w_\lambda$ be the longest element of $W_\lambda$ and let $v_\lambda$ be the maximal element of $W^\lambda$. As in (1.2), then $v_\lambda$ is the minimal length element of $W_{\mathrm{fin}}$ such that $v_\lambda\lambda$ is increasing with all entries $\leq 0$. Let

$$\varepsilon^+_\lambda = \frac{1}{\chi^+(T_{w_\lambda})} \sum_{w \in W_\lambda} \chi^+(T_w)T_w \tag{2.30}$$

so that $T_w\varepsilon^+_\lambda = \chi^+(T_w)\varepsilon^+_\lambda$, for $w \in W_\lambda$, and the coefficient of $T_{w_\lambda}$ in $\varepsilon^+_\lambda$ is 1.

Define $\rho, \omega, \pi \in (\mathbb{Z}^n_{\geq 0})^+$ by

$$\rho = (n, n-1, \dots, 2, 1), \qquad \omega = (1,1,\dots,1), \qquad \pi = (n-1,\dots,2,1,0). \tag{2.31}$$

The statement of the following proposition is designed to stress the analogies between the four symmetrizers $\varepsilon_+$, $\varepsilon_\pm$, $\varepsilon_\mp$ and $\varepsilon_-$. As in Remark (4.1), in practice, there are simplifications since the stabilizers $W_{\lambda+\rho}$ and $W_{\lambda+\pi}$ have order 1 or 2. Specifically,

$$W_{\lambda+\rho} = 1, \qquad W^{\lambda+\rho} = W_{\mathrm{fin}}, \quad \chi^+(T_{w_{\lambda+\rho}}) = 1 \quad \text{and} \quad \varepsilon^+_{\lambda+\rho} = 1;$$

$$\text{if } \lambda_n = 0 \text{ then} \qquad W_{\lambda+\pi} = \{1, s_n\}, \quad \chi^+(T_{w_{\lambda+\pi}}) = t_n^{\frac{1}{2}} \quad \text{and} \qquad \varepsilon^+_{\lambda+\pi} = T_n + t_n^{-\frac{1}{2}};$$

and

$$\text{if } \lambda_n \neq 0 \text{ then} \qquad W_{\lambda+\pi} = 1, \quad W^{\lambda+\pi} = W_{\mathrm{fin}}, \quad \chi^+(T_{w_{\lambda+\pi}}) = 1 \quad \text{and} \qquad \varepsilon^+_{\lambda+\pi} = 1.$$

**Proposition 2.4.** *Let $\lambda \in (\mathbb{Z}^n_{\geq 0})^+$. Then*

$$\chi^+(T_{w_0})\varepsilon_+ = \chi^+(w_\lambda)\left(\sum_{z\in W^\lambda} \kappa^+_{v_\lambda z}\eta_z\kappa^Y_z\right)\varepsilon^+_\lambda,$$

$$\chi^+(T_{w_0})\varepsilon_\pm = \chi^+(w_{\lambda+\omega})\left(\sum_{z\in W^{\lambda+\omega}} (-1)^{\ell_d(z)}\kappa^\pm_{v_{\lambda+\omega}z}\eta_z\kappa^Y_z\right)\varepsilon^+_{\lambda+\omega},$$

$$\chi^+(T_{w_0})\varepsilon_\mp = \chi^+(w_{\lambda+\pi})\left(\sum_{z\in W^{\lambda+\pi}} (-1)^{\ell_s(z)}\kappa^\mp_{v_{\lambda+\pi}z}\eta_z\kappa^Y_z\right)\varepsilon^+_{\lambda+\pi},$$

$$\chi^+(T_{w_0})\varepsilon_- = \chi^+(w_{\lambda+\rho})\left(\sum_{z\in W^{\lambda+\rho}} (-1)^{\ell(z)}\kappa^-_{v_{\lambda+\rho}z}\eta_z\kappa^Y_z\right)\varepsilon^+_{\lambda+\rho}.$$

*Proof.* We will prove the $\pm$ case. The proof for the other cases is similar.

Let

$$J_{\lambda+\omega} = \{\alpha \in (S^\vee)^+_{0,s} \mid \alpha^\vee \notin \mathrm{Inv}(w_{\lambda+\omega})\}$$

so that $J_{\lambda+\omega}$ is the complement of $\mathrm{Inv}(w_{\lambda+\omega})$ in $S^+_{0,s}$. If $v \in W_{\lambda+\omega}$ then $vJ_{\lambda+\omega} = J_{\lambda+\omega}$, since $v$ permutes the elements of $J_{\lambda+\omega}$.

$$\text{If } z \in W^{\lambda+\omega} \quad \text{then} \quad \kappa^{Y^{-1}}_{\mathrm{dr}} \eta_z \kappa^Y_{J_{\lambda+\omega}} = \kappa^{\pm}_{v_{\lambda+\omega} z} \eta_z \kappa^Y_z.$$

Then

$$\begin{aligned}
\chi^+(T_{w_0})\varepsilon_\pm = \kappa^{Y^{-1}}_{\mathrm{dr}} e^Y_\pm \kappa^Y_{\mathrm{st}} &= \kappa^{Y^{-1}}_{\mathrm{dr}} \left( \sum_{w \in W_{\mathrm{fin}}} (-1)^{\ell_d(w)} \eta_w \right) \kappa^Y_{\mathrm{st}} \\
&= \kappa^{Y^{-1}}_{\mathrm{dr}} \left( \sum_{z \in W^{\lambda+\omega}} \sum_{v \in W_{\lambda+\omega}} (-1)^{\ell_d(z)} \eta_z \eta_v \right) \kappa^Y_{\mathrm{st}} \\
&= \kappa^{Y^{-1}}_{\mathrm{dr}} \left( \sum_{z \in W^{\lambda+\omega}} (-1)^{\ell_d(z)} \eta_z \right) \left( \sum_{v \in W_{\lambda+\omega}} \eta_v \right) \kappa^Y_{J_{\lambda+\omega}} \kappa^Y_{w_{\lambda+\omega}} \\
&= \kappa^{Y^{-1}}_{\mathrm{dr}} \left( \sum_{z \in W^{\lambda+\omega}} (-1)^{\ell_d(z)} \eta_z \right) \kappa^Y_{J_{\lambda+\omega}} \left( \sum_{v \in W_{\lambda+\omega}} \eta_v \right) \kappa^Y_{w_{\lambda+\omega}} \\
&= \left( \sum_{z \in W^{\lambda+\omega}} (-1)^{\ell_d(z)} \kappa^{\pm}_{v_{\lambda+\omega} z} \eta_z \kappa^Y_z \right) \left( \sum_{v \in W_{\lambda+\omega}} \eta_v \right) \kappa^Y_{w_{\lambda+\omega}} \\
&= \left( \sum_{z \in W^{\lambda+\omega}} (-1)^{\ell_d(z)} \kappa^{\pm}_{v_{\lambda+\omega} z} \eta_z \kappa^Y_z \right) \chi^+(T_{w_{\lambda+\omega}}) \varepsilon^+_{\lambda+\omega}.
\end{aligned}$$

□

**Remark 2.5.** With $\omega = (1, 1, \ldots, 1)$ and $\pi = (n-1, \ldots, 2, 1, 0)$ and $v_\omega$ and $v_\pi$ as defined in (1.2) then

$$v_\omega(i) = -(n-i+1) \quad \text{so that} \quad \mathrm{Inv}_s(v_\omega) = \{\varepsilon_i + \varepsilon_j \mid i < j\} \quad \text{and} \quad \mathrm{Inv}_d(v_\omega) = \{\varepsilon_1, \ldots, \varepsilon_n\}$$

since, for example, $v_\omega(\varepsilon_1 - \varepsilon_n) = -\varepsilon_n + \varepsilon_1 = \varepsilon_1 - \varepsilon_n$. Then

$$v_\pi(i) = \begin{cases} -i, & \text{if } i \neq n, \\ n, & \text{if } i = n, \end{cases} \quad \text{so that} \quad \mathrm{Inv}_s(v_\pi) = \{\varepsilon_i \pm \varepsilon_j\} \quad \text{and} \quad \mathrm{Inv}_d(v_\pi) = \{\varepsilon_1, \ldots, \varepsilon_{n-1}\}.$$

If $\lambda \in (\mathbb{Z}^n_{\geq 0})^+$ then $W_{\lambda+\omega} \subseteq W_\omega$ and $W_{\lambda+\pi} \subseteq W_\pi$ giving

$$\mathrm{Inv}_d(v_{\lambda+\omega}) \supseteq \mathrm{Inv}_d(v_\omega) \qquad \text{and} \qquad \mathrm{Inv}_s(v_{\lambda+\pi}) \supseteq \mathrm{Inv}_s(v_\pi).$$

Thus

$$\mathrm{Inv}_d(v_{\lambda+\omega}) = \{\varepsilon_1, \ldots, \varepsilon_n\} \qquad \text{and} \qquad \mathrm{Inv}_s(v_{\lambda+\pi}) = \{\varepsilon_i \pm \varepsilon_j \mid i < j\}.$$

□

# 3 Electronic Macdonald-Koornwinder polynomials

The electronic Koornwinder polynomials $E_\mu$ form a basis of the polynomial ring $\mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. They are simultaneous eigenvectors for the Cherednik-Dunkl operators $Y_1, \ldots, Y_n$. This is in analogy to the way that, in quantum mechanics, Hermite polynomials are eigenfunctions of a Hamiltonian operator. In this section we set up the operators $Y_1, \ldots, Y_n$ on the polynomial ring, characterize the electronic Macdonald polynomials $E_\mu$ as eigenvectors, and provide recursive formulas for computing them.

## 3.1 Operators on polynomials

Let $\mathbb{K}[x] = \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Let $\mathbb{Z}^n$ denote the set of length $n$ sequences $\mu = (\mu_1, \ldots, \mu_n)$ of integers. The ring

$$\mathbb{K}[x] \text{ has basis } \quad \{x^\mu \mid \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n\}, \qquad \text{where} \qquad x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}.$$

Define operators $\xi_{s_0}, \xi_{s_1}, \ldots, \xi_{s_n}$ on $\mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ by

$$\begin{aligned}
(\xi_{s_0} f)(x_1, \ldots, x_n) &= f(q x_1^{-1}, x_2, \ldots, x_n), \\
(\xi_{s_i} f)(x_1, \ldots, x_n) &= f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n), \\
(\xi_{s_n} f)(x_1, \ldots, x_n) &= f(x_1, \ldots, x_{n-1}, x_n^{-1}),
\end{aligned}$$

for $f \in \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $i \in \{1, \ldots, n-1\}$. Define operators $X_1, \ldots, X_n$ on $\mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ by

$$X_j f = x_j f, \qquad \text{for } j \in \{1, \ldots, n\}. \tag{3.1}$$

Consider the induced representation

$$\mathrm{Ind}_{H_Y}^{\widetilde{H}}(\mathbf{1}_Y) = \widetilde{H}_{\mathrm{int}} \mathbf{1}_Y = \mathbb{K}\text{-span}\{X_1^{\mu_1} \cdots X_n^{\mu_n} \mathbf{1}_Y \mid \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n\}$$

determined by

$$T_{\alpha_0} \mathbf{1}_Y = t_0^{\frac{1}{2}} \mathbf{1}_Y, \qquad \text{and} \qquad T_i \mathbf{1}_Y = t^{\frac{1}{2}} \mathbf{1}_Y, \qquad \text{for } i \in \{1, \ldots, n\}.$$

Then the map

$$\begin{array}{ccc} \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] & \longrightarrow & \widetilde{H}_{\mathrm{int}} \mathbf{1}_Y \\ x_1^{\mu_1} \cdots x_n^{\mu_n} & \longmapsto & X_1^{\mu_1} \cdots X_n^{\mu_n} \mathbf{1}_Y \end{array} \qquad \text{is an } \widetilde{H}_{\mathrm{int}}\text{-module isomorphism.} \tag{3.2}$$

We shall often identify $\mathbb{K}[x] = \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $\widetilde{H}_{\mathrm{int}} \mathbf{1}_Y$ and $\mathbb{K}[X] = \mathbb{K}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ via this isomorphism.

## 3.2 The operators $T_{\alpha_0}, \ldots, T_{\alpha_n}$ and $Y_1, \ldots, Y_n$

Define operators $T_{\alpha_0}, T_{\alpha_1}, \ldots, T_{\alpha_n}$ by

$$T_{\alpha_i} = t_{\alpha_i}^{\frac{1}{2}} - c_{\alpha_i}^X (1 - \xi_{s_i}), \qquad \text{for } i \in \{0, 1, \ldots, n\}. \tag{3.3}$$

Define

$$Y_j = T_{\alpha_{j-1}}^{-1} \cdots T_{\alpha_1}^{-1} T_{\alpha_0} T_{\alpha_1} \cdots T_{\alpha_n} \cdots T_{\alpha_j}, \qquad \text{for } j \in \{1, \ldots, n\}.$$

Using (2.2) and (2.4),

$$
\begin{aligned}
t_0^{\frac{1}{2}} T_{\alpha_0} &= t_0 - \left( \frac{(x_1 + q^{\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{-\frac{1}{2}})(x_1 - q^{\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{\frac{1}{2}})}{x_1^2 - q} \right)(1 - \xi_{s_0}), \\
t_n^{\frac{1}{2}} T_{\alpha_n} &= t_n - \left( \frac{(1 + t_n^{\frac{1}{2}} u_n^{-\frac{1}{2}} x_n)(1 - t_n^{\frac{1}{2}} u_n^{\frac{1}{2}} x_n)}{1 - x_n^2} \right)(1 - \xi_{s_n}), \\
t^{\frac{1}{2}} T_{\alpha_i} &= t - \frac{x_{i+1} - t x_i}{x_{i+1} - x_i}(1 - \xi_{s_i}), \quad \text{for } i \in \{1, \dots, n-1\}
\end{aligned}
\tag{3.4}
$$

(see [Nou95, §3] and [Sah99, (13)] and [CGdGW16, (73)]).

### 3.3 Electronic Macdonald polynomials $E_\mu$

For $\mu \in \mathbb{Z}^n$ let $v_\mu \in W_{\mathrm{fin}}$ be the minimal length signed permutation such that $v_\mu \mu$ is weakly increasing with all entries $\leq 0$. The *electronic Macdonald polynomials*

$$
E_\mu(x_1, \dots, x_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}) \in \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \qquad \text{are indexed by } \mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n
$$

and are defined by the eigenvalue conditions

$$
Y_j E_\mu = q^{-\mu_j} t^{-v_\mu(j)} (t_0^{\frac{1}{2}} t_n^{\frac{1}{2}} t^n)^{\mathrm{sgn}(v_\mu(j))} E_\mu, \tag{3.5}
$$

for $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ and $j \in \{1, \dots, n\}$. The normalization of $E_\mu$ is such that the coefficient of $x^\mu$ in $E_\mu$ is 1.

Let $\mathbb{K}(Y)$ be the field of fractions of $\mathbb{K}(Y)$. For $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ define homomorphisms $\mathrm{ev}_\mu^t \colon \mathbb{K}[Y] \to \mathbb{K}[Y]$ by

$$
\mathrm{ev}_\mu^t(Y_i) = q^{-\mu_i} t^{-v_\mu(j)} (t_0^{\frac{1}{2}} t_n^{\frac{1}{2}} t^n)^{\mathrm{sgn}(v_\mu(j))}, \qquad \text{for } i \in \{1, \dots, n\}. \tag{3.6}
$$

Extend $\mathrm{ev}_\mu^t$ to those elements of the field $\mathbb{K}(Y)$ for which the evaluated denominator is nonzero. By (3.5)

$$
f E_\mu = \mathrm{ev}_\mu^t(f) E_\mu, \qquad \text{for } f \in \mathbb{K}[Y] \text{ and } \mu \in \mathbb{Z}^n. \tag{3.7}
$$

### 3.4 The recursion for the $E_\mu$

Although the eigenvalue conditions together with the normalization completely characterize the Koornwinder polynomials $E_\mu$, computing them by solving directly for eigenvectors is not efficient. Fortunately, the operators $\tau_i^\vee = \eta_{s_i^\vee} c_{\alpha_i^\vee}^Y$ from the algebra $\widetilde{H}_{\mathrm{loc}}$ provide a very nice recursive way of computing the $E_\mu$. This is analogous to the way that, in Schubert calculus, the Schubert polynomials are constructed recursively using divided-difference operators.

Define operators $T_{\alpha_0^\vee}, T_{\alpha_1^\vee}, \dots, T_{\alpha_n^\vee}$ on $\mathbb{K}[X]$ by

$$
(T_{\alpha_0^\vee})^{-1} = X_1 T_{\alpha_1} \cdots T_{\alpha_n} \cdots T_{\alpha_1} \qquad \text{and} \qquad T_{\alpha_i^\vee} = T_{\alpha_i} \qquad \text{for } i \in \{1, \dots, n\}. \tag{3.8}
$$

and define

$$
\tau_i^\vee = T_{\alpha_i^\vee} + (t_{\alpha_i^\vee}^{-\frac{1}{2}} - c_{\alpha_i^\vee}^Y) = (T_{\alpha_i^\vee})^{-1} + (t_{\alpha_i^\vee}^{\frac{1}{2}} - c_{\alpha_i^\vee}^Y), \qquad \text{for } i \in \{0, 1, \dots, n\}. \tag{3.9}
$$

By (2.16), $\tau_i^\vee = \eta_{s_i}^Y c_{\alpha_i^\vee}^Y$ so that

$$\tau_i^\vee Y^{\lambda^\vee} = Y^{s_i^\vee \lambda^\vee} \tau_i^\vee, \qquad \text{for } i \in \{0, 1, \ldots, n\}. \tag{3.10}$$

The group $W_X$ (generated by $s_0^\vee, \ldots, s_n^\vee$) acts on $\mathbb{Z}^n$ by

$$\begin{aligned} s_0^\vee(\mu_1, \ldots, \mu_n) &= (-\mu_1 + 1, \mu_2, \ldots, \mu_n), \\ s_i^\vee(\mu_1, \ldots, \mu_n) &= (\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \mu_i, \mu_{i+2}, \ldots, \mu_n), \qquad \text{for } i \in \{1, \ldots, n-1\}, \text{ and} \\ s_n^\vee(\mu_1, \ldots, \mu_n) &= (\mu_1, \ldots, \mu_{n-1}, -\mu_n). \end{aligned}$$

The relation (3.10) is the reason that the electronic Macdonald polynomials $E_\mu$ are equivalently defined by the following recursive relations:

(E0) $E_{(0,\ldots,0)} = 1$;

(E1) if $\mu_1 \leq 0$ then $E_{s_0^\vee \mu} = t^{n-1} t_n^{\frac{1}{2}} \tau_0^\vee E_\mu$;

(E2) if $i \in \{1, \ldots, n-1\}$ and $\mu_i > \mu_{i+1}$ then $E_{s_i^\vee \mu} = t^{\frac{1}{2}} \tau_i^\vee E_\mu$; and

(E3) if $\mu_n > 0$ then $E_{s_n^\vee \mu} = t_n^{\frac{1}{2}} \tau_n^\vee E_\mu$.

### 3.5 The creation formula for $E_\mu$

The recursion of the previous subsection can be packaged nicely as a single formula for creating the Koornwinder polynomial $E_\mu$. This is the creation formula in (3.11).

Let $\mu \in \mathbb{Z}^n$ and let $h_\mu \in W_X$ denote the corresponding translation. Let $u_\mu \in W_X$ and $v_\mu \in W_{\mathrm{fin}}$ be as defined in (1.2), so that

$$u_\mu \in W_X \quad \text{and} \quad v_\mu \in W_{\mathrm{fin}} \quad \text{and} \quad h_\mu = u_\mu v_\mu, \quad \text{with} \quad \ell(h_\mu) = \ell(u_\mu) + \ell(v_\mu).$$

Using the identification of $\mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ with $\widetilde{H}_{\mathrm{int}} \mathbf{1}_Y$ as in (3.2), the creation formula for $E_\mu$ is

$$E_\mu = \frac{1}{\chi^+(T_{v_\mu^{-1}})} \tau_{u_\mu}^\vee \mathbf{1}_Y, \qquad \text{where} \quad \tau_{u_\mu}^\vee = \tau_{i_1}^\vee \cdots \tau_{i_\ell}^\vee \tag{3.11}$$

if $u_\mu = s_{i_1} \cdots s_{i_\ell}$ is a reduced word for $u_\mu$.

*Proof.* Using

$$Y_i \mathbf{1}_Y = T_{\alpha_{i-1}}^{-1} \cdots T_{\alpha_1}^{-1} T_{\alpha_0} T_{\alpha_1} \cdots T_{\alpha_n} \cdots T_{\alpha_i} \mathbf{1}_Y = t^{-\frac{1}{2}(i-1)} t_0^{\frac{1}{2}} t^{\frac{1}{2}(n-1)} t_n^{\frac{1}{2}} t^{\frac{1}{2}(n-i)} \mathbf{1}_Y = t^{-i}(t_0^{\frac{1}{2}} t_n^{\frac{1}{2}} t^n) \mathbf{1}_Y,$$

and $h_\mu^{-1} \varepsilon_j^\vee = h_{-\mu} \varepsilon_j^\vee = \varepsilon_j^\vee - \langle \mu, -\varepsilon_j^\vee \rangle K = \varepsilon_j^\vee + \mu_j K$ gives

$$\begin{aligned} Y_j \tau_{u_\mu}^\vee \mathbf{1}_Y &= Y^{\varepsilon_j^\vee} \tau_{u_\mu}^\vee \mathbf{1}_Y = Y^{\varepsilon_j^\vee} \tau_{u_\mu}^\vee \mathbf{1}_Y = \tau_{u_\mu}^\vee Y^{u_\mu^{-1} \varepsilon_j^\vee} \mathbf{1}_Y = \tau_{u_\mu}^\vee Y^{v_\mu h_\mu^{-1} \varepsilon_j^\vee} \mathbf{1}_Y \\ &= \tau_{u_\mu}^\vee Y^{v_\mu(\varepsilon_j^\vee + \mu_j K)} \mathbf{1}_Y = q^{-\mu_j} \tau_{u_\mu}^\vee Y^{\varepsilon_{v_\mu(j)}^\vee} \mathbf{1}_Y \\ &= q^{-\mu_j} \tau_{u_\mu}^\vee Y_{v_\mu(j)} \mathbf{1}_Y = q^{-\mu_j} t^{-v_\mu(j)} (t_0^{\frac{1}{2}} t_n^{\frac{1}{2}} t^n)^{\mathrm{sgn}(v_\mu(j))} \tau_{u_\mu}^\vee \mathbf{1}_Y. \end{aligned}$$

Thus $\tau_{u_\mu}^\vee \mathbf{1}_Y$ is an eigenvector of $Y_j$ with eigenvalue $q^{-\mu_j} t^{-v_\mu(j)} (t_0^{\frac{1}{2}} t_n^{\frac{1}{2}} t^n)^{\mathrm{sgn}(v_\mu(j))}$.

