Clebsch-Gordan coefficients for Macdonald polynomials

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Abstract

In this paper we use the double affine Hecke algebra to compute the Macdonald polynomial products $E_\ell P_m$ and $P_\ell P_m$ for type SL_2 and type GL_2 Macdonald polynomials. Our method follows the ideas of Martha Yip but executes a compression to reduce the sum from $2\cdot 3^{\ell-1}$ signed terms to 2ℓ positive terms. We show that our rule for $P_\ell P_m$ is equivalent to a special case of the Pieri rule of Macdonald. Our method shows that computing $E_\ell \mathbf{1}_0$ and $\mathbf{1}_0 E_\ell \mathbf{1}_0$ in terms of a special basis of the double affine Hecke algebra provides universal compressed formulas for multiplication by E_ℓ and P_ℓ. The formulas for a specific products $E_\ell P_m$ and $P_\ell P_m$ are obtained by evaluating the universal formulas at $t^{-\frac{1}{2}}q^{-\frac{m}{2}}$.

Key words— Macdonald polynomials, symmetric functions, Hecke algebras

1 Introduction

The type SL_2 Macdonald polynomals $P_\ell(x)$ are special cases of the Askey-Wilson polynomlals, sometimes called the q-ultraspherical polynomials (see [Mac03, p.156-7]). The $P_\ell(x)$ are two-parameter q-t-generalizations of the characters of finite dimensional representations of SU_2; i.e. the characters of SU_2 which play a pivotal role in the "standard model" in particle physics and in the analysis of Heisenberg spin chains in mathematical physics. The polynomial representation of the double affine Hecke algebra, which is the source of the type SL_2 Macdonald polynomials, is a generalization of the "Dirac sea", a representation of the Heisenberg algebra which controls the mathematics behind the quantum harmonic oscillator (see (3.20) and Proposition 3.3 and compare to the discussion, for example, in the neighborhood of Figure 10.2 in [SF14]).

As in [Mac03, §6.1 and 6.3], we denote the electronic (nonsymmetric) Macdonald polynomials for type SL_2 by E_m, for $m \in \mathbb{Z}$, and the bosonic (symmetric) Macdonald polynomials for type SL_2 are denoted P_m, for $m \in \mathbb{Z}_{\geq 0}$ (see [CR22, §1] for the motivation for the terminology 'electronic' and 'bosonic'). The type SL_2 Macdonald polynomials are, by a coordinate transformation, "equivalent" to the type GL_2 Macdonald polynomials. We review this coordinate transformation in Section 2 and explain how a product rule for type SL_2 Macdonald polynomials translates to a product rule for type GL_2 Macdonald polynomials.

Following the development of [CR22, §3.4], we use the calculus of the bosonic symmetrizer $\mathbf{1}_0$ and the normalized intertwining operators η_{s_1}, η_π to compute the elements $E_\ell(X)\mathbf{1}_0$ and $\mathbf{1}_0 E_\ell(X)\mathbf{1}_0$ in terms of a special basis of the (localized) double affine Hecke algebra. Continuing the main conceptual idea of [HR22] the expansions of the elements $E_\ell(X)\mathbf{1}_0$ and $\mathbf{1}_0 E_\ell(X)\mathbf{1}_0$ function as universal formulas for multiplying Macdonald polynomials, since they contain enough information to compute arbitrary

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products $E_\ell P_m$ and $P_\ell P_m$. We review this calculus, in our SL_2 setting, in section 3. The use of this calculus enables to cast the product framework from [Yi10] in a form which is tractable for executing the compression from $2\cdot 3^{\ell-1}$ terms to 2ℓ terms.

The key computation for the proof of the product rules is done in sections 4 and 5. In section 4 we use the basic structural calculus reviewed in section 3 to compute recursions satsfied by the coefficients of the operators $E_\ell(X)\mathbf{1}_0$ and $\mathbf{1}_0E_\ell(X)\mathbf{1}_0$ when expanded in terms of the $\{\eta^\ell\mathbf{1}_0 \mid \ell\in\mathbb{Z}\}$ basis of the completed $\mathbf{1}_0$-projected double affine Hecke algebra. In section 5 we solve these recursions to provide product expressions for the coefficients similar to product expressions for binomial coefficients.

Yip [Yi10, Theorem 4.2 and Theorem 4.4] gives alcove walk expansions of the products $E_\ell P_m$ and $P_\ell P_m$. The illustrative [Yi10, Example 5.1] computes the alcove walk expansion of the product E_3P_m for the SL_2 case. In this example there are 18 alcove walks which, after simpification, produce 6 terms. In general, for the product $E_\ell P_m$ for type SL_2, the alcove walk expansion of Yip will be a sum over $2\cdot 3^{\ell-1}$ alcove walks which simplifies to 2ℓ terms.

The result of Theorem 6.2 of this paper provides an explicit closed formula for each of the 2ℓ terms which appear in the expansion of $E_\ell P_m$. To our knowledge, this formula for $E_\ell P_m$ is new, particularly the execution of the desired compression after executing the general double affine Hecke algebra method of deriving product rules given by Yip [Yi10]. The q-t-binomial coefficients are given by

$$\begin{bmatrix}\ell\\ j\end{bmatrix}_{q,t} = \frac{\frac{(q;q)_\ell}{(t;q)_\ell}}{\frac{(q;q)_j}{(t;q)_j}\frac{(q;q)_{\ell-j}}{(t;q)_{\ell-j}}}, \qquad \text{where} \qquad (a;q)_j = (1-a)(1-qa)(1-q^2a)\cdots(1-q^{j-1}a). \tag{1.1}$$

Then Theorem 6.2 proves that, for $\ell, m\in\mathbb{Z}_{>0}$,

$$\begin{aligned}
P_\ell P_m &= \sum_{j=0}^{\ell} c_j^{(\ell)}(q^m)\, P_{m+\ell-2j},\\
E_\ell P_m &= \sum_{j=0}^{\ell-1} a_j^{(\ell)}(q^m)E_{m+\ell-2j} + b_j^{(\ell)}(q^m)E_{-m+\ell-2j}, \qquad \text{and}\\
E_{-\ell}P_m &= \sum_{j=0}^{\ell} t\cdot b_j^{(\ell+1)}(q^m)E_{m-(\ell-2j)} + a_j^{(\ell+1)}(q^m)E_{-m-(\ell-2j)},
\end{aligned}$$

where

$$\begin{aligned}
c_j^{(\ell)}(q^m) &= \begin{bmatrix}\ell\\ j\end{bmatrix}_{q,t} \frac{(q^mq^{-(j-1)};q)_j}{(tq^mq^{-j};q)_j}\frac{(t^2q^mq^{\ell-2j};q)_j}{(tq^mq^{\ell-2j+1};q)_j}, \qquad (1.2)\\
a_j^{(\ell)}(q^m) &= c_j^{(\ell)}(q^m)\cdot\frac{(1-q^{\ell-j})}{(1-q^\ell)}\cdot\frac{(1-tq^mq^{\ell-j})}{(1-tq^mq^{\ell-2j})} \qquad \text{and}\\
b_j^{(\ell)}(q^m) &= c_{\ell-j}^{(\ell)}(q^m)\cdot q^j\cdot\frac{(1-q^{\ell-j})}{(1-q^\ell)}\cdot\frac{(1-tq^mq^{-(\ell-j)})}{(1-t^2q^mq^{-(\ell-2j)})}
\end{aligned}$$

The rule for the product $P_\ell P_m$ is equivalent to a special case of the Pieri formula given in [Mac, Ch. VI (6.24)]. The precise connection is derived in Proposition 2.1, reproducing work of Soojin Cho [Cho19]. Indeed, the rule for the product $P_\ell P_m$ is the "linearization formula" for q-ultraspherical polynomials and appears in [Is05, Theorem 13.3.2], where it is stated that it is an identity of Rogers from 1894.

We have taken some care to try to make our exposition so that it contains all necessary definitions and complete and thorough proofs of the results. Our goal, in hope that the powerful tools provided by the double affine Hecke will become broadly accessible and utilized, has been to make this paper so that it can be read from scratch with no previous knowledge of Macdonald polynomials or the double affine Hecke algebra. Section 7 provides explicit examples.

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2 Macdonald polynomials for SL_2 and for GL_2

In this section we introduce the electronic and bosonic Macdonald polynomials for types SL_2 and GL_2 and explain the relation between them. We show how product rules for type SL_2 Macdonald polynomials convert to product rules for type GL_2 Macdonald polynomials. In Section 2.4 we check that the Pieri rule for multiplying bosonic polynomials given in [Mac, Ch. VI (6.24)] matches with the rule for the product $P_\ell P_m$ stated in the introduction (and proved in Theorem 6.2).

For working with Macdonald polynomials, fix $q, t \in \mathbb{C}^\times$ such that the only pair of integers (a, b) for which $q^a t^b = 1$ is the pair $(a, b) = (0, 0)$. Alternatively, one may think of q and t as parameters and to work with polynomials over the coefficient ring $\mathbb{C}(q, t)$ instead of over the coefficient ring $\mathbb{C}$.

2.1 Macdonald polynomials for type SL_2

The electronic Macdonald polynomials for type SL_2,

$$E_\ell(x) \in \mathbb{C}[x, x^{-1}], \quad \text{are indexed by} \quad \ell \in \mathbb{Z},$$

and the bosonic Macdonald polynomials for type SL_2,

$$P_\ell(x) \in \mathbb{C}[x, x^{-1}], \quad \text{are indexed by} \quad \ell \in \mathbb{Z}_{\geq 0}.$$

Let $\ell \in \mathbb{Z}_{\geq 0}$. Using the notation of (1.1), the electronic Macdonald polynomials are given by

$$E_{-\ell}(x) = \sum_{j=0}^{\ell} \begin{bmatrix} \ell \\ j \end{bmatrix}_{q,t} \frac{(1-tq^j)}{(1-tq^\ell)} x^{\ell-2j} \qquad \text{and} \qquad E_\ell(x) = \sum_{j=0}^{\ell-1} \begin{bmatrix} \ell-1 \\ j \end{bmatrix}_{q,t} \frac{q^{\ell-1-j}(1-tq^j)}{(1-tq^{\ell-1})} x^{-\ell+2j+2},$$

and the bosonic Macdonald polynomials are given by

$$P_\ell(x) = \sum_{j=0}^{\ell} \begin{bmatrix} \ell \\ j \end{bmatrix}_{q,t} x^{\ell-2j}.$$

See §7.12 for the connection between these formulas and the formulas in [Mac03, (6.2.7), (6.2.8), (6.3.7)].

2.2 Macdonald polynomials for type GL_2

The electronic Macdonald polynomials for type GL_2,

$$E_{(\mu_1,\mu_2)}(x_1,x_2)\in\mathbb{C}[x_1,x_1^{-1},x_2,x_2^{-1}],\quad \text{are indexed by } (\mu_1,\mu_2)\in\mathbb{Z}^2,$$

and the bosonic Macdonald polynomials for type GL_2,

$$P_{(\lambda_1,\lambda_2)}(x_1,x_2)\in\mathbb{C}[x_1,x_1^{-1},x_2,x_2^{-1}],\quad \text{are indexed by } \lambda=(\lambda_1,\lambda_2) \text{ with } \lambda_1,\lambda_2\in\mathbb{Z} \text{ and } \lambda_1\ge\lambda_2.$$

The $E_{(\mu_1,\mu_2)}(x_1,x_2)$ and $P_{(\lambda_1,\lambda_2)}(x_1,x_2)$ are given, in terms of the Macdonald polynomials for type SL_2, by

$$\begin{aligned} E_{(\mu_1,\mu_2)}(x_1,x_2) &= (x_1^{\frac12}x_2^{\frac12})^{\mu_1+\mu_2}E_{\mu_1-\mu_2}(x_1^{\frac12}x_2^{-\frac12}) \qquad \text{and} \\ P_{(\lambda_1,\lambda_2)}(x_1,x_2) &= (x_1^{\frac12}x_2^{\frac12})^{\lambda_1+\lambda_2}P_{\lambda_1-\lambda_2}(x_1^{\frac12}x_2^{-\frac12}). \end{aligned} \tag{2.1}$$

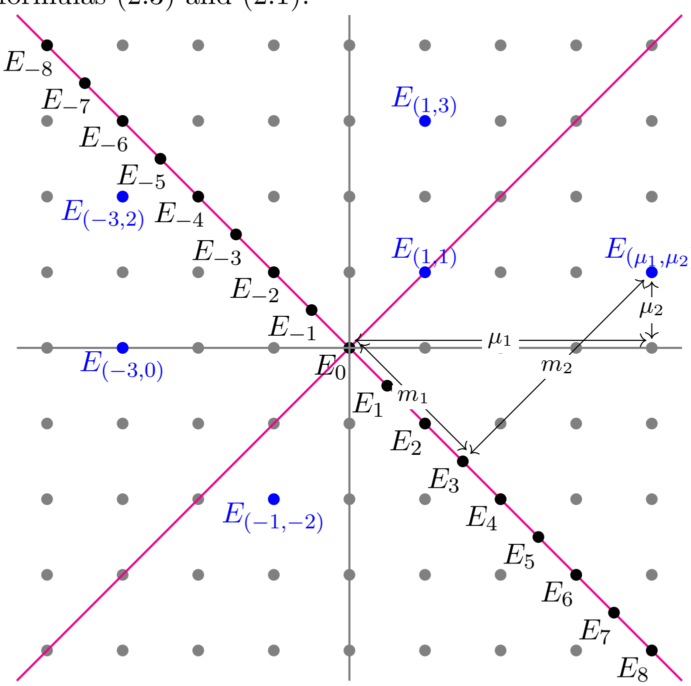
Equivalently, if $m_1,m_2\in\frac12\mathbb{Z}$ then $(x_1^{\frac12}x_2^{\frac12})^{2m_2}E_{2m_1}(x_1^{\frac12}x_2^{-\frac12}) = E_{(m_1+m_2,-m_1+m_2)}(x_1,x_2)$. Another way to express this conversion is to let

$$y = x_1^{\frac12}x_2^{\frac12} \text{ and } x = x_1^{\frac12}x_2^{-\frac12} \qquad \text{so that} \qquad x_1 = yx \text{ and } x_2 = yx^{-1}. \tag{2.2}$$

Then the Macdonald polynomials for type SL_2 are given in terms of the Macdonald polynomials for type GL_2 by

$$E_{2m_1}(x) = y^{-2m_2}E_{(m_1+m_2,-m_1+m_2)}(yx,yx^{-1}) = E_{(m_1+m_2,-m_1+m_2)}(x,x^{-1}), \tag{2.3}$$

for $m_1,m_2\in\frac12\mathbb{Z}$. The following picture illustrates the conversion between E_m and $E_{(\mu_1,\mu_2)}$ given by the formulas (2.3) and (2.1).



$(\mu_1,\mu_2)\in\mathbb{Z}^2,$
$m_1,m_2\in\frac12\mathbb{Z},$

$\mu_1 = m_1+m_2,$
$\mu_2 = m_1-m_2,$

$m_1 = \frac12(\mu_1-\mu_2),$
$m_2 = \frac12(\mu_1+\mu_2),$

2.3 Converting product rules for type SL_2 to product formulas for type GL_2

Assume that multiplication rules for multiplying type SL_2 Macdonald polynomials are given by

$$E_\ell(x)P_m(x) = \sum_{j=0}^{\ell-1} a_j^{(\ell)}(q^m)E_{m+\ell-2j}(x) + \sum_{j=0}^{\ell-1} b_j^{(\ell)}(q^m)E_{-m+\ell-2j}(x),$$

and

$$P_\ell(x)P_m(x) = \sum_{j=0}^{\ell} c_j^{(\ell)}(q^m)E_{m+\ell-2j}(x).$$

Assume $(\nu_1,\nu_2)\in\mathbb{Z}^2$ and $(\mu_1,\mu_2)\in\mathbb{Z}^2$ with $\mu_1\ge\mu_2$. Let

$$\ell=\nu_1-\nu_2 \qquad\text{and}\qquad m=\mu_1-\mu_2 \qquad\text{and}\qquad d=\mu_1+\mu_2+\nu_1+\nu_2.$$

Then

$$\begin{matrix} m+\ell-2j+d = 2(\mu_1+\nu_1-j), \\ -(m+\ell-2j)+d = 2(\mu_2+\nu_2+j), \end{matrix} \qquad\text{and}\qquad \begin{matrix} -m+\ell-2j+d = 2(\mu_2+\nu_1-j), \\ -(-m+\ell-2j)+d = 2(\mu_1+\nu_2+j). \end{matrix}$$

Thus, with $y = x_1^{\frac12}x_2^{\frac12}$ and $x = x_1^{\frac12}x_2^{-\frac12}$ as in (2.2), the conversions in (2.1) and (2.3) give

$$\begin{aligned} P_{(\nu_1,\nu_2)}(x_1,x_2)P_{(\mu_1,\mu_2)}(x_1,x_2) &= y^{\nu_1+\nu_2}P_\ell(x)y^{\mu_1+\mu_2}P_m(x) = \sum_{j=0}^{\nu_1-\nu_2} c_j^{(\ell)}(q^m)y^dP_{m+\ell-2j}(x) \\ &= \sum_{j=0}^{\nu_1-\nu_2} c_j^{(\nu_1-\nu_2)}(q^{\mu_1-\mu_2})P_{(\mu_1+\nu_1-j,\mu_2+\nu_2+j)}(x_1,x_2), \end{aligned}$$

and

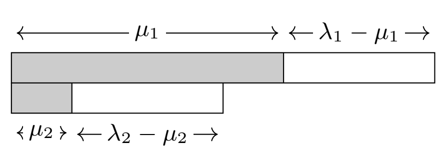
$$\begin{aligned} E_{(\nu_1,\nu_2)}(x_1,x_2)P_{(\mu_1,\mu_2)}(x_1,x_2) &= y^{\nu_1+\nu_2}E_{\nu_1-\nu_2}(x)y^{\mu_1+\mu_2}P_{\mu_1-\mu_2}(x) = y^dE_\ell(x)P_m(x) \\ &= \sum_{j=0}^{\nu_1-\nu_2-1} a_j^{(\ell)}(q^m)y^dE_{m+\ell-2j}(x) + \sum_{j=0}^{\nu_1-\nu_2-1} b_j^{(\ell)}(q^m)y^dE_{-m+\ell-2j}(x) \\ &= \sum_{j=0}^{\nu_1-\nu_2-1} a_j^{(\nu_1-\nu_2)}(q^{\mu_1-\mu_2})\,E_{(\mu_1+\nu_1-j,\mu_2+\nu_2+j)}(x_1,x_2) \\ &\quad + \sum_{j=0}^{\nu_1-\nu_2-1} b_j^{(\nu_1-\nu_2)}(q^{\mu_1-\mu_2})\,E_{(\mu_2+\nu_1-j,\mu_1+\nu_2+j)}(x_1,x_2), \end{aligned}$$

and these are the multiplication rules for type GL_2 Macdonald polynomials.

2.4 Comparison of the GL_2 case to Macdonald

A horizontal strip λ/μ of length ℓ is a pair $\lambda=(\lambda_1,\lambda_2)$ and $\mu=(\mu_1,\mu_2)$ of partitions such that

$$\mu_2\le\lambda_2\le\mu_1\le\lambda_1 \quad\text{and}\quad \lambda_1-\mu_1+\lambda_2-\mu_2=\ell.$$



Following [Mac, VI §6 Ex. 2a], define

$$\varphi_{\lambda/\mu} = \prod_{1\le i\le j\le \ell(\lambda)} \frac{f(q^{\lambda_i-\lambda_j}t^{j-i})}{f(q^{\lambda_i-\mu_j}t^{j-i})} \frac{f(q^{\mu_i-\mu_{j+1}}t^{j-i})}{f(q^{\mu_i-\lambda_{j+1}}t^{j-i})}, \qquad \text{where} \qquad f(u) = \frac{(tu;q)_\infty}{(qu;q)_\infty}$$

and $(z,q)_\infty = (1-z)(1-zq)(1-zq^2)\cdots$. Then [Mac, VI §6 Ex. 2a] gives that

$$\text{if} \qquad g_\ell = \frac{(t,q)_\ell}{(q;q)_\ell} P_{(\ell,0)}(x_1,x_2) \qquad \text{then} \qquad g_\ell P_{(\mu_1,\mu_2)} = \sum_\lambda \varphi_{\lambda/\mu} P_{(\lambda_1,\lambda_2)},$$

where the sum is over $\lambda = (\lambda_1,\lambda_2)$ such that λ/μ is a horizontal strip of length ℓ. Indeed this matches our results, in view of the following Proposition.

Proposition 2.1. *Let $c_j^{(\ell)}(q^m)$ be as defined in (1.2). Let $\lambda = (\lambda_1,\lambda_2)$ and $\mu = (\mu_1,\mu_2)$ be such that λ/μ is a horizontal strip of length ℓ and let $m = \mu_1 - \mu_2$ and $j = \lambda_2 - \mu_2$. Then*

$$\frac{(q,q)_\ell}{(t;q)_\ell}\varphi_{\lambda/\mu} = c_j^{(\ell)}(q^m).$$

Proof. Letting $\lambda_i = \mu_i + a_i$ gives

$$\begin{aligned}
\varphi_{\lambda/\mu} &= \prod_{1\le i\le j\le \ell(\lambda)} \frac{f(q^{a_i-a_j}q^{\mu_i-\mu_j}t^{j-i})}{f(q^{a_i}q^{\mu_i-\mu_j}t^{j-i})} \frac{f(t^{-1}q^{\mu_i-\mu_{j+1}}t^{j+1-i})}{f(t^{-1}q^{-a_{j+1}}q^{\mu_i-\mu_{j+1}}t^{j+1-i})} \\
&= \prod_{1\le i\le j\le \ell(\lambda)} \frac{(tq^{a_i-a_j}q^{\mu_i-\mu_j}t^{j-i};q)_\infty}{(qq^{a_i-a_j}q^{\mu_i-\mu_j}t^{j-i};q)_\infty} \frac{(qq^{a_i}q^{\mu_i-\mu_j}t^{j-i};q)_\infty}{(tq^{a_i}q^{\mu_i-\mu_j}t^{j-i};q)_\infty} \\
&\qquad \cdot \frac{(q^{\mu_i-\mu_{j+1}}t^{j+1-i};q)_\infty}{(qt^{-1}q^{\mu_i-\mu_{j+1}}t^{j+1-i};q)_\infty} \frac{(qt^{-1}q^{-a_{j+1}}q^{\mu_i-\mu_{j+1}}t^{j+1-i};q)_\infty}{(q^{-a_{j+1}}q^{\mu_i-\mu_{j+1}}t^{j+1-i};q)_\infty} \\
&= \prod_{1\le i\le j\le \ell(\lambda)} \frac{(tq^{a_i-a_j}q^{\mu_i-\mu_j}t^{j-i};q)_{a_j}}{(qq^{a_i-a_j}q^{\mu_i-\mu_j}t^{j-i};q)_{a_j}} \frac{(qt^{-1}q^{-a_{j+1}}q^{\mu_i-\mu_{j+1}}t^{j+1-i};q)_{a_{j+1}}}{(q^{-a_{j+1}}q^{\mu_i-\mu_{j+1}}t^{j+1-i};q)_{a_{j+1}}}
\end{aligned}$$

When $i = j$ the first factor is

$$\frac{(tq^{a_i-a_j}q^{\mu_i-\mu_j}t^{j-i};q)_{a_j}}{(qq^{a_i-a_j}q^{\mu_i-\mu_j}t^{j-i};q)_{a_j}} = \frac{(t;q)_{a_j}}{(q;q)_{a_j}},$$

and when $j+1 = n$ so that $a_{j+1} = 0$ and $\mu_{j+1} = 0$.

$$\frac{(qt^{-1}q^{-a_{j+1}}q^{\mu_i-\mu_{j+1}}t^{j+1-i};q)_{a_{j+1}}}{(q^{-a_{j+1}}q^{\mu_i-\mu_{j+1}}t^{j+1-i};q)_{a_{j+1}}} = \frac{(qt^{-1}q^{\mu_i}t^{j+1-i};q)_0}{(q^{\mu_i}t^{j+1-i};q)_0} = 1.$$

Thus, when $\ell(\lambda) = 2$,

$$\varphi_{\lambda/\mu} = \frac{(t;q)_{a_1}}{(q;q)_{a_1}} \frac{(t;q)_{a_2}}{(q;q)_{a_2}} \cdot \frac{(tq^{a_1-a_2}q^{\mu_1-\mu_2}t^{2-1};q)_{a_2}}{(q^{a_1-a_2+1}q^{\mu_1-\mu_2}t^{2-1};q)_{a_2}} \cdot \frac{(t^{-1}q^{-a_2+1}q^{\mu_1-\mu_2}t^{2-1};q)_{a_2}}{(q^{-a_2}q^{\mu_1-\mu_2}t^{2-1};q)_{a_2}}.$$

Since $m = \mu_1 - \mu_2$ and $j = \lambda_2 - \mu_2 = a_2$ then $a_1 - a_2 = (a_1+a_2) - 2a_2 = \ell - 2j$ and

$$\frac{(q;q)_\ell}{(tq;q)_\ell}\varphi_{\lambda/\mu} = \begin{bmatrix} \ell \\ j \end{bmatrix}_{q,t} \cdot \frac{(t^2q^mq^{\ell-2j};q)_j}{(tq^mq^{\ell-2j+1};q)_j} \frac{(q^mq^{-(j-1)};q)_j}{(tq^mq^{-j};q)_j} = c_j^{(\ell)}(q^m).$$

$\square$

3 DAHA for SL_2 and the polynomial representation

In this section we introduce the type SL_2 double affine Hecke algebra and its polynomial representation. The double affine Hecke algebra is a source for a myriad of operators acting on polynomials. In this section we carefully establish the identites between operators that will enable us to compute products of Macdonald polynomials.