Using the formulas (3.9) the product $\tau^\vee_{u_\mu}$ can be expanded in terms of the elements

$$\{X^\gamma T_v \mid \gamma \in \mathbb{Z}^n, v \in W_{\mathrm{fin}}\}.$$

Since $u_\mu = t_\mu v_\mu^{-1}$ then the top term in this expansion is $X^\mu T_{v_\mu^{-1}}$ and

$$X^\mu T_{v_\mu^{-1}} \mathbf{1}_Y = X^\mu \chi^+(T_{v_\mu^{-1}})\mathbf{1}_Y, \qquad \text{where} \quad \chi^+(T_{v_\mu^{-1}}) = t^{\frac{1}{2}\ell_s(v_\mu^{-1})} t_n^{\frac{1}{2}\ell_d(v_\mu^{-1})}.$$

Thus multiplying $\tau^\vee_{u_\mu}\mathbf{1}_Y$ by $\chi^+(T_{v_\mu^{-1}})^{-1}$ makes the coefficient of $X^\mu$ equal to 1. □

# 4 Bosonic, Fermionic and Mesonic

The Weyl character formula is the formula that expresses the Schur function (a symmetric polynomial) as a quotient of two determinants (antisymmetric polynomials). There are Weyl character formulas in the Koornwinder context as well (see Section 5.6). However, in the Koornwinder context, one finds that there are *four* Weyl character formulas, corresponding to the four symmetrizers $\varepsilon_+, \varepsilon_\pm, \varepsilon_\mp, \varepsilon_-$ introduced in Section 2.5.

This section sets up the components for Weyl character formulas in the Koornwinder context. There are four types of symmetrized Koornwinder polynomials: the bosonic (symmetric) Koornwinder polynomials, the fermionic (antisymmetric) Koornwinder polynomials and two types of mesonic (half symmetric-half antisymmetric) Koornwinder polynomials.

The denominator in the classical Weyl character formula is the Vandermonde determinant, an antisymmetric polynomial with a magical factorization. In the Koornwinder case the Weyl denominators also have magical factorizations. These Weyl denominators are presented in Section 4.2.

Every antisymmetric function can be obtained by multiplying a symmetric function by the Weyl denominators. In [CR22, §4.3] we viewed this correspondence between symmetric functions and antisymmetric functions as an analog of the Boson-Fermion correspondence relating the symmetric algebra realization of Fock space and exterior algebra realization of Fock space (a representation of a Heisenberg algebra, see [Kac, §14.10]). In the Koornwinder context there are four spaces: the bosonic space (symmetric functions), fermionic space (antisymmetrized functions) and two mesonic spaces (half symmetric-half antisymmetric functions). These four spaces are all isomorphic as vector spaces, the isomorphisms being given by multiplying by the different Weyl denominators. This structure is explained Section 4.3.

In Sections 4.4, 4.5 and 4.6, we use the symmetrizers to give formulas for the Poincaré polynomial of $W_{\mathrm{fin}}$, for the $E$-expansions of bosonic, fermionic and mesonic Koornwinder polynomials and formulas for the principal specializations. These results are Koornwinder analogues of the formulas in [CR22, Propositions 4,6 and 4.7 and Theorem 5.1]. All of these formulas are given, in an even more general setting, in [Mac03, (5,5,16), (5,7,8), (5.2.14),(5.3.9)].

## 4.1 Bosonic, Fermionic and Mesonic Macdonald-Koornwinder polynomials

Let

$$(\mathbb{Z}^n_{\geq 0})^+ = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n \geq 0\}.$$

For $\lambda \in (\mathbb{Z}^n_{\geq 0})^+$, let

$$W_\lambda = \{v \in W_{\mathrm{fin}} \mid v\lambda = \lambda\} \qquad \text{and} \qquad W_\lambda(t, t_n) = \sum_{v \in W_\lambda} t^{\ell_s(v)} t_n^{\ell_d(v)} = \sum_{v \in W_\lambda} \chi^+(T_v)^2.$$

Let $\rho, \omega, \pi \in (\mathbb{Z}^n_{\ge 0})^+$ be as defined in (2.31). Then, for $\lambda \in (\mathbb{Z}^n_{\ge 0})^+$, define the *bosonic and fermionic Macdonald-Koornwinder polynomials* are

$$P_\lambda = \frac{\chi^+(T_{w_0})}{W_\lambda(t,t_n)}\varepsilon_+ E_\lambda \qquad \text{and} \qquad A_{\lambda+\rho} = \frac{\chi^+(T_{w_0})}{W_{\lambda+\rho}(t,t_n)}\varepsilon_- E_{\lambda+\rho}, \tag{4.1}$$

and the *mesonic Koornwinder polynomials* are

$$A^\pm_{\lambda+\omega} = \frac{\chi^+(T_{w_0})}{W_{\lambda+\omega}(t,t_n)}\varepsilon_\pm E_{\lambda+\omega} \qquad \text{and} \qquad A^\mp_{\lambda+\pi} = \frac{\chi^+(T_{w_0})}{W_{\lambda+\pi}(t,t_n)}\varepsilon_\mp E_{\lambda+\pi}. \tag{4.2}$$

**Remark 4.1.** The notation in (4.1) and (4.2) displays the parallelism among the expressions. For computation it is useful to note that the denominators can be given very explicitly:

$$W_{\lambda+\rho}(t,t_n) = 1, \qquad W_{\lambda+\omega}(t,t_n) = \prod_i \frac{(1-t^{m_i})}{1-t}, \qquad W_{\lambda+\pi}(t,t_n) = \begin{cases} t_n+1, & \text{if } \lambda_n = 0, \\ 1, & \text{if } \lambda_n \neq 0, \end{cases}$$

where $m_i$ is the number of parts of size $i$ in $\lambda = (\lambda_1, \ldots, \lambda_n)$. In particular, $W_{\lambda+\pi}(t,t_n)$ depends only on $t_n$ and $W_{\lambda+\omega}(t,t_n)$ depends only on $t$. The factor $\chi^+(T_{w_0})$ guarantees that the coefficient of $X^{w_0\lambda} = X^{-\lambda}$ is equal to 1 in $P_\lambda$. Similarly, the coefficient of $X^{-(\lambda+\rho)}$ is equal to 1 in $A_{\lambda+\rho}$, the coefficient of $X^{-(\lambda+\omega)}$ is equal to 1 in $A^\pm_{\lambda+\omega}$ and the coefficient of $X^{-(\lambda+\pi)}$ is equal to 1 in $A^\mp_{\lambda+\pi}$. $\square$

## 4.2 Weyl denominators

Define $a^\pm_\omega, a^\mp_\pi, a_\rho, A^\pm_\omega, A^\mp_\pi, A_\rho \in \mathbb{K}[X]$ by

$$\begin{aligned}
a^\pm_\omega &= x^{-\omega}\prod_{i=1}^n (1-x_i^2), & A^\pm_\omega &= x^{-\omega}\left(\prod_{i=1}^n (1-t_n^{\frac12}u_n^{\frac12}x_i)(1+t_n^{\frac12}u_n^{-\frac12}x_i)\right), \\
a^\mp_\pi &= x^{-\pi}\prod_{1\le i<j\le n}(1-x_ix_j^{-1})(1-x_ix_j), & A^\mp_\pi &= x^{-\pi}\left(\prod_{1\le i<j\le n}(1-tx_ix_j)(1-tx_ix_j^{-1})\right), \\
a_\rho &= a^\mp_\pi a^\pm_\omega, & A_\rho &= A^\mp_\pi A^\pm_\omega.
\end{aligned} \tag{4.3}$$

Then

$$\begin{aligned}
a^\pm_\omega &= \tfrac{1}{n!}e^X_\pm x^\omega, & a^\mp_\pi &= \tfrac12 e^X_\mp x^\pi, & a_\rho &= e_- x^\rho, \\
A^\pm_\omega &= \frac{t^{\frac12 n(n-1)}t_n^{\frac12 n}}{[n]!}\varepsilon^X_\pm x^\omega, & A^\mp_\pi &= \frac{t^{\frac12 n(n-1)}t_n^{\frac12 n}}{(1+t_n)}\varepsilon^X_\mp x^\pi, & A_\rho &= t^{\frac12 n(n-1)}t_n^{\frac12 n}\varepsilon_- x^\rho,
\end{aligned}$$

where

$$t^{\frac12 n(n-1)}t_n^{\frac12 n} = \chi^+(T_{w_0}), \qquad [n]! = \prod_{i=1}^n \frac{(1-t^i)}{1-t} = W_\omega(t,t_n), \qquad 1+t_n = W_\pi(t,t_n).$$

By (2.11),

$$\frac{A^\pm_\omega}{a^\pm_\omega} = \kappa^X_{\mathrm{dr}}, \qquad \frac{A^\mp_\pi}{a^\mp_\pi} = \kappa^X_{\mathrm{st}}, \qquad \frac{A_\rho}{a_\rho} = \kappa^X_{w_0}.$$

Since

$$A_\rho = \frac{\chi^+(T_{w_0})}{W_{\lambda+\rho}(t,t_n)}\varepsilon_- x^\rho = \frac{\chi^+(T_{w_0})}{W_{\lambda+\rho}(t,t_n)}\varepsilon_- E_\rho,$$

$$A^\pm_\omega = \frac{\chi^+(T_{w_0})}{W_\omega(t,t_n)}\varepsilon_\pm x^\omega = \frac{\chi^+(T_{w_0})}{W_\omega(t,t_n)}\varepsilon_\pm E_\omega, \qquad \text{and} \qquad A^\mp_\pi = \frac{\chi^+(T_{w_0})}{W_\pi(t,t_n)}\varepsilon_\mp x^\pi = \frac{\chi^+(T_{w_0})}{W_\pi(t,t_n)}\varepsilon_\mp E_\pi,$$

there is no conflict of notation with the mesonic Macdonald polynomials introduced in (4.2).

## 4.3 Bosonic, fermionic and mesonic spaces

The polynomial ring $\mathbb{K}[X]$ is a module for the action of $\mathbb{K}[X]^{W_{\mathrm{fin}}}$ and the structure of $\mathbb{K}[X]$ as a $\mathbb{K}[X]^{W_{\mathrm{fin}}}$-module is of classical importance in the theory of reflection groups. In fact, there are two commuting actions on $\mathbb{K}[X]$, the action of $W_{\mathrm{fin}}$ and the action of $\mathbb{K}[X]^{W_{\mathrm{fin}}}$. The part of this picture that is captured by the $X$-symmetrizers can be stated as follows.

Define

$$\begin{aligned}\mathbb{K}[X]^{W_{\mathrm{fin}}} &= \{f \in \mathbb{K}[X] \mid \text{if } w \in W_{\mathrm{fin}} \text{ then } wf = f\},\\ \mathbb{K}[X]^{\pm} &= \{f \in \mathbb{K}[X] \mid \text{if } w \in W_{\mathrm{fin}} \text{ then } wf = (-1)^{\ell_d(w)} f\},\\ \mathbb{K}[X]^{\mp} &= \{f \in \mathbb{K}[X] \mid \text{if } w \in W_{\mathrm{fin}} \text{ then } wf = (-1)^{\ell_s(w)} f\},\\ \mathbb{K}[X]^{\det} &= \{f \in \mathbb{K}[X] \mid \text{if } w \in W_{\mathrm{fin}} \text{ then } wf = (-1)^{\ell_s(w)+\ell_d(w)} f\}.\end{aligned}$$

Then

$$\begin{aligned} e_+^X\mathbb{K}[X] &= \mathbb{K}[X]^{W_{\mathrm{fin}}}, & e_\pm^X\mathbb{K}[X] &= \mathbb{K}[X]^{\pm} = a_\omega^{\pm}\mathbb{K}[X]^{W_{\mathrm{fin}}},\\ e_-^X\mathbb{K}[X] &= \mathbb{K}[X]^{\det} = a_\rho\mathbb{K}[X]^{W_{\mathrm{fin}}}, & e_\mp^X\mathbb{K}[X] &= \mathbb{K}[X]^{\mp} = a_\pi^{\mp}\mathbb{K}[X]^{W_{\mathrm{fin}}}. \end{aligned} \tag{4.4}$$

Now we proceed to a $t$-analogue of the equalities in (4.4). In this case Hecke algebra $H_{\mathrm{fin}}$ replaces the finite Weyl group, and the actions of $H_{\mathrm{fin}}$ and $\mathbb{K}[X]^{W_{\mathrm{fin}}}$ are commuting actions on $\mathbb{K}[X]$. The part of this picture captured by the Hecke symmetrizers is the following.

The bosonic, fermionic, and mesonic spaces are

$$\begin{aligned}\mathbb{K}[X]^{\mathrm{Bos}} &= \{f \in \mathbb{K}[X] \mid T_n f = t_n^{\frac12} f \text{ and } T_i f = t^{\frac12} f \text{ for } i \in \{1,\ldots,n\}\},\\ \mathbb{K}[X]^{\mathrm{Fer}} &= \{f \in \mathbb{K}[X] \mid T_n f = -t_n^{-\frac12} f \text{ and } T_i f = -t^{-\frac12} f \text{ for } i \in \{1,\ldots,n\}\},\\ \mathbb{K}[X]^{\mathrm{Mes}\pm} &= \{f \in \mathbb{K}[X] \mid T_n f = -t_n^{-\frac12} f \text{ and } T_i f = t^{\frac12} f \text{ for } i \in \{1,\ldots,n\}\},\\ \mathbb{K}[X]^{\mathrm{Mes}\mp} &= \{f \in \mathbb{K}[X] \mid T_n f = t_n^{\frac12} f \text{ and } T_i f = -t^{-\frac12} f \text{ for } i \in \{1,\ldots,n\}\},\end{aligned}$$

With these definitions, the following proposition establishes $t$-analogues of the equalities in (4.4). The Weyl denominators of Section 4.2 are a key part of the structure.

**Proposition 4.2.** *Let $\varepsilon_+, \varepsilon_\pm, \varepsilon_\mp, \varepsilon_-$ be the symmetrizers defined in* (2.27)*.*

$$\begin{aligned}\mathbb{K}[X]^{\mathrm{Bos}} &= \varepsilon_+\mathbb{K}[X] = \mathbb{K}[X]^{W_{\mathrm{fin}}}, & \mathbb{K}[X]^{\mathrm{Fer}} &= \varepsilon_-\mathbb{K}[X] = A_\rho\mathbb{K}[X]^{W_{\mathrm{fin}}},\\ \mathbb{K}[X]^{\mathrm{Mes}\pm} &= \varepsilon_\pm\mathbb{K}[X] = A_\omega^{\pm}\mathbb{K}[X]^{W_{\mathrm{fin}}}, & \mathbb{K}[X]^{\mathrm{Mes}\mp} &= \varepsilon_\mp\mathbb{K}[X] = A_\pi^{\mp}\mathbb{K}[X]^{W_{\mathrm{fin}}},\end{aligned}$$

*Moreover, with $P_\lambda$, $A^{\pm}_{\lambda+\omega}$, $A^{\mp}_{\lambda+\pi}$ and $A_{\lambda+\rho}$ as in* (4.1) *and* (4.2)*,*

$$\begin{aligned}&\{P_\lambda \mid \lambda \in (\mathbb{Z}^n_{\ge 0})^+\} \text{ is a basis of } \varepsilon_+\mathbb{K}[X], && \{A_{\lambda+\rho} \mid \lambda \in (\mathbb{Z}^n_{\ge 0})^+\} \text{ is a basis of } \varepsilon_-\mathbb{K}[X],\\ &\{A^{\pm}_{\lambda+\omega} \mid \lambda \in (\mathbb{Z}^n_{\ge 0})^+\} \text{ is a basis of } \varepsilon_\pm\mathbb{K}[X], && \{A^{\mp}_{\lambda+\pi} \mid \lambda \in (\mathbb{Z}^n_{\ge 0})^+\} \text{ is a basis of } \varepsilon_\mp\mathbb{K}[X],\end{aligned}$$

*Proof.* We will give the proof for the $\pm$ case. The proofs for the other cases are similar.

Assume $f \in \varepsilon_\pm\mathbb{K}[X]$. Then there exists $g \in \mathbb{K}[X]$ such that $f = \varepsilon_\pm g$ and

$$T_n f = T_n\varepsilon_\pm g = -t_n^{-\frac12}\varepsilon_\pm g = -t_n^{-\frac12} f \qquad \text{and} \qquad T_i f = T_i\varepsilon_\pm g = t^{\frac12}\varepsilon_\pm g = t^{\frac12} f,$$

for $i \in \{1,\ldots,n-1\}$. So $f \in \mathbb{K}[X]^{\mathrm{Mes}\pm}$ and $\varepsilon_\pm\mathbb{K}[X] \subseteq \mathbb{K}[X]^{\mathrm{Mes}\pm}$.

If $f \in \mathbb{K}[X]^{\mathrm{Mes}\pm}$ then

$$f = \frac{\chi^{\pm}(T_{w_0})}{W_0(t, t_n^{-1})}\varepsilon_{\pm} f = \frac{\chi^{\pm}(T_{w_0})}{W_0(t, t_n^{-1})} c^X_{\mathrm{dr}} e^X_{\pm} c^{X^{-1}}_{\mathrm{st}} f = \frac{\chi^{\pm}(T_{w_0})}{W_0(t, t_n^{-1})} \frac{A^{\pm}_{\omega}}{a^{\pm}_{\omega}} e^X_{\pm} c^{X^{-1}}_{\mathrm{st}} f \in \frac{A^{\pm}_{\omega}}{a^{\pm}_{\omega}} \mathbb{K}[X]^{\pm}.$$

Since $\dfrac{A^{\pm}_{\omega}}{a^{\pm}_{\omega}}\mathbb{K}[X]^{\pm} = \dfrac{A^{\pm}_{\omega}}{a^{\pm}_{\omega}} a^{\pm}_{\omega}\mathbb{K}[X]^{W_{\mathrm{fin}}} = A^{\pm}_{\omega}\mathbb{K}[X]^{W_{\mathrm{fin}}}$ then $\mathbb{K}[X]^{\mathrm{Mes}\pm} \subseteq A^{\pm}_{\omega}\mathbb{K}[X]^{W_{\mathrm{fin}}}$.

Assume $f \in A^{\pm}_{\omega}\mathbb{K}[X]^{W_{\mathrm{fin}}}$. Then there exists $g \in \mathbb{K}[X]^{W_{\mathrm{fin}}}$ such that $f = A^{\pm}_{\omega} g$. Then

$$f = A^{\pm}_{\omega} g = \left(\frac{\chi^{+}(T_{w_0})}{W_{\omega}(t, t_n)}\varepsilon_{\pm} x^{\omega}\right) g = \varepsilon_{\pm}\left(\frac{\chi^{+}(T_{w_0})}{W_{\omega}(t, t_n)} x^{\omega} g\right) \in \varepsilon_{\pm}\mathbb{K}[X].$$

So $A^{\pm}_{\omega}\mathbb{K}[X]^{W_{\mathrm{fin}}} \subseteq \varepsilon_{\pm}\mathbb{K}[X]$. This completes the proof that $\mathbb{K}[X]^{\mathrm{Mes}\pm} = \varepsilon_{\pm}\mathbb{K}[X] = A^{\pm}_{\omega}\mathbb{K}[X]^{W_{\mathrm{fin}}}$.

Define

$$M^{\pm}_{\mu} = \varepsilon_{\pm} E_{\mu} \qquad \text{for } \mu \in \mathbb{Z}^n.$$

Let $i \in \{1, \ldots, n\}$. If $\mu_n = 0$ then $s_n\mu = \mu$ and $T_n E_{\mu} = t_n^{\frac{1}{2}} E_{\mu}$ so that

$$M^{\pm}_{\mu} = \varepsilon_{\pm} T_{\alpha_n^{\vee}} E_{\mu} = (-t_n)^{\frac{1}{2}} \varepsilon_{\pm} T_{\alpha_n^{\vee}} E_{\mu} = -M_{\mu}, \quad \text{which forces} \quad M^{\pm}_{\mu} = 0 \quad \text{when } \mu_n = 0.$$

If $i \in \{1, \ldots, n\}$ and $s_i\mu > \mu$ then

$$\begin{aligned} M^{\pm}_{s_i\mu} &= \varepsilon_{\pm} E_{s_i\mu} = \varepsilon_{\pm} t^{\frac{1}{2}} \tau_i^{\vee} E_{\mu} = \varepsilon_{\pm} t^{\frac{1}{2}} (T_{\alpha_i^{\vee}} + (t^{-\frac{1}{2}} - c^Y_{\alpha_i^{\vee}}) E_{\mu} \\ &= \varepsilon_{\pm} t^{\frac{1}{2}} (t^{\frac{1}{2}} + t^{-\frac{1}{2}} - c^Y_{\alpha_i^{\vee}}) E_{\mu} = \varepsilon_{\pm} t^{\frac{1}{2}} c^Y_{-\alpha_i^{\vee}} E_{\mu} = \mathrm{ev}^t_{\mu}(\kappa^Y_{-\alpha_i^{\vee}}) M_{\mu}, \end{aligned}$$

so that $M^{\pm}_{\mu}$ and $M^{\pm}_{s_i\mu}$ are linearly dependent. It follows that

$$\varepsilon_{\pm}\mathbb{K}[X] = \mathbb{K}\text{-span}\{M^{\pm}_{\lambda} \mid \lambda \in \mathbb{Z}^n\} = \mathbb{K}\text{-span}\{M^{\pm}_{\lambda+\omega} \mid \lambda \in (\mathbb{Z}^n_{\geq 0})^+\} = \mathbb{K}\text{-span}\{A^{\pm}_{\lambda+\omega} \mid \lambda \in (\mathbb{Z}^n_{\geq 0})^+\}$$

Since $A^{\pm}_{\lambda+\omega}$ has top coefficient $x^{-(\lambda+\omega)}$ (in the DBlex order, see Section 5.3) and the monomials are linear independent in $\mathbb{K}[X]$ then the set $\{A^{\pm}_{\lambda+\omega} \mid \lambda \in (\mathbb{Z}^n_{\geq 0})^+\}$ is linearly independent. □

## 4.4 Formulas for the Poincaré polynomial

Recall from (2.28) and (3.6) that the Poincaré polynomial for $W_{\mathrm{fin}}$ is

$$W_0(t, t_n) = \sum_{w \in W_{\mathrm{fin}}} \chi^{+}(T_w)^2 \qquad \text{and} \qquad \mathrm{ev}^t_0(Y_i) = t^{n-i} t_0^{\frac{1}{2}} t_n^{\frac{1}{2}}$$

defines the evaluation homomorphism $\mathrm{ev}^t_0 \colon \mathbb{K}[Y] \to \mathbb{K}$. The following Proposition gives four ways of looking at the Poincaré polynomial: as a sum, as a product, as an evaluation of $\kappa^Y_{w_0}$, and as a symmetrization of $\kappa^{X^{-1}}_{w_0}$.