3.1 The double affine Hecke algebra (DAHA) for type SL_2

Fix $q^{\frac{1}{2}}, t^{\frac{1}{2}} \in \mathbb{C}^\times$. Following [Mac03, (6.1.2), (6.1.3)], the double affine Hecke algebra for SL_2 is the $\mathbb{C}$-algebra $\tilde{H}_{\mathrm{int}}$ generated by $T_1^{\pm 1}, X^{\pm 1}, Y^{\pm 1}, T_\pi^{\pm 1}$ with relations $T_1 T_1^{-1} = T_1^{-1} T_1 = 1$, $XX^{-1} = X^{-1}X = 1$, $YY^{-1} = Y^{-1}Y = 1$, $T_\pi T_\pi^{-1} = T_\pi^{-1} T_\pi = 1$ and

$$T_\pi = Y T_1^{-1} = T_1 Y^{-1}, \qquad T_\pi X T_\pi^{-1} = q^{\frac{1}{2}} X^{-1},$$
$$T_1 X T_1 = X^{-1}, \qquad T_1 Y^{-1} T_1 = Y, \qquad (T_1 - t^{\frac{1}{2}})(T_1 + t^{-\frac{1}{2}}) = 0. \tag{3.1}$$

It follows from the relations $T_1 X T_1 = X^{-1}$ and $T_1 - T_1^{-1} = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ that

$$T_1 X^r = X^{-r} T_1 + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{X^r - X^{-r}}{1 - X^2}, \qquad \text{for } r \in \mathbb{Z}. \tag{3.2}$$

As a left module for the Laurent polynomial ring $\mathbb{C}[Y, Y^{-1}]$, the double affine Hecke algebra $\tilde{H}_{\mathrm{int}}$ has basis $\{X^k \mid k \in \mathbb{Z}\} \sqcup \{X^k T_1 \mid k \in \mathbb{Z}\}$. Letting $\mathbb{C}(Y)$ denote the field of fractions of $\mathbb{C}[Y, Y^{-1}]$, the *localized double affine Hecke algebra*,

$$\tilde{H} = \mathbb{C}(Y) \otimes_{\mathbb{C}[Y,Y^{-1}]} \tilde{H}_{\mathrm{int}},$$

is the algebra with $\mathbb{C}(Y)$-basis $\{X^k \mid k \in \mathbb{Z}\} \sqcup \{X^k T_1 \mid k \in \mathbb{Z}\}$ (as a left $\mathbb{C}(Y)$-module) and the relations in (3.1). Although the polynomial representation of $\tilde{H}_{\mathrm{int}}$ (which is where the Macdonald polynomials live, see §3.2) is not a $\tilde{H}$-module, there are operators on the polynomial representation which we can source from the larger algebra $\tilde{H}$. The operators which we wish to access are the intertwiners $\tau_\pi^\vee$ and $\tau_1^\vee$ and the normalized interwiners η_{s_1}, η_π, η and η^{-1}, which are defined below in (3.3), (3.9), (3.10) and (3.11).

3.1.1 Intertwiners and the bosonic symmetrizer

The *intertwiners* $\tau_1^\vee$ and $\tau_\pi^\vee$ and the *bosonic symmetrizer* $\mathbf{1}_0$ are defined by

$$\tau_1^\vee = T_1 + t^{-\frac{1}{2}} \frac{(1-t)}{(1-Y^{-2})}, \qquad \tau_\pi^\vee = X T_1, \qquad \text{and} \qquad \mathbf{1}_0 = T_1 + t^{-\frac{1}{2}}. \tag{3.3}$$

In [Mac03, (6.1.6) and (6.18)], $\tau_1^\vee$ is denoted α and $\tau_\pi^\vee$ is denoted β.

Proposition 3.1. *Then*

$$\tau_1^\vee Y = Y^{-1} \tau_1^\vee, \qquad \tau_\pi^\vee Y = Y^{-1} q^{-\frac{1}{2}} \tau_\pi^\vee, \tag{3.4}$$

$$(\tau_1^\vee)^2 = t^{-1} \frac{(1 - tY^2)(1 - tY^{-2})}{(1 - Y^2)(1 - Y^{-2})}, \qquad (\tau_\pi^\vee)^2 = 1, \tag{3.5}$$

$$T_1 \mathbf{1}_0 = \mathbf{1}_0 T_1 = t^{\frac{1}{2}} \mathbf{1}_0, \qquad \mathbf{1}_0 \tau_1^\vee = \mathbf{1}_0 t^{-\frac{1}{2}} \frac{(1 - tY^{-2})}{(1 - Y^{-2})}, \qquad \tau_1^\vee \mathbf{1}_0 = t^{-\frac{1}{2}} \frac{(1 - tY^{-2})}{(1 - Y^{-2})} \mathbf{1}_0, \tag{3.6}$$

$$\mathbf{1}_0^2 = \mathbf{1}_0 (t^{\frac{1}{2}} + t^{-\frac{1}{2}}) \qquad \textit{and} \qquad \mathbf{1}_0 = \tau_1^\vee + t^{-\frac{1}{2}} \frac{(1 - tY^2)}{(1 - Y^2)} = \tau_1^\vee + t^{\frac{1}{2}} \frac{(1 - t^{-1} Y^{-2})}{(1 - Y^{-2})}. \tag{3.7}$$

Proof. Using the relations in (3.1), $(\tau_\pi^\vee)^2 = XT_1XT_1 = XX^{-1} = 1$ and

$$\begin{aligned}\tau_\pi^\vee Y &= XT_1Y = T_1^{-1}X^{-1}Y = T_1^{-1}q^{-\frac{1}{2}}T_\pi X T_\pi^{-1}Y \\ &= T_1^{-1}q^{-\frac{1}{2}}T_\pi X T_\pi^{-1}T_\pi T_1 = q^{-\frac{1}{2}}T_1^{-1}T_\pi X T_1 = q^{-\frac{1}{2}}T_1^{-1}T_\pi \tau_\pi^\vee = q^{-\frac{1}{2}}Y^{-1}\tau_\pi^\vee.\end{aligned}$$

Using

$$\begin{aligned}\tau_1^\vee &= T_1 + t^{-\frac{1}{2}}\frac{(1-t)}{(1-Y^{-2})} = T_1^{-1} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) + t^{-\frac{1}{2}}\frac{(1-t)}{(1-Y^{-2})} \\ &= T_1^{-1} + (1-t)\frac{-t^{-\frac{1}{2}}(1-Y^{-2}) + t^{-\frac{1}{2}}}{(1-Y^{-2})} = T_1^{-1} + t^{-\frac{1}{2}}\frac{(1-t)Y^{-2}}{(1-Y^{-2})},\end{aligned} \tag{3.8}$$

then

$$\begin{aligned}\tau_1^\vee Y &= \Big(T_1^{-1} + t^{-\frac{1}{2}}\frac{(1-t)Y^{-2}}{(1-Y^{-2})}\Big)Y = T_1^{-1}Y + t^{-\frac{1}{2}}\frac{(1-t)Y^{-1}}{(1-Y^{-2})} \\ &= Y^{-1}T_1 + t^{-\frac{1}{2}}\frac{(1-t)Y^{-1}}{(1-Y^{-2})} = Y^{-1}\Big(T_1 + t^{-\frac{1}{2}}\frac{(1-t)}{(1-Y^{-2})}\Big) = Y^{-1}\tau_1^\vee\end{aligned}$$

and

$$\begin{aligned}(\tau_1^\vee)^2 &= \Big(T_1 + t^{-\frac{1}{2}}\frac{(1-t)}{(1-Y^{-2})}\Big)\tau_1^\vee = T_1\tau_1^\vee + \tau_1^\vee t^{-\frac{1}{2}}\frac{(1-t)}{(1-Y^2)} \\ &= T_1\Big(T_1^{-1} + t^{-\frac{1}{2}}\frac{(1-t)Y^{-2}}{(1-Y^{-2})}\Big) + \Big(T_1 + t^{-\frac{1}{2}}\frac{(1-t)}{(1-Y^{-2})}\Big)t^{-\frac{1}{2}}\frac{(1-t)}{(1-Y^2)} \\ &= 1 + t^{-1}\frac{(1-t)}{(1-Y^{-2})}\frac{(1-t)}{(1-Y^2)} = \frac{(1-Y^{-2}-Y^2+1+t^{-1}-2+t)}{(1-Y^2)(1-Y^{-2})} \\ &= t^{-1}\frac{(1-tY^2)(1-tY^{-2})}{(1-Y^2)(1-Y^{-2})}.\end{aligned}$$

Since $\mathbf{1}_0 = T_1 + t^{-\frac{1}{2}} = T_1 - T_1^{-1} + T_1^{-1} + t^{-\frac{1}{2}} = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) + T_1^{-1} + t^{-\frac{1}{2}} = T_1^{-1} + t^{\frac{1}{2}}$ then

$$\mathbf{1}_0 T_1 = (T_1^{-1} + t^{\frac{1}{2}})T_1 = 1 + t^{\frac{1}{2}}T_1 = t^{\frac{1}{2}}(T_1 + t^{-\frac{1}{2}}) = t^{\frac{1}{2}}\mathbf{1}_0 \quad \text{and} \quad \mathbf{1}_0^2 = \mathbf{1}_0(T_1 + t^{-\frac{1}{2}}) = \mathbf{1}_0(t^{\frac{1}{2}} + t^{-\frac{1}{2}}).$$

Similarly for the product $T_1\mathbf{1}_0$. Then

$$\mathbf{1}_0\tau_1^\vee = \mathbf{1}_0\Big(T_1 + t^{-\frac{1}{2}}\frac{(1-t)}{(1-Y^{-2})}\Big) = \mathbf{1}_0\Big(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\frac{(1-t)}{(1-Y^{-2})}\Big) = \mathbf{1}_0 t^{-\frac{1}{2}}\frac{(1-tY^{-2})}{(1-Y^{-2})}$$

and similarly for the product $\tau_1^\vee\mathbf{1}_0$. Finally

$$\mathbf{1}_0 = \tau_1^\vee - t^{-\frac{1}{2}}\frac{(1-t)}{(1-Y^{-2})} + t^{-\frac{1}{2}} = \tau_1^\vee + t^{-\frac{1}{2}}\frac{(t-Y^{-2})}{(1-Y^{-2})} = \tau_1^\vee + t^{-\frac{1}{2}}\frac{(1-tY^2)}{(1-Y^2)}.$$

□

3.1.2 Normalized intertwiners

Define normalized intertwiners

$$\eta_\pi = \tau_\pi^\vee \qquad \text{and} \qquad \eta_{s_1} = t^{\frac{1}{2}}\frac{(1-Y^{-2})}{(1-tY^{-2})}\tau_1^\vee. \tag{3.9}$$

Then define

$$\eta = \eta_\pi \eta_{s_1} = \tau_\pi^\vee t^{\frac{1}{2}} \frac{(1-Y^{-2})}{(1-tY^{-2})} \tau_1^\vee = t^{\frac{1}{2}} \frac{(1-Y^2 q)}{(1-tY^2 q)} \tau_\pi^\vee \tau_1^\vee \tag{3.10}$$

and

$$\eta^{-1} = \eta_{s_1} \eta_\pi = t^{\frac{1}{2}} \frac{(1-Y^{-2})}{(1-tY^{-2})} \tau_1^\vee \tau_\pi^\vee. \tag{3.11}$$

Warning: Although η and η^{-1} are inverses of each other as elements of $\tilde{H}$, and these are well defined operators on the polynomial representation (see Proposition 3.3), as operators on the polynomial representation η and η^{-1} are not invertible operators.

Proposition 3.2. *The following relations hold in $\tilde{H}$:*

$$\eta_\pi^2 = 1, \qquad \eta_{s_1}^2 = 1, \qquad \eta\eta_{s_1} = \eta_{s_1}\eta^{-1}. \tag{3.12}$$

$$\eta_\pi Y = Y^{-1} q^{-\frac{1}{2}} \eta_\pi, \qquad \eta_{s_1} Y = Y^{-1} \eta_{s_1}, \qquad \eta Y = Y q^{\frac{1}{2}} \eta, \tag{3.13}$$

$$\mathbf{1}_0 = (1+\eta_{s_1}) t^{-\frac{1}{2}} \frac{(1-tY^2)}{(1-Y^2)}, \qquad \eta_{s_1} \mathbf{1}_0 = \mathbf{1}_0, \qquad \mathbf{1}_0 \eta_{s_1} = \mathbf{1}_0 t^{-1} \frac{(1-tY^{-2})}{(1-t^{-1}Y^{-2})}. \tag{3.14}$$

Proof. From (3.5), $\eta_\pi^2 = (\tau_\pi^\vee)^2 = 1$. Using $\tau_1^\vee Y = Y^{-1}\tau_1^\vee$ and the formula for $(\tau_1^\vee)^2$ in (3.5) gives

$$\begin{aligned}
\eta_{s_1}^2 &= t^{\frac{1}{2}} \frac{(1-Y^{-2})}{(1-tY^{-2})} \tau_1^\vee t^{\frac{1}{2}} \frac{(1-Y^{-2})}{(1-tY^{-2})} \tau_1^\vee = t \frac{(1-Y^{-2})}{(1-tY^{-2})} \frac{(1-Y^2)}{(1-tY^2)} \tau_1^\vee \tau_1^\vee \\
&= t \frac{(1-Y^{-2})}{(1-tY^{-2})} \frac{(1-Y^2)}{(1-tY^2)} \cdot t^{-1} \frac{(1-tY^2)(1-tY^{-2})}{(1-Y^2)(1-Y^{-2})} = 1.
\end{aligned}$$

Then

$$\eta\eta_{s_1} = \eta_\pi \eta_{s_1} \eta_{s_1} = \eta_\pi = \eta_{s_1}\eta_{s_1}\eta_\pi = \eta_{s_1}\eta^{-1}.$$

The relations $\eta_\pi Y = Y^{-1} q^{-\frac{1}{2}} \eta_\pi$ and $\eta_{s_1} Y = Y^{-1}\eta_{s_1}$ follow from (3.4) and

$$\eta Y = \eta_\pi \eta_{s_1} Y = \eta_\pi Y^{-1} \eta_{s_1} = Y q^{\frac{1}{2}} \eta_\pi \eta_{s_1} = Y q^{\frac{1}{2}} \eta.$$

Using (3.7),

$$\mathbf{1}_0 = \tau_1^\vee + t^{-\frac{1}{2}} \frac{(1-tY^2)}{(1-Y^2)} = \Big(t^{\frac{1}{2}} \tau_1^\vee \frac{(1-Y^2)}{(1-tY^2)} + 1 \Big) \cdot t^{-\frac{1}{2}} \frac{(1-tY^2)}{(1-Y^2)} = (\eta_{s_1}+1) t^{-\frac{1}{2}} \frac{(1-tY^2)}{(1-Y^2)}.$$

By the last identity in (3.6),

$$\eta_{s_1}\mathbf{1}_0 = t^{\frac{1}{2}} \frac{(1-Y^{-2})}{(1-tY^{-2})} \tau_1^\vee \mathbf{1}_0 = t^{\frac{1}{2}} \frac{(1-Y^{-2})}{(1-tY^{-2})} \cdot t^{-\frac{1}{2}} \frac{(1-tY^{-2})}{(1-Y^{-2})} \mathbf{1}_0 = \mathbf{1}_0$$

and, by the second identity in (3.6),

$$\mathbf{1}_0 \eta_{s_1} = \mathbf{1}_0 t^{\frac{1}{2}} \frac{(1-Y^{-2})}{(1-tY^{-2})} \tau_1^\vee = \mathbf{1}_0 \tau_1^\vee t^{\frac{1}{2}} \frac{(1-Y^2)}{(1-tY^2)} = \mathbf{1}_0 t^{-\frac{1}{2}} \frac{(1-tY^{-2})}{(1-Y^{-2})} t^{\frac{1}{2}} \frac{(1-Y^2)}{(1-tY^2)} = \mathbf{1}_0 t^{-1} \frac{(1-tY^{-2})}{(1-t^{-1}Y^{-2})}.$$

□

3.2 The polynomial representation $\tilde{H}_{\mathrm{int}}\mathbf{1}_Y$

Let $\tilde{H}_{\mathrm{int}}\mathbf{1}_Y$ be the $\tilde{H}_{\mathrm{int}}$ module generated by a single generator $\mathbf{1}_Y$ with relations

$$T_1\mathbf{1}_Y = t^{\frac{1}{2}}\mathbf{1}_Y \qquad \text{and} \qquad T_\pi\mathbf{1}_Y = \mathbf{1}_Y.$$

Then $Y\mathbf{1}_Y = T_\pi T_1\mathbf{1}_Y = t^{\frac{1}{2}}\mathbf{1}_Y$ and

$$\tilde{H}_{\mathrm{int}}\mathbf{1}_Y \qquad \text{has } \mathbb{C}\text{-basis} \qquad \{X^k\mathbf{1}_Y \mid k \in \mathbb{Z}\}.$$

Using the second relation in (3.1) and (3.2), the action of $\tilde{H}_{\mathrm{int}}$ in the basis $\{X^k\mathbf{1}_Y \mid k \in \mathbb{Z}\}$ is given explicitly by

$$XX^r\mathbf{1}_Y = X^{r+1}\mathbf{1}_Y, \qquad T_\pi X^r\mathbf{1}_Y = q^{\frac{r}{2}}X^{-r}\mathbf{1}_Y \qquad \text{and}$$
$$T_1X^r\mathbf{1}_Y = t^{\frac{1}{2}}X^{-r}\mathbf{1}_Y + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\frac{X^r - X^{-r}}{1 - X^2}\mathbf{1}_Y, \qquad \text{for } r \in \mathbb{Z}.$$

The electronic Macdonald polynomials are the elements $E_m(X) \in \mathbb{C}[X, X^{-1}]$, $m \in \mathbb{Z}$, determined by

$$\begin{aligned} YE_m(X)\mathbf{1}_Y &= t^{-\frac{1}{2}}q^{\frac{-m}{2}}E_m(X)\mathbf{1}_Y, \quad \text{if } m \in \mathbb{Z}_{>0}, \text{ and} \\ YE_{-m}(X)\mathbf{1}_Y &= t^{\frac{1}{2}}q^{\frac{m}{2}}E_{-m}(X)\mathbf{1}_Y, \quad \text{if } m \in \mathbb{Z}_{\geq 0}, \end{aligned} \tag{3.15}$$

with normalization such that the coefficient of X^m in $E_m(X)$ is 1. The electronic Macdonald polynomials are given recursively (see [Mac03, (6.2.3)]) by $E_0(X) = 1$ and $E_1(X) = X$ and

$$\begin{aligned} \tau_1^\vee E_r(X)\mathbf{1}_Y &= t^{-\frac{1}{2}}E_{-r}(X)\mathbf{1}_Y, & \tau_1^\vee E_{-r}(X)\mathbf{1}_Y &= t^{-\frac{1}{2}}\frac{(1-tY^2)(1-tY^{-2})}{(1-Y^2)(1-Y^{-2})}E_r(X)\mathbf{1}_Y, \\ \tau_\pi^\vee E_r(X)\mathbf{1}_Y &= t^{-\frac{1}{2}}E_{-(r-1)}(X)\mathbf{1}_Y. & \tau_\pi^\vee E_{-r}(X)\mathbf{1}_Y &= t^{\frac{1}{2}}E_{r+1}(X)\mathbf{1}_Y, \end{aligned} \tag{3.16}$$

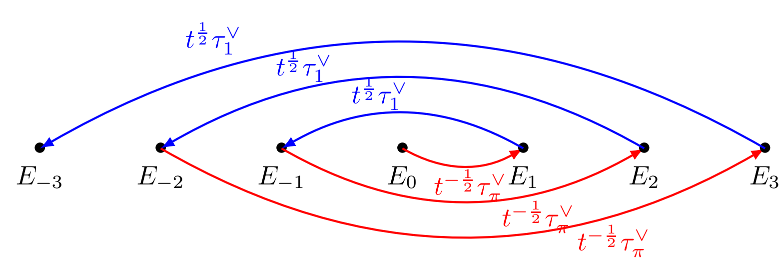
for $r \in \mathbb{Z}_{>0}$. Note that $\tau_\pi^\vee E_0(X)\mathbf{1}_Y = XT_1\mathbf{1}_Y = t^{\frac{1}{2}}X\mathbf{1}_Y = t^{\frac{1}{2}}E_1(X)\mathbf{1}_Y$ and

$$\tau_\pi^\vee E_1(X)\mathbf{1}_Y = XT_1X\mathbf{1}_Y = t^{-\frac{1}{2}}XT_1XT_1\mathbf{1}_Y = t^{-\frac{1}{2}}XX^{-1}\mathbf{1}_Y = t^{-\frac{1}{2}}E_0(X)\mathbf{1}_Y$$

and

$$\tau_1^\vee E_0(X)\mathbf{1}_Y = \Big(T_1 + t^{-\frac{1}{2}}\frac{(1-t)}{(1-Y^{-2})}\Big)\mathbf{1}_Y = \Big(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\frac{(1-t)}{(1-t^{-1})}\Big)\mathbf{1}_Y = (t^{\frac{1}{2}} - t^{\frac{1}{2}})\mathbf{1}_Y = 0. \tag{3.17}$$

As pictured below, the elements $\tau_1^\vee$ and $\tau_\pi^\vee$ can be used to recursively construct the electronic Macdonald polynomials.



The bosonic Macdonald polynomials $P_m(X) \in \mathbb{C}[X, X^{-1}]$, for $m \in \mathbb{Z}_{\geq 0}$, can be given [Mac03, (6.3.10)] by

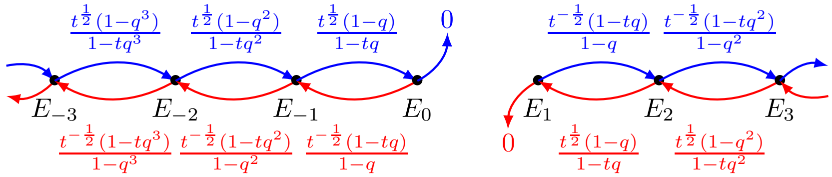
$$P_m(X)\mathbf{1}_Y = E_{-m}(X)\mathbf{1}_Y + \frac{t(1-q^m)}{(1-tq^m)}E_m(X)\mathbf{1}_Y. \tag{3.18}$$

Applying (3.15) to (3.18) and using $\tau_1^\vee E_r(X)\mathbf{1}_Y = t^{-\frac{1}{2}}E_{-r}(X)\mathbf{1}_Y$ gives, for $m \in \mathbb{Z}_{>0}$,

$$P_m(X)\mathbf{1}_Y = t^{\frac{1}{2}}\tau_1^\vee E_m(X)\mathbf{1}_Y + t\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}E_m(X)\mathbf{1}_Y = t^{\frac{1}{2}}\mathbf{1}_0 E_m(X)\mathbf{1}_Y, \tag{3.19}$$

where the last equality follows from (3.7).

The following Proposition analyzes the action of η and η^{-1} as operators on the polynomial representation. It shows that η acts as a raising operator with $\eta E_0(X)\mathbf{1}_Y = 0$ and that η^{-1} acts as a lowering operator with $\eta^{-1}E_1(X)\mathbf{1}_Y = 0$. The operator η is pictured in blue and the operator η^{-1} is pictured in red. The coefficients below and above the arrows provide the constants which appear in the formulas for ηE_m, η_{E_m}, $\eta^{-1}E_m$ and $\eta^{-1}E_m$ which are derived in (3.25), (3.26), (3.27) and (3.28).