**Proposition 4.3.** *The group $W_{\mathrm{fin}}$ acts on $\mathbb{K}[X]$ as in* (2.14)*. Let $w_0$ denote the longest element of $W_{\mathrm{fin}}$ and let $\kappa^Y_{w_0}$ and $\kappa^{X^{-1}}_{w_0}$ be the noramlized c-functions given in* (2.12) *and* (2.13)*. Then*

$$W_0(t, t_n) = \prod_{i=1}^{n} \frac{(1 - t^i)(1 + t_n t^{i-1})}{(1 - t)} = \mathrm{ev}^t_0(\kappa^Y_{w_0}) = \sum_{w \in W_{\mathrm{fin}}} w(\kappa^{X^{-1}}_{w_0}).$$

*Proof.* Since $\chi^+(T_w)^2 = t^{\ell_s(w)} t_n^{\ell_d(w)}$ then

$$\chi^+(T_{w_0})\varepsilon_+ \mathbf{1}_Y = \sum_{w \in W_{\mathrm{fin}}} \chi^+(T_w) T_w \mathbf{1}_Y = \sum_{w \in W_{\mathrm{fin}}} (\chi^+(T_w))^2 \mathbf{1}_Y = W_0(t, t_n) \mathbf{1}_Y.$$

Using the first formula in Proposition 2.3,

$$\chi^+(T_{w_0})\varepsilon_+ \mathbf{1}_Y = e_+^X \kappa_{w_0}^{X^{-1}} \mathbf{1}_Y = \sum_{w \in W_{\mathrm{fin}}} \xi_w \kappa_{w_0}^{X^{-1}} \mathbf{1}_Y = \sum_{w \in W_{\mathrm{fin}}} w(\kappa_{w_0}^{X^{-1}}).$$

Using the second formula in Proposition 2.3, $\varepsilon_+ \mathbf{1}_Y = e_+^Y \kappa_{w_0}^Y \mathbf{1}_Y$ and

$$\begin{aligned}
\chi^+(T_{w_0})\varepsilon_+ \mathbf{1}_Y &= \left( \sum_{w \in W_{\mathrm{fin}}} \eta_w \right) \kappa_{w_0}^Y \mathbf{1}_Y = \mathrm{ev}_0^t \left( \kappa_{w_0}^Y \right) \left( 1 + \sum_{w \in W_{\mathrm{fin}}, w \neq 1} \eta_w \right) \mathbf{1}_Y \\
&= \mathrm{ev}_0^t \left( \kappa_{w_0}^Y \right) (1 + 0) \mathbf{1}_Y = \mathrm{ev}_0^t \left( \kappa_{w_0}^Y \right) \mathbf{1}_Y.
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathrm{ev}_0^t \left( \kappa_{w_0}^Y \right) &= \mathrm{ev}_0^t \left( \prod_{i<j} \frac{(1 - tY_iY_j^{-1})(1 - tY_iY_j)}{(1 - Y_iY_j^{-1})(1 - Y_iY_j)} \right) \mathrm{ev}_0^t \left( \prod_{i=1}^n \frac{(1 - t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} Y_i)(1 + t_n^{\frac{1}{2}} t_0^{-\frac{1}{2}} Y_i)}{(1 - Y_i^2)} \right) \\
&= \left( \prod_{i<j} \frac{(1 - tt^{n-i} t_0^{\frac{1}{2}} t_n^{\frac{1}{2}} t^{-(n-j)} t_n^{-\frac{1}{2}} t_0^{-\frac{1}{2}})(1 - tt^{n-i} t_0^{\frac{1}{2}} t_n^{\frac{1}{2}} t^{n-j} t_0^{\frac{1}{2}} t_n^{\frac{1}{2}})}{(1 - t^{n-i} t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} t^{-(n-j)} t_n^{-\frac{1}{2}} t_0^{-\frac{1}{2}})(1 - t^{n-i} t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} t^{n-j} t_0^{\frac{1}{2}} t_n^{\frac{1}{2}})} \right) \\
&\qquad \left( \prod_{i=1}^n \frac{(1 - t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} t_0^{\frac{1}{2}} t_n^{\frac{1}{2}} t^{n-i})(1 + t_n^{\frac{1}{2}} t_0^{-\frac{1}{2}} t_0^{\frac{1}{2}} t_n^{\frac{1}{2}} t^{n-i})}{(1 - t_0 t_n t^{2(n-i)})} \right) \\
&= \left( \prod_{i<j} \frac{(1 - t^{j-i+1})(1 - t^{2n-j-i+1} t_n t_0)}{(1 - t^{j-i})(1 - t^{2n-j-i} t_n t_0)} \right) \left( \prod_{i=1}^n \frac{(1 - t_n t_0 t^{n-i})(1 + t_n t^{n-i})}{(1 - t_0 t_n t^{2(n-i)})} \right) \\
&= \left( \prod_{j=2}^n \frac{(1 - t^j)(1 - t^{2n-j} t_n t_0)}{(1 - t)(1 - t^{2(n-j)+1} t_n t_0)} \right) \left( \prod_{i=1}^n \frac{(1 - t_n t_0 t^{n-i})(1 + t_n t^{n-i})}{(1 - t_0 t_n t^{2(n-i)})} \right) \\
&= \left( \prod_{i=1}^n \frac{(1 - t^i)(1 + t_n t^i)}{(1 - t)} \right) \left( \prod_{j=2}^n \frac{(1 - t^{2n-j} t_n t_0)}{(1 - t^{2(n-j)+1} t_n t_0)} \right) \left( \prod_{i=1}^n \frac{(1 - t_n t_0 t^{n-i})}{(1 - t_0 t_n t^{2(n-i)})} \right) \\
&= \left( \prod_{i=1}^n \frac{(1 - t^i)(1 + t_n t^{i-1})}{(1 - t)} \right) \left( \prod_{i=1}^{2n} \frac{(1 - t^{i-1} t_n t_0)}{(1 - t^{i-1} t_n t_0)} \right) = \prod_{i=1}^n \frac{(1 - t^i)(1 + t_n t^{i-1})}{(1 - t)}.
\end{aligned}$$

$\square$

### 4.5 E-expansions

The following Proposition uses the formulas for symmetrizers in terms of $c$-functions from Proposition 2.4 to give explicit expansion of the bosonic, fermionic and mesonic Koornwinder polynomials in terms of the $E_\mu$. The coefficients in these expansions are evaluations of $c$-functions. This is an example of how the $c$-functions (which live in the field of fractions) appear in the structure even when doing expansions of polynomials that live in $\mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

**Proposition 4.4.** *(E-expansion formulas) Let $\lambda \in (\mathbb{Z}^n_{\geq 0})^+$ and let $P_\lambda$, $A^\pm_{\lambda+\omega}$, $A^\mp_{\lambda+\pi}$ and $A_{\lambda+\rho}$ be as defined in* (4.1) *and* (4.2)*. Let $\mathrm{ev}^t_\mu$ be the evaluation homomorphisms defined in* (3.6) *and let $\kappa^{\Xi}_{v_\mu}$ be the normalized c-functions defined in* (2.10) *and* (1.2)*. Then*

$$P_\lambda = \sum_{\mu\in W\lambda} \mathrm{ev}^t_\mu(\kappa^+_{v_\mu})E_\mu, \qquad A^\pm_{\lambda+\omega} = \sum_{\mu\in W(\lambda+\omega)} (-1)^{\ell_d(v_\mu)}\mathrm{ev}^t_\mu(\kappa^\pm_{v_\mu})E_\mu,$$

$$A_{\lambda+\rho} = \sum_{\mu\in W(\lambda+\rho)} (-1)^{\ell(v_\mu)}\mathrm{ev}^t_\mu(\kappa^-_{v_\mu})E_\mu, \qquad A^\mp_{\lambda+\pi} = \sum_{\mu\in W(\lambda+\pi)} (-1)^{\ell_s(v_\mu)}\mathrm{ev}^t_\mu(\kappa^\mp_{v_\mu})E_\mu.$$

*Proof.* Let us do the case $A^\pm_{\lambda+\omega}$. The other cases are similar.

Since $\tau^\vee_i = \eta^Y_{s_i}c^Y_{\alpha^\vee_i}$ and $t^{\frac12}\tau^\vee_i E_\mu = E_{s^\vee_i\mu}$ then

$$E_{s^\vee_i\mu} = t^{\frac12}\tau^\vee_i E_\mu = t^{\frac12}\eta^Y_{s_i}c^Y_{\alpha^\vee_i}E_\mu = \eta^Y_{s_i}\kappa^Y_{\alpha_i}E_\mu = \kappa^{Y^{-1}}_{\alpha_i}\eta^Y_{s_i}E_\mu.$$

If $z\in W^{\lambda+\omega}$ then

$$E_{z(\lambda+\omega)} = \eta_{s_{i_1}}\kappa^Y_{\alpha_{i_1}}\cdots\eta_{s_{i_\ell}}\kappa^Y_{\alpha_{i_\ell}}E_{\lambda+\omega} = \eta_z\kappa^Y_z E_{\lambda+\omega} = \kappa^{Y^{-1}}_{z^{-1}}\eta_z E_{\lambda+\omega}.$$

If $w\in W_{\lambda+\omega}$ then

$$T_w E_{\lambda+\omega} = \chi^+(T_w)E_{\lambda+\omega} \qquad \text{(in the same way that } T_w\mathbf{1}_Y = \chi^+(T_w)\mathbf{1}_Y \text{ for } w\in W_{\mathrm{fin}}).$$

This gives that

$$\varepsilon^+_{\lambda+\omega}E_{\lambda+\omega} = \frac{1}{\chi^+(T_{w_{\lambda+\omega}})}\sum_{w\in W_{\lambda+\omega}}\chi^+(T_w)^2E_{\lambda+\omega} = \frac{1}{\chi^+(T_{w_{\lambda+\omega}})}W_{\lambda+\omega}(t,t_n)E_{\lambda+\omega}.$$

Using Proposition (2.4) gives

$$\begin{aligned}
A^\pm_{\lambda+\omega} &= \frac{\chi^+(T_{w_0})}{W_{\lambda+\omega}(t,t_n)}\varepsilon_\pm E_{\lambda+\omega} = \frac{1}{W_{\lambda+\omega}(t,t_n)}\left(\sum_{v\in W^{\lambda+\omega}}(-1)^{\ell_d(z)}\kappa^\pm_{v_{\lambda+\omega}z}\eta^Y_z\kappa^Y_z\right)\chi^+(T_{w_{\lambda+\omega}})\varepsilon^+_{\lambda+\omega}E_{\lambda+\omega}\\
&= \left(\sum_{v\in W^{\lambda+\omega}}(-1)^{\ell_d(z)}\kappa^\pm_{v_{\lambda+\omega}z}\eta^Y_z\kappa^Y_z\right)E_{\lambda+\omega} = \sum_{v\in W^{\lambda+\omega}}(-1)^{\ell_d(z)}\kappa^\pm_{v_{\lambda+\omega}z}E_{z(\lambda+\omega)}\\
&= \sum_{z\in W_{\mathrm{fin}}}(-1)^{\ell_d(z)}\mathrm{ev}^t_{z(\lambda+\omega)}(\kappa^\pm_{v_{\lambda+\omega}z})E_{z(\lambda+\omega)}.
\end{aligned}$$

□

## 4.6 Principal specializations

One of the most pleasing combinatorial miracles in Lie theory is that principal specializations of Schur functions and Weyl characters factor as products (see [Kac, §10.9] and [Mac, Ch. I §3 Ex. 1]). This feature extends to Macdonald-Koornwinder polynomials, and the result in this subsection shows that the principal specializations of Macdonald-Koornwinder polynomials are evaluations of $c$-functions which come naturally out of the recursive construction of the electronic Macdonald polynomial $E_\mu$.

Define ring homomorphisms $\mathrm{ev}^t_0\colon \mathbb{K}[Y]\to\mathbb{K}$ and $\mathrm{ev}^{t^{-1}}_0\colon \mathbb{K}[Y]\to\mathbb{K}$ by

$$\mathrm{ev}^t_0(Y_i) = t^{n-i}t_0^{\frac12}t_n^{\frac12} \qquad\text{and}\qquad \mathrm{ev}^t_0(Y_i^{-1}) = t^{-(n-i)}t_0^{-\frac12}t_n^{-\frac12}, \qquad \text{for } i\in\{1,\dots,n\}.$$

**Theorem 4.5.** *Let $\mu, \lambda \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. Let $u_\mu$ and $h_\lambda$ be as defined in* (1.2) *and the normalized c-functions $\kappa_{u_\mu}^{Y^{-1}}$ and $\kappa_{h_\lambda}^{Y^{-1}}$ as in* (2.9)*. Let*

$$a_1 = t^{-(n-1)}(t_0t_n)^{-\frac{1}{2}}, \quad a_2 = t^{-(n-2)}(t_0t_n)^{-\frac{1}{2}}, \quad \ldots, \quad a_n = (t_0t_n)^{-\frac{1}{2}}.$$

*Then*

$$E_\mu(a_1, \ldots, a_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}) = \frac{1}{\chi^+(T_{v_\mu^{-1}})}\mathrm{ev}_0^t(c_{u_\mu}^{Y^{-1}}) \qquad \textit{and}$$

$$P_\lambda(a_1, \ldots, a_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}) = \mathrm{ev}_0^{t^{-1}}(c_{h_\lambda}^{Y^{-1}}).$$

*Proof.* For this proof use the realization of the polynomial representation $\mathbb{K}[X]$ as an induced module $\widetilde{H}\mathbf{1}_Y$ via the $\widetilde{H}$-module isomorphism of (3.2). Let $\mathbf{1}_X$ be a formal symbol which satisfies $\mathbf{1}_X T_j = t^{\frac{1}{2}}\mathbf{1}_X$ and $\mathbf{1}_X T_0^\vee = t_0^{\frac{1}{2}}\mathbf{1}_X$. Using $X_1 = (T_0^\vee)^{-1}T_1^{-1}\cdots T_n^{-1}\cdots T_1^{-1}$ and $X_{i+1} = T_iX_iT_i$ gives

$$\mathbf{1}_X X_i = t^{-(n-1)}t_n^{-\frac{1}{2}}t_0^{-\frac{1}{2}}t^{i-1}, \qquad \text{for } i \in \{1, \ldots, n\}.$$

Thus, if $\mu \in \mathbb{Z}^n$ then

$$\mathbf{1}_X E_\mu(x_1, \ldots, x_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}) = \mathbf{1}_X E_\mu(a_1, \ldots, a_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}).$$

For $i \in \{0, 1, \ldots, n\}$

$$\mathbf{1}_X\tau_i^\vee = \mathbf{1}_X\left(T_{\alpha_i^\vee} + (c_{\alpha_i^\vee}^{Y^{-1}} - t^{\frac{1}{2}})\right) = \mathbf{1}_X\left(t^{\frac{1}{2}} + (c_{\alpha_i^\vee}^{Y^{-1}} - t^{\frac{1}{2}})\right) = \mathbf{1}_X c_{\alpha_i^\vee}^{Y^{-1}}.$$

By (3.7),

$$c_{\alpha_i^\vee}^{Y^{-1}}\mathbf{1}_Y = \mathrm{ev}_0^t(c_{\alpha_i^\vee}^{Y^{-1}})\mathbf{1}_Y.$$

If $w \in W$ and $\ell(s_iw) > \ell(w)$ then

$$\mathbf{1}_X\tau_i^\vee\tau_w^\vee\mathbf{1}_Y = \mathbf{1}_X c_{\alpha_i^\vee}^{Y^{-1}}\tau_w^\vee\mathbf{1}_Y = \mathbf{1}_X\tau_w^\vee c_{w^{-1}\alpha_i^\vee}^{Y^{-1}}\mathbf{1}_Y = \mathrm{ev}_0^t(c_{w^{-1}\alpha_i^\vee}^{Y^{-1}})\mathbf{1}_X\tau_w^\vee\mathbf{1}_Y.$$

This is the induction step giving that if $w \in W$ and $w = s_{i_1}\cdots s_{i_\ell}$ is a reduced word for $w$ then

$$\mathbf{1}_X\tau_w^\vee\mathbf{1}_Y = \mathbf{1}_X\tau_{i_1}^\vee\cdots\tau_{i_\ell}^\vee\mathbf{1}_Y = \mathbf{1}_X\mathrm{ev}_0^t(c_w^{Y^{-1}})\mathbf{1}_Y = \mathrm{ev}_0^t(c_w^{Y^{-1}})\mathbf{1}_X\mathbf{1}_Y.$$

Thus

$$\mathbf{1}_X E_\mu\mathbf{1}_Y = \frac{1}{\chi^+(T_{v_\mu^{-1}})}\mathbf{1}_X\tau_{u_\mu}^\vee\mathbf{1}_Y = \frac{1}{\chi^+(T_{v_\mu^{-1}})}\mathrm{ev}_0^t(c_{u_\mu}^{Y^{-1}})\mathbf{1}_X\mathbf{1}_Y.$$

Using $\mathbf{1}_X\varepsilon_+ = \frac{1}{\chi^+(T_{w_0})}W_0(t, t_n)\mathbf{1}_X$ from (2.29) gives

$$\begin{aligned} P_\lambda(a_1, \ldots, a_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}})\mathbf{1}_X\mathbf{1}_Y &= \mathbf{1}_X P_\lambda\mathbf{1}_Y = \mathbf{1}_X\frac{\chi^+(T_{w_0})}{W_\lambda(t, t_n)}\varepsilon_+E_\lambda\mathbf{1}_Y \\ &= \frac{W_0(t, t_n)}{W_\lambda(t, t_n)}\mathbf{1}_X E_\lambda\mathbf{1}_Y = \frac{1}{\chi^+(T_{v_\lambda^{-1}})}\frac{W_0(t, t_n)}{W_\lambda(t, t_n)}\mathrm{ev}_0^t(c_{u_\lambda}^{Y^{-1}})\mathbf{1}_X\mathbf{1}_Y. \end{aligned}$$

Let

$$w_\lambda \in W_{\mathrm{fin}} \qquad \text{be the longest element of} \quad W_\lambda = \{v \in W_{\mathrm{fin}} \mid v\lambda = \lambda\}.$$

Then by Proposition 4.3

$$v_\lambda^{-1} = (w_0 w_\lambda)^{-1} = w_\lambda w_0 \qquad \text{and} \qquad W_\lambda(t, t_n) = \mathrm{ev}_0^t(\kappa_{w_\lambda}^Y) = \mathrm{ev}_0^{t^{-1}}(\kappa_{w_\lambda}^{Y^{-1}}).$$

Since $h_\lambda = u_\lambda v_\lambda$ then $u_\lambda = h_\lambda v_\lambda^{-1} = v_\lambda^{-1} h_{v_\lambda \lambda} = v_\lambda^{-1} h_{v_\lambda \lambda} = v_\lambda^{-1} h_{w_0 \lambda}$. Using this and $\mathrm{ev}_0^t(Y_i) = \mathrm{ev}_0^{t^{-1}}(Y_i^{-1}) = \mathrm{ev}_0^{t^{-1}}(Y_{-i}) = \mathrm{ev}_0^{t^{-1}}(Y_{w_0(i)})$ gives

$$\mathrm{ev}_0^t(c_{u_\lambda}^{Y^{-1}}) = \mathrm{ev}_0^{t^{-1}}(v_\lambda^{-1} c_{u_\lambda}^{Y^{-1}}).$$

Therefore

$$\begin{aligned}\frac{1}{\chi^+(T_{v_\lambda^{-1}})} \frac{W_0(t,t_n)}{W_\lambda(t,t_n)} \mathrm{ev}_0^t(c_{u_\lambda}^{Y^{-1}}) &= \frac{\chi^+(T_{w_\lambda})}{\chi^+(T_{w_0})} \mathrm{ev}_0^{t^{-1}}\left(\frac{\kappa_{w_0}^{Y^{-1}}}{\kappa_{w_\lambda}^{Y^{-1}}}\right) \mathrm{ev}_0^t(c_{u_\lambda}^{Y^{-1}}) = \mathrm{ev}_0^{t^{-1}}\left(\frac{c_{w_0}^{Y^{-1}}}{c_{w_\lambda}^{Y^{-1}}}\right) \mathrm{ev}_0^{t^{-1}}(v_\lambda^{-1} c_{u_\lambda}^{Y^{-1}}) \\ &= \mathrm{ev}_0^{t^{-1}}(c_{v_\lambda}^{Y^{-1}}) \mathrm{ev}_0^{t^{-1}}(v_\lambda^{-1} c_{u_\lambda}^{Y^{-1}}) = \mathrm{ev}_0^{t^{-1}}(c_{u_\lambda v_\lambda}^{Y^{-1}}) = \mathrm{ev}_0^{t^{-1}}(c_{h_\lambda}^{Y^{-1}}),\end{aligned}$$

which completes the proof of the second statement. □

**Remark 4.6.** The principal specializations of the fermionic and mesonic Macdonald polynomials are

$$A_{\lambda+\rho}(a_1, \ldots, a_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}) = 0,$$

$$A_{\lambda+\omega}^{\pm}(a_1, \ldots, a_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}) = 0, \qquad \text{and} \qquad A_{\lambda+\pi}^{\mp}(a_1, \ldots, a_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}) = 0.$$

To establish this for $A_{\lambda+\omega}^{\pm}$, use $\mathbf{1}_X \varepsilon_\pm = 0$ to get

$$A_{\lambda+\omega}^{\pm}(a_1, \ldots, a_n; q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}) \mathbf{1}_X \mathbf{1}_Y = \mathbf{1}_X A_{\lambda+\omega}^{\pm} \mathbf{1}_Y = \frac{\chi^+(T_{w_0})}{W_{\lambda+\omega}(t, t_n)} \mathbf{1}_X \varepsilon_\pm E_{\lambda+\omega} \mathbf{1}_Y = 0.$$

The proof for the other cases is similar. □

# 5 Orthogonality

In this section we study the Koornwinder polynomials as a family of orthogonal polynomials for a specific inner product. The inner product $(\ ,\ )_+$ is defined via multiplication by a kernel and taking the constant term (for those with an analytic bent, taking the constant term is an integral and the kernel is what defines the measure for the integral). The Macdonald-Koornwinder inner product is defined in Sections 5.1 and 5.2. The kernel is a huge product of $c$-functions, one for each positive root in the affine root system of type $CC_n$.

The Koornwinder polynomials are characterized by orthogonality with respect to this inner product and a triangular expansion in terms of monomials. In order to use the Hecke algebra as a tool in the inner product setting it is crucial to establish that the adjoints of operators that come from the Hecke algebra are tractable. This is done in Section 5.4. In particular, we find that the symmetrizers are self adjoint operators.