(3.20)

It is important to note that η and η^{-1} are not invertible as operators on the polynomial representation (even though they are inverses of each other as elements of $\tilde{H}$). This phenomenon is of the same nature as the fact that $(1-t^{-1}Y)$ is a well defined element of $\tilde{H}$ with inverse $\frac{1}{(1-t^{-1}Y)}$ in $\tilde{H}$, and $(1-t^{-1}Y)$ is a well defined operator on $\mathbb{C}[X,X^{-1}]$ that is not invertible as an operator on the polynomial representation $\mathbb{C}[X,X^{-1}]$.

The identities

$$\eta^{-(\ell-j)}\eta^j E_m(X)\mathbf{1}_Y = t^{\frac{1}{2}(\ell-2j)}\cdot \mathrm{ev}_m\Big(\frac{(t^{-1}Y^{-2}q^{-(\ell-2j)};q)_{\ell-j}}{(Y^{-2}q^{-(\ell-2j)};q)_{\ell-j}}\cdot\frac{(Y^{-2};q)_j}{(t^{-1}Y^{-2};q)_j}\Big)E_{m-(\ell-2j)}(X)\mathbf{1}_Y, \tag{3.21}$$

$$\eta^j\eta^{-(\ell-j)}E_{-m}(X)\mathbf{1}_Y = t^{-\frac{1}{2}(\ell-2j)}\cdot \mathrm{ev}_m\Big(\frac{(t^{-1}Y^{-2}q^{\ell-2j+1};q)_j}{(Y^{-2}q^{\ell-2j+1};q)_j}\frac{(Y^{-2}q;q)_{\ell-j}}{(t^{-1}Y^{-2}q;q)_{\ell-j}}\Big)E_{-m-(\ell-2j)}(X)\mathbf{1}_Y. \tag{3.22}$$

follow from (3.24) and (3.23) of the following Proposition by replacing j with $\ell-j$ (we keep the same conditions on j and ℓ as in Proposition 3.3). They will be used in the proof of Theorem 6.2.

Proposition 3.3. *As in* (3.10) *and* (3.11), *let*

$$\eta = t^{\frac{1}{2}}\frac{(1-Y^2q)}{(1-tY^2q)}\tau_\pi^\vee\tau_1^\vee \qquad \textit{and} \qquad \eta^{-1} = t^{\frac{1}{2}}\frac{(1-Y^{-2})}{(1-tY^{-2})}\tau_1^\vee\tau_\pi^\vee.$$

Let $\mathrm{ev}_m\colon \mathbb{C}[Y,Y^{-1}] \to \mathbb{C}$ *be the homomorphism given by* $\mathrm{ev}_m(Y) = t^{-\frac{1}{2}}q^{-\frac{1}{2}m}$ *and extend* ev_m *to elements of* $\mathbb{C}(Y)$ *such that the denominator does not evaluate to* 0.
If $\ell \in \mathbb{Z}_{\geq 0}$ *and* $m \in \mathbb{Z}_{>0}$ *and* $j \in \{0,\ldots,\ell\}$ *then*

$$\eta^{-j}\eta^{\ell-j}E_m(X)\mathbf{1}_Y = t^{-\frac{1}{2}(\ell-2j)}\cdot \mathrm{ev}_m\Big(\frac{(t^{-1}Y^{-2}q^{\ell-2j};q)_j}{(Y^{-2}q^{\ell-2j};q)_j}\cdot\frac{(Y^{-2};q)_{\ell-j}}{(t^{-1}Y^{-2};q)_{\ell-j}}\Big)E_{m+\ell-2j}(X)\mathbf{1}_Y, \tag{3.23}$$

$$\eta^{\ell-j}\eta^{-j}E_{-m}(X)\mathbf{1}_Y = t^{\frac{1}{2}(\ell-2j)}\cdot \mathrm{ev}_m\Big(\frac{(t^{-1}Y^{-2}q^{-(\ell-2j)+1};q)_{\ell-j}}{(Y^{-2}q^{-(\ell-2j)+1};q)_{\ell-j}}\frac{(Y^{-2}q;q)_j}{(t^{-1}Y^{-2}q;q)_j}\Big)E_{-m+\ell-2j}(X)\mathbf{1}_Y. \tag{3.24}$$

Proof. Assume $m \in \mathbb{Z}_{>0}$. By (3.16) and (3.15),

$$\begin{aligned}
\eta E_m(X)\mathbf{1}_Y &= t^{\frac{1}{2}}\frac{(1-Y^2q)}{(1-tY^2q)}\tau_\pi^\vee\tau_1^\vee E_m(X)\mathbf{1}_Y = t^{\frac{1}{2}}\frac{(1-Y^2q)}{(1-tY^2q)}\tau_\pi^\vee t^{-\frac{1}{2}}E_{-m}(X)\mathbf{1}_Y \\
&= t^{\frac{1}{2}}t^{-\frac{1}{2}}\frac{(1-Y^2q)}{(1-tY^2q)}t^{\frac{1}{2}}E_{m+1}(X)\mathbf{1}_Y = t^{\frac{1}{2}}\frac{(1-t^{-1}q^{-(m+1)}q)}{(1-tt^{-1}q^{-(m+1)}q)}E_{m+1}(X)\mathbf{1}_Y \\
&= t^{\frac{1}{2}}\frac{(1-t^{-1}q^{-m})}{(1-q^{-m})}E_{m+1}(X)\mathbf{1}_Y = t^{-\frac{1}{2}}\frac{(1-tq^m)}{(1-q^m)}E_{m+1}(X)\mathbf{1}_Y. \qquad (3.25)
\end{aligned}$$

Thus, for $\ell \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}_{>0}$,

$$\begin{aligned}
\eta^\ell E_m(X)\mathbf{1}_Y &= t^{-\frac{1}{2}\ell}\frac{(1-tq^m)(1-tq^{m+1})\cdots(1-tq^{m+\ell-1})}{(1-q^m)(1-q^{m+1})\cdots(1-q^{m+\ell-1})}E_{m+\ell}(X)\mathbf{1}_Y \\
&= t^{-\frac{1}{2}\ell}\mathrm{ev}_m\Big(\frac{(Y^{-2};q)_\ell}{(t^{-1}Y^{-2};q)_\ell}\Big)E_{m+\ell}(X)\mathbf{1}_Y.
\end{aligned}$$

Assume $m \in \mathbb{Z}_{>0}$. Using (3.16), (3.4) and (3.15),

$$\begin{aligned}
\eta E_{-m}(X)\mathbf{1}_Y &= t^{\frac{1}{2}}\frac{(1-Y^2q)}{(1-tY^2q)}\tau_\pi^\vee\tau_1^\vee E_{-m}(X)\mathbf{1}_Y = t^{\frac{1}{2}}\frac{(1-Y^2q)}{(1-tY^2q)}\tau_\pi^\vee t^{-\frac{1}{2}}\frac{(1-tY^2)(1-tY^{-2})}{(1-Y^2)(1-Y^{-2})}E_m(X)\mathbf{1}_Y \\
&= \frac{(1-q^{-m})(1-t^2q^m)}{(1-t^{-1}q^{-m})(1-tq^m)}\frac{(1-Y^2q)}{(1-tY^2q)}\tau_\pi^\vee E_m(X)\mathbf{1}_Y \\
&= \frac{(1-q^{-m})(1-t^2q^m)}{(1-t^{-1}q^{-m})(1-tq^m)}\frac{(1-Y^2q)}{(1-tY^2q)}t^{-\frac{1}{2}}E_{-(m-1)}\mathbf{1}_Y \\
&= \frac{(1-q^{-m})(1-t^2q^m)}{(1-t^{-1}q^{-m})(1-tq^m)}\frac{(1-tq^m)}{(1-t^2q^m)}t^{-\frac{1}{2}}E_{-(m-1)}\mathbf{1}_Y = t^{\frac{1}{2}}\frac{(1-q^m)}{(1-tq^m)}E_{-m+1}(X)\mathbf{1}_Y. \qquad (3.26)
\end{aligned}$$

By (3.17),

$$\eta E_0(X)\mathbf{1}_Y = t^{\frac{1}{2}}\frac{(1-Y^2q)}{(1-tY^2q)}\tau_\pi^\vee\tau_1^\vee E_0(X)\mathbf{1}_Y = 0 = t^{\frac{1}{2}}\frac{(1-q^0)}{(1-tq^0)}E_{-0+1}(X)\mathbf{1}_Y.$$

Thus, for $\ell \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned}
\eta^\ell E_{-m}(X)\mathbf{1}_Y &= t^{\frac{1}{2}\ell}\frac{(1-q^{m-\ell+1..m})}{(1-tq^{m-\ell+1..m})}E_{-m+\ell}(X)\mathbf{1}_Y \\
&= t^{\frac{1}{2}\ell}\mathrm{ev}_m\Big(\frac{(1-q^{m-\ell+1})\cdots(1-q^{m-1})(1-q^m))}{(1-tq^{m-\ell+1})\cdots(1-tq^{m-1})(1-tq^m)}\Big)E_{-m+\ell}(X)\mathbf{1}_Y \\
&= t^{\frac{1}{2}\ell}\mathrm{ev}_m\Big(\frac{(t^{-1}Y^{-2}q^{-(\ell-1)};q)_\ell}{(Y^{-2}q^{-(\ell-1)};q)_\ell}\Big)E_{-m+\ell}(X)\mathbf{1}_Y,
\end{aligned}$$

where the right hand side evaluates to 0 if $\ell > m$ (because the denominator factors are all nonzero and the numerator contains a factor of $(1-q^0) = 1-1 = 0$).

Assume $m \in \mathbb{Z}_{>0}$. By (3.16) and (3.15),

$$\begin{aligned}
\eta^{-1} E_m(X)\mathbf{1}_Y &= t^{\frac{1}{2}} \frac{(1-Y^{-2})}{(1-tY^{-2})} \tau_1^\vee \tau_\pi^\vee E_m(X)\mathbf{1}_Y = t^{\frac{1}{2}} \frac{(1-Y^{-2})}{(1-tY^{-2})} \tau_1^\vee t^{-\frac{1}{2}} E_{-(m-1)}(X)\mathbf{1}_Y \\
&= \frac{(1-Y^{-2})}{(1-tY^{-2})} t^{-\frac{1}{2}} \frac{(1-tY^{2})(1-tY^{-2})}{(1-Y^{2})(1-Y^{-2})} E_{m-1}(X)\mathbf{1}_Y \\
&= \frac{(1-tq^{m-1})}{(1-t^2q^{m-1})} t^{-\frac{1}{2}} \frac{(1-q^{-(m-1)})(1-t^2q^{m-1})}{(1-t^{-1}q^{-(m-1)})(1-tq^{m-1})} E_{m-1}(X)\mathbf{1}_Y \\
&= t^{\frac{1}{2}} \frac{(1-q^{m-1})}{(1-tq^{m-1})} E_{m-1}(X)\mathbf{1}_Y.
\end{aligned} \tag{3.27}$$

In particular, $\eta^{-1}E_1(X)\mathbf{1}_Y = 0$. Thus, for $\ell \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}_{>0}$,

$$\begin{aligned}
\eta^{-\ell} E_m(X)\mathbf{1}_Y &= t^{\frac{1}{2}\ell} \frac{(1-q^{m-\ell})\cdots(1-q^{m-2})(1-q^{m-1})}{(1-tq^{m-\ell})\cdots(1-tq^{m-2})(1-tq^{m-1})} E_{m-\ell}(X)\mathbf{1}_Y \\
&= t^{\frac{1}{2}} \mathrm{ev}_m\Big(\frac{(t^{-1}Y^{-2}q^{-\ell};q)_\ell}{(Y^{-2}q^{-\ell};q)_\ell}\Big) E_{m-\ell}(X)\mathbf{1}_Y,
\end{aligned}$$

where the right hand side evaluates to 0 if $\ell \geq m$.

Assume $m \in \mathbb{Z}_{\geq 0}$. By (3.16) and (3.15),

$$\begin{aligned}
\eta^{-1} E_{-m}(X)\mathbf{1}_Y &= t^{\frac{1}{2}} \frac{(1-Y^{-2})}{(1-tY^{-2})} \tau_1^\vee \tau_\pi^\vee E_{-m}(X)\mathbf{1}_Y = t^{\frac{1}{2}} \frac{(1-Y^{-2})}{(1-tY^{-2})} \tau_1^\vee t^{\frac{1}{2}} E_{m+1}(X)\mathbf{1}_Y \\
&= t \frac{(1-Y^{-2})}{(1-tY^{-2})} t^{-\frac{1}{2}} E_{-(m+1)}(X)\mathbf{1}_Y = t^{\frac{1}{2}} \frac{(1-t^{-1}q^{-(m+1)})}{(1-tt^{-1}q^{-(m+1)})} E_{-(m+1)}(X)\mathbf{1}_Y \\
&= t^{\frac{1}{2}} \frac{(1-t^{-1}q^{-m-1})}{(1-q^{-m-1})} E_{-m-1}(X)\mathbf{1}_Y = t^{-\frac{1}{2}} \frac{(1-tq^{m+1})}{(1-q^{m+1})} E_{-m-1}(X)\mathbf{1}_Y.
\end{aligned} \tag{3.28}$$

Thus, for $\ell \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned}
\eta^{-\ell} E_{-m}(X)\mathbf{1}_Y &= t^{-\frac{1}{2}\ell} \cdot \frac{(1-tq^{m+1})(1-q^{m+2})\cdots(1-q^{m+\ell})}{(1-q^{m+1})(1-q^{m+2})\cdots(1-q^{m+\ell})} E_{-m-\ell}(X)\mathbf{1}_Y \\
&= t^{-\frac{1}{2}\ell} \mathrm{ev}_m\Big(\frac{(Y^{-2}q;q)_\ell}{(t^{-1}Y^{-2}q;q)_\ell}\Big) E_{-m-\ell}(X)\mathbf{1}_Y.
\end{aligned}$$

□

3.3 Some identities in $\tilde{H}$

Proposition 3.4. *Let $\ell \in \mathbb{Z}_{>0}$. As elements of $\tilde{H}_{\mathrm{int}}$,*

$$E_{-\ell}(X)\mathbf{1}_0 = t^{\frac{1}{2}}\Big(T_1^{-1} + t^{-\frac{1}{2}} \frac{(1-t)q^\ell t}{(1-q^\ell t)}\Big) E_\ell(X)\mathbf{1}_0, \qquad E_{\ell+1}(X)\mathbf{1}_0 = t^{-\frac{1}{2}} \tau_\pi^\vee E_{-\ell}(X)\mathbf{1}_0, \tag{3.29}$$

and

$$P_\ell(X)\mathbf{1}_0 = t^{\frac{1}{2}} \mathbf{1_0} E_\ell(X)\mathbf{1}_0. \tag{3.30}$$

Additionally, $E_1(X)\mathbf{1}_0 = t^{-\frac{1}{2}} \tau_\pi^\vee E_0(X)\mathbf{1}_0$.

Proof. (a) Let $Q_1(X), Q_2(X) \in \mathbb{C}[X, X^{-1}]$ be such that

$$t^{\frac{1}{2}}\Big(T_1^{-1} + t^{-\frac{1}{2}}\frac{(1-t)q^\ell t}{(1-q^\ell t)}\Big)E_\ell(X) = Q_1(X)T_1 + Q_2(X).$$

Then, by (3.16),

$$\begin{aligned} E_{-\ell}(X)\mathbf{1}_Y &= t^{\frac{1}{2}}\tau_1^\vee E_\ell(X)\mathbf{1}_Y = t^{\frac{1}{2}}\Big(T_1^{-1} + t^{-\frac{1}{2}}\frac{(1-t)Y^{-2}}{(1-Y^{-2})}\Big)E_\ell(X)\mathbf{1}_Y \\ &= t^{\frac{1}{2}}\Big(T_1^{-1} + t^{-\frac{1}{2}}\frac{(1-t)q^\ell t}{(1-q^\ell t)}\Big)E_\ell(X)\mathbf{1}_Y = (Q_1(X)t^{\frac{1}{2}} + Q_2(X))\mathbf{1}_Y. \end{aligned}$$

Since $\{X^k\mathbf{1}_Y \mid k \in \mathbb{Z}\}$ is a basis of $\tilde{H}\mathbf{1}_Y$ then $E_{-\ell}(X) = Q_1(X)t^{\frac{1}{2}} + Q_2(X)$. So

$$\begin{aligned} E_{-\ell}(X)\mathbf{1}_0 &= (Q_1(X)t^{\frac{1}{2}} + Q_2(X))\mathbf{1}_0 \\ &= (Q_1(X)T_1 + Q_2(X))\mathbf{1}_0 = t^{\frac{1}{2}}\Big(T_1^{-1} + t^{-\frac{1}{2}}\frac{(1-t)q^\ell t}{(1-q^\ell t)}\Big)E_\ell(X)\mathbf{1}_0. \end{aligned}$$

(b) Let $Q_1(X), Q_2(X) \in \mathbb{C}[X, X^{-1}]$ such that $t^{-\frac{1}{2}}\tau_\pi^\vee E_{-\ell}(X) = XT_1E_{-\ell}(X) = Q_1(X)T_1 + Q_2(X)$. Then, by (3.16),

$$E_{\ell+1}(X)\mathbf{1}_Y = t^{-\frac{1}{2}}\tau_\pi^\vee E_{-\ell}(X)\mathbf{1}_Y = (Q_1(X)T_1 + Q_2(X))\mathbf{1}_Y = (Q_1(X)t^{\frac{1}{2}} + Q_2(X))\mathbf{1}_Y.$$

Since $\{X^k\mathbf{1}_Y \mid k \in \mathbb{Z}\}$ is a basis of $\tilde{H}\mathbf{1}_Y$ then $E_{\ell+1}(X) = (Q_1(X)t^{\frac{1}{2}} + Q_2(X))$. So

$$E_{\ell+1}(X)\mathbf{1}_0 = (Q_1(X)t^{\frac{1}{2}} + Q_2(X))\mathbf{1}_0 = (Q_1(X)T_1 + Q_2(X))\mathbf{1}_0 = t^{-\frac{1}{2}}\tau_\pi^\vee E_{-\ell}(X)\mathbf{1}_0.$$

(c) Let $Q_1(X), Q_2(X)$ be such that $t^{\frac{1}{2}}\mathbf{1}_0E_\ell(X) = Q_1(X)T_1 + Q_2(X)$. Then

$$P_\ell(X)\mathbf{1}_Y = t^{\frac{1}{2}}\mathbf{1}_0E_\ell(X)\mathbf{1}_Y = (Q_1(X)T_1 + Q_2(X))\mathbf{1}_Y = (Q_1(X)t^{\frac{1}{2}} + Q_2(X))\mathbf{1}_Y.$$

So $P_\ell(X) = (Q_1(X)t^{\frac{1}{2}} + Q_2(X))$ and

$$P_\ell(X)\mathbf{1}_0 = (Q_1(X)t^{\frac{1}{2}} + Q_2(X))\mathbf{1}_0 = (Q_1(X)T_1 + Q_2(X))\mathbf{1}_0 = t^{\frac{1}{2}}\mathbf{1}_0E_\ell(X)\mathbf{1}_0.$$

□

Proposition 3.5. *Let*

$$c(Y) = \frac{(1-tY^2)}{(1-Y^2)} \qquad \textit{and} \qquad F_\ell(Y) = \frac{(1-t)}{(1-tq^\ell)}\frac{(1-tY^2q^\ell)}{(1-Y^2)}, \quad \textit{for } \ell \in \mathbb{Z}_{>0}.$$

Then, as elements of $\tilde{H}$,

$$E_{-\ell}(X)\mathbf{1}_0 = (\eta_{s_1}c(Y) + F_\ell(Y))E_\ell(X)\mathbf{1}_0 \quad \textit{and} \quad E_{\ell+1}(X)\mathbf{1}_0 = t^{-\frac{1}{2}}\eta_\pi E_{-\ell}(X)\mathbf{1}_0. \tag{3.31}$$

Proof. Using the first identity in (3.29), (3.8) and (3.9),

$$
\begin{aligned}
E_{-\ell}(X)\mathbf{1}_0 &= t^{\frac12}\Big(T_1^{-1} + t^{-\frac12}\frac{(1-t)q^\ell t}{(1-q^\ell t)}\Big)E_\ell(X)\mathbf{1}_0 \\
&= t^{\frac12}\Big(\tau_1^\vee - t^{-\frac12}\frac{(1-t)Y^{-2}}{(1-Y^{-2})} + t^{-\frac12}\frac{(1-t)q^\ell t}{(1-q^\ell t)}\Big)E_\ell(X)\mathbf{1}_0 \\
&= t^{\frac12}\Big(t^{-\frac12}\frac{(1-tY^{-2})}{(1-Y^{-2})}\eta_{s_1} + t^{-\frac12}\frac{(1-t)(-Y^{-2}+q^\ell tY^{-2}+q^\ell t-q^\ell tY^{-2})}{(1-Y^{-2})(1-q^\ell t)}\Big)E_\ell(X)\mathbf{1}_0 \\
&= t^{\frac12}\Big(t^{-\frac12}\eta_{s_1}\frac{(1-tY^2)}{(1-Y^2)} + t^{-\frac12}\frac{(1-t)(-1+q^\ell tY^2)}{(Y^2-1)(1-q^\ell t)}\Big)E_\ell(X)\mathbf{1}_0 \\
&= \Big(\eta_{s_1}\frac{(1-tY^2)}{(1-Y^2)} + \frac{(1-t)(1-tY^2q^\ell)}{(1-tq^\ell)(1-Y^2)}\Big)E_\ell(X)\mathbf{1}_0,
\end{aligned}
$$

where the next to last equality follows from the second identity in (3.13). Then, by (3.9), the second identity in (3.29) and (3.10),

$$
E_{\ell+1}(X)\mathbf{1}_0 = t^{-\frac12}\eta_\pi E_{-\ell}(X)\mathbf{1}_0 = t^{-\frac12}\Big(\eta\frac{(1-tY^2)}{(1-Y^2)} + \eta_\pi\frac{(1-t)(1-tY^2q^\ell)}{(1-tq^\ell)(1-Y^2)}\Big)E_\ell(X)\mathbf{1}_0.
$$

□

4 Operator expansions

Let $\mathbb{C}(Y)$ be the field of fractions of $\mathbb{C}[Y, Y^{-1}]$. As indicated in Section 3.1, as a left $\mathbb{C}(Y)$-module, the localised double affine Hecke algebra

$$
\tilde{H} \quad \text{has } \mathbb{C}(Y)\text{-basis} \qquad \{X^k \mid k\in\mathbb{Z}\}\cup\{X^kT_1 \mid k\in\mathbb{Z}\}.
$$

Then $\tilde{H}\mathbf{1}_0$ is a $\mathbb{C}(Y)$-subspace of $\tilde{H}$ and

$$
\tilde{H}\mathbf{1}_0 \quad \text{has } \mathbb{C}(Y)\text{-basis} \qquad \{X^k\mathbf{1}_0 \mid k\in\mathbb{Z}\},
$$

since, by the first relation in (3.6), $T_1\mathbf{1}_0 = t^{\frac12}\mathbf{1}_0$. The sets

$$
\{E_k(X)\mathbf{1}_0 \mid k\in\mathbb{Z}\} \qquad \text{and} \qquad \{\eta^k(X)\mathbf{1}_0 \mid k\in\mathbb{Z}\} \qquad \text{are also } \mathbb{C}(Y)\text{-bases of } \tilde{H}\mathbf{1}_0,
$$

and the results of this section and the next provide explicit product formulas for the transition coefficients between these bases.

4.1 Definition of $D_j^{(\ell)}(Y)$ and $K_j^{(\ell)}(Y)$

Define functions $D_j^{(\ell-1)}(Y)$ for $\ell\in\mathbb{Z}_{>0}$ and $D_j^{(-\ell)}(Y)$ for $\ell\in\mathbb{Z}_{\ge0}$ by the expansions

$$
E_\ell(X)\mathbf{1}_0 = \sum_{j=0}^{\ell-1}\eta^{\ell-2j}D_j^{(\ell-1)}(Y)\mathbf{1}_0 \qquad \text{and} \qquad E_{-\ell}(X)\mathbf{1}_0 = \sum_{j=0}^{\ell}\eta^{-\ell+2j}D_j^{(-\ell)}(Y)\mathbf{1}_0. \tag{4.1}
$$

Define $K_j^{(\ell)}(Y)$ for $\ell\in\mathbb{Z}_{\ge0}$ and $j\in\{0,1,\ldots,\ell\}$ by

$$
\mathbf{1}_0E_\ell(X)\mathbf{1}_0 = \sum_{j=0}^{\ell}\mathbf{1}_0\eta^{\ell-2j}K_j^{(\ell)}(Y). \tag{4.2}
$$

For example,

$$E_3(X)\mathbf{1}_0 = \eta^3 D_0^{(2)}(Y)\mathbf{1}_0 + \eta D_1^{(2)}(Y)\mathbf{1}_0 + \eta^{-1} D_2^{(2)}(Y)\mathbf{1}_0,$$

$$E_{-3}(X)\mathbf{1}_0 = \eta^3 D_3^{(-3)}(Y)\mathbf{1}_0 + \eta D_2^{(-3)}(Y)\mathbf{1}_0 + \eta^{-1} D_1^{(-3)}(Y)\mathbf{1}_0 + \eta^{-3} D_0^{(-3)}(Y)\mathbf{1}_0,$$

and

$$\mathbf{1}_0 E_3(X)\mathbf{1}_0 = \mathbf{1}_0\eta^3 K_0^{(3)}(Y) + \mathbf{1}_0\eta K_1^{(3)}(Y) + \mathbf{1}_0\eta^{-1} K_2^{(3)}(Y) + \mathbf{1}_0\eta^{-3} K_3^{(3)}(Y).$$

See Section 7.4 for examples of the first few of the functions $D_j^{(\ell)}(Y)$ and $K_j^{(\ell)}(Y)$. Proposition 4.2 below provides a formula for the $K_j^{(\ell)}(Y)$ in terms of the $D_j^{(\ell)}(Y)$.

4.2 A recursion for the $D_j^{(\ell)}(Y)$

The following Proposition provides recursions determining the $D_j^{(\ell)}(Y)$, showing that the $D_j^{(\ell)}$ for $\ell \in \mathbb{Z}_{\geq 0}$ and $j \in \{0, \ldots, \ell-1\}$ form something like a Pascal triangle,

$$\begin{array}{ccccccc}
 & & & D_0^{(0)} & & & \\
 & & D_0^{(1)} & & D_1^{(1)} & & \\
 & D_0^{(2)} & & D_1^{(2)} & & D_2^{(2)} & \\
D_0^{(3)} & & D_1^{(3)} & & D_2^{(3)} & & D_3^{(3)}
\end{array}$$

Proposition 4.1. *Let $D_j^{(\ell)}(Y)$ and $D_j^{(-\ell)}(Y)$ be as defined in* (4.1)*.*

(a) If $\ell \in \mathbb{Z}_{>0}$ and $j \in \{0, \ldots, \ell\}$ then $D_j^{(-\ell)}(Y) = t^{\frac{1}{2}} D_j^{(\ell)}(Y^{-1})$.

(b) The $D_j^{(\ell)}(Y)$ satisfy, and are determined by, $D_0^{(0)}(Y) = t^{-\frac{1}{2}}$ and the recursion

$$D_0^{(\ell)} = t^{\frac{1}{2}} \frac{(1-t^{-1}Y^{-2}q^{\ell})}{(1-Y^{-2}q^{\ell})} D_0^{(\ell-1)}(Y), \qquad D_\ell^{(\ell)} = t^{-\frac{1}{2}} \frac{(1-t)(1-tY^{-2})}{(1-tq^{\ell})(1-Y^{-2}q^{-\ell})} D_0^{(\ell-1)}(Y^{-1}),$$

$$D_j^{(\ell)}(Y) = t^{\frac{1}{2}} \frac{(1-t^{-1}Y^{-2}q^{\ell-2j})}{(1-Y^{-2}q^{\ell-2j})} D_j^{(\ell-1)}(Y) + t^{-\frac{1}{2}} \frac{(1-t)(1-q^{\ell}tY^{-2}q^{\ell-2j})}{(1-tq^{\ell})(1-Y^{-2}q^{\ell-2j})} D_{\ell-j}^{(\ell-1)}(Y^{-1}). \tag{4.3}$$

Proof. (a) Using the first relation in (4.1), the second relation in (3.29), the second relation in (4.1),

$$\sum_{j=0}^{\ell} \eta^{\ell+1-2j} D_j^{(\ell)}(Y)\mathbf{1}_0 = E_{\ell+1}(X)\mathbf{1}_0 = t^{-\frac{1}{2}}\tau_\pi^\vee E_{-\ell}(X)\mathbf{1}_0 = t^{-\frac{1}{2}}\eta_\pi \sum_{j=0}^{\ell} \eta^{-\ell+2j} D_j^{(-\ell)}(Y)\mathbf{1}_0.$$

By (3.12), (3.13), and the second relation in (3.14),

$$\begin{aligned}
\eta_\pi \eta^{-\ell+2j} D_j^{(-\ell)}(Y)\mathbf{1}_0 &= \eta_\pi\eta_{s_1}\eta_{s_1}\eta^{-\ell+2j} D_j^{(-\ell)}(Y)\mathbf{1}_0 = \eta\eta_{s_1}\eta^{-\ell+2j} D_j^{(-\ell)}(Y)\mathbf{1}_0 \\
&= \eta\eta^{\ell-2j} D_j^{(-\ell)}(Y^{-1})\eta_{s_1}\mathbf{1}_0 = \eta^{\ell-2j+1} D_j^{(-\ell)}(Y^{-1})\mathbf{1}_0.
\end{aligned}$$

So $\sum_{j=0}^{\ell} \eta^{\ell+1-2j} D_j^{(\ell)}(Y)\mathbf{1}_0 = t^{-\frac{1}{2}} \sum_{j=0}^{\ell} \eta^{\ell-2j+1} D_j^{(-\ell)}(Y^{-1})\mathbf{1}_0$, giving $D_j^{(\ell)}(Y) = t^{-\frac{1}{2}} D_j^{(-\ell)}(Y^{-1})$.

(b) Let

$$c(Y) = \frac{(1-tY^2)}{(1-Y^2)} \qquad \text{and} \qquad F_\ell(Y) = \frac{(1-t)}{(1-tq^\ell)} \frac{(1-q^\ell tY^2)}{(1-Y^2)}, \quad \text{for } \ell \in \mathbb{Z}_{>0}.$$

By (4.1) and the first relation in (3.31),

$$\sum_{j=0}^{\ell}\eta^{-\ell+2j}D_j^{(-\ell)}(Y)\mathbf{1}_0 = E_{-\ell}(X)\mathbf{1}_0 = (\eta_{s_1}c(Y)+F_\ell(Y))E_\ell(X)\mathbf{1}_0$$
$$= (\eta_{s_1}c(Y)+F_\ell(Y))\sum_{j=0}^{\ell-1}\eta^{\ell-2j}D_j^{(\ell-1)}(Y)\mathbf{1}_0$$

By (3.13) and the last relation in (3.12),

$$\eta_{s_1}c(Y)\eta^{\ell-2j}D_j^{(\ell-1)}(Y) = \eta_{s_1}\eta^{\ell-2j}c(q^{-\frac{1}{2}(\ell-2j)}Y)D_j^{(\ell-1)}(Y) = \eta^{-(\ell-2j)}\eta_{s_1}c(q^{-\frac{1}{2}(\ell-2j)}Y)D_j^{(\ell-1)}(Y)$$
$$= \eta^{-(\ell-2j)}c(q^{-\frac{1}{2}(\ell-2j)}Y^{-1})D_j^{(\ell-1)}(Y^{-1})\eta_{s_1}$$

and $F_\ell(Y)\eta^{\ell-2j}D_j^{(\ell-1)}(Y) = \eta^{\ell-2j}F_\ell(q^{-\frac{1}{2}(\ell-2j)}Y)D_j^{(\ell-1)}(Y)$. So

$$\sum_{j=0}^{\ell}\eta^{-(\ell-2j)}D_j^{(-\ell)}(Y)\mathbf{1}_0 = (\eta_{s_1}c(Y)+F_\ell(Y))\sum_{j=0}^{\ell-1}\eta^{\ell-2j}D_j^{(\ell-1)}(Y)\mathbf{1}_0$$
$$= \sum_{j=0}^{\ell-1}\eta^{-(\ell-2j)}c(q^{-\frac{1}{2}(\ell-2j)}Y^{-1})D_j^{(\ell-1)}(Y^{-1})\mathbf{1}_0 + \sum_{j=0}^{\ell-1}\eta^{\ell-2j}F_\ell(q^{-\frac{1}{2}(\ell-2j)}Y)D_j^{(\ell-1)}(Y)\mathbf{1}_0$$

where the last equality uses the relation $\eta_{s_1}\mathbf{1}_0 = \mathbf{1}_0$ from (3.14). Putting $k=\ell-j$ in the second sum makes $j=\ell-k$ and

$$\sum_{j=0}^{\ell-1}\eta^{\ell-2(\ell-k)}F_\ell(q^{-\frac{1}{2}(\ell-2(\ell-k))}Y)D_j^{(\ell-1)}(Y)\mathbf{1}_0 = \sum_{k=1}^{\ell}\eta^{-(\ell-2k)}F_\ell(q^{\frac{1}{2}(\ell-2k)}Y)D_{\ell-k}^{(\ell-1)}(Y)\mathbf{1}_0$$

Thus $\sum_{j=0}^{\ell}\eta^{-\ell+2j}D_j^{(-\ell)}(Y)\mathbf{1}_0$ is equal to

$$\eta^{-\ell}c(q^{-\frac{1}{2}\ell}Y^{-1})D_0^{(\ell-1)}(Y^{-1})\mathbf{1}_0 + \eta^{\ell}F_\ell(q^{-\frac{1}{2}\ell}Y)D_0^{(\ell-1)}(Y)\mathbf{1}_0$$
$$+\sum_{j=1}^{\ell-1}\eta^{-(\ell-2j)}\big(c(q^{-\frac{1}{2}(\ell-2j)}Y^{-1})D_j^{(\ell-1)}(Y^{-1}) + F_\ell(q^{\frac{1}{2}(\ell-2j)}Y)D_{\ell-j}^{(\ell-1)}(Y)\big)\mathbf{1}_0,$$

which implies

$$D_0^{(-\ell)}(Y) = c(q^{-\frac{1}{2}\ell}Y^{-1})D_0^{(\ell-1)}(Y^{-1}), \qquad D_\ell^{(-\ell)}(Y) = F_\ell(q^{-\frac{1}{2}\ell}Y)D_0^{(\ell-1)}(Y), \qquad \text{and}$$
$$D_j^{(-\ell)}(Y) = c(q^{-\frac{1}{2}(\ell-2j)}Y^{-1})D_j^{(\ell-1)}(Y^{-1}) + F_\ell(q^{\frac{1}{2}(\ell-2j)}Y)D_{\ell-j}^{(\ell-1)}(Y),$$

for $j\in\{1,\ldots,\ell-1\}$. Combining this with the identity $D_j^{(\ell)}(Y) = t^{-\frac{1}{2}}D_j^{(-\ell)}(Y^{-1})$ from (a) gives

$$D_0^{(\ell)}(Y) = t^{-\frac{1}{2}}c(q^{-\frac{1}{2}\ell}Y)D_0^{(\ell)}(Y), \qquad D_\ell^{(\ell)}(Y) = t^{-\frac{1}{2}}F_\ell(q^{-\frac{1}{2}\ell}Y^{-1})D_0^{(\ell-1)}(Y^{-1}), \qquad \text{and}$$
$$D_j^{(\ell)}(Y) = t^{-\frac{1}{2}}c(q^{-\frac{1}{2}(\ell-2j)}Y)D_j^{(\ell-1)}(Y) + t^{-\frac{1}{2}}F_\ell(q^{\frac{1}{2}(\ell-2j)}Y^{-1})D_{\ell-j}^{(\ell-1)}(Y^{-1}).$$

□

4.3 A formula for $K_j^{(\ell)}(Y)$ in terms of the $D_j^{(\ell)}(Y)$

Since $E_0(X)=1$, equation (4.2) and the first relation in (3.7) give $\mathbf{1}_0E_0(X)\mathbf{1}_0=\mathbf{1}_0^2=\mathbf{1}_0(t^{\frac12}+t^{-\frac12})$ so that

$$K_0^{(0)}=t^{\frac12}+t^{-\frac12}.$$

Proposition 4.2. *Let $\ell\in\mathbb{Z}_{>0}$ and let $K_j^{(\ell)}(Y)$ and $D_j^{(\ell-1)}(Y)$ be as defined in (4.2) and (4.1). Then*

$$K_0^{(\ell)}(Y)=t^{-\frac12}D_0^{(\ell-1)}(Y)\frac{(1-tY^2)}{(1-Y^2)},\qquad K_\ell^{(\ell)}(Y)=t^{\frac12}D_0^{(\ell-1)}(Y^{-1})\frac{(1-t^{-1}Y^2q^\ell)(1-tY^2)}{(1-tY^2q^\ell)(1-Y^2)},$$

and, for $j\in\{1,\dots,\ell-1\}$,

$$K_j^{(\ell)}(Y)=t^{\frac12}D_j^{(\ell-1)}(Y)\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}+t^{-\frac12}D_{\ell-j}^{(\ell-1)}(Y^{-1})\frac{(1-tY^{-2}q^{\ell-2j})}{(1-t^{-1}Y^{-2}q^{\ell-2j})}\frac{(1-t^{-1}Y^2)}{(1-Y^{-2})}. \tag{4.4}$$

Proof. Let

$$c(Y)=\frac{(1-tY^2)}{(1-Y^2)}=t\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}.$$

Then, by (4.1), the first relation in (3.14), the second relation in (3.13) and the relation $\eta\eta_{s_1}=\eta_{s_1}\eta^{-1}$ from (3.12),

$$\begin{aligned}\mathbf{1}_0E_\ell(X)\mathbf{1}_0&=\mathbf{1}_0\sum_{j=0}^{\ell-1}\eta^{\ell-2j}D_j^{(\ell-1)}(Y)\mathbf{1}_0=\mathbf{1}_0\sum_{j=0}^{\ell-1}\eta^{\ell-2j}D_j^{(\ell-1)}(Y)(1+\eta_{s_1})t^{-\frac12}c(Y)\\&=\sum_{j=0}^{\ell-1}\mathbf{1}_0\eta^{\ell-2j}D_j^{(\ell-1)}(Y)t^{-\frac12}c(Y)+\sum_{j=0}^{\ell-1}\mathbf{1}_0\eta_{s_1}\eta^{-(\ell-2j)}D_j^{(\ell-1)}(Y^{-1})t^{-\frac12}c(Y).\end{aligned}$$

By the last relation in (3.14) and the last relation in (3.13),

$$\mathbf{1}_0\eta_{s_1}\eta^{-(\ell-2j)}=\mathbf{1}_0t^{-1}\frac{(1-tY^{-2})}{(1-t^{-1}Y^{-2})}\eta^{-(\ell-2j)}=\mathbf{1}_0\eta^{-(\ell-2j)}t^{-1}\frac{(1-tY^{-2}q^{-(\ell-2j)})}{(1-t^{-1}Y^{-2}q^{-(\ell-2j)})}$$

and thus, by reindexing with $k=\ell-j$ and using $-(\ell-2j)=\ell-2k$, the second sum is

$$\sum_{j=0}^{\ell-1}\mathbf{1}_0\eta_{s_1}\eta^{-(\ell-2j)}D_j^{(\ell-1)}(Y^{-1})t^{-\frac12}c(Y)=\sum_{k=1}^{\ell}\mathbf{1}_0\eta^{\ell-2k}t^{-1}\frac{(1-tY^{-2}q^{\ell-2k})}{(1-t^{-1}Y^{-2}q^{\ell-2k})}D_{\ell-k}^{(\ell-1)}(Y^{-1})t^{-\frac12}c(Y).$$

Hence, by (4.2),

$$\begin{aligned}&\sum_{j=0}^{\ell}\mathbf{1}_0\eta^{\ell-2j}K_j^{(\ell)}(Y)=\mathbf{1}_0E_\ell(X)\mathbf{1}_0\\&\quad=\sum_{j=0}^{\ell-1}\mathbf{1}_0t^{-\frac12}\eta^{\ell-2j}D_j^{(\ell-1)}(Y)c(Y)+\sum_{k=1}^{\ell}\mathbf{1}_0t^{-\frac32}\eta^{\ell-2k}\frac{(1-tY^{-2}q^{\ell-2k})}{(1-t^{-1}Y^{-2}q^{\ell-2k})}D_{\ell-k}^{(\ell-1)}(Y^{-1})c(Y).\end{aligned}$$

$\square$

5 Product expressions for $D_j^{(\ell)}(Y)$ and $K_j^{(\ell)}(Y)$

In this section we establish product formulas for the coefficients in the operators

$$E_\ell(X)\mathbf{1}_0 = \sum_{j=0}^{\ell-1} \eta^{\ell-2j} D_j^{(\ell-1)}(Y)\mathbf{1}_0, \qquad E_{-\ell}(X)\mathbf{1}_0 = \sum_{j=0}^{\ell} \eta^{-\ell+2j} D_j^{(-\ell)}(Y)\mathbf{1}_0 \qquad \text{and}$$

$$\mathbf{1}_0 E_\ell(X)\mathbf{1}_0 = \sum_{j=0}^{\ell} \mathbf{1}_0 \eta^{\ell-2j} K_j^{(\ell)}(Y).$$

These coefficients turn out to be something like generalized binomial coefficients, determined by the recursions that were established in Proposition 4.1 and 4.2. These product formulas provide a kind of binomial theorem for operators $E_\ell(X)\mathbf{1}_0$ and $\mathbf{1}_0 E_\ell(X)\mathbf{1}_0$, as elements of the double affine Hecke algebra $\tilde{H}$. The final result is Theorem 5.4.

5.1 q-t-binomial coefficients

For $\ell \in \mathbb{Z}_{\ge 0}$ and $j \in \{0, \ldots, \ell\}$ define

$$(z;q)_j = (1-z)(1-zq)(1-zq^2)\cdots(1-zq^{j-1}) \qquad \text{and} \qquad \begin{bmatrix} \ell \\ j \end{bmatrix}_{q,t} = \frac{\frac{(q;q)_\ell}{(t;q)_\ell}}{\frac{(q;q)_j}{(t;q)_j}\frac{(q;q)_{\ell-j}}{(t;q)_{\ell-j}}}. \tag{5.1}$$

For $a, b \in \mathbb{Z}$ with $a \le b$ let

$$(1-zq^{a..b}) = (1-zq^a)(1-zq^{a+1})\cdots(1-zq^{b-1})(1-zq^b).$$

With this notation,

$$\begin{bmatrix} \ell \\ j \end{bmatrix}_{q,t} = \frac{(1-q^{1..\ell})}{(1-q^{1..j})(1-q^{1..\ell-j})} \frac{(1-tq^{0..j-1})(1-tq^{0..\ell-j-1})}{(1-tq^{0..\ell-1})} = \frac{(1-q^{j+1..\ell})}{(1-q^{1..\ell-j})} \frac{(1-tq^{0..\ell-j-1})}{(1-tq^{j..\ell-1})}.$$

Then

$$\begin{bmatrix} \ell \\ \ell-j \end{bmatrix}_{q,t} = \begin{bmatrix} \ell \\ j \end{bmatrix}_{q,t} \qquad \text{and} \qquad \begin{bmatrix} \ell+1 \\ j \end{bmatrix}_{q,t} = \begin{bmatrix} \ell \\ j \end{bmatrix}_{q,t} \cdot \frac{(1-q^{\ell+1})(1-tq^{\ell-j})}{(1-q^{\ell+1-j})(1-tq^\ell)}. \tag{5.2}$$

5.2 Y-binomial coefficients

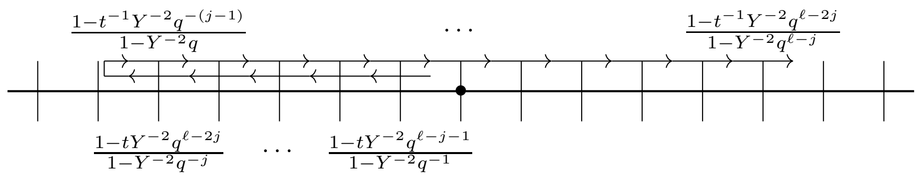
For $\ell \in \mathbb{Z}_{\ge 0}$ and $j \in \{0, 1, \ldots, \ell\}$ define a rational function in Y by

$$\binom{\ell}{j}_Y = \frac{(t^{-1}Y^{-2}q^{-(j-1)};q)_{\ell-j}(tY^{-2}q^{\ell-2j};q)_j}{(Y^{-2}q;q)_{\ell-j}(Y^{-2}q^{-j};q)_j}. \tag{5.3}$$

An alternative expression is

$$\binom{\ell}{j}_Y = \frac{(1-t^{-1}Y^{-2}q^{-(j-1)..\ell-2j})(1-tY^{-2}q^{\ell-2j..\ell-j-1})}{(1-Y^{-2}q^{1..\ell-j})(1-Y^{-2}q^{-j..-1})}.$$

which, in the alcove walk point of view of [Yi10], might be thought of as a weighted alcove walk of total length ℓ with j left moving crossings and $\ell-j$ right moving crossings.



Then (see the examples in Section 7.3)

$$\begin{aligned}
\binom{\ell}{\ell-j}_{Y^{-1}} &= \frac{(1-t^{-1}Y^2q^{-(\ell-j-1)..\ell-2(\ell-j)})(1-tY^2q^{\ell-2(\ell-j)..\ell-(\ell-j)-1})}{(1-Y^2q^{1..\ell-(\ell-j)})(1-Y^2q^{-(\ell-j)..-1})} \\
&= \frac{(1-t^{-1}Y^2q^{-(\ell-j-1)..-(\ell-2j)})(1-tY^2q^{-(\ell-2j)..j-1})}{(1-Y^2q^{1..j})(1-Y^2q^{-(\ell-j)..-1})} \\
&= \frac{(1-tY^{-2}q^{\ell-2j..\ell-j-1})(1-t^{-1}Y^{-2}q^{-(j-1)..\ell-2j})}{(1-Y^{-2}q^{-j..-1})(1-Y^{-2}q^{1..\ell-j})} \\
&\qquad \cdot \frac{(t^{-1}Y^2)^j(q^{-(\ell-j)})^j q^{\frac12 j(j-1)}(tY^2)^{\ell-j}(q^{-(\ell-2j)-1})^{\ell-j}q^{\frac12(\ell-j)(\ell-j-1)}}{Y^{2j}q^{\frac12 j(j-1)}Y^{2(\ell-j)}(q^{-(\ell-j+1)})^{\ell-j}q^{\frac12(\ell-j)(\ell-j-1)}} \\
&= t^{\ell-2j}q^0\cdot\binom{\ell}{j}_Y.
\end{aligned} \tag{5.4}$$

Also

$$\begin{aligned}
\binom{\ell+1}{j}_Y &= \frac{(1-t^{-1}Y^{-2}q^{-(j-1)..\ell+1-2j})(1-tY^{-2}q^{\ell+1-2j..\ell+1-j-1})}{(1-Y^{-2}q^{1..\ell+1-j})(1-Y^{-2}q^{-j..-1})} \\
&= \binom{\ell}{j}_Y\cdot\frac{(1-t^{-1}Y^{-2}q^{\ell+1-2j})(1-tY^{-2}q^{\ell-j})}{(1-tY^{-2}q^{\ell-2j})(1-Y^{-2}q^{\ell+1-j})}
\end{aligned} \tag{5.5}$$

5.3 Definition of the products $\tilde K_j^{(\ell)}(Y)$ and $\tilde D_j^{(\ell-1)}(Y)$

For $\ell\in\mathbb{Z}_{>0}$ and $j\in\{0,\ldots,\ell\}$ define

$$\tilde D_j^{(\ell)}(Y) = t^{-\frac12(\ell+1)}\cdot t^{\ell-j}\cdot\begin{bmatrix}\ell\\ j\end{bmatrix}_{q,t}\cdot\binom{\ell}{j}_Y\cdot\frac{(1-tq^{\ell-j})}{(1-tq^{\ell})}\cdot\frac{(1-tY^{-2}q^{\ell-j})}{(1-tY^{-2}q^{\ell-2j})} \tag{5.6}$$

$$\tilde D_j^{(-\ell)}(Y) = t^{-\frac12\ell}\cdot(qt)^j\cdot\begin{bmatrix}\ell\\ j\end{bmatrix}_{q,t}\cdot\binom{\ell}{\ell-j}_Y\cdot\frac{(1-tq^{\ell-j})}{(1-tq^{\ell})}\cdot\frac{(1-t^{-1}Y^{-2}q^{-(\ell-j)})}{(1-t^{-1}Y^{-2}q^{-(\ell-2j)})} \tag{5.7}$$

$$\tilde K_j^{(\ell)}(Y) = t^{-\frac12(\ell-1)}\cdot t^{\ell-1-j}\cdot\begin{bmatrix}\ell\\ j\end{bmatrix}_{q,t}\cdot\binom{\ell}{j}_Y\cdot\frac{(1-Y^{-2}q^{\ell-2j})}{(1-t^{-1}Y^{-2}q^{\ell-2j})}\cdot\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})} \tag{5.8}$$

The following Proposition provides useful relationships between these expressions which follow from (5.2), (5.3) and (5.4).