The proof of the Weyl character formulas and the norm formulas for Koornwinder polynomials rely on a shift of parameters coming from multiplying by the Weyl denominators. These going up a level formulas, derived in Section 5.5, are the key to establishing recursive relations for computing norms. The recursive relations are derived in Section 5.7 and the norm formula for $(P_\lambda, P_\lambda)_+$ is established in Section 5.9.

In the same way that there are four symmetrizers, there are four going up a level formulas, four Weyl character formulas, four types of recursion relations. In each case, one of the four formulas is usually a triviality, but we have included these trivial formulas in our exposition each time in order to highlight the underlying symmetry of the structures. In the end, the various formulas combine and complement each other to provide the inductive structure for computing norms in terms of $c$-functions.

## 5.1 The kernel $\Delta_{CC}^+$

For an affine root $\beta$, define

$$\kappa_\beta^X = t_\beta^{\frac{1}{2}} c_\beta^X = \frac{(1 - t_\beta^{\frac{1}{2}} u_\beta^{\frac{1}{2}} X^\beta)(1 + t_\beta^{\frac{1}{2}} u_\beta^{-\frac{1}{2}} X^\beta)}{(1 - X^{2\beta})}. \tag{5.1}$$

More specifically, if $i, j \in \{1, \ldots, n\}$ with $i < j$ and $r \in \mathbb{Z}_{\geq 0}$ then

$$\kappa^X_{\varepsilon_i - \varepsilon_j + (r+1)\delta} = \frac{1 - tq^{r+1} X_i X_j^{-1}}{1 - q^{r+1} X_i X_j^{-1}}, \qquad \kappa^X_{\varepsilon_i + (r+1)\delta} = \frac{(1 - q^{r+1} t_n^{\frac{1}{2}} u_n^{\frac{1}{2}} X_i)(1 + q^{r+1} t_n^{\frac{1}{2}} u_n^{-\frac{1}{2}} X_i)}{1 - q^{2r+2} X_i^2},$$

$$\kappa^X_{\varepsilon_i + \varepsilon_j + (r+1)\delta} = \frac{1 - tq^{r+1} X_i X_j}{1 - q^{r+1} X_i X_j}, \qquad \kappa^X_{\varepsilon_i + (r+\frac{1}{2})\delta} = \frac{(1 - q^{r+\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{\frac{1}{2}} X_i)(1 + q^{r+\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{-\frac{1}{2}} X_i)}{1 - q^{2r+1} X_i^2}.$$

Let $S^+$ be the set of positive roots for the affine root system so that

$$S^+ = S^+_{g,+} \cup S^+_{g,-} \cup S^+_{s,+} \cup S^+_{s,-} \cup S^+_{d,+} \cup S^+_{d,-} \cup S^+_{0,s} \cup S^+_{0,d},$$

where

$$S^+_{s,+} = \left\{ \begin{array}{l} (\varepsilon_i - \varepsilon_j) + (r+1)\delta, \\ (\varepsilon_i + \varepsilon_j) + (r+1)\delta \end{array} \;\middle|\; \begin{array}{l} i, j \in \{1, \ldots, n\}, \\ i < j \text{ and } r \in \mathbb{Z}_{\geq 0} \end{array} \right\},$$

$$S^+_{s,-} = \left\{ \begin{array}{l} -(\varepsilon_i - \varepsilon_j) + (r+1)\delta, \\ -(\varepsilon_i + \varepsilon_j) + (r+1)\delta \end{array} \;\middle|\; \begin{array}{l} i, j \in \{1, \ldots, n\}, \\ i < j \text{ and } r \in \mathbb{Z}_{\geq 0} \end{array} \right\},$$

$$S^+_{g,+} = \left\{ \varepsilon_i + (r + \tfrac{1}{2})\delta \;\middle|\; \begin{array}{c} i \in \{1, \ldots, n\} \\ r \in \mathbb{Z}_{\geq 0} \end{array} \right\}, \qquad S^+_{d,+} = \left\{ \varepsilon_i + (r+1)\delta \;\middle|\; \begin{array}{c} i \in \{1, \ldots, n\} \\ r \in \mathbb{Z}_{\geq 0} \end{array} \right\},$$

$$S^+_{g,-} = \left\{ -\varepsilon_i + (r + \tfrac{1}{2})\delta \;\middle|\; \begin{array}{c} i \in \{1, \ldots, n\} \\ r \in \mathbb{Z}_{\geq 0} \end{array} \right\}, \qquad S^+_{d,-} = \left\{ -\varepsilon_i + (r+1)\delta \;\middle|\; \begin{array}{c} i \in \{1, \ldots, n\} \\ r \in \mathbb{Z}_{\geq 0} \end{array} \right\},$$

$$S^+_{0,s} = \{\varepsilon_i \pm \varepsilon_j \mid i, j \in \{1, \ldots, n\} \text{ with } i < j\}, \qquad S^+_{0,d} = \{\varepsilon_i \mid i \in \{1, \ldots, n\}\}.$$

Then define

$$\Delta^+_{CC} = \prod_{\beta \in S^+} \kappa^X_\beta. \tag{5.2}$$

and

$$\Delta^X_g = \prod_{\beta \in S^+_{g,+}} \kappa^X_\beta, \qquad \Delta^X_s = \prod_{\beta \in S^+_{s,+}} \kappa^X_\beta, \qquad \Delta^X_d = \prod_{\beta \in S^+_{d,+}} \kappa^X_\beta, \qquad \Delta^X_{0,s} = \prod_{\beta \in S^+_{0,s}} \kappa^X_\beta,$$

$$\Delta^{X^{-1}}_g = \prod_{\beta \in S^+_{g,-}} \kappa^X_\beta, \qquad \Delta^{X^{-1}}_s = \prod_{\beta \in S^+_{s,-}} \kappa^X_\beta, \qquad \Delta^{X^{-1}}_d = \prod_{\beta \in S^+_{d,-}} \kappa^X_\beta, \qquad \Delta^X_{0,d} = \prod_{\beta \in S^+_{0,d}} \kappa^X_\beta,$$

so that

$$\Delta^+_{CC} = \Delta^X_g \Delta^{X^{-1}}_g \Delta^X_s \Delta^{X^{-1}}_s \Delta^X_d \Delta^{X^{-1}}_d \Delta^X_{0,s} \Delta^X_{0,d}.$$

**Remark 5.1.** In terms of the Askey-Wilson parameters $a, b, c, d$ (see (0.1)), the expression $\Delta^+_{CC}$ used to define the inner product is

$$\Delta^+_{CC} = \Delta^{(1)} \Delta^{(2)},$$

where
$$\Delta^{(2)}=\prod_{1\le i<j\le n}\frac{(tx_ix_j^{-1};q)_\infty(tx_ix_j;q)_\infty(tqx_i^{-1}x_j;q)_\infty(tqx_i^{-1}x_j^{-1};q)_\infty}{(x_ix_j^{-1};q)_\infty(x_ix_j;q)_\infty(qx_i^{-1}x_j;q)_\infty(qx_i^{-1}x_j^{-1};q)_\infty}$$
and
$$\Delta^{(1)}=\prod_{i=1}^n\frac{(cx_i;q)_\infty(qcx_i^{-1};q)_\infty(dx_i;q)_\infty(qdx_i^{-1};q)_\infty(ax_i;q)_\infty(ax_i^{-1};q)_\infty(bx_i;q)_\infty(bx_i^{-1};q)_\infty}{(x_i^2;q)_\infty(qx_i^{-2};q)_\infty}.$$
This is verified by noting that $\Delta^{(2)}=\Delta_s^X\Delta_{0,s}^X\Delta_s^{X^{-1}}$, and that $\Delta^{(1)}=\Delta_g^X\Delta_g^{X^{-1}}\Delta_d^X\Delta_d^{X^{-1}}\Delta_{0,d}^X$ since
$$\prod_{i=1}^n\prod_{r\in\mathbb{Z}_{\ge0}}\frac{(cx_i;q)_\infty(dx_i;q)_\infty}{(x_i^2;q^2)_\infty}=\prod_{i=1}^n\prod_{r\in\mathbb{Z}_{\ge0}}\frac{(1-t_n^{\frac12}u_n^{\frac12}q^rx_i)(1+t_n^{\frac12}u_n^{-\frac12}q^rx_i)}{(1-q^{2r}x_i^2)}=\Delta_d^X\Delta_{0,d}^X,$$
$$\prod_{i=1}^n\prod_{r\in\mathbb{Z}_{\ge0}}\frac{(qcx_i^{-1};q)_\infty(qdx_i^{-1};q)_\infty}{(q^2x_i^{-2};q^2)_\infty}=\prod_{i=1}^n\prod_{r\in\mathbb{Z}_{\ge0}}\frac{(1-t_n^{\frac12}u_n^{\frac12}q^{r+1}x_i^{-1})(1+t_n^{\frac12}u_n^{-\frac12}q^{r+1}x_i^{-1})}{(1-q^{2r+2}x_i^{-2})}=\Delta_d^{X^{-1}},$$
$$\prod_{i=1}^n\prod_{r\in\mathbb{Z}_{\ge0}}\frac{(ax_i;q)_\infty(bx_i;q)_\infty}{(qx_i;q^2)_\infty}=\prod_{i=1}^n\prod_{r\in\mathbb{Z}_{\ge0}}\frac{(1-t_0^{\frac12}u_0^{\frac12}q^{r+\frac12}x_i)(1+t_0^{\frac12}u_0^{-\frac12}q^{r+\frac12}x_i)}{(1-q^{2r+1}x_i^2)}=\Delta_g^X,$$
$$\prod_{i=1}^n\prod_{r\in\mathbb{Z}_{\ge0}}\frac{(ax_i^{-1};q)_\infty(bx_i^{-1};q)_\infty}{(qx_i^{-2};q^2)_\infty}=\prod_{i=1}^n\prod_{r\in\mathbb{Z}_{\ge0}}\frac{(1-t_0^{\frac12}u_0^{\frac12}q^{r+\frac12}x_i^{-1})(1+t_0^{\frac12}u_0^{-\frac12}q^{r+\frac12}x_i^{-1})}{(1-q^{2r+1}x_i^{-2})}=\Delta_g^{X^{-1}}.$$
□

### 5.2 Definition of the inner product

Let $\mathbb{K}[x]=\mathbb{K}[x_1^{\pm1},\dots,x_n^{\pm1}]$. Define an involution $\overline{\phantom{x}}\colon\mathbb{K}[x]\to\mathbb{K}[x]$ by
$$\overline{f}(x_1,\dots,x_n;q,t,t_0^{\frac12},u_0^{\frac12},t_n^{\frac12},u_n^{\frac12})=f(x_1^{-1},\dots,x_n^{-1};q^{-1},t^{-1},t_0^{-\frac12},u_0^{-\frac12},t_n^{-\frac12},u_n^{-\frac12}).\tag{5.3}$$

Let $\Delta_{CC}^+$ be as defined in (5.2). Define a scalar product $(\ ,\ )_+\colon\mathbb{K}[x]\times\mathbb{K}[x]\to\mathbb{K}$ by
$$(f_1,f_2)_+=\mathrm{ct}\left(\frac{f_1\overline{f_2}}{\Delta_{CC}^+}\right),\qquad\text{where}\quad\mathrm{ct}(f)=(\text{constant term in }f),\quad\text{for }f\in\mathbb{K}[x].\tag{5.4}$$

### 5.3 The inner product characterization of $E_\mu$ and $P_\lambda$

Define
$$(\mathbb{Z}^n)^+=\{(\gamma_1,\dots,\gamma_n)\in\mathbb{Z}^n\mid\gamma_1\ge\cdots\ge\gamma_n\ge0\}.$$
The elements of $(\mathbb{Z}^n)^+$ are partially ordered by the dominance order: For $\lambda,\mu\in(\mathbb{Z}^n_{\ge0})^+$,
$$\lambda<\mu\qquad\text{if}\qquad\lambda_1+\cdots+\lambda_i\le\mu_1+\cdots+\mu_i,\quad\text{for }i\in\{1,\dots,n\}.$$
The elements of $\mathbb{Z}^n$ are partially ordered by the DBlex order: For $\lambda,\mu\in\mathbb{Z}^n$,
$$\lambda\le\mu\quad\text{if}\quad\begin{array}{c}\lambda^+<\mu^+\text{ in dominance order}\\ \text{or}\\ \lambda^+=\mu^+\text{ and }z_\lambda<z_\mu\text{ in Bruhat order on }W_{\mathrm{fin}},\end{array}$$

where $\lambda^+ \in W_{\mathrm{fin}}\lambda \cap (\mathbb{Z}^n_{\geq 0})^+$ and $z_\lambda$ is the minimal length element of $W_{\mathrm{fin}}$ so that $\lambda = z_\lambda \lambda^+$.

For $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ write $x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$ and for $\gamma \in (\mathbb{Z}^n)^+$, define the *monomial symmetric function* $m_\gamma$ by

$$m_\gamma = \sum_{\mu \in W_{\mathrm{fin}}\gamma} x^\mu, \qquad \text{where the sum is over all elements of the orbit } W_{\mathrm{fin}}\text{-orbit of } \gamma.$$

With these definitions we have the following characterizations of the $E_\mu$ and the $P_\lambda$. The proofs of Propositions 5.2 and 5.3 are exactly as in [Mac03, (5.2.1) and (5.3.1)] and [CR22, Prop. 6.2 and 6.3].

**Proposition 5.2.** *Let $\mu \in \mathbb{Z}^n$. The electronic Macdonald polynomial $E_\mu$ is the unique element of $\mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ such that*

*(a) $E_\mu = x^\mu + (\text{lower terms})$;*

*(b) If $\nu \in \mathbb{Z}^n$ and $\nu < \mu$ then $(E_\mu, x^\nu)_+ = 0$.*

**Proposition 5.3.** *Let $\lambda \in (\mathbb{Z}^n)^+$. The bosonic Macdonald polynomial $P_\lambda$ is the unique element of $\mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{W_{\mathrm{fin}}}$ such that*

*(a) $P_\lambda = m_\lambda + (\text{lower terms})$;*

*(b) If $\gamma \in (\mathbb{Z}^n)^+$ and $\gamma < \lambda$ then $(P_\lambda, m_\gamma)_+ = 0$.*

### 5.4 Adjoints and orthogonality

For a linear operator $M\colon \mathbb{K}[X] \to \mathbb{K}[X]$, the *adjoint of $M$* is the linear operator $M^*\colon \mathbb{K}[X] \to \mathbb{K}[X]$ determined by

$$(Mf_1, f_2)_+ = (f_1, M^* f_2)_+, \qquad \text{for } f_1, f_2 \in \mathbb{K}[X],$$

where the inner product on $\mathbb{K}[X]$ is as defined in (5.4).

The following Proposition computes the adjoints of operators on $\mathbb{K}[X]$ which come from $\widetilde{H}_{\mathrm{loc}}$.

**Proposition 5.4.** *Let $i \in \{1, \ldots, n\}$ and $k \in \{0, 1, \ldots, n\}$. Then, as operators on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$,*

$$x_i^* = x_i^{-1}, \qquad T_{\alpha_k}^* = T_{\alpha_k}^{-1}, \qquad Y_i^* = Y_i^{-1}, \qquad \xi_{s_k}^* = \frac{\kappa^X_{\alpha_k}}{\kappa^X_{-\alpha_k}} \xi_{s_k}.$$

*Proof.* Let $J = \dfrac{1}{\Delta^+_{CC}}$.

- Adjoint of multiplication by $x_i$:

$$(x_i f, g)_+ = \mathrm{ct}(x_i f \cdot \overline{g} \cdot J) = \mathrm{ct}(f \cdot \overline{x_i^{-1} g} \cdot J) = \big(f, x_i^{-1} g\big)_+ .$$

- Adjoint of $\xi_{s_k}$: With $\kappa_\beta$ as in (5.1),

$$\begin{aligned}
(\kappa^X_\beta)^* = \overline{\kappa^X_\beta} &= \frac{(1 - t_\beta^{-\frac12} u_\beta^{-\frac12} X^{-\beta})(1 + t_\beta^{-\frac12} u_\beta^{\frac12} X^{-\beta})}{(1 - X^{-2\beta})} \\
&= \frac{t_\beta^{-\frac12} u_\beta^{-\frac12} X^{-\beta} t_\beta^{-\frac12} u_\beta^{\frac12} X^{-\beta} (1 - t_\beta^{\frac12} u_\beta^{\frac12} X^{\beta})(1 + t_\beta^{\frac12} u_\beta^{-\frac12} X^{\beta})}{X^{-2\beta}(1 - X^{2\beta})} = t_\beta^{-1} \kappa_\beta.
\end{aligned}$$

Since $s_kS^+ = (S^+ - \{\alpha_k\}) \cup \{-\alpha_k\}$, we have

$$\xi_{s_k}\Delta^+_{CC} = \xi_{s_k}\left(\prod_{\beta\in S^+}\kappa_\beta\right) = \left(\prod_{\beta\in S^+-\{\alpha_k\}}\kappa_\beta\right)\kappa_{-\alpha_k} = \Delta^+_{CC}\frac{\kappa_{-\alpha_k}}{\kappa_{\alpha_k}}. \tag{5.5}$$

Using $\mathrm{ct}(\xi_{s_k}g) = \mathrm{ct}(g)$ and the formula from (5.5),

$$\begin{aligned}(\xi_{s_k}f, g)_+ &= \mathrm{ct}\left((\xi_{s_k}f)\overline{g}J\right) = \mathrm{ct}\left(\xi_{s_k}(f(\xi_{s_k}(\overline{g}J)))\right) = \mathrm{ct}\left(f(\xi_{s_k}(\overline{g}J))\right)\\ &= \mathrm{ct}\left(f(\xi_{s_k}\overline{g})J\frac{\kappa_{\alpha_k}}{\kappa_{-\alpha_k}}\right) = \mathrm{ct}\left(f\overline{\frac{\kappa_{\alpha_k}}{\kappa_{-\alpha_k}}(\xi_{s_k}g)}J\right) = \left(f, \frac{\kappa_{\alpha_k}}{\kappa_{-\alpha_k}}(\xi_{s_k}g)\right)_+.\end{aligned}$$

- Adjoint of $T_{\alpha_k}$: Using the formula for $T_{\alpha_k}$ in (2.16) and recalling that $(\ ,\ )_+$ is sesquilinear with respect to the involution $\overline{\phantom{x}}\colon \mathbb{K}\to\mathbb{K}$,

$$\begin{aligned}(T_{\alpha_k})^* &= \left(-t_{\alpha_k}^{-\frac12} + (1+\xi_{s_k})c^{X^{-1}}_{\alpha_k}\right)^* = \left(-t_{\alpha_k}^{-\frac12} + (1+\xi_{s_k})t_{\alpha_k}^{-\frac12}\kappa_{-\alpha_k}\right)^* = -t_{\alpha_k}^{\frac12} + t_{\alpha_k}^{\frac12}\overline{\kappa_{-\alpha_k}}(1+\xi_{s_k}^*)\\ &= -t_{\alpha_k}^{\frac12} + t_{\alpha_k}^{\frac12}t_{\alpha_k}^{-1}\kappa_{-\alpha_k}\left(1+\frac{\kappa_{\alpha_k}}{\kappa_{-\alpha_k}}\xi_{s_k}\right) = -t_{\alpha_k}^{\frac12} + t_{\alpha_k}^{-\frac12}\left(\kappa_{-\alpha_k} + \kappa_{\alpha_k}\xi_{s_k}\right)\\ &= -t_{\alpha_k}^{\frac12} + (1+\xi_{s_k})c^X_{-\alpha_k} = T_{\alpha_k}^{-1}.\end{aligned}$$

- Adjoint of $Y_j$:

$$Y_1^* = (T_0T_1\cdots T_n\cdots T_1)^* = T_1^{-1}\cdots T_n^{-1}\cdots T_1^{-1}T_0^{-1} = (T_0T_1\cdots T_n\cdots T_1)^{-1} = Y_1^{-1},$$

and if $j\in\{2,\dots,n\}$ then

$$Y_j^* = (T_{j-1}^{-1}Y_{j-1}T_{j-1}^{-1})^* = T_{j-1}Y_{j-1}^{-1}T_{j-1} = (T_{j-1}^{-1}Y_{j-1}T_{j-1}^{-1})^{-1} = Y_j^{-1}.$$

□

Since $T_j^{-1}\varepsilon_\pm^* = T_j^*\varepsilon_\pm^* = (\varepsilon_\pm T_j)^*$ for $j\in\{1,\dots,n\}$ then $T_n^{-1}\varepsilon_\pm^* = (-t_n)^{\frac12}\varepsilon_\pm^*$ and $T_i^{-1}\varepsilon_\pm^* = t^{-\frac12}\varepsilon_\pm^*$ for $i\in\{1,\dots,n-1\}$. Since

$$\varepsilon_\pm^* = T_{w_0}^{-1} + (\text{lower terms}) = T_{w_0} + (\text{lower terms}) \qquad \text{then} \qquad \varepsilon_\pm^* = \varepsilon_\pm.$$

A similar argument applies to the other symmetrizers to show that

$$\varepsilon_\Xi^* = \varepsilon_\Xi, \qquad \text{for} \qquad \Xi\in\{+,\pm,\mp,-\}. \tag{5.6}$$

The relations $Y_i^* = Y_i^{-1}$ in combination with the knowledge of the eigenvalues for the action of the $Y_i$ on the $E_\mu$ give the following orthogonality relations for Macdonald polynomials. The proof is exactly as in [Mac03, (5.7.11)] and [CR22, Prop, 7.2].

**Proposition 5.5.** *Let $(\mathbb{Z}^n_{\ge0})^+ = \{\lambda = (\lambda_1,\dots,\lambda_n) \mid \lambda_1\ge\cdots\ge\lambda_n\ge0\}$ and, for $\lambda\in(\mathbb{Z}^n_{\ge0})^+$, let $P_\lambda$, $A^\pm_{\lambda+\omega}$, $A^\mp_{\lambda+\pi}$ and $A_{\lambda+\rho}$ be as defined in (4.1) and (4.2).*

*(a) Let $\lambda,\mu\in\mathbb{Z}^n$. If $\mu\ne\lambda$ then $(E_\lambda,E_\mu)_+ = 0$.*

*(b) Let $\lambda,\mu\in(\mathbb{Z}^n_{\ge0})^+$. If $\mu\ne\lambda$ then*

$$\begin{aligned}(P_\lambda,P_\mu)_+ &= 0, & (A^\pm_{\lambda+\omega},A^\pm_{\mu+\omega})_+ &= 0,\\ (A_{\lambda+\rho},A_{\mu+\rho})_+ &= 0, & (A^\mp_{\lambda+\pi},A^\mp_{\mu+\pi})_+ &= 0.\end{aligned}$$

### 5.5 Going up a level

We describe four slightly different collections of 5 parameters by the brief notations

$$t^+ = (t, t_0^{\frac12}, u_0^{\frac12}, t_n^{\frac12}, u_n^{\frac12}), \qquad t^\pm = (t, t_0^{\frac12}, u_0^{\frac12}, qt_n^{\frac12}, u_n^{\frac12}),$$
$$t^- = (qt, t_0^{\frac12}, u_0^{\frac12}, qt_n^{\frac12}, u_n^{\frac12}), \qquad t^\mp = (qt, t_0^{\frac12}, u_0^{\frac12}, t_n^{\frac12}, u_n^{\frac12}),$$

and define

$$(f,g)_+ = (f,g)_{q,t^+} \qquad (f,g)_\pm = (f,g)_{q,t^\pm}$$
$$(f,g)_- = (f,g)_{q,t^-} \qquad (f,g)_\mp = (f,g)_{q,t^\mp}.$$

The following Proposition shows that the norms of polynomials in the fermionic, bosonic and mesonic spaces can be computed as norms of symmetric polynomials, but *with shifted parameters.* Alternatively, in the world of norms for symmetric polynomials, the shifted parameters are a residue arising from the effect of multiplying by the Weyl denominators $A_\rho$, $A_\omega^\pm$, $A_\pi^\mp$.