Proposition 5.1.

$$\frac{\tilde{D}^{(\ell-1)}_j(Y)}{\tilde{K}^{(\ell)}_j(Y)} = t^{-\frac{1}{2}} \cdot \frac{(1-q^{\ell-j})}{(1-q^{\ell})} \cdot \frac{(1-Y^{-2}q^{\ell-j})}{(1-Y^{-2}q^{\ell-2j})} \cdot \frac{(1-Y^{-2})}{(1-t^{-1}Y^{-2})} \tag{5.9}$$

$$\frac{\tilde{D}^{(\ell-1)}_{\ell-j}(Y^{-1})}{\tilde{K}^{(\ell)}_j(Y)} = t^{\frac{1}{2}}q^{\ell-j} \cdot \frac{(1-q^{j})}{(1-q^{\ell})} \cdot \frac{(1-Y^{-2})}{(1-t^{-1}Y^{-2})} \cdot \frac{(1-t^{-1}Y^{-2}q^{\ell-2j})}{(1-tY^{-2}q^{\ell-2j})} \cdot \frac{(1-Y^{-2}q^{-j})}{(1-Y^{-2}q^{\ell-2j})}. \tag{5.10}$$

$$\frac{\tilde{D}^{(\ell)}_j(Y)}{\tilde{D}^{(\ell+1)}_j(Y)} = t^{-\frac{1}{2}} \cdot \frac{(1-tq^{\ell+1})}{(1-tq^{\ell+1-j})} \cdot \frac{(1-q^{\ell+1-j})}{(1-q^{\ell+1})} \cdot \frac{(1-tY^{-2}q^{\ell+1-2j})}{(1-tY^{-2}q^{\ell+1-j})} \cdot \frac{(1-Y^{-2}q^{\ell+1-j})}{(1-t^{-1}Y^{-2}q^{\ell+1-2j})}. \tag{5.11}$$

$$\frac{\tilde{D}^{(\ell)}_{\ell+1-j}(Y^{-1})}{\tilde{D}^{(\ell+1)}_j(Y)} = t^{\frac{1}{2}}q^{\ell+1-j} \cdot \frac{(1-tq^{\ell+1})}{(1-tq^{\ell+1-j})} \cdot \frac{(1-q^{j})}{(1-q^{\ell+1})} \cdot \frac{(1-Y^{-2}q^{-j})}{(1-tY^{-2}q^{\ell+1-j})} \tag{5.12}$$

$$\frac{\tilde{D}^{(\ell)}_{\ell+1-j}(Y^{-1})}{\tilde{D}^{(\ell)}_j(Y)} = tq^{\ell+1-j} \cdot \frac{(1-q^{j})}{(1-q^{\ell+1-j})} \cdot \frac{(1-Y^{-2}q^{-j})(1-t^{-1}Y^{-2}q^{\ell+1-2j})}{(1-tY^{-2}q^{\ell+1-2j})(1-Y^{-2}q^{\ell+1-j})}, \tag{5.13}$$

Proof. All of these are proved by using the relations (5.2), (5.3) and (5.4) and cancelling all common factors from the numerator and denominators. The proof of (5.9) is similar to the proofs of (5.11), (5.12) and (5.13) and (5.10), which are as follows.

Using (5.6), (5.5) and the second relation in (5.2) gives (5.11):

$$\begin{aligned}
\tilde{D}^{(\ell)}_j(Y) &= t^{-\frac{1}{2}(\ell+1)} \cdot t^{\ell-j} \cdot \begin{bmatrix} \ell \\ j \end{bmatrix}_{q,t} \cdot \frac{(1-tq^{\ell-j})}{(1-tq^{\ell})} \cdot \binom{\ell}{j}_Y \cdot \frac{(1-tY^{-2}q^{\ell-j})}{(1-tY^{-2}q^{\ell-2j})} \\
&= t^{\frac{1}{2}}t^{-\frac{1}{2}(\ell+2)} \cdot t^{-1}t^{\ell+1-j} \cdot \begin{bmatrix} \ell+1 \\ j \end{bmatrix}_{q,t} \cdot \frac{(1-q^{\ell+1-j})\cancel{(1-tq^{\ell})}}{(1-q^{\ell+1})\cancel{(1-tq^{\ell-j})}} \cdot \frac{\cancel{(1-tq^{\ell-j})}}{\cancel{(1-tq^{\ell})}} \cdot \\
&\quad \cdot \binom{\ell+1}{j}_Y \cdot \frac{\cancel{(1-tY^{-2}q^{\ell-2j})}(1-Y^{-2}q^{\ell+1-j})}{(1-t^{-1}Y^{-2}q^{\ell+1-2j})\cancel{(1-tY^{-2}q^{\ell-j})}} \cdot \frac{\cancel{(1-tY^{-2}q^{\ell-j})}}{\cancel{(1-tY^{-2}q^{\ell-2j})}} \\
&= t^{-\frac{1}{2}}\tilde{D}^{(\ell+1)}_j(Y) \cdot \frac{(1-tq^{\ell+1})}{(1-tq^{\ell+1-j})} \cdot \frac{(1-q^{\ell+1-j})}{(1-q^{\ell+1})} \cdot \frac{(1-tY^{-2}q^{\ell+1-2j})}{(1-tY^{-2}q^{\ell+1-j})} \cdot \frac{(1-Y^{-2}q^{\ell+1-j})}{(1-t^{-1}Y^{-2}q^{\ell+1-2j})}.
\end{aligned}$$

Using (5.6), (5.4) and the first relation in (5.2) gives (5.12):

$$
\begin{aligned}
t^{\frac{1}{2}(\ell+1)}\tilde{D}^{(\ell)}_{\ell+1-j}(Y^{-1}) &= t^{\ell-(\ell+1-j)}\cdot \begin{bmatrix} \ell \\ \ell+1-j \end{bmatrix}_{q,t} \cdot \frac{(1-tq^{\ell-(\ell+1-j)})}{(1-tq^{\ell})} \cdot \begin{pmatrix} \ell \\ \ell+1-j \end{pmatrix}_{Y^{-1}} \cdot \frac{(1-tY^2q^{\ell-(\ell+1-j)})}{(1-tY^2q^{\ell-2(\ell+1-j)})} \\
&= t^{j-1}\cdot \begin{bmatrix} \ell+1 \\ \ell+1-j \end{bmatrix}_{q,t} \cdot \frac{(1-q^{\ell+1-(\ell+1-j)})\cancel{(1-tq^{\ell})}}{(1-q^{\ell+1})\cancel{(1-tq^{\ell-(\ell+1-j)})}} \cdot \frac{\cancel{(1-tq^{j-1})}}{\cancel{(1-tq^{\ell})}} \\
&\qquad \cdot \begin{pmatrix} \ell+1 \\ \ell+1-j \end{pmatrix}_{Y^{-1}} \cdot \frac{\cancel{(1-tY^2q^{\ell-2(\ell+1-j)})}(1-Y^2q^{\ell+1-(\ell+1-j)})}{(1-t^{-1}Y^2q^{\ell+1-2(\ell+1-j)})\cancel{(1-tY^2q^{\ell-(\ell+1-j)})}} \cdot \frac{\cancel{(1-tY^2q^{\ell-(\ell+1-j)})}}{\cancel{(1-tY^2q^{\ell-2(\ell+1-j)})}} \\
&= t^{j-1}\cdot \begin{bmatrix} \ell+1 \\ j \end{bmatrix}_{q,t} \cdot \frac{(1-q^{j})}{(1-q^{\ell+1})} \cdot t^{\ell+1-2j} \cdot \begin{pmatrix} \ell+1 \\ j \end{pmatrix}_{Y} \cdot \frac{(1-Y^2q^{j})}{(1-t^{-1}Y^2q^{-(\ell+1-2j)})} \\
&= t^{\ell-j}\cdot \begin{bmatrix} \ell+1 \\ j \end{bmatrix}_{q,t} \cdot \frac{(1-q^{j})}{(1-q^{\ell+1})} \cdot \begin{pmatrix} \ell+1 \\ j \end{pmatrix}_{Y} \cdot \frac{(1-Y^{-2}q^{-j})}{(1-tY^{-2}q^{\ell+1-2j})} \cdot tq^{j+\ell+1-2j} \\
&= q^{\ell+1-j}\cdot t^{\frac{1}{2}(\ell+2)}\tilde{D}^{(\ell+1)}_{j}(Y) \cdot \frac{(1-tq^{\ell+1})}{(1-tq^{\ell+1-j})} \cdot \frac{(1-q^{j})}{(1-q^{\ell+1})} \cdot \frac{\cancel{(1-tY^{-2}q^{\ell+1-2j})}}{(1-tY^{-2}q^{\ell+1-j})} \cdot \frac{(1-Y^{-2}q^{-j})}{\cancel{(1-tY^{-2}q^{\ell+1-2j})}} \\
&= q^{\ell+1-j}\cdot t^{\frac{1}{2}(\ell+2)}\tilde{D}^{(\ell+1)}_{j}(Y) \cdot \frac{(1-tq^{\ell+1})}{(1-tq^{\ell+1-j})} \cdot \frac{(1-q^{j})}{(1-q^{\ell+1})} \cdot \frac{(1-Y^{-2}q^{-j})}{(1-tY^{-2}q^{\ell+1-j})}.
\end{aligned}
$$

Equation (5.13) follows from (5.12) and (5.11). Using (5.13) and (5.9)

$$
\begin{aligned}
D^{(\ell-1)}_{\ell-j}(Y^{-1}) &= tq^{\ell-j}D^{(\ell-1)}_{j}(Y)\cdot \frac{(1-q^{j})}{(1-q^{\ell-j})}\frac{(1-t^{-1}Y^{-2}q^{\ell-2j})}{(1-tY^{-2}q^{\ell-2j})} \cdot \frac{(1-Y^{-2}q^{-j})}{(1-Y^{-2}q^{\ell-j})} \\
&= t^{\frac{1}{2}}q^{\ell-j}\cdot K^{(\ell)}_{j}(Y) \cdot \frac{\cancel{(1-q^{\ell-j})}}{(1-q^{\ell})} \cdot \frac{\cancel{(1-Y^{-2}q^{\ell-j})}}{(1-Y^{-2}q^{\ell-2j})} \cdot \frac{(1-Y^{-2})}{(1-t^{-1}Y^{-2})} \\
&\qquad \cdot \frac{(1-q^{j})}{\cancel{(1-q^{\ell-j})}}\frac{(1-t^{-1}Y^{-2}q^{\ell-2j})}{(1-tY^{-2}q^{\ell-2j})} \cdot \frac{(1-Y^{-2}q^{-j})}{\cancel{(1-Y^{-2}q^{\ell-j})}}
\end{aligned}
$$

which gives equation (5.10). ☐

5.4 A recursion for the $\tilde{D}^{(\ell)}_{j}(Y)$

Proposition 5.2. *Let $\ell \in \mathbb{Z}_{\geq 0}$ and let $\tilde{D}^{(\ell)}_{j}(Y)$ and $\tilde{D}^{(-\ell)}_{j}(Y)$ be as defined in (4.1).*

(a) If $\ell \in \mathbb{Z}_{>0}$ and $j \in \{0, \ldots, \ell\}$ then $\tilde{D}^{(-\ell)}_{j}(Y) = t^{\frac{1}{2}}\tilde{D}^{(\ell)}_{j}(Y^{-1})$.

(b) The $\tilde{D}^{(\ell)}_{j}(Y)$ satisfy, and are determined by, $\tilde{D}^{(0)}_{0}(Y) = t^{-\frac{1}{2}}$ and the recursions

$$
\tilde{D}^{(\ell)}_{0}(Y) = t^{\frac{1}{2}}\frac{(1-t^{-1}Y^{-2}q^{\ell})}{(1-Y^{-2}q^{\ell})}\tilde{D}^{(\ell-1)}_{0}(Y), \qquad \tilde{D}^{(\ell)}_{\ell}(Y) = t^{-\frac{1}{2}}\frac{(1-t)(1-tY^{-2})}{(1-tq^{\ell})(1-Y^{-2}q^{-\ell})}\tilde{D}^{(\ell-1)}_{0}(Y^{-1}),
$$

and

$$
\tilde{D}^{(\ell)}_{j}(Y) = t^{\frac{1}{2}}\frac{(1-t^{-1}Y^{-2}q^{\ell-2j})}{(1-Y^{-2}q^{\ell-2j})}\tilde{D}^{(\ell-1)}_{j}(Y) + t^{-\frac{1}{2}}\frac{(1-t)(1-q^{\ell}tY^{-2}q^{\ell-2j})}{(1-tq^{\ell})(1-Y^{-2}q^{\ell-2j})}\tilde{D}^{(\ell-1)}_{\ell-j}(Y^{-1}). \tag{5.14}
$$

for $j \in \{1, \ldots, \ell-1\}$.

Proof. (a) Using (5.4),

$$\begin{aligned}
t^{\ell-j}\cdot\binom{\ell}{j}_{Y^{-1}}\cdot\frac{(1-tY^2q^{\ell-j})}{(1-tY^2q^{\ell-2j})} &= t^{\ell-j}\cdot t^{-(\ell-2j)}\binom{\ell}{\ell-j}_Y\cdot\frac{(1-t^{-1}Y^{-2}q^{-(\ell-j)})}{(1-t^{-1}Y^{-2}q^{-(\ell-2j)})}\cdot q^j \\
&= (qt)^j\binom{\ell}{\ell-j}_Y\cdot\frac{(1-t^{-1}Y^{-2}q^{-(\ell-j)})}{(1-t^{-1}Y^{-2}q^{-(\ell-2j)})}.
\end{aligned}$$

Thus, by (5.6),

$$t^{\frac12}D_j^{(\ell)}(Y^{-1}) = t^{-\frac12\ell}\cdot(qt)^j\cdot\begin{bmatrix}\ell\\ j\end{bmatrix}_{q,t}\cdot\frac{(1-tq^{\ell-j})}{(1-tq^{\ell})}\cdot\binom{\ell}{\ell-j}_Y\cdot\frac{(1-t^{-1}Y^{-2}q^{-(\ell-j)})}{(1-t^{-1}Y^{-2}q^{-(\ell-2j)})}, \tag{5.15}$$

which is equal to $\tilde D_j^{(-\ell)}(Y)$ as defined in (5.7).

(b) The first two identities are special cases of (5.11) and (5.12).

Assume $j\in\{1,\ldots,\ell-1\}$. From (5.6), (5.11) and (5.12),

$$t^{\frac12}\frac{\tilde D_j^{(\ell)}(Y)}{\tilde D_j^{(\ell+1)}(Y)} = \frac{(1-tq^{\ell+1})}{(1-tq^{\ell+1-j})}\cdot\frac{(1-q^{\ell+1-j})}{(1-q^{\ell+1})}\cdot\frac{(1-tY^{-2}q^{\ell+1-2j})}{(1-tY^{-2}q^{\ell+1-j})}\cdot\frac{(1-Y^{-2}q^{\ell+1-j})}{(1-t^{-1}Y^{-2}q^{\ell+1-2j})}$$

and

$$t^{-\frac12}\frac{\tilde D_{\ell+1-j}^{(\ell)}(Y^{-1})}{\tilde D_j^{(\ell+1)}(Y)} = q^{\ell+1-j}\cdot\frac{(1-tq^{\ell+1})}{(1-tq^{\ell+1-j})}\cdot\frac{(1-q^j)}{(1-q^{\ell+1})}\cdot\frac{(1-Y^{-2}q^{-j})}{(1-tY^{-2}q^{\ell+1-j})}.$$

Thus

$$\begin{aligned}
&t^{\frac12}\frac{(1-t^{-1}Y^{-2}q^{\ell+1-2j})}{(1-Y^{-2}q^{\ell+1-2j})}\frac{\tilde D_j^{(\ell)}(Y)}{\tilde D_j^{(\ell+1)}(Y)} + t^{-\frac12}\frac{(1-t)(1-q^{\ell+1}tY^{-2}q^{\ell+1-2j})}{(1-tq^{\ell+1})(1-Y^{-2}q^{\ell+1-2j})}\frac{\tilde D_{\ell+1-j}^{(\ell)}(Y^{-1})}{\tilde D_j^{(\ell+1)}(Y)} \\
&= \frac{\cancel{(1-t^{-1}Y^{-2}q^{\ell+1-2j})}}{(1-Y^{-2}q^{\ell+1-2j})}\frac{(1-tq^{\ell+1})(1-q^{\ell+1-j})}{(1-tq^{\ell+1-j})(1-q^{\ell+1})}\frac{(1-tY^{-2}q^{\ell+1-2j})}{(1-tY^{-2}q^{\ell+1-j})}\frac{(1-Y^{-2}q^{\ell+1-j})}{\cancel{(1-t^{-1}Y^{-2}q^{\ell+1-2j})}} \\
&\quad+\frac{(1-t)(1-q^{\ell+1}tY^{-2}q^{\ell+1-2j})}{\cancel{(1-tq^{\ell+1})}(1-Y^{-2}q^{\ell+1-2j})}q^{\ell+1-2j}\frac{\cancel{(1-tq^{\ell+1})}(1-q^j)}{(1-tq^{\ell+1-j})(1-q^{\ell+1})}\frac{(1-Y^{-2}q^{-j})}{(1-tY^{-2}q^{\ell+1-j})},
\end{aligned}$$

which is equal to

$$\frac{\left(\begin{matrix}(1-q^{\ell+1-j})(1-tq^{\ell+1})(1-Y^{-2}q^{\ell+1-j})(1-tY^{-2}q^{\ell+1-2j}) \\ +q^{\ell+1-j}(1-t)(1-q^j)(1-tY^{-2}q^{2(\ell+1-j)})(1-Y^{-2}q^{-j})\end{matrix}\right)}{(1-tq^{\ell+1-j})(1-q^{\ell+1})(1-Y^{-2}q^{\ell+1-2j})(1-tY^{-2}q^{\ell+1-j})} = 1,$$

since the numerator has roots at $tq^{\ell+1-j}=1$, $q^{\ell+1}=1$, $Y^{-2}q^{\ell+1-2j}=1$ and $tY^{-2}q^{\ell+1-j}=1$. ☐

5.5 The $\tilde K_j^{(\ell)}(Y)$ in terms of the $\tilde D_j^{(\ell)}(Y)$

Proposition 5.3. *Let $\tilde K_j^{(\ell)}(Y)$ and $\tilde D_j^{(\ell-1)}(Y)$ be as defined in (5.8) and (5.6). Then*

$$\tilde K_0^{(\ell)}(Y) = t^{\frac12}\tilde D_0^{(\ell-1)}(Y)\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}, \qquad \tilde K_\ell^{(\ell)}(Y) = t^{-\frac12}\tilde D_0^{(\ell-1)}(Y^{-1})\frac{(1-tY^{-2}q^{-\ell})}{(1-t^{-1}Y^{-2}q^{-\ell})}\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})},$$

and, for $j\in\{1,\ldots,\ell-1\}$,

$$\tilde K_j^{(\ell)}(Y) = t^{\frac12}\tilde D_j^{(\ell-1)}(Y)\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})} + t^{-\frac12}\tilde D_{\ell-j}^{(\ell-1)}(Y^{-1})\frac{(1-tY^{-2}q^{\ell-2j})}{(1-t^{-1}Y^{-2}q^{\ell-2j})}\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}. \tag{5.16}$$

Proof. The first two identities are special cases of (5.9) and (5.10).
Let $j \in \{1, \dots, \ell-1\}$. Using (5.13) gives

$$1 + t^{-1}\frac{\tilde{D}^{(\ell)}_{\ell+1-j}(Y^{-1})}{\tilde{D}^{(\ell)}_j(Y)} \cdot \frac{(1-tY^{-2}q^{\ell+1-2j})}{(1-t^{-1}Y^{-2}q^{\ell+1-2j})} = 1 + q^{\ell+1-j}\frac{(1-q^j)}{(1-q^{\ell+1-j})}\frac{(1-Y^{-2}q^{-j})}{(1-Y^{-2}q^{\ell+1-j})}$$
$$= \frac{(1-\cancel{Y^{-2}q^{\ell+1-j}} - \cancel{q^{\ell+1-j}} + Y^{-2}q^{2(\ell+1-j)} + \cancel{q^{\ell+1-j}} - q^{\ell+1} - Y^{-2}q^{\ell+1-2j} + \cancel{Y^{-2}q^{\ell+1-j}})}{(1-q^{\ell+1-j})(1-Y^{-2}q^{\ell+1-j})}$$
$$= \frac{(1-q^{\ell+1})(1-Y^{-2}q^{\ell+1-2j})}{(1-q^{\ell+1-j})(1-Y^{-2}q^{\ell+1-j})}.$$

Multiplying both sides by $t^{\frac{1}{2}}\tilde{D}^{(\ell)}_j(Y)\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}$ gives

$$t^{\frac{1}{2}}\tilde{D}^{(\ell)}_j(Y)\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})} + t^{-\frac{1}{2}}\tilde{D}^{(\ell)}_{\ell+1-j}(Y^{-1})\frac{(1-tY^{-2}q^{\ell+1-2j})}{(1-t^{-1}Y^{-2}q^{\ell+1-2j})}\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}$$
$$= t^{\frac{1}{2}}\tilde{D}^{(\ell)}_j(Y)\cdot\frac{(1-q^{\ell+1})(1-Y^{-2}q^{\ell+1-2j})}{(1-q^{\ell+1-j})(1-Y^{-2}q^{\ell+1-j})}\cdot\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})} = \tilde{K}^{(\ell+1)}_j,$$

where the last equality is (5.9). □

Theorem 5.4. *As in* (4.1) *and* (4.2), *let* $D^{(\ell)}_j(Y)$ *and* $K^{(\ell)}_j(Y)$ *be defined by the expansions*

$$E_\ell(X)\mathbf{1}_0 = \sum_{j=0}^{\ell-1}\eta^{\ell-2j}D^{(\ell-1)}_j(Y)\mathbf{1}_0 \qquad \text{and} \qquad E_{-\ell}(X)\mathbf{1}_0 = \sum_{j=0}^{\ell}\eta^{-\ell+2j}D^{(-\ell)}_j(Y)\mathbf{1}_0$$

$$\text{and} \qquad \mathbf{1}_0E_\ell(X)\mathbf{1}_0 = \sum_{j=0}^{\ell}\mathbf{1}_0\eta^{\ell-2j}K^{(\ell)}_j(Y)$$

in the localized double affine Hecke algebra $\tilde{H}$. *Then*

$$D^{(\ell)}_j(Y) = t^{-\frac{1}{2}(\ell+1)}\cdot t^{\ell-j}\cdot\begin{bmatrix}\ell\\ j\end{bmatrix}_{q,t}\cdot\binom{\ell}{j}_Y\cdot\frac{(1-tq^{\ell-j})}{(1-tq^\ell)}\cdot\frac{(1-tY^{-2}q^{\ell-j})}{(1-tY^{-2}q^{\ell-2j})},$$
$$D^{(-\ell)}_j(Y) = t^{-\frac{1}{2}\ell}\cdot(qt)^j\cdot\begin{bmatrix}\ell\\ j\end{bmatrix}_{q,t}\cdot\binom{\ell}{\ell-j}_Y\cdot\frac{(1-tq^{\ell-j})}{(1-tq^\ell)}\cdot\frac{(1-t^{-1}Y^{-2}q^{-(\ell-j)})}{(1-t^{-1}Y^{-2}q^{-(\ell-2j)})} \quad \text{and}$$
$$K^{(\ell)}_j(Y) = t^{-\frac{1}{2}(\ell-1)}\cdot t^{\ell-1-j}\cdot\begin{bmatrix}\ell\\ j\end{bmatrix}_{q,t}\cdot\binom{\ell}{j}_Y\cdot\frac{(1-Y^{-2}q^{\ell-2j})}{(1-t^{-1}Y^{-2}q^{\ell-2j})}\cdot\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}.$$

6 Products for type SL_2 Macdonald polynomials

The formulas for $E_\ell(X)\mathbf{1}_0$ and $E_{-\ell}(X)\mathbf{1}_0$ in Theorem 5.4 serve as universal formulas for products, containing, for all m at once, the information of the products $E_\ell P_m$ and $E_{-\ell}P_m$ expanded in terms of electronic Macdonald polynomials. In the same way the expansion of $\mathbf{1}_0E_\ell(X)\mathbf{1}_0$ in $\tilde{H}$ which is given in Theorem 5.4 is a universal formula for the products $P_\ell P_m$ expanded in terms of the bosonic Macdonald polynomials $P_r(x)$. In this section we use the results of Theorem 5.4 to derive these products, thus accomplishing our goal, in the SL_2 case, of using double affine Hecke algebra tools to compute compact formulas for products of Macdonald polynomials.

6.1 The universal coefficients $A_j^{(\ell)}(Y)$, $B_j^{(\ell)}(Y)$ and $C_j^{(\ell)}(Y)$

Define

$$C_j^{(\ell)}(Y) = \begin{bmatrix} \ell \\ j \end{bmatrix}_{q,t} \cdot \frac{(t^{-1}Y^{-2}q^{-(j-1)};q)_j(tY^{-2}q^{\ell-2j};q)_j}{(Y^{-2}q^{\ell-2j+1};q)_j(Y^{-2}q^{-j};q)_j}$$

$$A_j^{(\ell)}(Y) = \begin{bmatrix} \ell \\ j \end{bmatrix}_{q,t} \cdot \frac{(1-q^{\ell-j})}{(1-q^\ell)} \cdot \frac{(t^{-1}Y^{-2}q^{-(j-1)};q)_j(tY^{-2}q^{\ell-2j};q)_j}{(Y^{-2}q^{-j};q)_j(Y^{-2}q^{\ell-2j};q)_j}$$

$$B_j^{(\ell)}(Y) = q^j \cdot \begin{bmatrix} \ell \\ j \end{bmatrix}_{q,t} \cdot \frac{(1-q^{\ell-j})}{(1-q^\ell)} \cdot \frac{(t^{-1}Y^{-2}q^{-(\ell-j-1)};q)_{\ell-j}(tY^{-2}q^{-(\ell-2j-1)};q)_{\ell-j-1}}{(Y^{-2}q^{-(\ell-2j-1)};q)_{\ell-j}(Y^{-2}q^{-(\ell-j-1)};q)_{\ell-j-1}}$$

Then

$$A_j^{(\ell)}(Y) = C_j^{(\ell)}(Y) \cdot \frac{(1-q^{\ell-j})}{(1-q^\ell)} \cdot \frac{(1-Y^{-2}q^{\ell-j})}{(1-Y^{-2}q^{\ell-2j})} \qquad \text{and} \tag{6.1}$$

$$\begin{aligned} B_{\ell-j}^{(\ell)}(Y) &= q^{\ell-j} \cdot \begin{bmatrix} \ell \\ j \end{bmatrix}_{q,t} \cdot \frac{(1-q^j)}{(1-q^\ell)} \cdot \frac{(t^{-1}Y^{-2}q^{-(j-1)};q)_j(tY^{-2}q^{\ell-2j+1};q)_{j-1}}{(Y^{-2}q^{\ell-2j+1};q)_j(Y^{-2}q^{-(j-1)};q)_{j-1}} \\ &= C_j^{(\ell)}(Y) \cdot q^{\ell-j} \cdot \frac{(1-q^j)}{(1-q^\ell)} \cdot \frac{(1-Y^{-2}q^{-j})}{(1-tY^{-2}q^{\ell-2j})} \end{aligned} \tag{6.2}$$

The following proposition gives formulas for $A_j^{(\ell)}$, $B_j^{(\ell)}$ and $C_j^{(\ell)}$ in terms of $D_j^{(\ell)}$ and $K_j^{(\ell)}$.

Proposition 6.1.

$$t^{-\frac{1}{2}}C_j^{(\ell)}(Y) = K_j^{(\ell)}(Y) \cdot t^{-\frac{1}{2}(\ell-2j)} \cdot \frac{(t^{-1}Y^{-2}q^{\ell-2j};q)_j}{(Y^{-2}q^{\ell-2j};q)_j} \cdot \frac{(Y^{-2};q)_{\ell-j}}{(t^{-1}Y^{-2};q)_{\ell-j}}. \tag{6.3}$$

$$A_j^{(\ell)}(Y) = D_j^{(\ell-1)}(Y) \cdot t^{-\frac{1}{2}(\ell-2j)} \cdot \frac{(t^{-1}Y^{-2}q^{\ell-2j};q)_j}{(Y^{-2}q^{\ell-2j};q)_j} \cdot \frac{(Y^{-2};q)_{\ell-j}}{(t^{-1}Y^{-2};q)_{\ell-j}} \cdot t\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})} \tag{6.4}$$

$$B_j^{(\ell)}(Y) = D_j^{(\ell-1)}(Y^{-1})t^{\frac{1}{2}(\ell-2j)} \cdot \frac{(t^{-1}Y^{-2}q^{-(\ell-2j)+1};q)_{\ell-j}}{(Y^{-2}q^{-(\ell-2j)+1};q)_{\ell-j}} \cdot \frac{(Y^{-2}q;q)_j}{(t^{-1}Y^{-2}q;q)_j} \tag{6.5}$$

Proof. (a) Using (5.8) gives

$$
\begin{aligned}
&K_j^{(\ell)}(Y)\cdot t^{-\frac12(\ell-2j)}\cdot\frac{(t^{-1}Y^{-2}q^{\ell-2j};q)_j}{(Y^{-2}q^{\ell-2j};q)_j}\cdot\frac{(Y^{-2};q)_{\ell-j}}{(t^{-1}Y^{-2};q)_{\ell-j}}\\
&=t^{-\frac12(\ell-1)}\cdot t^{\ell-1-j}\cdot\begin{bmatrix}\ell\\ j\end{bmatrix}_{q,t}\cdot\binom{\ell}{j}_Y\cdot\frac{(1-Y^{-2}q^{\ell-2j})}{(1-t^{-1}Y^{-2}q^{\ell-2j})}\cdot\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}\\
&\qquad\cdot t^{-\frac12(\ell-2j)}\cdot\frac{(t^{-1}Y^{-2}q^{\ell-2j};q)_j}{(Y^{-2}q^{\ell-2j};q)_j}\cdot\frac{(Y^{-2};q)_{\ell-j}}{(t^{-1}Y^{-2};q)_{\ell-j}}\\
&=t^{-\frac12}\begin{bmatrix}\ell\\ j\end{bmatrix}_{q,t}\cdot\frac{(t^{-1}Y^{-2}q^{-(j-1)};q)_{\ell-j}(tY^{-2}q^{\ell-2j};q)_j}{(Y^{-2}q;q)_{\ell-j}(Y^{-2}q^{-j};q)_j}\cdot\frac{(t^{-1}Y^{-2}q^{\ell-2j};q)_j}{(Y^{-2}q^{\ell-2j};q)_j}\cdot\frac{(Y^{-2};q)_{\ell-j}}{(t^{-1}Y^{-2};q)_{\ell-j}}\\
&\qquad\cdot\frac{(1-Y^{-2}q^{\ell-2j})}{(1-t^{-1}Y^{-2}q^{\ell-2j})}\cdot\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}\\
&=t^{-\frac12}\begin{bmatrix}\ell\\ j\end{bmatrix}_{q,t}\cdot\frac{(t^{-1}Y^{-2}q^{-(j-1)};q)_{\ell-j}(1-t^{-1}Y^{-2})(t^{-1}Y^{-2}q^{\ell-2j};q)_j}{(t^{-1}Y^{-2};q)_{\ell-j}(1-t^{-1}Y^{-2}q^{\ell-2j})}\\
&\qquad\cdot\frac{(Y^{-2};q)_{\ell-j}}{(1-Y^{-2})(Y^{-2}q;q)_{\ell-j}}\cdot\frac{(1-Y^{-2}q^{\ell-2j})}{(Y^{-2}q^{\ell-2j};q)_j}\cdot\frac{1}{(Y^{-2}q^{-j};q)_j}\cdot(tY^{-2}q^{\ell-2j};q)_j\\
&=t^{-\frac12}\begin{bmatrix}\ell\\ j\end{bmatrix}_{q,t}\cdot\frac{(t^{-1}Y^{-2}q^{-(j-1)};q)_{\ell-j-1}(t^{-1}Y^{-2}q^{\ell-2j};q)_j}{(t^{-1}Y^{-2}q;q)_{\ell-j-1}}\\
&\qquad\cdot\frac{1}{(1-Y^{-2}q^{\ell-j})}\cdot\frac{1}{(Y^{-2}q^{\ell-2j+1};q)_{j-1}}\cdot\frac{1}{(Y^{-2}q^{-j};q)_j}\cdot(tY^{-2}q^{\ell-2j};q)_j\\
&=t^{-\frac12}\begin{bmatrix}\ell\\ j\end{bmatrix}_{q,t}\cdot\frac{(t^{-1}Y^{-2}q^{-(j-1)};q)_j(tY^{-2}q^{\ell-2j};q)_j}{(Y^{-2}q^{\ell-2j+1};q)_j(Y^{-2}q^{-j};q)_j}=t^{-\frac12}C_j^{(\ell)}.
\end{aligned}
$$

(b) Using (5.9) and (6.3) and (6.1), gives

$$
\begin{aligned}
&D_j^{(\ell-1)}(Y)\cdot t^{-\frac12(\ell-2j)}\cdot\frac{(t^{-1}Y^{-2}q^{\ell-2j};q)_j}{(Y^{-2}q^{\ell-2j};q)_j}\cdot\frac{(Y^{-2};q)_{\ell-j}}{(t^{-1}Y^{-2};q)_{\ell-j}}\cdot t\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}\\
&\quad=t^{-\frac12}K_j^{(\ell)}(Y)\cdot t^{-\frac12(\ell-2j)}\cdot\frac{(1-q^{\ell-j})}{(1-q^{\ell})}\cdot\frac{(1-Y^{-2}q^{\ell-j})}{(1-Y^{-2}q^{\ell-2j})}\cdot\cancel{\frac{(1-Y^{-2})}{(1-t^{-1}Y^{-2})}}\\
&\qquad\cdot\frac{(t^{-1}Y^{-2}q^{\ell-2j};q)_j}{(Y^{-2}q^{\ell-2j};q)_j}\cdot\frac{(Y^{-2};q)_{\ell-j}}{(t^{-1}Y^{-2};q)_{\ell-j}}\cdot t\cdot\cancel{\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}}\\
&\quad=t^{\frac12}\cdot t^{-\frac12}C_j^{(\ell)}(Y)\cdot\frac{(1-q^{\ell-j})}{(1-q^{\ell})}\cdot\frac{(1-Y^{-2}q^{\ell-j})}{(1-Y^{-2}q^{\ell-2j})}=A_j^{(\ell)}(Y).
\end{aligned}
$$

(c) Using (5.10) and (6.3) and (6.2) gives

$$
\begin{aligned}
&D_{\ell-j}^{(\ell-1)}(Y^{-1})\cdot t^{-\frac{1}{2}(\ell-2j)}\cdot\frac{(t^{-1}Y^{-2}q^{(\ell-2j)+1};q)_j}{(Y^{-2}q^{(\ell-2j)+1};q)_j}\cdot\frac{(Y^{-2}q;q)_{\ell-j}}{(t^{-1}Y^{-2}q;q)_{\ell-j}}\\
&=K_j^{(\ell)}(Y)\cdot t^{-\frac{1}{2}(\ell-2j)}\cdot\frac{(t^{-1}Y^{-2}q^{(\ell-2j)+1};q)_j}{(Y^{-2}q^{(\ell-2j)+1};q)_j}\cdot\frac{(Y^{-2}q;q)_{\ell-j}}{(t^{-1}Y^{-2}q;q)_{\ell-j}}\cdot\frac{(1-t^{-1}Y^{-2}q^{\ell-2j})}{(1-t^{-1}Y^{-2})}\cdot\frac{(1-Y^{-2})}{(1-Y^{-2}q^{\ell-2j})}\\
&\qquad\cdot t^{\frac{1}{2}}q^{\ell-j}\cdot\frac{(1-q^j)}{(1-q^\ell)}\cdot\frac{(1-Y^{-2}q^{-j})}{(1-tY^{-2}q^{\ell-2j})}\\
&=K_j^{(\ell)}(Y)\cdot t^{-\frac{1}{2}(\ell-2j)}\cdot\frac{(t^{-1}Y^{-2}q^{(\ell-2j)};q)_j}{(Y^{-2}q^{(\ell-2j)};q)_j}\cdot\frac{(Y^{-2};q)_{\ell-j}}{(t^{-1}Y^{-2};q)_{\ell-j}}\cdot\frac{(1-t^{-1}Y^{-2}q^{\ell-j})}{(1-t^{-1}Y^{-2}q^{\ell-j})}\cdot\frac{(1-Y^{-2}q^{\ell-j})}{(1-Y^{-2}q^{\ell-j})}\\
&\qquad\cdot t^{\frac{1}{2}}q^{\ell-j}\cdot\frac{(1-q^j)}{(1-q^\ell)}\cdot\frac{(1-Y^{-2}q^{-j})}{(1-tY^{-2}q^{\ell-2j})}\\
&=t^{-\frac{1}{2}}C_j^{(\ell)}(Y)\cdot t^{\frac{1}{2}}q^{\ell-j}\cdot\frac{(1-q^j)}{(1-q^\ell)}\cdot\frac{(1-Y^{-2}q^{-j})}{(1-tY^{-2}q^{\ell-2j})}=B_{\ell-j}^{(\ell)}(Y).
\end{aligned}
$$

Then, replacing j with $\ell-j$ gives

$$B_j^{(\ell)}(Y)=D_j^{(\ell-1)}(Y^{-1})\cdot t^{\frac{1}{2}(\ell-2j)}\cdot\frac{(t^{-1}Y^{-2}q^{-(\ell-2j)+1};q)_{\ell-j}}{(Y^{-2}q^{-(\ell-2j)+1};q)_{\ell-j}}\cdot\frac{(Y^{-2}q;q)_j}{(t^{-1}Y^{-2}q;q)_j}.$$

□

6.2 Product formulas for type SL_2 Macdonald polynomials

The following theorem provides formulas for the products $E_\ell(x)E_m(x)$, $E_{-\ell}(x)P_m(x)$ and $P_\ell(x)P_m(x)$ expanded in terms of Macdonald polynomials. It is useful to note that, in Theorem 6.2,

$$
\begin{aligned}
&\mathrm{ev}_m(A_j^{(\ell)}(Y))=0\ \text{ if } m+\ell-2j<0, && \mathrm{ev}_m(tB_j^{(\ell+1)}(Y))=0\ \text{ if } m-(\ell-2j)<0,\\
&\mathrm{ev}_m(B_j^{(\ell)}(Y))=0\ \text{ if } -m+\ell-2j>0, && \mathrm{ev}_m(A_j^{(\ell+1)}(Y))=0\ \text{ if } -m-(\ell-2j)>0,
\end{aligned}
$$

and

$$\mathrm{ev}_m(C_j^{(\ell)}(Y))=0\ \text{ if } m+\ell-2j<0,$$

since a factor in the numerator of each of these expressions evaluates to $(1-1)=0$.

Theorem 6.2. *Let $\ell,m\in\mathbb{Z}_{>0}$. Let $\mathrm{ev}_m\colon\mathbb{C}[Y,Y^{-1}]\to\mathbb{C}$ be the homomorphism given by $\mathrm{ev}_m(Y)=t^{-\frac{1}{2}}q^{-\frac{1}{2}m}$ and extend ev_m to elements of $\mathbb{C}(Y)$ such that the denominator does not evaluate to 0. Then*

$$P_\ell(x)P_m(x)=\sum_{j=0}^{\ell}\mathrm{ev}_m(C_j^{(\ell)}(Y))P_{m+\ell-2j}(x),$$

$$E_\ell(x)P_m(x)=\sum_{j=0}^{\ell-1}\mathrm{ev}_m(A_j^{(\ell)}(Y))E_{m+\ell-2j}(x)+\sum_{j=0}^{\ell-1}\mathrm{ev}_m(B_j^{(\ell)}(Y))E_{-m+\ell-2j}(x)\qquad\textit{and}$$

$$E_{-\ell}(x)P_m(x)=\sum_{j=0}^{\ell}\mathrm{ev}_m(tB_j^{(\ell+1)}(Y))E_{m-(\ell-2j)}(x)+\sum_{j=0}^{\ell}\mathrm{ev}_m(A_j^{(\ell+1)}(Y))E_{-m-(\ell-2j)}(x).$$

Proof. By (3.15), if $f \in \mathbb{C}(Y)$ such that $\mathrm{ev}_m(f(Y))$ is defined then $f(Y)E_m(X)\mathbf{1}_Y = \mathrm{ev}_m(f(Y)E_m(X)\mathbf{1}_Y$.
(a) By (3.19), (3.30), (4.2) and (3.15),

$$\begin{aligned}
P_\ell(X)P_m(X)\mathbf{1}_Y &= P_\ell(X)t^{\frac{1}{2}}\mathbf{1}_0E_m(X)\mathbf{1}_Y = t\mathbf{1}_0E_\ell(X)\mathbf{1}_0E_m(X)\mathbf{1}_Y \\
&= t\Big(\sum_{j=0}^{\ell}\mathbf{1}_0\eta^{\ell-2j}K_j^{(\ell)}(Y)\Big)E_m(X)\mathbf{1}_Y
\end{aligned}$$

and, using (3.23),

$$\begin{aligned}
&\mathbf{1}_0\eta^{\ell-2j}tK_j^{(\ell)}(Y)E_m(X)\mathbf{1}_Y = t\cdot\mathrm{ev}_m(K_j^{(\ell)}(Y))\mathbf{1}_0\eta^{-j}\eta^{\ell-j}E_m(X)\mathbf{1}_Y \\
&\quad= t\cdot\mathrm{ev}_m\Big(K_j^{(\ell)}(Y)t^{-\frac{1}{2}(\ell-2j)}\frac{(t^{-1}Y^{-2}q^{\ell-2j};q)_j}{(Y^{-2}q^{\ell-2j};q)_j}\cdot\frac{(Y^{-2};q)_{\ell-j}}{(t^{-1}Y^{-2};q)_{\ell-j}}\Big)\mathbf{1}_0E_{m+\ell-2j}(X)\mathbf{1}_Y \\
&\quad= t\cdot\mathrm{ev}_m\Big(K_j^{(\ell)}(Y)t^{-\frac{1}{2}(\ell-2j)}\frac{(t^{-1}Y^{-2}q^{\ell-2j};q)_j}{(Y^{-2}q^{\ell-2j};q)_j}\cdot\frac{(Y^{-2};q)_{\ell-j}}{(t^{-1}Y^{-2};q)_{\ell-j}}\Big)t^{-\frac{1}{2}}P_{m+\ell-2j}(X)\mathbf{1}_Y \\
&\quad= t^{\frac{1}{2}}\mathrm{ev}_m(t^{-\frac{1}{2}}C_j^{(\ell)}(Y))P_{m+\ell-2j}(X)\mathbf{1}_Y,
\end{aligned}$$

where the last equality is (6.3).
(b) By (3.19), (4.1) and (3.18),

$$\begin{aligned}
E_\ell(X)P_m(X)\mathbf{1}_Y &= E_\ell(X)t^{\frac{1}{2}}\mathbf{1}_0E_m(X)\mathbf{1}_Y = t^{\frac{1}{2}}\Big(\sum_{j=0}^{\ell-1}\eta^{\ell-2j}D_j^{(\ell-1)}(Y)\Big)\mathbf{1}_0E_m(X)\mathbf{1}_Y \\
&= t^{\frac{1}{2}}\Big(\sum_{j=0}^{\ell-1}\eta^{\ell-2j}D_j^{(\ell-1)}(Y)\Big)t^{-\frac{1}{2}}P_m(X)\mathbf{1}_Y \\
&= \Big(\sum_{j=0}^{\ell-1}\eta^{\ell-2j}D_j^{(\ell-1)}(Y)\Big)\Big(\frac{t(1-q^m)}{(1-tq^m)}E_m(X)+E_{-m}(X)\Big)\mathbf{1}_Y.
\end{aligned}$$

Using (3.23),

$$\begin{aligned}
&\eta^{\ell-2j}D_j^{(\ell-1)}(Y)t\frac{(1-q^m)}{(1-tq^m)}E_m(X)\mathbf{1}_Y = \mathrm{ev}_m\big(D_j^{(\ell-1)}(Y)\big)t\frac{(1-q^m)}{(1-tq^m)}\eta^{-j}\eta^{\ell-j}E_m(X)\mathbf{1}_Y \\
&\quad= \mathrm{ev}_m\Big(D_j^{(\ell-1)}(Y)t\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}t^{-\frac{1}{2}(\ell-2j)}\frac{(t^{-1}Y^{-2}q^{\ell-2j};q)_j}{(Y^{-2}q^{\ell-2j};q)_j}\cdot\frac{(Y^{-2};q)_{\ell-j}}{(t^{-1}Y^{-2};q)_{\ell-j}}\Big)E_{m+\ell-2j}(X)\mathbf{1}_Y \\
&\quad= \mathrm{ev}_m(A_j^{(\ell)}(Y))E_{m+\ell-2j}(X)\mathbf{1}_Y,
\end{aligned}$$

where the last equality follows from (6.4). Using (3.15) and (3.24),

$$\begin{aligned}
&\eta^{\ell-2j}D_j^{(\ell-1)}(Y)E_{-m}(X)\mathbf{1}_Y = \mathrm{ev}_m(D_j^{(\ell-1)}(Y^{-1}))\eta^{\ell-j}\eta^{-j}E_{-m}(X)\mathbf{1}_Y \\
&\quad= \mathrm{ev}_m\Big(D_j^{(\ell-1)}(Y^{-1})t^{\frac{1}{2}(\ell-2j)}\frac{(t^{-1}Y^{-2}q^{-(\ell-2j)+1};q)_{\ell-j}}{(Y^{-2}q^{-(\ell-2j)+1};q)_{\ell-j}}\frac{(Y^{-2}q;q)_j}{(t^{-1}Y^{-2}q;q)_j}\Big)E_{-m+\ell-2j}(X)\mathbf{1}_Y \\
&\quad= \mathrm{ev}_m(B_j^{(\ell)}(Y))E_{-m+\ell-2j}(X)\mathbf{1}_Y,
\end{aligned}$$

where the last equality is (6.5).