**Proposition 5.6.** *(Going up a level) Let $f, g \in \mathbb{K}[X]^{W_{\mathrm{fin}}}$ so that $f$ and $g$ are symmetric functions. If $P_0 = 1$ and $A_\omega^\pm$, $A_\pi^\mp$ and $A_\rho$ are the Weyl denominators defined in* (4.3) *then*

$$(f,g)_+ = \frac{W_0(t,t_n)}{W_0(t,t_n)}(P_0 f, P_0 g)_+, \qquad (f,g)_\pm = \frac{W_0(t,qt_n)}{W_0(t,t_n^{-1})}(A_\omega^\pm f, A_\omega^\pm g)_+,$$
$$(f,g)_- = \frac{W_0(qt,qt_n)}{W_0(t^{-1},t_n^{-1})}(A_\rho f, A_\rho g)_+, \qquad (f,g)_\mp = \frac{W_0(qt,t_n)}{W_0(t^{-1},t_n)}(A_\pi^\mp f, A_\pi^\mp g)_+.$$

*Proof.* Let $t^\pm$ be the 5-tuple of parameters $t^\pm = (t, t_0^{\frac12}, u_0^{\frac12}, qt_n^{\frac12}, u_n^{\frac12})$ and let

$$\Delta_{CC}^\pm = \Delta_g^X(t_0^{\frac12}, u_0^{\frac12})\Delta_g^{X^{-1}}(t_0^{\frac12}, u_0^{\frac12})\Delta_s^X(t)\Delta_s^{X^{-1}}(t)\Delta_d^X(qt_n^{\frac12}, u_n^{\frac12})\Delta_d^{X^{-1}}(qt_n^{\frac12}, u_n^{\frac12})\Delta_{0,s}^X(t)\Delta_{0,d}^X(qt_n^{\frac12}, u_n^{\frac12}).$$

Since

$$\begin{aligned}
\Delta_d^X(t_n^{\frac12}, u_n^{\frac12}) &= \Delta_d^X(qt_n^{\frac12}, u_n^{\frac12}) \cdot \left(\prod_{1\le i\le n}(1 - qt_n^{\frac12}u_n^{\frac12}x_i)(1 + qt_n^{\frac12}u_n^{-\frac12}x_i)\right) \\
&= \Delta_d^X(qt_n^{\frac12}, u_n^{\frac12}) \cdot x^\omega A_\omega^\pm(x, qt_n^{\frac12}, u_n^{\frac12}) \\
\Delta_{0,d}^X(t_n^{\frac12}, u_n^{\frac12}) &= \Delta_{0,d}^X(qt_n^{\frac12}, u_n^{\frac12}) \cdot \left(\prod_{1\le i\le n}\frac{(1 - t_n^{\frac12}u_n^{\frac12}x_i)(1 + t_n^{\frac12}u_n^{-\frac12}x_i)}{(1 - qt_n^{\frac12}u_n^{\frac12}x_i)(1 + qt_n^{\frac12}u_n^{-\frac12}x_i)}\right) \\
&= \Delta_{0,d}^X(qt_n^{\frac12}, u_n^{\frac12})\frac{x^\omega A_\omega^\pm(x, t_n^{\frac12}, u_n^{\frac12})}{x^\omega A_\omega^\pm(x, qt_n^{\frac12}, u_n^{\frac12})} \\
\Delta_d^{X^{-1}}(t_n^{\frac12}, u_n^{\frac12}) &= \Delta_d^{X^{-1}}(qt_n^{\frac12}, u_n^{\frac12}) \cdot \left(\prod_{1\le i\le n}(1 - qt_n^{\frac12}u_n^{\frac12}x_i^{-1})(1 + qt_n^{\frac12}u_n^{-\frac12}x_i^{-1})\right) \\
&= \Delta_d^{X^{-1}}(qt_n^{\frac12}, u_n^{\frac12})x^{-\omega}A_\omega^\pm(x^{-1}, qt_n^{\frac12}, u_n^{\frac12})
\end{aligned}$$

then

$$\frac{\Delta_{CC}^+}{\Delta_{CC}^\pm} = A_\omega^\pm(x, t_n^{\frac12}, u_n^{\frac12})A_\omega^\pm(x^{-1}, qt_n^{\frac12}, u_n^{\frac12}).$$

Using Proposition 4.3 and that $\mathrm{Card}(W_{\mathrm{fin}}) = 2^n n!$ gives that for $h \in \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{W_{\mathrm{fin}}}$,

$$\mathrm{ct}(h\kappa_{w_0}^{X^{-1}}) = \frac{1}{2^n n!}\mathrm{ct}\left(\sum_{w\in W_{\mathrm{fin}}} w(h\kappa_{w_0}^{X^{-1}})\right) = \frac{1}{2^n n!}\mathrm{ct}\left(h\sum_{w\in W_{\mathrm{fin}}} w(\kappa_{w_0}^{X^{-1}})\right) = \frac{W_0(t,t_n)}{2^n n!}\mathrm{ct}(h).$$

Let $\Delta_\infty^X = \Delta_g^X \Delta_s^X \Delta_d^X$ and $\Delta_0^X = \Delta_{0,s}(t)\Delta_{0,d}^X$ so that

$$\Delta_{CC}^{+} = \Delta_\infty^X \Delta_\infty^{X^{-1}} \Delta_0^X \qquad \text{and} \qquad \Delta_0^X = \frac{A_\omega^{\pm}(x, t_n^{\frac{1}{2}} u_n^{\frac{1}{2}})}{a_\omega^{\pm}(x)} \frac{A_\pi^{\mp}(x,t)}{a_\pi^{\mp}(x)}.$$

Then

$$\begin{aligned}
(A_\omega^{\pm} f, A_\omega^{\pm} g)_+ &= \mathrm{ct}\left(\frac{f\overline{g}}{\Delta_{CC}^{+}} A_\omega^{\pm}(x, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}})\overline{A_\omega^{\pm}(x, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}})}\right)\\
&= \mathrm{ct}\left(\frac{f\overline{g}}{\Delta_\infty^X \Delta_\infty^{X^{-1}}} \frac{A_\omega^{\pm}(x, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}})}{\Delta_0^X} A_\omega^{\pm}(x^{-1}, t_n^{-\frac{1}{2}}, u_n^{-\frac{1}{2}})\right)\\
&= \mathrm{ct}\left(\frac{f\overline{g}}{\Delta_\infty^X \Delta_\infty^{X^{-1}}} \frac{a_\omega^{\pm}(x) a_\pi^{\mp}(x)}{A_\pi^{\mp}(x,t)} A_\omega^{\pm}(x^{-1}, t_n^{-\frac{1}{2}}, u_n^{-\frac{1}{2}})\right)\\
&= \mathrm{ct}\left(\frac{f\overline{g}}{\Delta_\infty^X \Delta_\infty^{X^{-1}}} \frac{a_\omega^{\pm}(x) a_\pi^{\mp}(x) a_\omega^{\pm}(x^{-1}) a_\pi^{\mp}(x^{-1})}{A_\pi^{\mp}(x,t) A_\pi^{\mp}(x^{-1},t)} \cdot \frac{A_\omega^{\pm}(x^{-1}, t_n^{-\frac{1}{2}}, u_n^{-\frac{1}{2}}) A_\pi^{\mp}(x^{-1},t)}{a_\omega^{\pm}(x^{-1}) a_\pi^{\mp}(x^{-1})}\right)\\
&= \mathrm{ct}\left(\frac{f\overline{g}}{\Delta_\infty^X \Delta_\infty^{X^{-1}}} \frac{a_\omega^{\pm}(x) a_\pi^{\mp}(x) a_\omega^{\pm}(x^{-1}) a_\pi^{\mp}(x^{-1})}{A_\pi^{\mp}(x,t) A_\pi^{\mp}(x^{-1},t)} \cdot \kappa_{w_0}^{X^{-1}}(t, t_n^{-\frac{1}{2}}, u_n^{-\frac{1}{2}})\right)\\
&= \frac{W_0(t, t_n^{-1})}{2^n n!}\mathrm{ct}\left(\frac{f\overline{g}}{\Delta_\infty^X \Delta_\infty^{X^{-1}}} \frac{a_\omega^{\pm}(x) a_\pi^{\mp}(x) a_\omega^{\pm}(x^{-1}) a_\pi^{\mp}(x^{-1})}{A_\pi^{\mp}(x,t) A_\pi^{\mp}(x^{-1},t)}\right)
\end{aligned}$$

and

$$\begin{aligned}
(f,g)_{\pm} &= \mathrm{ct}\left(\frac{f\overline{g}}{\Delta_{CC}^{\pm}}\right) = \mathrm{ct}\left(\frac{f\overline{g}}{\Delta_{CC}^{+}} A_\omega^{\pm}(x, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}) A_\omega^{\pm}(x^{-1}, qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}})\right)\\
&= \mathrm{ct}\left(\frac{f\overline{g}}{\Delta_\infty^X \Delta_\infty^{X^{-1}}} \frac{A_\omega^{\pm}(x, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}})}{\Delta_0^X} A_\omega^{\pm}(x^{-1}, qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}})\right)\\
&= \mathrm{ct}\left(\frac{f\overline{g}}{\Delta_\infty^X \Delta_\infty^{X^{-1}}} \frac{a_\omega^{\pm}(x) a_\pi^{\mp}(x)}{A_\pi^{\mp}(x,t)} A_\omega^{\pm}(x^{-1}, qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}})\right)\\
&= \mathrm{ct}\left(\frac{f\overline{g}}{\Delta_\infty^X \Delta_\infty^{X^{-1}}} \frac{a_\omega^{\pm}(x) a_\pi^{\mp}(x) a_\omega^{\pm}(x^{-1}) a_\pi^{\mp}(x^{-1})}{A_\pi^{\mp}(x,t) A_\pi^{\mp}(x^{-1},t)} \cdot \frac{A_\omega^{\pm}(x^{-1}, qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}) A_\pi^{\mp}(x^{-1},t)}{a_\omega^{\pm}(x^{-1}) a_\pi^{\mp}(x^{-1})}\right)\\
&= \mathrm{ct}\left(\frac{f\overline{g}}{\Delta_\infty^X \Delta_\infty^{X^{-1}}} \frac{a_\omega^{\pm}(x) a_\pi^{\mp}(x) a_\omega^{\pm}(x^{-1}) a_\pi^{\mp}(x^{-1})}{A_\pi^{\mp}(x,t) A_\pi^{\mp}(x^{-1},t)} \cdot \kappa_{w_0}^{X^{-1}}(t, qt_n^{\frac{1}{2}}, u_n^{\frac{1}{2}})\right)\\
&= \frac{W_0(t, qt_n)}{2^n n!}\mathrm{ct}\left(\frac{f\overline{g}}{\Delta_\infty^X \Delta_\infty^{X^{-1}}} \frac{a_\omega^{\pm}(x) a_\pi^{\mp}(x) a_\omega^{\pm}(x^{-1}) a_\pi^{\mp}(x^{-1})}{A_\pi^{\mp}(x,t) A_\pi^{\mp}(x^{-1},t)}\right).
\end{aligned}$$

which proves the $\pm$ case. The other cases are similar. $\square$

## 5.6 Weyl character formulas

As in (2.31), let $0, \omega, \pi, \rho \in \mathbb{Z}^n$ by $0 = (0, 0, \dots, 0)$,

$$\omega = (1, 1, \dots, 1), \qquad \pi = (n-1, n-2, \dots, 2, 1, 0), \qquad \rho = (n, n-1, \dots, 3, 2, 1).$$

**Theorem 5.7.** *(Weyl character formulas) Let $\lambda \in \mathbb{Z}^n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$. Then*

$$P_\lambda(q, t^+) = \frac{P_{\lambda+0}(q, t^+)}{P_0(q, t^+)}, \qquad P_\lambda(q, t^\pm) = \frac{A^\pm_{\lambda+\omega}(q, t^+)}{A^\omega_\pi(q, t^+)},$$

$$P_\lambda(q, t^-) = \frac{A_{\lambda+\rho}(q, t^+)}{A_\rho(q, t^+)}, \qquad P_\lambda(q, t^\mp) = \frac{A^\mp_{\lambda+\pi}(q, t^+)}{A^\mp_\pi(q, t^+)},$$

*Proof.* The following is the proof for the case $P_\lambda(q, t^\pm)$. The proof for the other cases is similar. Since $A^\pm_{\lambda+\omega} = t^{\frac{1}{2}\ell(w_0)}\varepsilon_\pm E_{\lambda+\omega}$ then $A^\pm_{\lambda+\omega} \in \mathbb{C}[X]^{\mathrm{Mes}\pm}$. Thus, by Proposition 4.2,

$$\text{there exists} \qquad f \in \mathbb{K}[X]^{W_{\mathrm{fin}}} \qquad \text{such that} \qquad A^\pm_{\lambda+\omega} = A^\pm_\omega f.$$

Since $\lambda+\omega$ is dominant then $w_0(\lambda+\omega) = -(\lambda+\omega)$ is antidominant (weakly increasing with all entries $\leq 0$). If $\mu \in \mathbb{Z}^n$ is such that the coefficient of $x^\mu$ in $A^\pm_{\lambda+\omega}$ is nonzero then $\mu \leq -(\lambda+\omega)$ in the DBlex order. Thus, using the notations of Section 5.3,

$$f = m_\lambda + (\text{lower terms}).$$

The $E$-expansion for $A^\pm_{\lambda+\omega}$ in Proposition 4.4 gives that

$$A^\pm_{\lambda+\omega} = \sum_{\mu \in W_{\mathrm{fin}}(\lambda+\pi)} d^\mu_{\lambda+\omega} E_\mu = E_{w_0(\lambda+\omega)} + (\text{lower terms})$$

and, from the definitions of $A^\pm_\omega$ and $m_\nu$,

$$A^\pm_\omega m_\nu = x^{w_0(\nu+\omega)} + (\text{lower terms}).$$

Since $(E_{w_0(\lambda+\omega)}, x^\gamma)_+ = 0$ for $\gamma < w_0(\lambda+\omega)$ then

$$(A^\pm_\omega f, A^\pm_\omega m_\nu)_+ = (A^\pm_{\lambda+\omega}, A^\pm_\omega m_\nu)_+ = 0, \qquad \text{for } \nu \in (\mathbb{Z}^n)^+ \text{ with } \nu < \lambda.$$

Using Proposition 5.6, since $f \in \mathbb{K}[X]^{W_{\mathrm{fn}}}$ and $m_\nu \in \mathbb{K}[X]^{W_{\mathrm{fin}}}$ then

$$(f, m_\nu)_\pm = \frac{W_0(t, qt_n)}{W_0(t, t_n^{-1})}(A^\pm_\omega f, A^\pm_\omega m_\nu)_+ = 0, \qquad \text{for } \nu \in (\mathbb{Z}^n)^+ \text{ with } \nu < \lambda.$$

Thus, by Proposition 5.3, $f = P_\lambda(q, t^\pm)$. $\square$

## 5.7 Reductions for norms

The following proposition shows that the $c$-functions provide an effective framework for describing the differences between the norms of the various flavors of Koornwinder polynomials.

**Proposition 5.8.**

*(a) Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$. Then*

$$\frac{(E_\mu, E_\mu)_+}{(1, 1)_+} = \mathrm{ev}^t_0(c^Y_{u_\mu} c^{Y^{-1}}_{u_\mu}).$$

*(b) Let* $\lambda \in (\mathbb{Z}^n_{\geq 0})^+$*.*

$$\begin{aligned}
\frac{(P_\lambda, P_\lambda)_+}{(E_\lambda, E_\lambda)_+} &= \frac{W_0(t,t_n)}{W_\lambda(t,t_n)} \mathrm{ev}^t_\lambda(\kappa^+_{v_\lambda}), \\
\frac{(A^\pm_{\lambda+\omega}, A^\pm_{\lambda+\omega})_+}{(E_{\lambda+\omega}, E_{\lambda+\omega})_+} &= \frac{W_0(t,t_n^{-1})}{W_{\lambda+\omega}(t,t_n)} t_n^n \mathrm{ev}^t_{\lambda+\omega}(\kappa^\pm_{v_{\lambda+\omega}}), \\
\frac{(A^\mp_{\lambda+\pi}, A^\mp_{\lambda+\pi})_+}{(E_{\lambda+\pi}, E_{\lambda+\pi})_+} &= \frac{W_0(t^{-1},t_n)}{W_{\lambda+\pi}(t,t_n)} t^{n(n-1)} \mathrm{ev}^t_{\lambda+\pi}(\kappa^\mp_{v_{\lambda+\pi}}), \\
\frac{(A_{\lambda+\rho}, A_{\lambda+\rho})_+}{(E_{\lambda+\rho}, E_{\lambda+\rho})_+} &= \frac{W_0(t^{-1},t_n^{-1})}{W_{\lambda+\rho}(t,t_n)} t^{n(n-1)} t_n^n \mathrm{ev}^t_{\lambda+\rho}(\kappa^-_{v_{\lambda+\rho}}).
\end{aligned}$$

*(c) Let* $\lambda \in (\mathbb{Z}^n_{\geq 0})^+$*.*

$$\begin{aligned}
\frac{(A^\pm_{\lambda+\omega}, A^\pm_{\lambda+\omega})_+}{(P_{\lambda+\omega}, P_{\lambda+\omega})_+} &= \frac{W_0(t,t_n^{-1})}{W_0(t,t_n)} t_n^n \mathrm{ev}^t_{\lambda+\omega}\left(\frac{\kappa^\pm_{v_{\lambda+\omega}}}{\kappa^+_{v_{\lambda+\omega}}}\right), \\
\frac{(A^\mp_{\lambda+\pi}, A^\mp_{\lambda+\pi})_+}{(P_{\lambda+\pi}, P_{\lambda+\pi})_+} &= \frac{W_0(t^{-1},t_n)}{W_0(t,t_n)} t^{n(n-1)} \mathrm{ev}^t_{\lambda+\pi}\left(\frac{\kappa^\mp_{v_{\lambda+\pi}}}{\kappa^+_{v_{\lambda+\pi}}}\right), \\
\frac{(A_{\lambda+\rho}, A_{\lambda+\rho})_+}{(P_{\lambda+\rho}, P_{\lambda+\rho})_+} &= \frac{W_0(t^{-1},t_n^{-1})}{W_0(t,t_n)} t^{n(n-1)} t_n^n \mathrm{ev}^t_{\lambda+\rho}\left(\frac{\kappa^-_{w_0}}{\kappa^+_{w_0}}\right).
\end{aligned}$$

*(d) Let* $\lambda \in (\mathbb{Z}^n_{\geq 0})^+$*. Then*

$$\begin{aligned}
\frac{(P_\lambda(q,t^\pm), P_\lambda(q,t^\pm))_\pm}{(P_{\lambda+\omega}(q,t^+), P_{\lambda+\omega}(q,t^+))_+} &= \frac{W_0(t,qt_n)}{W_0(t,t_n)} t_n^{\ell_d(w_0)} \mathrm{ev}^t_{\lambda+\omega}\left(\frac{\kappa^\pm_{v_{\lambda+\omega}}}{\kappa^+_{v_{\lambda+\omega}}}\right) \\
\frac{(P_\lambda(q,t^\mp), P_\lambda(q,t^\mp))_\mp}{(P_{\lambda+\pi}(q,t^+), P_{\lambda+\pi}(q,t^+))_+} &= \frac{W_0(qt,t_n)}{W_0(t,t_n)} t^{\ell_s(w_0)} \mathrm{ev}^t_{\lambda+\pi}\left(\frac{\kappa^\mp_{v_{\lambda+\pi}}}{\kappa^+_{v_{\lambda+\pi}}}\right) \\
\frac{(P_\lambda(q,t^-), P_\lambda(q,t^-))_-}{(P_{\lambda+\rho}(q,t^+), P_{\lambda+\rho}(q,t^+))_+} &= \frac{W_0(qt,qt_n)}{W_0(t,t_n)} t^{\ell_s(w_0)} t_n^{\ell_d(w_0)} \mathrm{ev}^t_{\lambda+\rho}\left(\frac{\kappa^-_{w_0}}{\kappa^+_{w_0}}\right).
\end{aligned}$$

*Proof.* (a) Using the creation formula for $E_\mu$,

$$\begin{aligned}
(E_\mu, E_\mu)_+ &= (t^{-\frac{1}{2}\ell(v_\mu^{-1})} \tau^\vee_{u_\mu} \mathbf{1}_Y, t^{-\frac{1}{2}\ell(v_\mu^{-1})} \tau^\vee_{u_\mu} \mathbf{1}_Y)_+ = (\tau^\vee_{u_\mu^{-1}} \tau^\vee_{u_\mu} \mathbf{1}_Y, \mathbf{1}_Y)_+ \\
&= (c^Y_{u_\mu} c^{Y^{-1}}_{u_\mu} \mathbf{1}_Y, \mathbf{1}_Y)_+ = \mathrm{ev}^t_0(c^Y_{u_\mu} c^{Y^{-1}}_{u_\mu}) \cdot (1,1)_+.
\end{aligned}$$

(b) Using Proposition 4.4 ($E$-expansion formulas),

$$A^\pm_{\lambda+\omega} = \frac{\chi^\pm(T_{w_0})}{W_{\lambda+\omega}(t,t_n)} \varepsilon_\pm E_{\lambda+\omega} = \sum_{\mu \in W_{\mathrm{fin}}(\lambda+\omega)} d^\mu_{\lambda+\omega} E_\mu, \qquad \text{with} \qquad d^{\lambda+\omega}_{\lambda+\omega} = (-1)^{\ell_d(v_{\lambda+\omega})} \mathrm{ev}^t_{\lambda+\omega}(\kappa^\pm_{v_{\lambda+\omega}}),$$

and (2.29)

$$\varepsilon_\pm^2 = \frac{1}{\chi^\pm(T_{w_0})} W_0(t,t_n^{-1}) \varepsilon_\pm,$$

gives

$$
\begin{aligned}
(A^{\pm}_{\lambda+\omega}, A^{\pm}_{\lambda+\omega})_+ &= \left(\frac{\chi^+(T_{w_0})}{W_{\lambda+\omega}(t,t_n)}\varepsilon_\pm E_{\lambda+\omega}, \frac{\chi^+(T_{w_0})}{W_{\lambda+\omega}(t,t_n)}\varepsilon_\pm E_{\lambda+\omega}\right)_+ && \text{(by (4.2))} \\
&= \frac{1}{W_{\lambda+\omega}(t,t_n)W_{\lambda+\omega}(t^{-1},t_n^{-1})}\left(\varepsilon_\pm^2 E_{\lambda+\omega}, E_{\lambda+\omega}\right)_+ && \text{(by (5.6))} \\
&= \frac{1}{\chi^\pm(T_{w_0})}\frac{W_0(t,t_n^{-1})}{W_{\lambda+\omega}(t,t_n)W_{\lambda+\omega}(t^{-1},t_n^{-1})}\left(\varepsilon_\pm E_{\lambda+\omega}, E_{\lambda+\omega}\right)_+ && \text{(by (2.29))} \\
&= \frac{\chi^+(T_{w_{\lambda+\omega}})^2}{\chi^\pm(T_{w_0})\chi^+(T_{w_0})}\frac{W_0(t,t_n^{-1})}{W_{\lambda+\omega}(t,t_n)}\left(A^\pm_{\lambda+\omega}, E_{\lambda+\omega}\right)_+ && \text{(by (4.2))} \\
&= \frac{t^{\ell_d(w_0)}}{(-1)^{\ell_d(w_0)}}\frac{W_0(t,t_n^{-1})}{W_{\lambda+\omega}(t^{-1},t_n^{-1})}d^{\lambda+\omega}_{\lambda+\omega}(E_{\lambda+\omega}, E_{\lambda+\omega})_+ \\
&= \frac{t_n^n}{(-1)^{\ell_d(w_0)}}\frac{W_0(t,t_n^{-1})}{W_{\lambda+\omega}(t,t_n)}(-1)^{\ell_d(v_{\lambda+\omega})}\mathrm{ev}^t_{\lambda+\omega}(\kappa^\pm_{v_{\lambda+\omega}})(E_{\lambda+\omega}, E_{\lambda+\omega})_+ \\
&= \frac{W_0(t,t_n^{-1})}{W_{\lambda+\omega}(t,t_n)}t_n^n\mathrm{ev}^t_{\lambda+\omega}(\kappa^\pm_{v_{\lambda+\omega}})(E_{\lambda+\omega}, E_{\lambda+\omega})_+,
\end{aligned}
$$

because $(-1)^{\ell_d(v_{\lambda+\omega})} = (-1)^{\ell_d(w_0)}$. This proves the $\pm$ case. The proof for the other cases is similar.