(c) By (3.19), (4.1), (3.18) and Proposition 4.1(a),

$$\begin{aligned}
E_{-\ell}(X)P_m(X)\mathbf{1}_Y &= E_{-\ell}(X)t^{\frac12}\mathbf{1}_0E_m(X)\mathbf{1}_Y = t^{\frac12}\Big(\sum_{j=0}^{\ell}\eta^{-(\ell-2j)}D_j^{(-\ell)}(Y)\Big)\mathbf{1}_0E_m(X)\mathbf{1}_Y \\
&= t^{\frac12}\Big(\sum_{j=0}^{\ell}\eta^{-(\ell-2j)}D_j^{(-\ell)}(Y)\Big)t^{-\frac12}P_m(X)\mathbf{1}_Y \\
&= \Big(\sum_{j=0}^{\ell}\eta^{-(\ell-2j)}D_j^{(-\ell)}(Y)\Big)\Big(\frac{t(1-q^m)}{(1-tq^m)}E_m(X)+E_{-m}(X)\Big)\mathbf{1}_Y. \\
&= \Big(\sum_{j=0}^{\ell}\eta^{-(\ell-2j)}t^{\frac12}D_j^{(\ell)}(Y^{-1})\Big)\Big(\frac{t(1-q^m)}{(1-tq^m)}E_m(X)+E_{-m}(X)\Big)\mathbf{1}_Y.
\end{aligned}$$

Using

$$\begin{aligned}
&\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}\frac{(t^{-1}Y^{-2}q^{-(\ell-2j)};q)_{\ell-j}}{(Y^{-2}q^{-(\ell-2j)};q)_{\ell-j}}\cdot\frac{(Y^{-2};q)_j}{(t^{-1}Y^{-2};q)_j} \\
&\qquad= \frac{(t^{-1}Y^{-2}q^{-(\ell-2j)};q)_{\ell+1-j}}{(Y^{-2}q^{-(\ell-2j)};q)_{\ell+1-j}}\cdot\frac{(1-Y^{-2}q^j)}{(1-t^{-1}Y^{-2}q^j)}\cdot\frac{(Y^{-2}q;q)_{j-1}}{(t^{-1}Y^{-2}q;q)_{j-1}} \\
&\qquad= \frac{(t^{-1}Y^{-2}q^{-(\ell-2j)};q)_{\ell+1-j}}{(Y^{-2}q^{-(\ell-2j)};q)_{\ell+1-j}}\cdot\frac{(Y^{-2}q;q)_j}{(t^{-1}Y^{-2}q;q)_j}
\end{aligned}$$

and (3.21)gives

$$\begin{aligned}
&\eta^{-(\ell-2j)}t^{\frac12}D_j^{(\ell)}(Y^{-1})t\frac{(1-q^m)}{(1-tq^m)}E_m(X)\mathbf{1}_Y = \mathrm{ev}_m\Big(t^{\frac12}D_j^{(\ell)}(Y^{-1})t\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}\Big)\eta^{-(\ell-j)}\eta^jE_m(X)\mathbf{1}_Y \\
&= \mathrm{ev}_m\Big(t^{\frac12}D_j^{(\ell)}(Y^{-1})t\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}t^{\frac12(\ell-2j)}\frac{(t^{-1}Y^{-2}q^{-(\ell-2j)};q)_{\ell-j}}{(Y^{-2}q^{-(\ell-2j)};q)_{\ell-j}}\cdot\frac{(Y^{-2};q)_j}{(t^{-1}Y^{-2};q)_j}\Big)E_{m-(\ell-2j)}(X)\mathbf{1}_Y \\
&= \mathrm{ev}_m\Big(tD_j^{(\ell)}(Y^{-1})\cdot t^{\frac12(\ell+1-2j)}\cdot\frac{(t^{-1}Y^{-2}q^{-(\ell-2j)};q)_{\ell+1-j}}{(Y^{-2}q^{-(\ell-2j)};q)_{\ell+1-j}}\cdot\frac{(Y^{-2}q;q)_j}{(t^{-1}Y^{-2}q;q)_j}\Big)E_{m-(\ell-2j)}(X)\mathbf{1}_Y \\
&= \mathrm{ev}_m(tB_j^{(\ell+1)}(Y))E_{m-(\ell-2j)}(X)\mathbf{1}_Y,
\end{aligned}$$

where the last equality is (6.5). Using (3.15) and (3.22) gives

$$\begin{aligned}
&\eta^{-(\ell-2j)}t^{\frac12}D_j^{(\ell)}(Y^{-1})E_{-m}(X)\mathbf{1}_Y = \mathrm{ev}_m(t^{\frac12}D_j^{(\ell)}(Y))\eta^j\eta^{-(\ell-j)}E_{-m}(X)\mathbf{1}_Y \\
&= \mathrm{ev}_m\Big(t^{\frac12}D_j^{(\ell)}(Y)t^{-\frac12(\ell-2j)}\frac{(t^{-1}Y^{-2}q^{\ell-2j+1};q)_j}{(Y^{-2}q^{\ell-2j+1};q)_j}\frac{(Y^{-2}q;q)_{\ell-j}}{(t^{-1}Y^{-2}q;q)_{\ell-j}}\Big)E_{-m-(\ell-2j)}(X)\mathbf{1}_Y \\
&= \mathrm{ev}_m\Big(D_j^{(\ell)}(Y)t^{-\frac12(\ell+1-2j)}\frac{(t^{-1}Y^{-2}q^{\ell-2j+1};q)_j}{(Y^{-2}q^{\ell-2j+1};q)_j}\frac{(Y^{-2};q)_{\ell+1-j}}{(t^{-1}Y^{-2};q)_{\ell+1-j}}\cdot t\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}\Big)E_{-m-(\ell-2j)}(X)\mathbf{1}_Y \\
&= \mathrm{ev}(A_j^{(\ell+1)}(Y))E_{-m-(\ell-2j)}(X)\mathbf{1}_Y,
\end{aligned}$$

where the last equality is (6.4). $\square$

Remark 6.3. After replacing Y^{-2} by $X_1X_2^{-1}$ the expression for $C_j^{(\ell)}(Y)$ coincides with the expression for the Macdonald Littlewood-Richardson coefficient given in [MW23, Theorem 1.4].

7 Examples

For $j \in \mathbb{Z}_{>0}$ and $a, b \in \mathbb{Z}$ with $a \le b$ define

$$(z;q)_j = (1-z)(1-qz)(1-q^2z)\cdots(1-q^{j-1}z) \qquad \text{and}$$
$$(1-zq^{a..b}) = (1-zq^a)(1-zq^{a+1})\cdots(1-zq^{b-1})(1-zq^b) \quad \text{so that} \quad (1-zq^{a..b}) = (1-zq^a)_{b-a+1}.$$

7.1 Examples of the q-t-binomial coefficients

Let

$$\begin{bmatrix} k \\ j \end{bmatrix}_{q,t} = \frac{\frac{(q;q)_k}{(t;q)_k}}{\frac{(q;q)_j}{(t;q)_j}\frac{(q;q)_{k-j}}{(t;q)_{k-j}}} = \frac{(1-q^{j+1..k})}{(1-q^{1..k-j})}\frac{(1-tq^{0..k-j-1})}{(1-tq^{j..k-1})}.$$

Then

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{q,t} = 1,$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{q,t} = 1, \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{q,t} = 1,$$

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix}_{q,t} = 1 \qquad \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{q,t} = \frac{(1-q^2)(1-t)}{(1-q)(1-tq)}, \qquad \begin{bmatrix} 2 \\ 2 \end{bmatrix}_{q,t} = 1,$$

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix}_{q,t} = 1, \qquad \begin{bmatrix} 3 \\ 1 \end{bmatrix}_{q,t} = \frac{(1-t)(1-q^3)}{(1-q)(1-tq^2)}, \qquad \begin{bmatrix} 3 \\ 2 \end{bmatrix}_{q,t} = \frac{(1-t)(1-q^3)}{(1-q)(1-tq^2)}, \qquad \begin{bmatrix} 3 \\ 3 \end{bmatrix}_{q,t} = 1$$

7.2 Examples of the shifted q-t-binomial coefficients

Let

$$\begin{Bmatrix} k \\ j \end{Bmatrix}_{q,t} = \begin{bmatrix} k \\ j \end{bmatrix}_{q,t} \frac{(1-tq^{k-j})}{(1-tq^k)} = \frac{(1-q^{j+1..k})}{(1-q^{1..k-j})}\frac{(1-tq^{1..k-j-1})}{(1-tq^{j..k})}.$$

Then

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}_{q,t} = 1,$$

$$\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}_{q,t} = 1, \qquad \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}_{q,t} = \frac{(1-t)}{(1-tq)},$$

$$\begin{Bmatrix} 2 \\ 0 \end{Bmatrix}_{q,t} = 1 \qquad \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}_{q,t} = \frac{(1-q^2)(1-t)}{(1-q)(1-tq^2)}, \qquad \begin{Bmatrix} 2 \\ 2 \end{Bmatrix}_{q,t} = \frac{(1-t)}{(1-tq^2)},$$

$$\begin{Bmatrix} 3 \\ 0 \end{Bmatrix}_{q,t} = 1, \qquad \begin{Bmatrix} 3 \\ 1 \end{Bmatrix}_{q,t} = \frac{(1-t)(1-q^3)}{(1-q)(1-tq^3)}, \qquad \begin{Bmatrix} 3 \\ 2 \end{Bmatrix}_{q,t} = \frac{(1-t)(1-tq)}{(1-q)(1-tq^2)}, \qquad \begin{Bmatrix} 3 \\ 3 \end{Bmatrix}_{q,t} = \frac{(1-t)}{(1-tq^3)}.$$

7.3 Examples of the Y-binomial coefficients

$$\binom{\ell}{j}_Y = \frac{(1-t^{-1}Y^{-2}q^{-(j-1)..\ell-2j})(1-tY^{-2}q^{\ell-2j..\ell-j-1})}{(1-Y^{-2}q^{1..\ell-j})(1-Y^{-2}q^{-j..-1})} = \frac{(t^{-1}Y^{-2}q^{-(j-1)};q)_{\ell-j}(tY^{-2}q^{\ell-2j};q)_j}{(Y^{-2}q;q)_{\ell-j}(Y^{-2}q^{-j};q)_j}.$$

Then

$$\binom{0}{0}_Y = 1,$$

$$\binom{1}{0}_Y = \frac{(1-t^{-1}Y^{-2}q)}{(1-Y^{-2}q)} = t^{-1}\cdot\frac{(1-tY^{2}q^{-1})}{(1-Y^{2}q^{-1})}$$
$$\binom{1}{1}_Y = \frac{(1-tY^{-2}q^{-1})}{(1-Y^{-2}q^{-1})} = t\cdot\frac{(1-t^{-1}Y^{2}q)}{(1-Y^{2}q)}$$

$$\binom{2}{0}_Y = \frac{(1-t^{-1}Y^{-2}q)(1-t^{-1}Y^{-2}q^2)}{(1-Y^{-2}q)(1-Y^{-2}q^2)} = t^{-2}\cdot\frac{(1-tY^{2}q^{-1})(1-tY^{2}q^{-2})}{(1-Y^{2}q^{-1})(1-Y^{2}q^{-2})}$$
$$\binom{2}{1}_Y = \frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2}q)}\cdot\frac{(1-tY^{-2})}{(1-Y^{-2}q^{-1})} = \frac{(1-tY^{2})}{(1-Y^{2}q^{-1})}\cdot\frac{(1-t^{-1}Y^{2})}{(1-Y^{2}q)}$$
$$\binom{2}{2}_Y = \frac{(1-tY^{-2}q^{-2})(1-tY^{-2}q^{-1})}{(1-Y^{-2}q^{-2})(1-Y^{-2}q^{-1})} = t^{2}\cdot\frac{(1-t^{-1}Y^{2}q^{2})(1-t^{-1}Y^{2}q)}{(1-Y^{2}q^{2})(1-Y^{2}q)}$$

$$\binom{3}{0}_Y = \frac{(1-t^{-1}Y^{-2}q)(1-t^{-1}Y^{-2}q^2)(1-t^{-1}Y^{-2}q^3)}{(1-Y^{-2}q)(1-Y^{-2}q^2)(1-Y^{-2}q^3)}$$
$$\binom{3}{1}_Y = \frac{(1-t^{-1}Y^{-2})(1-t^{-1}Y^{-2}q)}{(1-Y^{-2}q)(1-Y^{-2}q^2)}\cdot\frac{(1-tY^{-2}q)}{(1-Y^{-2}q^{-1})}$$
$$\binom{3}{2}_Y = \frac{(1-t^{-1}Y^{-2}q^{-1})}{(1-Y^{-2}q)}\cdot\frac{(1-tY^{-2}q^{-1})(1-tY^{-2})}{(1-Y^{-2}q^{-2})(1-Y^{-2}q^{-1})}$$
$$\binom{3}{3}_Y = \frac{(1-tY^{-2}q^{-3})(1-tY^{-2}q^{-2})(1-tY^{-2}q^{-1})}{(1-Y^{-2}q^{-3})(1-Y^{-2}q^{-2})(1-Y^{-2}q^{-1})}$$

7.4 Examples of the $D_j^{(\ell-1)}(Y)$

The general product formula for the $D_j^{(\ell)}(Y)$ is

$$D_j^{(\ell)}(Y) = t^{-\frac{1}{2}(\ell+1)} \cdot t^{\ell-j} \cdot \begin{bmatrix} \ell \\ j \end{bmatrix}_{q,t} \cdot \frac{(1-tq^{\ell-j})}{(1-tq^{\ell})} \cdot \binom{\ell}{j}_Y \cdot \frac{(1-tY^{-2}q^{\ell-j})}{(1-tY^{-2}q^{\ell-2j})}$$

The first few of the $D_j^{(\ell-1)}(Y)$ are

$$D_0^{(0)}(Y) = t^{-\frac{1}{2}} \cdot 1,$$

$$\begin{aligned}
D_0^{(1)}(Y) &= t^{-\frac{2}{2}} \cdot \frac{(1-tY^2q^{-1})}{(1-Y^2q^{-1})} = t^{-\frac{2}{2}} \cdot t \cdot \frac{(1-t^{-1}Y^{-2}q)}{(1-Y^{-2}q)} \\
D_1^{(1)}(Y) &= t^{-\frac{2}{2}} \cdot qt \cdot \frac{(1-t)}{(1-tq)} \cdot \frac{(1-t^{-1}Y^2)}{(1-Y^2q)} = t^{-\frac{2}{2}} \cdot \frac{(1-t)}{(1-tq)} \cdot \frac{(1-tY^{-2})}{(1-Y^{-2}q^{-1})},
\end{aligned}$$

$$\begin{aligned}
D_0^{(2)}(Y) &= t^{-\frac{3}{2}} \cdot \frac{(1-tY^2q^{-2})(1-tY^2q^{-1})}{(1-Y^2q^{-2})(1-Y^2q^{-1})} = t^{-\frac{3}{2}} \cdot t^2 \cdot \frac{(1-t^{-1}Y^{-2}q)(1-t^{-1}Y^{-2}q^2)}{(1-Y^{-2}q)(1-Y^{-2}q^2)} \\
D_1^{(2)}(Y) &= t^{-\frac{3}{2}} \cdot qt \cdot \frac{(1-q^2)(1-t)}{(1-q)(1-tq^2)} \cdot \frac{(1-tY^2)}{(1-Y^2q^{-1})} \cdot \frac{(1-t^{-1}Y^2q^{-1})}{(1-Y^2q)}, \\
&= t^{-\frac{3}{2}} \cdot t \cdot \frac{(1-q^2)(1-t)}{(1-q)(1-tq^2)} \cdot \frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2}q)} \cdot \frac{(1-tY^{-2}q)}{(1-Y^{-2}q^{-1})} \\
D_2^{(2)}(Y) &= t^{-\frac{3}{2}} \cdot q^2t^2 \cdot \frac{(1-t)}{(1-tq^2)} \cdot \frac{(1-t^{-1}Y^2)(1-t^{-1}Y^2q)}{(1-Y^2q)(1-Y^2q^2)} \\
&= t^{-\frac{3}{2}} \cdot \frac{(1-t)}{(1-tq^2)} \cdot \frac{(1-tY^{-2}q^{-1})(1-tY^{-2})}{(1-Y^{-2}q^{-2})(1-Y^{-2}q^{-1})}.
\end{aligned}$$

7.5 Examples of the $K_j^{(\ell)}(Y)$

The general product formula for the $K_j^{(\ell)}(Y)$ for $\ell \in \mathbb{Z}_{>0}$ is

$$K_j^{(\ell)}(Y) = t^{-\frac{1}{2}(\ell-1)} \cdot t^{\ell-1-j} \cdot \begin{bmatrix} \ell \\ j \end{bmatrix}_{q,t} \cdot \binom{\ell}{j}_Y \cdot \frac{(1-Y^{-2}q^{\ell-2j})}{(1-t^{-1}Y^{-2}q^{\ell-2j})} \cdot \frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}$$

The first few of the $K_j^{(\ell)}(Y)$ are

$$K_0^{(0)}(Y) = (t^{\frac{1}{2}} + t^{-\frac{1}{2}}) \qquad (\text{since } \mathbf{1}_0 E_0(X)\mathbf{1}_0 = \mathbf{1}_0^2 = \mathbf{1}_0(t^{\frac{1}{2}} + t^{-\frac{1}{2}})),$$

$$\begin{aligned}
K_0^{(1)}(Y) &= 1 \cdot 1 \cdot \frac{\cancel{(1-t^{-1}Y^{-2}q)}}{\cancel{(1-Y^{-2}q)}} \cdot \frac{\cancel{(1-Y^{-2}q)}}{\cancel{(1-t^{-1}Y^{-2}q)}} \cdot \frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}, \\
K_1^{(1)}(Y) &= 1 \cdot t^{-1} \cdot \frac{(1-tY^{-2}q^{-1})}{\cancel{(1-Y^{-2}q^{-1})}} \cdot \frac{\cancel{(1-Y^{-2}q^{-1})}}{(1-t^{-1}Y^{-2}q^{-1})} \cdot \frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})},
\end{aligned}$$

$$\begin{aligned}
K_0^{(2)}(Y) &= t^{-\frac{1}{2}} \cdot t \cdot \frac{(1-t^{-1}Y^{-2}q)\cancel{(1-t^{-1}Y^{-2}q^2)}}{(1-Y^{-2}q)\cancel{(1-Y^{-2}q^2)}} \cdot \frac{\cancel{(1-Y^{-2}q^2)}}{\cancel{(1-t^{-1}Y^{-2}q^2)}} \cdot \frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})} \\
K_1^{(2)}(Y) &= t^{-\frac{1}{2}} \cdot 1 \cdot \frac{(1-q^2)(1-t)}{(1-q)(1-tq)} \cdot \frac{\cancel{(1-t^{-1}Y^{-2})}}{(1-Y^{-2}q)} \cdot \frac{(1-tY^{-2})}{(1-Y^{-2}q^{-1})} \cdot \frac{\cancel{(1-Y^{-2})}}{\cancel{(1-t^{-1}Y^{-2})}} \cdot \frac{(1-t^{-1}Y^{-2})}{\cancel{(1-Y^{-2})}} \\
K_2^{(2)}(Y) &= t^{-\frac{1}{2}} \cdot t^{-1} \cdot \frac{(1-tY^{-2}q^{-2})(1-tY^{-2}q^{-1})}{\cancel{(1-Y^{-2}q^{-2})}(1-Y^{-2}q^{-1})} \cdot \frac{\cancel{(1-Y^{-2}q^{-2})}}{(1-t^{-1}Y^{-2}q^{-2})} \cdot \frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}
\end{aligned}$$

$$\begin{aligned}
K_0^{(3)}(Y) &= t^{-1} \cdot t^2 \cdot \frac{(1-t^{-1}Y^{-2}q)(1-t^{-1}Y^{-2}q^2)(1-t^{-1}Y^{-2}q^3)}{(1-Y^{-2}q)(1-Y^{-2}q^2)(1-Y^{-2}q^3)} \cdot \frac{(1-Y^{-2}q^3)}{(1-t^{-1}Y^{-2}q^3)} \cdot \frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})} \\
K_1^{(3)}(Y) &= t^{-1} \cdot t \cdot \frac{(1-t)(1-q^3)}{(1-q)(1-tq^2)} \cdot \frac{(1-t^{-1}Y^{-2})(1-t^{-1}Y^{-2}q)}{(1-Y^{-2}q)(1-Y^{-2}q^2)} \cdot \frac{(1-tY^{-2}q)}{(1-Y^{-2}q^{-1})} \\
&\qquad \cdot \frac{(1-Y^{-2}q)}{(1-t^{-1}Y^{-2}q)} \cdot \frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})} \\
K_2^{(3)}(Y) &= t^{-1} \cdot 1 \cdot \frac{(1-t)(1-q^3)}{(1-q)(1-tq^2)} \cdot \frac{(1-t^{-1}Y^{-2}q^{-1})}{(1-Y^{-2}q)} \cdot \frac{(1-tY^{-2}q^{-1})(1-tY^{-2})}{(1-Y^{-2}q^{-2})(1-Y^{-2}q^{-1})} \\
&\qquad \cdot \frac{(1-Y^{-2}q^{-1})}{(1-t^{-1}Y^{-2}q^{-1})} \cdot \frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})} \\
K_3^{(3)}(Y) &= t^{-1} \cdot t^{-1} \cdot \frac{(1-tY^{-2}q^{-3})(1-tY^{-2}q^{-2})(1-tY^{-2}q^{-1})}{(1-Y^{-2}q^{-3})(1-Y^{-2}q^{-2})(1-Y^{-2}q^{-1})} \cdot \frac{(1-Y^{-2}q^{-3})}{(1-t^{-1}Y^{-2}q^{-3})} \cdot \frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}
\end{aligned}$$

7.6 Examples of $K_j^{(\ell)}(Y)$ to $C_j^{(\ell)}(Y)$

The following are examples of the identity (6.3) from Proposition 6.1 which says

$$K_j^{(\ell)}(Y) \cdot t^{-\frac{1}{2}(\ell-2j)} \cdot \frac{(t^{-1}Y^{-2}q^{\ell-2j};q)_j}{(Y^{-2}q^{\ell-2j};q)_j} \cdot \frac{(Y^{-2};q)_{\ell-j}}{(t^{-1}Y^{-2};q)_{\ell-j}} = C_j^{(\ell)}(Y),$$

where

$$C_j^{(\ell)}(Y) = \begin{bmatrix} \ell \\ j \end{bmatrix}_{q,t} \cdot \frac{(t^{-1}Y^{-2}q^{-(j-1)};q)_j(tY^{-2}q^{\ell-2j};q)_j}{(Y^{-2}q^{\ell-2j+1};q)_j(Y^{-2}q^{-j};q)_j}.$$

The case $\ell = 0$.

$$\begin{aligned} K_0^{(0)}(Y) \cdot t^{-\frac{1}{2}(0-2\cdot 0)} \cdot \frac{(t^{-1}Y^{-2}q^{\ell-2j};q)_0}{(Y^{-2}q^{\ell-2j};q)_0} \cdot \frac{(Y^{-2};q)_0}{(t^{-1}Y^{-2};q)_0} \\ = (t^{\frac{1}{2}} + t^{-\frac{1}{2}}) \cdot 1 \cdot \frac{1}{1} \cdot \frac{1}{1} = t^{\frac{1}{2}} + t^{-\frac{1}{2}}. \end{aligned}$$

The case $\ell = 1$.

$$\begin{aligned} K_0^{(1)}(Y) \cdot t^{-\frac{1}{2}(1-2\cdot 0)} \cdot \frac{(t^{-1}Y^{-2}q^{\ell-2j};q)_0}{(Y^{-2}q^{\ell-2j};q)_0} \cdot \frac{(Y^{-2};q)_1}{(t^{-1}Y^{-2};q)_1} \\ = \frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})} \cdot t^{-\frac{1}{2}} \frac{1}{1} \cdot \frac{(1-Y^{-2})}{(1-t^{-1}Y^{-2})} = t^{-\frac{1}{2}}. \end{aligned}$$

$$\begin{aligned} K_1^{(1)}(Y) \cdot t^{-\frac{1}{2}(1-2\cdot 1)} \cdot \frac{(t^{-1}Y^{-2}q^{1-2\cdot 1};q)_1}{(Y^{-2}q^{1-2\cdot 1};q)_1} \cdot \frac{(Y^{-2};q)_0}{(t^{-1}Y^{-2};q)_0} \\ = t^{-1} \frac{(1-tY^{-2}q^{-1})}{(1-t^{-1}Y^{-2}q^{-1})} \frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})} \cdot t^{\frac{1}{2}} \frac{(1-t^{-1}Y^{-2}q^{-1})}{(1-Y^{-2}q^{-1})} \cdot \frac{1}{1} \\ = t^{-\frac{1}{2}} \frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})} \frac{(1-tY^{-2}q^{-1})}{(1-Y^{-2}q^{-1})}. \end{aligned}$$

The case $\ell = 2$.