(c) By part (b),

$$
\frac{(A^\pm_{\lambda+\omega}, A^\pm_{\lambda+\omega})_+}{(E_{\lambda+\omega}, E_{\lambda+\omega})_+} = \frac{W_0(t,t_n^{-1})}{W_{\lambda+\omega}(t,t_n)}t_n^n\mathrm{ev}^t_{\lambda+\omega}(\kappa^\pm_{v_{\lambda+\omega}}) \quad \text{and} \quad \frac{(P_{\lambda+\omega}, P_{\lambda+\omega})_+}{(E_{\lambda+\omega}, E_{\lambda+\omega})_+} = \frac{W_0(t,t_n)}{W_{\lambda+\omega}(t,t_n)}\mathrm{ev}^t_{\lambda+\omega}(\kappa^+_{v_{\lambda+\omega}})
$$

gives the relation between $(A^\pm_{\lambda+\omega}, A^\pm_{\lambda+\omega})_+$ and $(P_{\lambda+\omega}, P_{\lambda+\omega})_+$. This proves the $\pm$ case. The proof for the other cases is similar.

(d) Using Proposition 5.6 (going up a level) and Proposition 5.7 (Weyl character formula) gives

$$
\begin{aligned}
(P_\lambda(q,t^\pm), P_\lambda(q,t^\pm))_\pm &= \frac{W_0(t,qt_n)}{W_0(t,t_n^{-1})}(A^\pm_\omega P_\lambda(q,t^\pm), A^\pm_\omega P_\lambda(q,t^\pm))_+ \\
&= \frac{W_0(t,qt_n)}{W_0(t,t_n^{-1})}(A^\pm_{\lambda+\omega}(q,t^+), A^\pm_{\lambda+\omega}(q,t^+))_+ \\
&= \frac{W_0(t,qt_n)}{W_0(t,t_n)}t_n^n\mathrm{ev}^t_{\lambda+\omega}\left(\frac{\kappa^\pm_{v_{\lambda+\omega}}}{\kappa^+_{v_{\lambda+\omega}}}\right)(P_{\lambda+\omega}(q,t^+), P_{\lambda+\omega}(q,t^+))_+
\end{aligned}
$$

This proves the $\pm$ case. The proof for the other cases is similar. □

## 5.8 The symmetric inner product

In this subsection we define, for symmetric functions, a slightly different inner product $\langle\ ,\ \rangle_+$ that has more symmetry than the original inner product $(\ ,\ )_+$. Fortunately, the difference between these inner products is only a factor of $W_0(t,t_n)$, which makes the reduction relations of the previous subsection even simpler. The symmetric inner product $\langle\ ,\ \rangle_+$ has another very useful advantage: in terms of the Askey-Wilson parameters $a,b,c,d$, the inner product $\langle\ ,\ \rangle_+$ is completely symmetric in $a,b,c$ and $d$.

Define involutions $\overline{\phantom{x}}\colon \mathbb{K}[X] \to \mathbb{K}[X]$ and ${}^{\sigma}\colon \mathbb{K}[X] \to \mathbb{K}[X]$ and ${}^{t}\colon \mathbb{K}[X] \to \mathbb{K}[X]$ by

$$\begin{aligned}
\bar{f}(x_1,\ldots,x_n;q,t,t_0^{\frac12},u_0^{\frac12},t_n^{\frac12},u_n^{\frac12}) &= f(x_1^{-1},\ldots,x_n^{-1};q^{-1},t^{-1},t_0^{-\frac12},u_0^{-\frac12},t_n^{-\frac12},u_n^{-\frac12}),\\
f^{\sigma}(x_1,\ldots,x_n;q,t,t_0^{\frac12},u_0^{\frac12},t_n^{\frac12},u_n^{\frac12}) &= f(x_1^{-1},\ldots,x_n^{-1};q,t,t_0^{\frac12},u_0^{\frac12},t_n^{\frac12},u_n^{\frac12}),\\
f^{t}(x_1,\ldots,x_n;q,t,t_0^{\frac12},u_0^{\frac12},t_n^{\frac12},u_n^{\frac12}) &= f(x_1,\ldots,x_n;q^{-1},t^{-1},t_0^{-\frac12},u_0^{-\frac12},t_n^{-\frac12},u_n^{-\frac12}).
\end{aligned}$$

Let $\kappa_{w_0}^{X^{-1}}$ be as given in (2.13) and define

$$\nabla_{CC} = \Delta_{CC}^{+}\kappa_{w_0}^{X^{-1}}. \tag{5.7}$$

Then define a new scalar product $\langle\ ,\ \rangle_+\colon \mathbb{K}[X]\times\mathbb{K}[X]\to\mathbb{K}$ by

$$\langle f_1, f_2\rangle_+ = \frac{1}{2^n n!}\mathrm{ct}\left(f_1\overline{f_2}\frac{1}{\nabla_{CC}}\right), \qquad \text{where}\quad 2^n n! = \mathrm{Card}(W_{\mathrm{fin}}). \tag{5.8}$$

The following result provides a comparison of $(\ ,\ )_+$ and $\langle\ ,\ \rangle_+$ as inner products on symmetric polynomials.

**Proposition 5.9.** *Let $f,g\in\mathbb{K}[x_1^{\pm},\ldots,x_n^{\pm1}]^{W_{\mathrm{fin}}}$. Then*

$$\langle f,g\rangle_+ = \frac{1}{W_0(t,t_n)}(f,g^t)_+.$$

*Proof.* Let $f,g\in\mathbb{K}[x_1^{\pm},\ldots,x_n^{\pm1}]^{W_{\mathrm{fin}}}$. Since $f$, $\bar{g}^t$ and $\nabla_{CC}$ are all invariant under the action of $W_{\mathrm{fin}}$ then

$$\begin{aligned}
\langle f,g\rangle_+ = \frac{1}{2^n n!}\mathrm{ct}\left(fg^{\sigma}\frac{1}{\nabla_{CC}}\right) &= \frac{1}{2^n n!\cdot W_0(t,t_n)}\mathrm{ct}\left(f\bar{g}^t\frac{1}{\nabla_{CC}}W_0(t,t_n)\right)\\
&= \frac{1}{2^n n!\cdot W_0(t,t_n)}\mathrm{ct}\left(f\bar{g}^t\frac{1}{\nabla_{CC}}\left(\sum_{w\in W_{\mathrm{fin}}} w(\kappa_{w_0}^{X^{-1}})\right)\right)\\
&= \frac{1}{2^n n!\cdot W_0(t,t_n)}\mathrm{ct}\left(\sum_{w\in W_{\mathrm{fin}}} w\left(f\bar{g}^t\frac{1}{\nabla_{CC}}\kappa_{w_0}^{X^{-1}}\right)\right)\\
&= \frac{1}{2^n n!\cdot W_0(t,t_n)}\mathrm{ct}\left(\sum_{w\in W_{\mathrm{fin}}} w\left(f\bar{g}^t\frac{1}{\Delta_{CC}^{+}}\right)\right)\\
&= \frac{1}{W_0(t,t_n)}\mathrm{ct}\left(f\bar{g}^t\frac{1}{\Delta_{CC}^{+}}\right) = \frac{1}{W_0(t,t_n)}(f,g^t)_+,
\end{aligned}$$

where the third equality uses Proposition 4.3. $\square$

The following corollary of Proposition 5.8 records the norm comparisons in terms of the symmetric inner product $\langle\ ,\ \rangle_+$.

**Corollary 5.10.** *Let $\lambda\in(\mathbb{Z}^n_{\geq 0})^+$. Then*

$$\begin{aligned}
\frac{\langle P_\lambda(q,t^{\pm}),P_\lambda(q,t^{\pm})\rangle_{\pm}}{\langle P_{\lambda+\omega}(q,t^{+}),P_{\lambda+\omega}(q,t^{+})\rangle_{+}} &= t_n^{\ell_d(w_0)}\mathrm{ev}^t_{\lambda+\omega}\left(\frac{\kappa^{\pm}_{v_{\lambda+\omega}}}{\kappa^{+}_{v_{\lambda+\omega}}}\right)\\
\frac{\langle P_\lambda(q,t^{\mp}),P_\lambda(q,t^{\mp})\rangle_{\mp}}{\langle P_{\lambda+\pi}(q,t^{+}),P_{\lambda+\pi}(q,t^{+})\rangle_{+}} &= t^{\ell_s(w_0)}\mathrm{ev}^t_{\lambda+\pi}\left(\frac{\kappa^{\mp}_{v_{\lambda+\pi}}}{\kappa^{+}_{v_{\lambda+\pi}}}\right)\\
\frac{\langle P_\lambda(q,t^{-}),P_\lambda(q,t^{-})\rangle_{-}}{\langle P_{\lambda+\rho}(q,t^{+}),P_{\lambda+\rho}(q,t^{+})\rangle_{+}} &= t^{\ell_s(w_0)}t_n^{\ell_d(w_0)}\mathrm{ev}^t_{\lambda+\rho}\left(\frac{\kappa^{-}_{w_0}}{\kappa^{+}_{w_0}}\right).
\end{aligned}$$

**Remark 5.11.** Using Remark 5.1 then

$$\nabla_{CC} = \nabla^{(1)}\nabla^{(2)}, \qquad \text{where} \quad \nabla^{(2)} = \prod_{1\le i<j\le n} \frac{(tx_ix_j^{-1};q)_\infty (tx_ix_j;q)_\infty (tx_i^{-1}x_j;q)_\infty (tx_i^{-1}x_j^{-1};q)_\infty}{(x_ix_j^{-1};q)_\infty (x_ix_j;q)_\infty (x_i^{-1}x_j;q)_\infty (x_i^{-1}x_j^{-1};q)_\infty}$$

and

$$\nabla^{(1)} = \prod_{i=1}^n \frac{(cx_i;q)_\infty (cx_i^{-1};q)_\infty (dx_i;q)_\infty (dx_i^{-1};q)_\infty (ax_i;q)_\infty (ax_i^{-1};q)_\infty (bx_i;q)_\infty (bx_i^{-1};q)_\infty}{(x_i^2;q)_\infty (x_i^{-2};q)_\infty}.$$

This formula shows that $\nabla_{CC}$, and thus $\langle\ ,\ \rangle_+$ and $P_\lambda(q,t^+)$, are completely symmetric in the Askey-Wilson parameters $a, b, c$ and $d$. □

**Remark 5.12.** Using Remark 2.5,

$$\frac{\kappa^\pm_{v_{\lambda+\omega}}}{\kappa^+_{v_{\lambda+\omega}}} = \frac{\kappa^\pm_{v_\omega}}{\kappa^+_{v_\omega}} = \prod_{i=1}^n \frac{\kappa^{Y^{-1}}_{\varepsilon_i}}{\kappa^Y_{\varepsilon_i}} = \frac{\kappa^\pm_{w_0}}{\kappa^+_{w_0}} \qquad \text{and} \qquad \frac{\kappa^\mp_{v_{\lambda+\pi}}}{\kappa^+_{v_{\lambda+\pi}}} = \frac{\kappa^\mp_{v_\pi}}{\kappa^+_{v_\pi}} = \prod_{1\le i<j\le n} \frac{\kappa^{Y^{-1}}_{\varepsilon_i-\varepsilon_j}\kappa^{Y^{-1}}_{\varepsilon_i+\varepsilon_j}}{\kappa^Y_{\varepsilon_i-\varepsilon_j}\kappa^Y_{\varepsilon_i+\varepsilon_j}} = \frac{\kappa^\mp_{w_0}}{\kappa^+_{w_0}}$$

and

$$\frac{\kappa^\pm_{v_\omega}}{\kappa^+_{v_\omega}} \cdot \frac{\kappa^\mp_{v_\omega}}{\kappa^+_{v_\omega}} = \frac{\kappa^-_{w_0}}{\kappa^+_{w_0}} = \left(\prod_{i=1}^n \frac{\kappa^{Y^{-1}}_{\varepsilon_i}}{\kappa^Y_{\varepsilon_i}}\right)\left(\prod_{1\le i<j\le n} \frac{\kappa^{Y^{-1}}_{\varepsilon_i-\varepsilon_j}\kappa^{Y^{-1}}_{\varepsilon_i+\varepsilon_j}}{\kappa^Y_{\varepsilon_i-\varepsilon_j}\kappa^Y_{\varepsilon_i+\varepsilon_j}}\right)$$

□

## 5.9 The norm formula

In this section we use the recursive relations of Corollary 5.10 to derive a $c$-function formula for $\langle P_\lambda(q,t^+), P_\lambda(q,t^+)\rangle_+$.

Introduce $Y$-versions of the $c$-function products from Remark 5.1 as follows:

$$\begin{aligned}
\Delta^Y_s(t^+)\Delta^Y_{0,s}(t^+) &= \prod_{1\le i<j\le n} \frac{(tY_iY_j^{-1};q)_\infty (tY_iY_j;q)_\infty}{(Y_iY_j^{-1};q)_\infty (Y_iY_j;q)_\infty} \\
\Delta^Y_d(t^+)\Delta^Y_{0,d}(t^+) &= \prod_{i=1}^n \frac{(t_n^{\frac12}t_0^{\frac12}Y_i;q)_\infty (-t_n^{\frac12}t_0^{-\frac12}Y_i;q)_\infty}{(Y_i^2;q^2)_\infty} \\
\Delta^Y_g(t^+) &= \prod_{i=1}^n \frac{(u_n^{\frac12}u_0^{\frac12}q^{\frac12}Y_i;q)_\infty (-u_n^{\frac12}u_0^{-\frac12}q^{\frac12}Y_i;q)_\infty}{(qY_i^2;q^2)_\infty} \\
\Delta^{Y^{-1}}_s((t^+)^{-1}) &= \prod_{1\le i<j\le n} \frac{(t^{-1}qY_i^{-1}Y_j;q)_\infty (t^{-1}qY_i^{-1}Y_j^{-1};q)_\infty}{(qY_i^{-1}Y_j;q)_\infty (qY_i^{-1}Y_j^{-1};q)_\infty} \\
\Delta^{Y^{-1}}_d((t^+)^{-1}) &= \prod_{i=1}^n \frac{(t_n^{-\frac12}t_0^{-\frac12}qY_i^{-1};q)_\infty (-t_n^{-\frac12}t_0^{\frac12}qY_i^{-1};q)_\infty}{(q^2Y_i^{-2};q^2)_\infty} \\
\Delta^{Y^{-1}}_g((t^+)^{-1}) &= \prod_{i=1}^n \frac{(u_n^{-\frac12}u_0^{-\frac12}q^{\frac12}Y_i^{-1};q)_\infty (-u_n^{-\frac12}u_0^{\frac12}q^{\frac12}Y_i^{-1};q)_\infty}{(qY_i^{-2};q^2)_\infty}.
\end{aligned}$$

Define homomorphisms $\mathrm{ev}_{q^\lambda t^\rho}\colon \mathbb{K}[Y] \to \mathbb{K}$ and $\mathrm{ev}_{q^{-\lambda}t^{-\rho}}\colon \mathbb{K}[Y] \to \mathbb{K}$ by

$$\mathrm{ev}_{q^\lambda t^\rho}(Y_i) = q^{\lambda_i}t^{n-i}t_0^{\frac12}t_n^{\frac12} \qquad \text{and} \qquad \mathrm{ev}_{q^{-\lambda}t^{-\rho}}(Y_i) = q^{-\lambda_i}t^{-(n-i)}t_0^{-\frac12}t_n^{-\frac12} = \mathrm{ev}_{q^\lambda t^\rho}(Y_i^{-1}).$$

The correspondence between the parameters $q, t, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}$ and the Askey-Wilson parameters is given by

$$q = q, \quad a = q^{\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{\frac{1}{2}}, \quad b = -q^{\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{-\frac{1}{2}}, \quad c = t_n^{\frac{1}{2}} u_n^{\frac{1}{2}}, \quad d = -t_n^{\frac{1}{2}} u_n^{-\frac{1}{2}}.$$

To control the spacing of the formulas introduce a notation

$$(z_1, z_2, \ldots, z_k; q)_\infty = (z_1; q)_\infty (z_2; q)_\infty \cdots (z_k; q)_\infty \tag{5.9}$$

Then $\mathrm{ev}_{q^\lambda t^\rho}(\Delta_d^Y(t^+)\Delta_{0,d}^Y(t^+)\Delta_g^Y(t^+))$ is equal to

$$\begin{aligned}
&\prod_{i=1}^n \frac{(t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} q^{\lambda_i} t^{n-i} t_n^{\frac{1}{2}} t_0^{\frac{1}{2}}, -t_n^{\frac{1}{2}} t_0^{-\frac{1}{2}} q^{\lambda_i} t^{n-i} t_n^{\frac{1}{2}} t_0^{\frac{1}{2}}, u_n^{\frac{1}{2}} u_0^{\frac{1}{2}} q^{\frac{1}{2}} q^{\lambda_i} t^{n-i} t_n^{\frac{1}{2}} t_0^{\frac{1}{2}}, -u_n^{\frac{1}{2}} u_0^{-\frac{1}{2}} q^{\frac{1}{2}} q^{\lambda_i} t^{n-i} t_n^{\frac{1}{2}} t_0^{\frac{1}{2}}; q)_\infty}{(q^{2\lambda_i} t^{2(n-i)} t_n t_0, q q^{2\lambda_i} t^{2(n-i)} t_n t_0; q^2)_\infty} \\
&= \prod_{i=1}^n \frac{(t_n t_0 q^{\lambda_i} t^{n-i}, -t_n q^{\lambda_i} t^{n-i}, t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} u_n^{\frac{1}{2}} u_0^{\frac{1}{2}} q^{\frac{1}{2}} q^{\lambda_i} t^{n-i}, -t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} u_n^{\frac{1}{2}} u_0^{-\frac{1}{2}} q^{\frac{1}{2}} q^{\lambda_i} t^{n-i}; q)_\infty}{(q^{2\lambda_i} t^{2(n-i)} t_n t_0; q)_\infty} \\
&= \prod_{i=1}^n \frac{(q^{-1} abcd q^{\lambda_i} t^{n-i}, cd q^{\lambda_i} t^{n-i}, ac q^{\lambda_i} t^{n-i}, bc q^{\lambda_i} t^{n-i}; q)_\infty}{(q^{-1} abcd q^{2\lambda_i} t^{2(n-i)}; q)_\infty}
\end{aligned} \tag{5.10}$$

and $\mathrm{ev}_{q^{-\lambda} t^{-\rho}}(\Delta_d^{Y^{-1}}((t^+)^{-1})\Delta_g^{Y^{-1}}((t^+)^{-1})$ is equal to

$$\begin{aligned}
&\prod_{i=1}^n \frac{(t_n^{-\frac{1}{2}} t_0^{-\frac{1}{2}} q q^{\lambda_i} t^{n-i} t_n^{\frac{1}{2}} t_0^{\frac{1}{2}}, -t_n^{-\frac{1}{2}} t_0^{\frac{1}{2}} q q^{\lambda_i} t^{n-i} t_n^{\frac{1}{2}} t_0^{\frac{1}{2}}, u_n^{-\frac{1}{2}} u_0^{-\frac{1}{2}} q^{\frac{1}{2}} q^{\lambda_i} t^{n-i} t_n^{\frac{1}{2}} t_0^{\frac{1}{2}}, -u_n^{-\frac{1}{2}} u_0^{\frac{1}{2}} q^{\frac{1}{2}} q^{\lambda_i} t^{n-i} t_n^{\frac{1}{2}} t_0^{\frac{1}{2}}; q)_\infty}{(q^2 q^{2\lambda_i} t^{2(n-i)} t_n t_0, q q^{2\lambda_i} t^{2(n-i)} t_n t_0; q^2)_\infty} \\
&= \prod_{i=1}^n \frac{(q q^{\lambda_i} t^{n-i}, -t_0 q q^{\lambda_i} t^{n-i}, t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} u_n^{-\frac{1}{2}} u_0^{-\frac{1}{2}} q^{\frac{1}{2}} q^{\lambda_i} t^{n-i}, -t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} u_n^{-\frac{1}{2}} u_0^{\frac{1}{2}} q^{\frac{1}{2}} q^{\lambda_i} t^{n-i}; q)_\infty}{(q q^{2\lambda_i} t^{2(n-i)} t_n t_0; q)_\infty} \\
&= \prod_{i=1}^n \frac{(q q^{\lambda_i} t^{n-i}, ab q^{\lambda_i} t^{n-i}, bd q^{\lambda_i} t^{n-i}, ad q^{\lambda_i} t^{n-i}; q)_\infty}{(abcd q^{2\lambda_i} t^{2(n-i)}; q)_\infty}.
\end{aligned} \tag{5.11}$$

**Theorem 5.13.** *Define $N_\lambda(q, t^+)$ to be the product*

$$N_\lambda(q, t^+) = \mathrm{ev}_{q^\lambda t^\rho}\left(\frac{1}{\Delta_g^Y(t^+)\Delta_s^Y(t^+)\Delta_d^Y(t^+)\Delta_{0,s}^Y(t^+)\Delta_{0,d}^Y(t^+)}\right) \cdot \mathrm{ev}_{q^{-\lambda} t^{-\rho}}\left(\frac{1}{\Delta_g^{Y^{-1}}((t^+)^{-1})\Delta_s^{Y^{-1}}((t^+)^{-1})\Delta_d^{Y^{-1}}((t^+)^{-1})}\right).$$

*Then*

$$\langle P_\lambda(q, t^+), P_\lambda(q, t^+)\rangle_+ = N_\lambda(q, t^+).$$

*Proof.* The proof is accomplished by verifying the following properties

(a) $N_\lambda(q, 1, 1, 1, 1, 1) = 1$,

(b) $N_\lambda(q, t^+)$ is symmetric in the parameters $a, b, c, d$,

(c) $N_\lambda(q, t^+)$ satisfies the recursions

$$\frac{N_\lambda(q, t^\pm)}{N_{\lambda+\omega}(q, t^+)} = t_n^n \mathrm{ev}_{\lambda+\omega}^t\left(\frac{\kappa_{v_{\lambda+\omega}}^\pm}{\kappa_{v_{\lambda+\omega}}^+}\right) \qquad \text{and} \qquad \frac{N_\lambda(q, t^\mp)}{N_{\lambda+\pi}(q, t^+)} = t^{n(n-1)} \mathrm{ev}_{\lambda+\pi}^t\left(\frac{\kappa_{v_{\lambda+\pi}}^\mp}{\kappa_{v_{\lambda+\pi}}^+}\right)$$

Property (a) follows from the fact that $\Delta_d^Y(1,1,1,1,1)$, $\Delta_{0,d}^Y(1,1,1,1,1)$, $\Delta_g^Y(1,1,1,1,1)$, $\Delta_d^{Y^{-1}}(1,1,1,1,1)$ and $\Delta_g^{Y^{-1}}(1,1,1,1,1)$ are all equal to 1.

Property (b) follows from the fact that the product of the expressions in (5.10) and (5.11) is symmetric in the parameters $a,b,c,d$.

Property (c) Changing from the parameters $t^+ = (t, t_0^{\frac12}, u_0^{\frac12}, t_n^{\frac12}, u_n^{\frac12})$ to the parameters $t^\pm = (t, t_0^{\frac12}, u_0^{\frac12}, qt_n^{\frac12}, u_n^{\frac12})$ replaces $t_n^{\frac12}$ by $qt_n^{\frac12}$ so that $c$ gets replaced by $qc$ and $d$ gets replaced by $qd$ and $a$ and $b$ and $t$ stay fixed. Write

$$N_\lambda^{gd}(q,t^+) = \mathrm{ev}_{q^\lambda t^\rho}\left(\frac{1}{\Delta_g^Y(t^+)\Delta_d^Y(t^+)\Delta_{0,d}^Y(t^+)}\right)\cdot \mathrm{ev}_{q^{-\lambda}t^{-\rho}}\left(\frac{1}{\Delta_g^{Y^{-1}}((t^+)^{-1})\Delta_d^{Y^{-1}}((t^+)^{-1})}\right),$$

$$N_\lambda^{s}(q,t^+) = \mathrm{ev}_{q^\lambda t^\rho}\left(\frac{1}{\Delta_s^Y(t^+)\Delta_{0,s}^Y(t^+)}\right)\cdot \mathrm{ev}_{q^{-\lambda}t^{-\rho}}\left(\frac{1}{\Delta_s^{Y^{-1}}((t^+)^{-1})}\right),$$

so that $N_\lambda(q,t^+) = N_\lambda^{gd}(q,t^+)N_\lambda^s(q,t^+)$. Using the notation of (5.9),

$$N_\lambda^{gd}(q,t^\pm) = \prod_{i=1}^n \frac{(q^2abcdq^{2\lambda_i}t^{2(n-i)}, q^2q^{-1}abcdq^{2\lambda_i}t^{2(n-i)};q)_\infty}{\left(\begin{matrix} qq^{\lambda_i}t^{n-i}, q^2q^{-1}abcdq^{\lambda_i}t^{n-i}, abq^{\lambda_i}t^{n-i}, qacq^{\lambda_i}t^{n-i}, \\ qadq^{\lambda_i}t^{n-i}, qbcq^{\lambda_i}t^{n-i}, qbdq^{\lambda_i}t^{n-i}, q^2cdq^{\lambda_i}t^{n-i}\end{matrix};q\right)_\infty}$$

and

$$N_{\lambda+\omega}^{gd}(q,t^+) = \prod_{i=1}^n \frac{(abcdq^{2(\lambda_i+1)}t^{2(n-i)}, q^{-1}abcdq^{2(\lambda_i+1)}t^{2(n-i)};q)_\infty}{\left(\begin{matrix} qq^{\lambda_i+1}t^{n-i}, q^{-1}abcdq^{\lambda_i+1}t^{n-i}, abq^{\lambda_i+1}t^{n-i}, acq^{\lambda_i+1}t^{n-i}, \\ adq^{\lambda_i+1}t^{n-i}, bcq^{\lambda_i+1}t^{n-i}, bdq^{\lambda_i+1}t^{n-i}, cdq^{\lambda_i+1}t^{n-i}\end{matrix};q\right)_\infty}$$

and

$$\frac{N_\lambda^{gd}(q,t^\pm)}{N_{\lambda+\omega}^{gd}(q,t^+)} = \prod_{i=1}^n \frac{(1-abcdq^{\lambda_i}t^{n-i})(1-cdq^{\lambda_i+1}t^{n-i})}{(1-qq^{\lambda_i}t^{n-i})(1-abq^{\lambda_i}t^{n-i})} = \prod_{i=1}^n \frac{(1-abcdq^{\lambda_i}t^{n-i})(1-cdq^{\lambda_i+1}t^{n-i})}{(1-q^{\lambda_i+1}t^{n-i})(1-abq^{\lambda_i}t^{n-i})}.$$

Since

$$\mathrm{ev}_{q^\lambda t^\rho}(\Delta_s^Y(t^+)\Delta_{0,s}^Y(t^+)) = \prod_{1\le i<j\le n} \frac{(q^{\lambda_i-\lambda_j}t^{j-i+1};q)_\infty (q^{\lambda_i+\lambda_j}t^{2n-i-j+1}t_0t_n;q)_\infty}{(q^{\lambda_i-\lambda_j}t^{j-i};q)_\infty (q^{\lambda_i+\lambda_j}t^{2n-i-j}t_0t_n;q)_\infty}$$

and

$$\mathrm{ev}_{q^{-\lambda}t^{-\rho}}(\Delta_s^{Y^{-1}}(t^+)^{-1}) = \prod_{1\le i<j\le n} \frac{(q^{\lambda_i-\lambda_j+1}t^{j-i-1};q)_\infty (q^{\lambda_i+\lambda_j+1}t^{2n-i-j-1}t_0t_n;q)_\infty}{(q^{\lambda_i-\lambda_j+1}t^{j-i};q)_\infty (q^{\lambda_i+\lambda_j+1}t^{2n-i-j}t_0t_n;q)_\infty}$$

then

$$N_\lambda^s(q,t^\pm) = \prod_{1\le i<j\le n} \frac{\left(\begin{matrix} q^{\lambda_i-\lambda_j}t^{j-i}, q^{\lambda_i+\lambda_j}t^{2n-i-j}t_0q^2t_n, \\ q^{\lambda_i-\lambda_j+1}t^{j-i}, q^{\lambda_i+\lambda_j+1}t^{2n-i-j}t_0q^2t_n\end{matrix};q\right)_\infty}{\left(\begin{matrix} q^{\lambda_i-\lambda_j}t^{j-i+1}, q^{\lambda_i+\lambda_j}t^{2n-i-j+1}t_0q^2t_n, \\ q^{\lambda_i-\lambda_j+1}t^{j-i-1}, q^{\lambda_i+\lambda_j+1}t^{2n-i-j-1}t_0q^2t_n\end{matrix};q\right)_\infty}$$

and

$$N_{\lambda+\omega}^s(q,t^+) = \prod_{1\le i<j\le n} \frac{\left(\begin{matrix} q^{\lambda_i-\lambda_j}t^{j-i}, q^{\lambda_i+\lambda_j+2}t^{2n-i-j}t_0t_n, \\ q^{\lambda_i-\lambda_j+1}t^{j-i}, q^{\lambda_i+\lambda_j+3}t^{2n-i-j}t_0t_n\end{matrix};q\right)_\infty}{\left(\begin{matrix} q^{\lambda_i-\lambda_j}t^{j-i+1}, q^{\lambda_i+\lambda_j+2}t^{2n-i-j+1}t_0t_n, \\ q^{\lambda_i-\lambda_j+1}t^{j-i-1}, q^{\lambda_i+\lambda_j+3}t^{2n-i-j-1}t_0t_n\end{matrix};q\right)_\infty},$$

since $\omega_i = 1$. So

$$\frac{N^s_\lambda(q,t^\pm)}{N^s_{\lambda+\omega}(q,t^+)} = 1.$$

Since

$$\begin{aligned}
t_n^n \cdot \mathrm{ev}^t_{\lambda+\omega}\left(\frac{\kappa^\pm_{v_{\lambda+\omega}}}{\kappa^+_{v_{\lambda+\omega}}}\right) &= \mathrm{ev}^t_{\lambda+\omega}\left(\prod_{i=1}^n \frac{\kappa^{Y^{-1}}_{\varepsilon_i}}{\kappa^Y_{\varepsilon_i}}\right)\\
&= \prod_{i=1}^n \mathrm{ev}^t_{\lambda+\omega}\left(t_n \frac{(1-t_n^{\frac12}t_0^{\frac12}Y_i^{-1})(1+t_n^{\frac12}t_0^{-\frac12}Y_i^{-1})}{(1-Y_i^{-2})}\frac{(1-Y_i^2)}{(1-t_n^{\frac12}t_0^{\frac12}Y_i)(1+t_n^{\frac12}t_0^{-\frac12}Y_i)}\right)\\
&= \prod_{i=1}^n \mathrm{ev}^t_{\lambda+\omega}\left(\frac{(1-t_n^{\frac12}t_0^{\frac12}Y_i^{-1})(1+t_n^{\frac12}t_0^{-\frac12}Y_i^{-1})}{(1-t_n^{-\frac12}t_0^{-\frac12}Y_i^{-1})(1+t_n^{-\frac12}t_0^{\frac12}Y_i^{-1})}\right)\\
&= \prod_{i=1}^n \left(\frac{(1-t_n^{\frac12}t_0^{\frac12}q^{\lambda_i+1}t^{n-i}t_0^{\frac12}t_n^{\frac12})(1+t_n^{\frac12}t_0^{-\frac12}q^{\lambda_i+1}t^{n-i}t_0^{\frac12}t_n^{\frac12})}{(1-t_n^{-\frac12}t_0^{-\frac12}q^{\lambda_i+1}t^{n-i}t_0^{\frac12}t_n^{\frac12})(1+t_n^{-\frac12}t_0^{\frac12}q^{\lambda_i+1}t^{n-i}t_0^{\frac12}t_n^{\frac12})}\right)\\
&= \prod_{i=1}^n \left(\frac{(1-t_nt_0q^{\lambda_i+1}t^{n-i})(1+t_nq^{\lambda_i+1}t^{n-i})}{(1-q^{\lambda_i+1}t^{n-i})(1+t_0q^{\lambda_i+1}t^{n-i})}\right)\\
&= \prod_{i=1}^n \left(\frac{(1-q^{-1}abcdq^{\lambda_i+1}t^{n-i})(1-cdq^{\lambda_i+1}t^{n-i})}{(1-q^{\lambda_i+1}t^{n-i})(1-q^{-1}abq^{\lambda_i+1}t^{n-i})}\right)\\
&= \prod_{i=1}^n \left(\frac{(1-abcdq^{\lambda_i}t^{n-i})(1-cdq^{\lambda_i+1}t^{n-i})}{(1-q^{\lambda_i+1}t^{n-i})(1-abq^{\lambda_i}t^{n-i})}\right)
\end{aligned}$$

then

$$\frac{N_\lambda(q,t^\pm)}{N_{\lambda+\omega}(q,t^+)} = \frac{N^{gd}_\lambda(q,t^\pm)}{N^{gd}_{\lambda+\omega}(q,t^+)}\cdot\frac{N^s_\lambda(q,t^\pm)}{N^s_{\lambda+\omega}(q,t^+)} = \frac{N^{gd}_\lambda(q,t^\pm)}{N^{gd}_{\lambda+\omega}(q,t^+)}\cdot 1 = t_n^n\cdot \mathrm{ev}^t_{\lambda+\omega}\left(\frac{\kappa^\pm_{v_{\lambda+\omega}}}{\kappa^+_{v_{\lambda+\omega}}}\right).$$

The proof of the second equality in (c) is similar as follows.

$$N^{gd}_\lambda(q,t^\mp) = \prod_{i=1}^n \frac{(q^2abcdq^{2\lambda_i}(qt)^{2(n-i)}, q^2q^{-1}abcdq^{2\lambda_i}(qt)^{2(n-i)};q)_\infty}{\left(\begin{matrix} qq^{\lambda_i}(qt)^{n-i}, q^2q^{-1}abcdq^{\lambda_i}(qt)^{n-i}, abq^{\lambda_i}(qt)^{n-i}, qacq^{\lambda_i}(qt)^{n-i},\\ qadq^{\lambda_i}(qt)^{n-i}, qbcq^{\lambda_i}(qt)^{n-i}, qbdq^{\lambda_i}(qt)^{n-i}, q^2cdq^{\lambda_i}(qt)^{n-i}\end{matrix};q\right)_\infty}$$

and

$$N^{gd}_{\lambda+\pi}(q,t^+) = \prod_{i=1}^n \frac{(abcdq^{2(\lambda_i+n-i)}t^{2(n-i)}, q^{-1}abcdq^{2(\lambda_i+n-i)}t^{2(n-i)};q)_\infty}{\left(\begin{matrix} qq^{\lambda_i+n-i}t^{n-i}, q^{-1}abcdq^{\lambda_i+n-i}t^{n-i}, abq^{\lambda_i+n-i}t^{n-i}, acq^{\lambda_i+n-i}t^{n-i},\\ adq^{\lambda_i+n-i}t^{n-i}, bcq^{\lambda_i+n-i}t^{n-i}, bdq^{\lambda_i+n-i}t^{n-i}, cdq^{\lambda_i+n-i}t^{n-i}\end{matrix};q\right)_\infty}$$

so that

$$\frac{N^{gd}_\lambda(q,t^\mp)}{N^{gd}_\lambda(q,t^+)} = 1.$$

Then

$$N^s_\lambda(q,t^\mp) = \prod_{1\le i<j\le n}\frac{\left(\begin{matrix} q^{\lambda_i-\lambda_j+j-i}t^{j-i}, q^{\lambda_i+\lambda_j+2n-i-j}t^{2n-i-j}t_0t_n,\\ q^{\lambda_i-\lambda_j+j-i+1}t^{j-i}, q^{\lambda_i+\lambda_j+1+2n-i-j}t^{2n-i-j}t_0t_n\end{matrix};q\right)_\infty}{\left(\begin{matrix} q^{\lambda_i-\lambda_j+j-i+1}t^{j-i+1}, q^{\lambda_i+\lambda_j+2n-i-j+1}t^{2n-i-j+1}t_0t_n,\\ q^{\lambda_i-\lambda_j+j-i}t^{j-i-1}, q^{\lambda_i+\lambda_j+2n-i-j}t^{2n-i-j-1}t_0t_n\end{matrix};q\right)_\infty}$$

and

$$N^s_{\lambda+\pi}(q,t^+) = \prod_{1\le i<j\le n} \frac{\left(\begin{matrix} q^{\lambda_i-\lambda_j+j-i}t^{j-i}, q^{\lambda_i+\lambda_j+2n-i-j}t^{2n-i-j}t_0t_n, \\ q^{\lambda_i-\lambda_j+j-i+1}t^{j-i}, q^{\lambda_i+\lambda_j+1+2n-i-j}t^{2n-i-j}t_0t_n \end{matrix} ;q\right)_\infty}{\left(\begin{matrix} q^{\lambda_i-\lambda_j+j-i}t^{j-i+1}, q^{\lambda_i+\lambda_j+2n-i-j}t^{2n-i-j+1}t_0t_n, \\ q^{\lambda_i-\lambda_j+j-i+1}t^{j-i-1}, q^{\lambda_i+\lambda_j+1+2n-i-j}t^{2n-i-j-1}t_0t_n \end{matrix} ;q\right)_\infty},$$

since $\pi_i = n-i$. So

$$\frac{N^s_\lambda(q,t^\mp)}{N^s_{\lambda+\pi}(t^+)} = \prod_{1\le i<j\le n} \frac{(1-q^{\lambda_i-\lambda_j+j-i}t^{j-i+1})(1-q^{\lambda_i+\lambda_j+2n-i-j}t^{2n-i-j+1}t_0t_n)}{(1-q^{\lambda_i-\lambda_j+j-i}t^{j-i-1})(1-q^{\lambda_i+\lambda_j+2n-i-j}t^{2n-i-j-1}t_0t_n)}$$

and

$$\frac{N_\lambda(q,t^\mp)}{N_{\lambda+\pi}(t^+)} = \frac{N^{gd}_\lambda(q,t^\mp)}{N^{gd}_{\lambda+\pi}(t^+)}\cdot\frac{N^s_\lambda(q,t^\mp)}{N^s_{\lambda+\pi}(t^+)} = 1\cdot\frac{N^s_\lambda(q,t^\mp)}{N^s_{\lambda+\pi}(t^+)} = t^{n(n-1)}\mathrm{ev}^t_{\lambda+\pi}\left(\frac{\kappa^\mp_{v_{\lambda+\pi}}}{\kappa^+_{v_{\lambda+\pi}}}\right),$$

since

$$\begin{aligned}
t^{n(n-1)}\mathrm{ev}^t_{\lambda+\pi}\left(\frac{\kappa^\mp_{v_{\lambda+\pi}}}{\kappa^+_{v_{\lambda+\pi}}}\right) &= \mathrm{ev}^t_{\lambda+\pi}\left(\prod_{1\le i<j\le n} t^2\frac{\kappa^{Y^{-1}}_{\varepsilon_i-\varepsilon_j}\kappa^{Y^{-1}}_{\varepsilon_i+\varepsilon_j}}{\kappa^{Y}_{\varepsilon_i-\varepsilon_j}\kappa^{Y}_{\varepsilon_i+\varepsilon_j}}\right)\\
&= \mathrm{ev}^t_{\lambda+\pi}\left(\prod_{1\le i<j\le n} t^2\frac{(1-tY_i^{-1}Y_j)(1-tY_i^{-1}Y_j^{-1})}{(1-Y_i^{-1}Y_j)(1-Y_i^{-1}Y_j^{-1})}\frac{(1-Y_iY_j^{-1})(1-Y_iY_j)}{(1-tY_iY_j^{-1})(1-tY_iY_j)}\right)\\
&= \mathrm{ev}^t_{\lambda+\pi}\left(\prod_{1\le i<j\le n} \frac{(1-tY_i^{-1}Y_j)(1-tY_i^{-1}Y_j^{-1})}{(1-t^{-1}Y_i^{-1}Y_j)(1-t^{-1}Y_i^{-1}Y_j^{-1})}\right)\\
&= \prod_{1\le i<j\le n} \frac{(1-tq^{\lambda_i+(n-i)-\lambda_j-(n-j)}t^{-(n-j)+(n-i)})(1-tq^{\lambda_i+(n-i)+\lambda_j+(n-j)}t^{(n-i)+(n-j)}t_0t_n)}{(1-t^{-1}q^{\lambda_i+(n-i)-\lambda_j-(n-j)}t^{-(n-j)+(n-i)})(1-t^{-1}q^{\lambda_i+(n-i)+\lambda_j+(n-j)}t^{(n-i)+(n-j)}t_0t_n)}\\
&= \prod_{1\le i<j\le n} \frac{(1-tq^{\lambda_i-\lambda_j+j-i}t^{j-i})(1-tq^{\lambda_i+\lambda_j+2n-i-j)}t^{2n-i-j}t_0t_n)}{(1-t^{-1}q^{\lambda_i-\lambda_j+j-i}t^{j-i})(1-t^{-1}q^{\lambda_i+\lambda_j+2n-i-j}t^{2n-i-j}t_0t_n)}.
\end{aligned}$$

□

## 5.10 The constant term

To get the constant term of $\dfrac{1}{\Delta^+_{CC}}$ specialize $\lambda = 0$. Then

$$N^{gd}_0(t^+) = \prod_{i=1}^n \frac{(abcdt^{2(n-i)}, q^{-1}abcdt^{2(n-i)};q)_\infty}{(qt^{n-i}, q^{-1}abcdt^{n-i}, abt^{n-i}, act^{n-i}, adt^{n-i}, bct^{n-i}, bdt^{n-i}, cdt^{n-i};q)_\infty}$$

and

$$N^s_0(t^+) = \prod_{1\le i<j\le n} \frac{(t^{j-i}, t^{2n-i-j}t_0t_n, qt^{j-i}, qt^{2n-i-j}t_0t_n;q)_\infty}{(tt^{j-i}, tt^{2n-i-j}t_0t_n, qt^{-1}t^{j-i}, qt^{-1}t^{2n-i-j}t_0t_n;q)_\infty},$$

so that

$$W_0(t,t_n)\mathrm{ct}\left(\frac{1}{\Delta^+_{CC}}\right) = W_0(t,t_n)(1,1)_+ = \langle 1,1\rangle_+ = N_0(t^+) = N^{gd}_0(t^+)N^s_0(t^+).$$

# 6 Appendix: Examples for $n = 2$

## 6.1 The finite Weyl group and the roots in $S_0^+$

The finite Weyl group is $W_{\text{fin}} = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, s_1s_2s_1s_2\}$ where

$$s_1^2 = s_2^2 = 1, \quad s_1s_2s_1s_2 = s_2s_1s_2s_1.$$

Then

$$S_{0,d}^+ = \{\varepsilon_1, \varepsilon_2\} \qquad \text{and} \qquad S_{0,s}^+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 + \varepsilon_2\}$$

with $s_{\varepsilon_1} = s_1s_2s_1$, $s_{\varepsilon_2} = s_2$, $s_{\varepsilon_1-\varepsilon_2} = s_1$, $s_{\varepsilon_1+\varepsilon_2} = s_2s_1s_2$.

## 6.2 Hecke symmetrizers

The Hecke symmetrizers are

$$\begin{aligned}
\varepsilon_+ &= T_1T_2T_1T_2 + t^{-\frac{1}{2}}T_2T_1T_2 + t_n^{-\frac{1}{2}}T_1T_2T_1 \\
&\qquad + t^{-\frac{1}{2}}t_n^{-\frac{1}{2}}T_1T_2 + t^{-\frac{1}{2}}t_n^{-\frac{1}{2}}T_2T_1 + t^{-1}t_n^{-\frac{1}{2}}T_2 + t^{-\frac{1}{2}}t_n^{-1}T_1 + t^{-1}t_n^{-1}, \\
\varepsilon_\pm &= T_1T_2T_1T_2 + t^{-\frac{1}{2}}T_2T_1T_2 - t_n^{\frac{1}{2}}T_1T_2T_1 \\
&\qquad - t^{-\frac{1}{2}}t_n^{\frac{1}{2}}T_1T_2 - t^{-\frac{1}{2}}t_n^{\frac{1}{2}}T_2T_1 - t^{-1}t_n^{\frac{1}{2}}T_2 + t^{-\frac{1}{2}}t_nT_1 + t^{-1}t_n, \\
\varepsilon_\mp &= T_1T_2T_1T_2 - t^{\frac{1}{2}}T_2T_1T_2 + t_n^{-\frac{1}{2}}T_1T_2T_1 \\
&\qquad - t^{\frac{1}{2}}t_n^{-\frac{1}{2}}T_1T_2 - t^{\frac{1}{2}}t_n^{-\frac{1}{2}}T_2T_1 + tt_n^{-\frac{1}{2}}T_2 - t^{\frac{1}{2}}t_n^{-1}T_1 + tt_n^{-1}, \\
\varepsilon_- &= T_1T_2T_1T_2 - t^{\frac{1}{2}}T_2T_1T_2 - t_n^{\frac{1}{2}}T_1T_2T_1 + t^{\frac{1}{2}}t_n^{\frac{1}{2}}T_1T_2 + t^{\frac{1}{2}}t_n^{\frac{1}{2}}T_2T_1 - tt_n^{\frac{1}{2}}T_2 - t^{\frac{1}{2}}t_nT_1 + tt_n.
\end{aligned}$$

Since

$$\varepsilon_\pm = (T_1 + t^{-\frac{1}{2}})(T_2T_1T_2 - t_n^{\frac{1}{2}}T_2T_1 - t^{-\frac{1}{2}}t_n^{\frac{1}{2}}T_2 + t^{-\frac{1}{2}}t_n) = (T_1 + t^{-\frac{1}{2}})(T_2T_1 - t^{-\frac{1}{2}}t_n^{\frac{1}{2}})(T_2 - t_n^{\frac{1}{2}})$$

and $(T_1 - t^{\frac{1}{2}})(T_1 + t^{-\frac{1}{2}}) = 0$ then $T_1\varepsilon_\pm = t^{\frac{1}{2}}\varepsilon_\pm$. Similarly, since

$$\varepsilon_\pm = (T_2 - t_n^{\frac{1}{2}})(T_1T_2T_1 + t^{-\frac{1}{2}}T_1T_2 - t^{-\frac{1}{2}}t_n^{\frac{1}{2}}T_1 - t^{-1}t_n^{\frac{1}{2}}) = (T_2 - t_n^{\frac{1}{2}})(T_1T_2 - t^{-\frac{1}{2}}t_n^{\frac{1}{2}})(T_1 + t^{-\frac{1}{2}})$$

and $(T_2 + t_n^{-\frac{1}{2}})(T_2 - t_n^{\frac{1}{2}}) = 0$ then $T_2\varepsilon_\pm = -t_n^{-\frac{1}{2}}\varepsilon_\pm$. Then

$$\begin{aligned}
\varepsilon_\pm^2 &= (tt_n^{-1} + t_n^{-1} + t + 1 + 1 + t^{-1} + t_n + t^{-1}t_n)\varepsilon_\pm \\
&= t^{-1}t_2(t^2t_2^{-2} + tt_2^{-2} + t^2t_2^{-1} + tt_2^{-1} + tt_2^{-1} + t_2^{-1} + t + 1)\varepsilon_\pm \\
&= t^{-1}t_2W_0(t, t_2^{-1})\varepsilon_\pm.
\end{aligned}$$

since

$$W_0(t, t_n) = 1 + t_2 + t + tt_2 + tt_2 + tt_2^2 + t^2t_2 + t^2t_2^2 = (1 + t_2)(1 + t)(1 + tt_2).$$

### 6.3 $c$-functions

The $c$-functions in $X$ for roots in $S_{0,s}^{+}$ and $S_{0,d}^{+}$ are

$$c_{\varepsilon_1-\varepsilon_2}^{X} = \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} X_1 X_2^{-1}}{1 - X_1 X_2^{-1}}, \qquad c_{\varepsilon_1+\varepsilon_2}^{X} = \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} X_1 X_2}{1 - X_1 X_2},$$

$$c_{\varepsilon_1}^{X} = t_2^{-\frac{1}{2}} \frac{(1 - t_2^{\frac{1}{2}} u_2^{\frac{1}{2}} X_1)(1 + t_2^{\frac{1}{2}} u_2^{-\frac{1}{2}} X_1)}{1 - X_1^2}, \qquad c_{\varepsilon_2}^{X} = t_2^{-\frac{1}{2}} \frac{(1 - t_2^{\frac{1}{2}} u_2^{\frac{1}{2}} X_2)(1 + t_2^{\frac{1}{2}} u_2^{-\frac{1}{2}} X_2)}{1 - X_2^2}.$$

Then

$$c_{\mathrm{dr}}^{X^{-1}} = c_{\varepsilon_1}^{X^{-1}} c_{\varepsilon_2}^{X^{-1}} = t_2^{-\frac{2}{2}} \frac{(1 - t_2^{\frac{1}{2}} u_2^{\frac{1}{2}} X_1^{-1})(1 + t_2^{\frac{1}{2}} u_2^{-\frac{1}{2}} X_1^{-1})}{(1 - X_1^{-2})} \frac{(1 - t_2^{\frac{1}{2}} u_2^{\frac{1}{2}} X_2^{-1})(1 + t_2^{\frac{1}{2}} u_2^{-\frac{1}{2}} X_2^{-1})}{(1 - X_2^{-2})}$$

and

$$c_{\mathrm{st}}^{X^{-1}} = c_{\varepsilon_1-\varepsilon_2}^{X^{-1}} c_{\varepsilon_1+\varepsilon_2}^{X^{-1}} = \frac{(t^{-\frac{1}{2}} - t^{\frac{1}{2}} X_1^{-1} X_2)}{(1 - X_1^{-1} X_2)} \frac{(t^{-\frac{1}{2}} - t^{\frac{1}{2}} X_1^{-1} X_2^{-1})}{(1 - X_1^{-1} X_2^{-1})}.$$

### 6.4 Weyl denominators

Since $\omega = (1,1)$, $\pi = (1,0)$ and $\rho = (2,1)$ then

$$x^{\omega} = x_1 x_2, \qquad x^{\pi} = x_1, \qquad x^{\rho} = x_1^2 x_2.$$

Since

$$\begin{aligned}
e_{\pm} &= s_1 s_2 s_1 s_2 - s_1 s_2 s_1 + s_2 s_1 s_2 - s_1 s_2 - s_2 s_1 + s_1 - s_2 + 1, \\
e_{\mp} &= s_1 s_2 s_1 s_2 + s_1 s_2 s_1 - s_2 s_1 s_2 - s_1 s_2 - s_2 s_1 - s_1 + s_2 + 1, \\
e_{-} &= s_1 s_2 s_1 s_2 - s_1 s_2 s_1 - s_2 s_1 s_2 + s_1 s_2 + s_2 s_1 - s_1 - s_2 + 1,
\end{aligned}$$

then the Weyl denominators for $n = 2$ are

$$\begin{aligned}
a_{\omega}^{\pm} &= \tfrac{1}{2} e_{\pm} x^{\omega} = e_{\pm} x_1 x_2 = x_1 x_2 - x_1 x_2^{-1} + x_1^{-1} x_2 - x_1^{-1} x_2^{-1} = x_1^{-1} x_2^{-1} (1 - x_1^2)(1 - x_2^2), \\
a_{\pi}^{\mp} &= \tfrac{1}{2} e_{\mp} x^{\delta} = e_{\mp} x_1 = x_1 - x_2 - x_2^{-1} + x_1^{-1} = x_1^{-1} (1 - x_1 x_2)(1 - x_1 x_2^{-1}), \\
a_{\rho} &= e_{+} x^{\rho} = e_{+} x_1^2 x_2 = x_1^2 x_2 - x_1 x_2^2 - x_1^2 x_2^{-1} + x_1 x_2^{-2} + x_1^{-2} x_2 - x_1^{-1} x_2^2 - x_1^{-1} x_2^{-2} + x_1^{-2} x_2^{-1} \\
&= x_1^{-2} x_2^{-1} (1 - x_1 x_2)(1 - x_1 x_2^{-1})(1 - x_1^2)(1 - x_2^2) = a_{\omega}^{\pm} a_{\pi}^{\mp},
\end{aligned}$$

and

$$\begin{aligned}
P_0 &= 1, \\
A_{\omega}^{\pm} &= A_{-\varepsilon_1-\varepsilon_2}^{\pm} = x_1^{-1} x_2^{-1} (1 - t_2^{\frac{1}{2}} u_2^{\frac{1}{2}} x_1)(1 + t_2^{\frac{1}{2}} u_2^{-\frac{1}{2}} x_1)(1 - t_2^{\frac{1}{2}} u_2^{\frac{1}{2}} x_2)(1 + t_2^{\frac{1}{2}} u_2^{-\frac{1}{2}} x_2) \\
A_{\pi}^{\mp} &= A_{-\varepsilon_1}^{\mp} = x_1^{-1} (1 - t x_1 x_2^{-1})(1 - t x_1 x_2) = x_1^{-1} - t x_2^{-1} - t x_2 + t^2 x_1, \\
A_{\rho} &= x_1^{-2} x_2^{-1} (1 - t x_1 x_2^{-1})(1 - t x_1 x_2)(1 - t_2^{\frac{1}{2}} u_2^{\frac{1}{2}} x_1)(1 + t_2^{\frac{1}{2}} u_2^{-\frac{1}{2}} x_1)(1 - t_2^{\frac{1}{2}} u_2^{\frac{1}{2}} x_2)(1 + t_2^{\frac{1}{2}} u_2^{-\frac{1}{2}} x_2).
\end{aligned}$$

### 6.5 Examples of Proposition 2.3, 2.4 and 4.4

Since $\chi^+(T_{s_1s_2s_1s_2}) = tt_2$ then

$$\begin{aligned}
tt_n\varepsilon_+ &= (\eta_{s_1s_2s_1s_2} + \eta_{s_1s_2s_1} + \eta_{s_2s_1s_2} + \eta_{s_1s_2} + \eta_{s_2s_1} + \eta_{s_1} + \eta_{s_2} + 1)\kappa^Y_{\varepsilon_1}\kappa^Y_{\varepsilon_2}\kappa^Y_{\varepsilon_1-\varepsilon_2}\kappa^Y_{\varepsilon_1+\varepsilon_2} \\
&= (\eta_{s_2s_1s_2} + \eta_{s_2s_1s_2} + \eta_{s_1s_2} + \eta_{s_2} + 1)\kappa^Y_{\varepsilon_1}\kappa^Y_{\varepsilon_2}\kappa^Y_{\varepsilon_1+\varepsilon_2}(1+\eta_{s_1})\kappa^Y_{\varepsilon_1-\varepsilon_2} \\
&= \begin{pmatrix} \eta_{s_2s_1s_2}\kappa^Y_{\varepsilon_1}\kappa^Y_{\varepsilon_2}\kappa^Y_{\varepsilon_1+\varepsilon_2} + \kappa^Y_{\varepsilon_2}\eta_{s_1s_2}\kappa^Y_{\varepsilon_2}\kappa^Y_{\varepsilon_1+\varepsilon_2} \\ +\kappa^Y_{\varepsilon_1}\kappa^Y_{\varepsilon_1+\varepsilon_2}\eta_{s_2}\kappa^{Y^{-1}}_{\varepsilon_2} + \kappa^Y_{\varepsilon_1}\kappa^Y_{\varepsilon_2}\kappa^Y_{\varepsilon_1+\varepsilon_2} \end{pmatrix} \chi^+(T_{s_1})\varepsilon^+_\omega, \\
tt_n\varepsilon_\pm &= \kappa^{Y^{-1}}_{\varepsilon_1}\kappa^{Y^{-1}}_{\varepsilon_2}(\eta_{s_1s_2s_1s_2} - \eta_{s_1s_2s_1} + \eta_{s_2s_1s_2} - \eta_{s_1s_2} - \eta_{s_2s_1} + \eta_{s_1} - \eta_{s_2} + 1)\kappa^Y_{\varepsilon_1-\varepsilon_2}\kappa^Y_{\varepsilon_1+\varepsilon_2} \\
&= \kappa^{Y^{-1}}_{\varepsilon_1}\kappa^{Y^{-1}}_{\varepsilon_2}(\eta_{s_2s_1s_2} - \eta_{s_1s_2} - \eta_{s_2} + 1)\kappa^Y_{\varepsilon_1+\varepsilon_2}(1+\eta_{s_1})\kappa^Y_{\varepsilon_1-\varepsilon_2} \\
&= \begin{pmatrix} \eta_{s_2s_1s_2}\kappa^Y_{\varepsilon_1}\kappa^Y_{\varepsilon_2}\kappa^Y_{\varepsilon_1+\varepsilon_2} - \kappa^{Y^{-1}}_{\varepsilon_2}\eta_{s_1s_2}\kappa^Y_{\varepsilon_2}\kappa^Y_{\varepsilon_1+\varepsilon_2} \\ -\kappa^{Y^{-1}}_{\varepsilon_1}\kappa^Y_{\varepsilon_1-\varepsilon_2}\eta_{s_2}\kappa^Y_{\varepsilon_2} + \kappa^{Y^{-1}}_{\varepsilon_1}\kappa^{Y^{-1}}_{\varepsilon_2}\kappa^Y_{\varepsilon_1+\varepsilon_2} \end{pmatrix} \chi^+(T_{s_1})\varepsilon^+_\omega, \\
tt_n\varepsilon_\mp &= \kappa^{Y^{-1}}_{\varepsilon_1^\vee-\varepsilon_2^\vee}\kappa^{Y^{-1}}_{\varepsilon_1^\vee+\varepsilon_2^\vee}(\eta_{s_1s_2s_1s_2} + \eta_{s_1s_2s_1} - \eta_{s_2s_1s_2} - \eta_{s_1s_2} - \eta_{s_2s_1} - \eta_{s_1} + \eta_{s_2} + 1)\kappa^Y_{\varepsilon_1^\vee}\kappa^Y_{\varepsilon_2^\vee} \\
&= \kappa^{Y^{-1}}_{\varepsilon_1^\vee-\varepsilon_2^\vee}\kappa^{Y^{-1}}_{\varepsilon_1^\vee+\varepsilon_2^\vee}(\eta_{s_1s_2s_1} - \eta_{s_2s_1} - \eta_{s_1} + 1)\kappa^Y_{\varepsilon_1^\vee}(1+\eta_{s_2})\kappa^Y_{\varepsilon_2^\vee} \\
&= \begin{pmatrix} \eta_{s_1s_2s_1}\kappa^Y_{\varepsilon_1^\vee}\kappa^Y_{\varepsilon_1^\vee-\varepsilon_2^\vee}\kappa^Y_{\varepsilon_1^\vee+\varepsilon_2^\vee} - \kappa^{Y^{-1}}_{\varepsilon_1^\vee-\varepsilon_2^\vee}\eta_{s_2s_1}\kappa^Y_{\varepsilon_1^\vee}\kappa^Y_{\varepsilon_1^\vee-\varepsilon_2^\vee} \\ -\kappa^{Y^{-1}}_{\varepsilon_1^\vee+\varepsilon_2^\vee}\kappa^Y_{\varepsilon_2^\vee}\eta_{s_1}\kappa^Y_{\varepsilon_1^\vee-\varepsilon_2^\vee} + \kappa^{Y^{-1}}_{\varepsilon_1^\vee-\varepsilon_2^\vee}\kappa^{Y^{-1}}_{\varepsilon_1^\vee+\varepsilon_2^\vee}\kappa^Y_{\varepsilon_1^\vee} \end{pmatrix} \cdot \chi^+(T_{s_2})\varepsilon^+_\pi, \\
tt_n\varepsilon_- &= \kappa^{Y^{-1}}_{\varepsilon_1}\kappa^{Y^{-1}}_{\varepsilon_2}\kappa^{Y^{-1}}_{\varepsilon_1-\varepsilon_2}\kappa^{Y^{-1}}_{\varepsilon_1+\varepsilon_2}(\eta_{s_1s_2s_1s_2} - \eta_{s_1s_2s_1} - \eta_{s_2s_1s_2} + \eta_{s_1s_2} + \eta_{s_2s_1} - \eta_{s_1} - \eta_{s_2} + 1) \\
&= \eta_{s_1s_2s_1s_2}\kappa^Y_{\varepsilon_1}\kappa^Y_{\varepsilon_2}\kappa^Y_{\varepsilon_1-\varepsilon_2}\kappa^Y_{\varepsilon_1+\varepsilon_2} - \kappa^{Y^{-1}}_{\varepsilon_2}\eta_{s_1s_2s_1}\kappa^Y_{\varepsilon_1}\kappa^Y_{\varepsilon_1-\varepsilon_2}\kappa^Y_{\varepsilon_1+\varepsilon_2} - \kappa^{Y^{-1}}_{\varepsilon_1-\varepsilon_2}\eta_{s_2s_1s_2}\kappa^Y_{\varepsilon_1}\kappa^Y_{\varepsilon_2}\kappa^Y_{\varepsilon_1+\varepsilon_2} \\
&\quad + \kappa^{Y^{-1}}_{\varepsilon_1}\kappa^{Y^{-1}}_{\varepsilon_1-\varepsilon_2}\eta_{s_1s_2}\kappa^Y_{\varepsilon_2}\kappa^Y_{\varepsilon_1+\varepsilon_2} + \kappa^{Y^{-1}}_{\varepsilon_2}\kappa^{Y^{-1}}_{\varepsilon_1+\varepsilon_2}\eta_{s_2s_1}\kappa^Y_{\varepsilon_1}\kappa^Y_{\varepsilon_1-\varepsilon_2} \\
&\quad - \kappa^{Y^{-1}}_{\varepsilon_2}\kappa^{Y^{-1}}_{\varepsilon_1-\varepsilon_2}\kappa^{Y^{-1}}_{\varepsilon_1+\varepsilon_2}\eta_{s_1}\kappa^Y_{\varepsilon_1-\varepsilon_2} - \kappa^{Y^{-1}}_{\varepsilon_1}\kappa^{Y^{-1}}_{\varepsilon_1-\varepsilon_2}\kappa^{Y^{-1}}_{\varepsilon_1+\varepsilon_2}\eta_{s_2}\kappa^Y_{\varepsilon_2} + \kappa^{Y^{-1}}_{\varepsilon_1}\kappa^{Y^{-1}}_{\varepsilon_2}\kappa^{Y^{-1}}_{\varepsilon_1-\varepsilon_2}\kappa^{Y^{-1}}_{\varepsilon_1+\varepsilon_2}.
\end{aligned}$$

Since

$$\begin{aligned}
\kappa^Y_{s_1} &= \kappa^Y_{\varepsilon_1^\vee-\varepsilon_2^\vee}, & \kappa^Y_{s_1s_2} &= \kappa^Y_{\varepsilon_2^\vee}\kappa^Y_{\varepsilon_1^\vee+\varepsilon_2^\vee}, & \kappa^Y_{s_1s_2s_1} &= \kappa^Y_{\varepsilon_1^\vee-\varepsilon_2^\vee}\kappa^Y_{\varepsilon_1^\vee}\kappa^Y_{\varepsilon_1^\vee+\varepsilon_2^\vee}. \\
\kappa^Y_{s_2} &= \kappa^Y_{\varepsilon_2^\vee}, & \kappa^Y_{s_2s_1} &= \kappa^Y_{\varepsilon_1^\vee-\varepsilon_2^\vee}\kappa^Y_{\varepsilon_1^\vee}, & \kappa^Y_{s_2s_1s_2} &= \kappa^Y_{\varepsilon_2^\vee}\kappa^Y_{\varepsilon_1^\vee+\varepsilon_2^\vee}\kappa^Y_{\varepsilon_1^\vee},
\end{aligned}$$

then

$$\begin{aligned}
A^\pm_\omega &= E_{-\varepsilon_1-\varepsilon_2} - \mathrm{ev}^t_{-\varepsilon_1+\varepsilon_2}(\kappa^{Y^{-1}}_{\varepsilon_2^\vee})E_{-\varepsilon_1+\varepsilon_2} - \mathrm{ev}^t_{\varepsilon_1-\varepsilon_2}(\kappa^{Y^{-1}}_{\varepsilon_2^\vee}\kappa^Y_{\varepsilon_1^\vee-\varepsilon_2^\vee})E_{\varepsilon_1-\varepsilon_2} \\
&\qquad + \mathrm{ev}^t_{\varepsilon_1+\varepsilon_2}(\kappa^{Y^{-1}}_{\varepsilon_1^\vee}\kappa^{Y^{-1}}_{\varepsilon_2^\vee}\kappa^Y_{\varepsilon_1^\vee+\varepsilon_2^\vee})E_{\varepsilon_1+\varepsilon_2}, \qquad \text{and} \\
A^\mp_\pi &= E_{-\varepsilon_1} - \mathrm{ev}^t_{-\varepsilon_2}(\kappa^{Y^{-1}}_{\varepsilon_1^\vee-\varepsilon_2^\vee})E_{-\varepsilon_2} - \mathrm{ev}^t_{\varepsilon_2}(\kappa^{Y^{-1}}_{\varepsilon_1^\vee+\varepsilon_2^\vee}\kappa^Y_{\varepsilon_2^\vee})E_{\varepsilon_2} + \mathrm{ev}^t_{\varepsilon_1}(\kappa^{Y^{-1}}_{\varepsilon_1^\vee-\varepsilon_2^\vee}\kappa^{Y^{-1}}_{\varepsilon_1^\vee+\varepsilon_2^\vee}\kappa^Y_{\varepsilon_1^\vee})E_{\varepsilon_1}.
\end{aligned}$$

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