$$\begin{aligned} K_0^{(2)}(Y) \cdot t^{-\frac{1}{2}(2-2\cdot 0)} \cdot \frac{(t^{-1}Y^{-2}q^{\ell-2j};q)_0}{(Y^{-2}q^{\ell-2j};q)_0} \cdot \frac{(Y^{-2};q)_2}{(t^{-1}Y^{-2};q)_2} \\ = t^{\frac{1}{2}} \frac{(1-t^{-1}Y^{-2}q)(1-t^{-1}Y^{-2})}{(1-Y^{-2}q)(1-Y^{-2})} \cdot t^{-1} \cdot \frac{1}{1} \cdot \frac{(1-Y^{-2})(1-Y^{-2}q)}{(1-t^{-1}Y^{-2})(1-t^{-1}Y^{-2}q)} = t^{-\frac{1}{2}}. \end{aligned}$$

$$\begin{aligned} K_1^{(2)}(Y) \cdot t^{-\frac{1}{2}(2-2\cdot 1)} \cdot \frac{(t^{-1}Y^{-2}q^{\ell-2j};q)_1}{(Y^{-2}q^{\ell-2j};q)_1} \cdot \frac{(Y^{-2};q)_1}{(t^{-1}Y^{-2};q)_1} \\ = t^{-\frac{1}{2}} \frac{(1-q^2)(1-t)}{(1-q)(1-tq)} \frac{(1-tY^{-2})(1-t^{-1}Y^{-2})}{(1-Y^{-2}q)(1-Y^{-2}q^{-1})} \cdot 1 \cdot \frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})} \cdot \frac{(1-Y^{-2})}{(1-t^{-1}Y^{-2})} \\ = t^{-\frac{1}{2}} \frac{(1-q^2)(1-t)}{(1-q)(1-tq)} \frac{(1-tY^{-2})(1-t^{-1}Y^{-2})}{(1-Y^{-2}q)(1-Y^{-2}q^{-1})}. \end{aligned}$$

$$\begin{aligned} K_2^{(2)}(Y) \cdot t^{-\frac{1}{2}(2-2\cdot 2)} \cdot \frac{(t^{-1}Y^{-2}q^{2-2\cdot 2};q)_2}{(Y^{-2}q^{2-2\cdot 2};q)_2} \cdot \frac{(Y^{-2};q)_0}{(t^{-1}Y^{-2};q)_0} \\ = t^{-\frac{3}{2}} \frac{(1-tY^{-2}q^{-2})(1-tY^{-2}q^{-1})(1-t^{-1}Y^{-2})}{(1-Y^{-2}q^{-1})(1-t^{-1}Y^{-2}q^{-2})(1-Y^{-2})} \cdot t^{-1} \cdot \frac{(1-t^{-1}Y^{-2}q^{-2})(1-t^{-1}Y^{-2}q^{-1})}{(1-Y^{-2}q^{-2})(1-Y^{-2}q^{-1})} \cdot \frac{1}{1} \\ = t^{-\frac{1}{2}} \frac{(1-tY^{-2}q^{-2})(1-tY^{-2}q^{-1})}{(1-Y^{-2}q^{-2})(1-Y^{-2}q^{-1})} \cdot \frac{(1-t^{-1}Y^{-2}q^{-1})(1-t^{-1}Y^{-1})}{(1-Y^{-2}q^{-1})(1-Y^{-2})} \end{aligned}$$

7.7 Examples of $E_\ell(x)$ and $P_\ell(x)$

$$
\begin{aligned}
&\vdots \\
E_{-2}(x) &= x^{-2} + \frac{(1-t)(1-q^2)}{(1-q)(1-q^2t)} + \frac{(1-t)}{(1-q^2t)}x^2 \\
E_{-1}(x) &= x^{-1} + \frac{1-t}{1-qt}x, \\
E_0(x) &= 1, \\
E_1(x) &= x, \\
E_2(x) &= x^2 + q\frac{(1-t)}{(1-qt)}, \\
E_3(x) &= x^3 + \Big(\frac{(1-t)q}{(1-tq)} + \frac{(1-t)q^2}{(1-tq^2)}\frac{(1-t)}{(1-tq)}\Big)x + \frac{(1-t)q^2}{(1-tq^2)}x^{-1}, \\
&\vdots
\end{aligned}
$$

and

$$
\begin{aligned}
P_0(x) &= 1, \\
P_1(x) &= x + x^{-1}, \\
P_2(x) &= (x^2 + x^{-2}) + \frac{(1-q^2)(1-t)}{(1-q)(1-qt)}, \\
P_3(x) &= (x^3 + x^{-3}) + \frac{(1-q^3)(1-t)}{(1-q^2t)(1-q)}(x + x^{-1}), \\
P_4(x) &= (x^4 + x^{-4}) + \frac{(1-q^4)(1-t)}{(1-q^3t)(1-q)}(x^2 + x^{-2}) + \frac{(1-q^4)(1-q^3)(1-qt)(1-t)}{(1-q^3t)(1-q^2t)(1-q^2)(1-q)}, \\
&\vdots
\end{aligned}
$$

7.8 Examples of products $E_\ell P_m$

$$E_1P_m = E_{m+1} + \frac{(1-q^m)}{(1-tq^m)}E_{-m+1} = E_{m+1} + \mathrm{ev}_m\Big(\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}\Big)E_{-m+1},$$

$$\begin{aligned}E_2P_m &= E_{m+2} + \frac{(1-t)}{(1-tq)}\cdot\frac{(1-q^m)}{(1-tq^{m-1})}\cdot\frac{(1-t^2q^m)}{(1-tq^m)}E_m \\ &\quad + \frac{(1-q^{m-1})}{(1-tq^{m-1})}\frac{(1-q^m)}{(1-tq^m)}\cdot\frac{(1-t^2q^{m-1})}{(1-tq^{m-1})}E_{-m+2} + q\frac{(1-t)}{(1-tq)}\cdot\frac{(1-q^m)}{(1-tq^{m+1})}E_{-m}\end{aligned}$$

$$\begin{aligned}&= E_{m+2} + \begin{Bmatrix}1\\1\end{Bmatrix}_{q,t}\mathrm{ev}_m\Big(\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2}q^{-1})}\cdot\frac{(1-tY^{-2})}{(1-Y^{-2})}\Big)E_m \\ &\quad + \mathrm{ev}_m\Big(\frac{(1-t^{-1}Y^{-2}q^{-1})}{(1-Y^{-2}q^{-1})}\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}\cdot\frac{(1-tY^{-2}q^{-1})}{(1-Y^{-2}q^{-1})}\Big)E_{-m+2} \\ &\quad + q\begin{Bmatrix}1\\1\end{Bmatrix}_{q,t}\mathrm{ev}_m\Big(\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2}q)}\Big)E_{-m},\end{aligned}$$

$$\begin{aligned}E_3P_m &= E_{m+3} + \frac{(1-t)(1-q^2)}{(1-q)(1-tq^2)}\cdot\frac{(1-q^m)}{(1-tq^{m-1})}\cdot\frac{(1-t^2q^{m+1})}{(1-tq^{m+1})}\ E_{m+1} \\ &\quad + \frac{(1-t)}{(1-tq^2)}\cdot\frac{(1-q^{m-1})(1-q^m)}{(1-tq^{m-2})(1-tq^{m-1})}\cdot\frac{(1-t^2q^{m-1})(1-t^2q^m)}{(1-tq^{m-1})(1-tq^m)}\ E_{m-1} \\ &\quad + \frac{(1-q^{m-2})(1-q^{m-1})(1-q^m)}{(1-tq^{m-2})(1-tq^{m-1})(1-tq^m)}\cdot\frac{(1-t^2q^{m-2})(1-t^2q^{m-1})}{(1-tq^{m-2})(1-tq^{m-1})}\ E_{-m+3} \\ &\quad + q\frac{(1-t)(1-q^2)}{(1-q)(1-tq^2)}\cdot\frac{(1-q^{m-1})(1-q^m)}{(1-tq^m)(1-tq^{m+1})}\cdot\frac{(1-t^2q^m)}{(1-tq^{m-1})}\ E_{-m+1} \\ &\quad + q^2\frac{(1-t)}{(1-tq^2)}\cdot\frac{(1-q^m)}{(1-tq^{m+2})}\ E_{-m-1}\end{aligned}$$

$$\begin{aligned}&= E_{m+3} + \begin{Bmatrix}2\\1\end{Bmatrix}_{q,t}\mathrm{ev}_m\Big(\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2}q^{-1})}\cdot\frac{(1-tY^{-2}q)}{(1-Y^{-2}q)}\Big)E_{m+1} \\ &\quad + \begin{Bmatrix}2\\2\end{Bmatrix}_{q,t}\mathrm{ev}_m\Big(\frac{(1-t^{-1}Y^{-2}q^{-1})(1-t^{-1}Y^{-2})}{(1-Y^{-2}q^{-2})(1-Y^{-2}q^{-1})}\cdot\frac{(1-tY^{-2}q^{-1})(1-tY^{-2})}{(1-Y^{-2}q^{-1})(1-Y^{-2})}\Big)E_{m-1} \\ &\quad + \mathrm{ev}_m\Big(\frac{(1-t^{-1}Y^{-2}q^{-2})(1-t^{-1}Y^{-2}q^{-1})(1-t^{-1}Y^{-2})}{(1-Y^{-2}q^{-2})(1-Y^{-2}q^{-1})(1-Y^{-2})}\cdot\frac{(1-tY^{-2}q^{-2})(1-tY^{-2}q^{-1})}{(1-Y^{-2}q^{-2})(1-Y^{-2}q^{-1})}\Big)\ E_{-m+3} \\ &\quad + q\begin{Bmatrix}2\\1\end{Bmatrix}_{q,t}\mathrm{ev}_m\Big(\frac{(1-t^{-1}Y^{-2}q^{-1})(1-t^{-1}Y^{-2})}{(1-Y^{-2})(1-Y^{-2}q)}\cdot\frac{(1-tY^{-2})}{(1-Y^{-2}q^{-1})}\Big)\ E_{-m+1} \\ &\quad + q^2\begin{Bmatrix}2\\2\end{Bmatrix}_{q,t}\mathrm{ev}_m\Big(\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2}q^2)}\Big)E_{-m-1}.\end{aligned}$$

7.9 General formulas for $E_\ell P_m$ and $E_{-\ell}P_m$

The general formula for $E_\ell P_m$ with $\ell \in \mathbb{Z}_{>0}$ is

$$\begin{aligned}
&E_\ell(x)P_m(x)\\
&= \sum_{j=0}^{\ell-1} \begin{Bmatrix} \ell-1 \\ j \end{Bmatrix}_{q,t} \mathrm{ev}_m\Big(\frac{1-t^{-1}Y^{-2}q^{-(j-1)..0})(1-tY^{-2}q^{\ell-2j..\ell-1-j})}{(1-Y^{-2}q^{-j..-1})(1-Y^{-2}q^{\ell-2j..\ell-1-j})}\Big)E_{m+\ell-2j}(x)\\
&\quad + \sum_{j=0}^{\ell-1} q^j \begin{Bmatrix} \ell-1 \\ j \end{Bmatrix}_{q,t} \mathrm{ev}_m\Big(\frac{1-t^{-1}Y^{-2}q^{-(\ell-j-1)..0})(1-tY^{-2}q^{-(\ell-2j)..j-1})}{(1-Y^{-2}q^{j-(\ell-j-1)..j})(1-Y^{-2}q^{-(\ell-j-1)..-1})}\Big)E_{-m+\ell-2j}(x)\\
&= \sum_{j=0}^{\ell-1} \begin{Bmatrix} \ell-1 \\ j \end{Bmatrix}_{q,t} \mathrm{ev}_m\Big(\frac{(t^{-1}Y^{-2}q^{-(j-1)};q)_j(tY^{-2}q^{\ell-2j};q)_j}{(Y^{-2}q^{-j};q)_j(Y^{-2}q^{\ell-2j})_j}\Big)E_{m+\ell-2j}(x)\\
&\quad + \sum_{j=0}^{\ell-1} q^j \begin{Bmatrix} \ell-1 \\ j \end{Bmatrix}_{q,t} \mathrm{ev}_m\Big(\frac{(t^{-1}Y^{-2}q^{-(\ell-j-1)};q)_{\ell-j}(tY^{-2}q^{-(\ell-2j-1)};q)_{\ell-j-1}}{(Y^{-2}q^{-(\ell-2j-1)};q)_{\ell-j}(Y^{-2}q^{-(\ell-j-1)};q)_{\ell-j-1}}\Big)E_{-m+\ell-2j}(x)\\
&= \sum_{j=0}^{\ell-1} \mathrm{ev}_m(A_m^{(\ell)}(Y))E_{m+\ell-2j}(x) + \sum_{j=0}^{\ell-1} \mathrm{ev}_m(B_j^{(\ell)}(Y))E_{-m+\ell-2j}(x),
\end{aligned}$$

where

$$\begin{aligned}
A_j^{(\ell)}(Y) &= \begin{Bmatrix} \ell-1 \\ j \end{Bmatrix}_{q,t} \frac{(t^{-1}Y^{-2}q^{-(j-1)};q)_j(tY^{-2}q^{\ell-2j};q)_j}{(Y^{-2}q^{-j};q)_j(Y^{-2}q^{\ell-2j})_j} \qquad \text{and}\\
B_j^{(\ell)}(Y) &= q^j \begin{Bmatrix} \ell-1 \\ j \end{Bmatrix}_{q,t} \frac{(t^{-1}Y^{-2}q^{-(\ell-j-1)};q)_{\ell-j}(tY^{-2}q^{-(\ell-2j-1)};q)_{\ell-j-1}}{(Y^{-2}q^{-(\ell-2j-1)};q)_{\ell-j}(Y^{-2}q^{-(\ell-j-1)};q)_{\ell-j-1}}.
\end{aligned}$$

The general formula for $E_{-\ell}P_m$ with $\ell \in \mathbb{Z}_{\geq 0}$ is

$$\begin{aligned}
&E_{-\ell}(x)P_m(x)\\
&= \sum_{j=0}^{\ell} \begin{Bmatrix} \ell \\ j \end{Bmatrix}_{q,t} \mathrm{ev}_m\Big(\frac{1-t^{-1}Y^{-2}q^{-(j-1)..0})(1-tY^{-2}q^{\ell+1-2j..\ell-j})}{(1-Y^{-2}q^{-j..-1})(1-Y^{-2}q^{\ell+1-2j..\ell-j})}\Big)E_{-m-\ell+2j}(x)\\
&\quad + \sum_{j=0}^{\ell} t\cdot q^j \begin{Bmatrix} \ell \\ j \end{Bmatrix}_{q,t} \mathrm{ev}_m\Big(\frac{1-t^{-1}Y^{-2}q^{-(\ell-j)..0})(1-tY^{-2}q^{-(\ell+1-2j)..j-1})}{(1-Y^{-2}q^{j-(\ell-j)..j})(1-Y^{-2}q^{-(\ell-j)..-1})}\Big)E_{m-\ell+2j}(x)\\
&= \sum_{j=0}^{\ell} \begin{Bmatrix} \ell \\ j \end{Bmatrix}_{q,t} \mathrm{ev}_m\Big(\frac{(t^{-1}Y^{-2}q^{-(j-1)};q)_j(tY^{-2}q^{\ell+1-2j};q)_j}{(Y^{-2}q^{-j};q)_j(Y^{-2}q^{\ell+1-2j})_j}\Big)E_{-m-\ell+2j}(x)\\
&\quad + \sum_{j=0}^{\ell} t\cdot q^j \begin{Bmatrix} \ell \\ j \end{Bmatrix}_{q,t} \mathrm{ev}_m\Big(\frac{(t^{-1}Y^{-2}q^{-(\ell-j)};q)_{\ell+1-j}(tY^{-2}q^{-(\ell-2j)};q)_{\ell-j}}{(Y^{-2}q^{-(\ell-2j)};q)_{\ell-j}(Y^{-2}q^{-(\ell-j)};q)_{\ell-j}}\Big)E_{m-\ell+2j}(x)\\
&= \sum_{j=0}^{\ell} \mathrm{ev}_m(A_j^{(\ell+1)}(Y))E_{-m-\ell+2j}(x) + \sum_{j=0}^{\ell} t\cdot \mathrm{ev}_m(B_j^{(\ell+1)}(Y))E_{m-\ell+2j}(x).
\end{aligned}$$

7.10 Examples of products $E_{-\ell+1}P_m$

$$E_0P_m = E_{-m} + t\frac{(1-q^m)}{(1-tq^m)}E_m = E_{-m} + t\cdot \mathrm{ev}_m\Big(\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}\Big)E_m,$$

$$\begin{aligned}
E_{-1}P_m = E_{-m-1} &+ \frac{(1-t)}{(1-tq)}\cdot\frac{(1-q^m)}{(1-tq^{m-1})}\cdot\frac{(1-t^2q^m)}{(1-tq^m)}E_{-m+1} \\
&+ t\frac{(1-q^{m-1})}{(1-tq^{m-1})}\frac{(1-q^m)}{(1-tq^m)}\cdot\frac{(1-t^2q^{m-1})}{(1-tq^{m-1})}E_{m-1} + tq\frac{(1-t)}{(1-tq)}\cdot\frac{(1-q^m)}{(1-tq^{m+1})}E_{m+1}
\end{aligned}$$

$$\begin{aligned}
= E_{-m-1} &+ \begin{Bmatrix}1\\1\end{Bmatrix}_{q,t}\mathrm{ev}_m\Big(\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2}q^{-1})}\cdot\frac{(1-tY^{-2})}{(1-Y^{-2})}\Big)E_{-m+1} \\
&+ t\cdot\mathrm{ev}_m\Big(\frac{(1-t^{-1}Y^{-2}q^{-1})}{(1-Y^{-2}q^{-1})}\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}\cdot\frac{(1-tY^{-2}q^{-1})}{(1-Y^{-2}q^{-1})}\Big)E_{m-1} \\
&+ t\cdot q\begin{Bmatrix}1\\1\end{Bmatrix}_{q,t}\mathrm{ev}_m\Big(\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2}q)}\Big)E_{m+1},
\end{aligned}$$

$$\begin{aligned}
E_{-2}P_m = E_{-m-2} &+ \frac{(1-t)(1-q^2)}{(1-q)(1-tq^2)}\cdot\frac{(1-q^m)}{(1-tq^{m-1})}\cdot\frac{(1-t^2q^{m+1})}{(1-tq^{m+1})}\ E_{-m} \\
&+ \frac{(1-t)}{(1-tq^2)}\cdot\frac{(1-q^{m-1})(1-q^m)}{(1-tq^{m-2})(1-tq^{m-1})}\cdot\frac{(1-t^2q^{m-1})(1-t^2q^m)}{(1-tq^{m-1})(1-tq^m)}\ E_{-m+2} \\
&+ t\cdot\frac{(1-q^{m-2})(1-q^{m-1})(1-q^m)}{(1-tq^{m-2})(1-tq^{m-1})(1-tq^m)}\cdot\frac{(1-t^2q^{m-2})(1-t^2q^{m-1})}{(1-tq^{m-2})(1-tq^{m-1})}\ E_{m-2} \\
&+ t\cdot q\frac{(1-t)(1-q^2)}{(1-q)(1-tq^2)}\cdot\frac{(1-q^{m-1})(1-q^m)}{(1-tq^{m-1})(1-tq^m)}\cdot\frac{(1-t^2q^m)}{(1-tq^{m+1})}\ E_m \\
&+ t\cdot q^2\frac{(1-t)}{(1-tq^2)}\cdot\frac{(1-q^m)}{(1-tq^{m+2})}\ E_{m+2}
\end{aligned}$$

$$\begin{aligned}
= E_{-m-2} &+ \begin{Bmatrix}2\\1\end{Bmatrix}_{q,t}\mathrm{ev}_m\Big(\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2}q^{-1})}\cdot\frac{(1-tY^{-2}q)}{(1-Y^{-2}q)}\Big)E_{-m} \\
&+ \begin{Bmatrix}2\\2\end{Bmatrix}_{q,t}\mathrm{ev}_m\Big(\frac{(1-t^{-1}Y^{-2}q^{-1})(1-t^{-1}Y^{-2})}{(1-Y^{-2}q^{-2})(1-Y^{-2}q^{-1})}\cdot\frac{(1-tY^{-2}q^{-1})(1-tY^{-2})}{(1-Y^{-2}q^{-1})(1-Y^{-2})}\Big)E_{-m+2} \\
&+ t\cdot\mathrm{ev}_m\Big(\frac{(1-t^{-1}Y^{-2}q^{-2})(1-t^{-1}Y^{-2}q^{-1})(1-t^{-1}Y^{-2})}{(1-Y^{-2}tq^{-2})(1-Y^{-2}tq^{-1})(1-Y^{-2})}\cdot\frac{(1-tY^{-2}q^{-2})(1-tY^{-2}q^{-1})}{(1-Y^{-2}q^{-2})(1-Y^{-2}q^{-1})}\Big)\ E_{m-2} \\
&+ t\cdot q\begin{Bmatrix}2\\1\end{Bmatrix}_{q,t}\mathrm{ev}_m\Big(\frac{(1-t^{-1}Y^{-2}q^{-1})(1-t^{-1}Y^{-2})}{(1-Y^{-2}q^{-1})(1-Y^{-2})}\cdot\frac{(1-tY^{-2})}{(1-Y^{-2}q)}\Big)\ E_m \\
&+ t\cdot q^2\begin{Bmatrix}2\\2\end{Bmatrix}_{q,t}\mathrm{ev}_m\Big(\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2}q^2)}\Big)E_{m+2}.
\end{aligned}$$

7.11 Examples of products $P_\ell P_m$

The general formula is

$$P_\ell(x)P_m(x) = \sum_{j=0}^{\ell} \mathrm{ev}_m\big(C_j^{(\ell)}(Y)\big)P_{m+\ell-2j}(x),$$

where

$$C_j^{(\ell)}(Y) = \begin{bmatrix} \ell \\ j \end{bmatrix}_{q,t} \cdot \frac{(1-t^{-1}Y^{-2}q^{-(j-1)..0})(1-tY^{-2}q^{\ell-2j..\ell-j-1})}{(1-Y^{-2}q^{\ell-2j+1..\ell-j})(1-Y^{-2}q^{-j..-1})}.$$

The first few cases are

$$\begin{aligned}
P_1P_m &= P_{m+1} + \frac{(1-q^m)}{(1-tq^m)}\frac{(1-t^2q^{m-1})}{(1-tq^{m-1})}P_{m-1} \\
&= P_{m+1} + \mathrm{ev}\Big(\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}\frac{(1-tY^{-2}q^{-1})}{(1-Y^{-2}q^{-1})}\Big)P_{m-1},
\end{aligned}$$

$$\begin{aligned}
P_2P_m &= P_{m+2} + \frac{(1-q^2)(1-t)}{(1-tq)(1-q)}\cdot\frac{(1-q^m)}{(1-tq^{m+1})}\cdot\frac{(1-t^2q^m)}{(1-tq^{m-1})}P_m \\
&\quad + \frac{(1-q^{m-1})(1-q^m)}{(1-tq^{m-1})(1-tq^m)}\cdot\frac{(1-t^2q^{m-2})(1-t^2q^{m-1})}{(1-tq^{m-2})(1-tq^{m-1})}P_{m-2} \\
&= P_{m+2} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{q,t}\mathrm{ev}\Big(\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2}q)}\cdot\frac{(1-tY^{-2})}{(1-Y^{-2}q^{-1})}\Big)P_m \\
&\quad + \mathrm{ev}\Big(\frac{(1-t^{-1}Y^{-2}q^{-1})(1-t^{-1}Y^{-2})}{(1-Y^{-2}q^{-1})(1-Y^{-2})}\cdot\frac{(1-tY^{-2}q^{-2})(1-tY^{-2}q^{-1})}{(1-Y^{-2}q^{-2})(1-Y^{-2}q^{-1})}\Big)P_{m-2},
\end{aligned}$$

$$\begin{aligned}
P_3P_m &= P_{m+3} + \frac{(1-t)(1-q^3)}{(1-q)(1-tq^2)}\cdot\frac{(1-q^m)}{(1-tq^{m+2})}\cdot\frac{(1-t^2q^{m+1})}{(1-tq^{m-1})}\,P_{m+1} \\
&\quad + \frac{(1-t)(1-q^3)}{(1-q)(1-tq^2)}\cdot\frac{(1-q^{m-1})(1-q^m)}{(1-tq^m)(1-tq^{m+1})}\cdot\frac{(1-t^2q^{m-1})(1-t^2q^m)}{(1-tq^{m-2})(1-tq^{m-1})}\,P_{m-1} \\
&\quad + \frac{(1-q^{m-2})(1-q^{m-1})(1-q^m)}{(1-tq^{m-2})(1-tq^{m-1})(1-tq^m)}\cdot\frac{(1-t^2q^{m-3})(1-t^2q^{m-2})(1-t^2q^{m-1})}{(1-tq^{m-3})(1-tq^{m-2})(1-tq^{m-1})}P_{m-3} \\
&= P_{m+3} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}_{q,t}\mathrm{ev}\Big(\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2}q^2)}\cdot\frac{(1-tY^{-2}q)}{(1-Y^{-2}q^{-1})}\Big)P_{m+1} \\
&\quad + \begin{bmatrix} 3 \\ 2 \end{bmatrix}_{q,t}\mathrm{ev}\Big(\frac{(1-t^{-1}Y^{-2}q^{-1})(1-t^{-1}Y^{-2})}{(1-Y^{-2})(1-Y^{-2}q)}\cdot\frac{(1-tY^{-2}q^{-1})(1-tY^{-2})}{(1-Y^{-2}q^{-2})(1-Y^{-2}q^{-1})}\Big)P_{m-1} \\
&\quad + \mathrm{ev}\Big(\frac{(1-t^{-1}Y^{-2}q^{-2..0})}{(1-Y^{-2}q^{-2..0})}\cdot\frac{(1-tY^{-2}q^{-3..-1})}{(1-Y^{-2}q^{-3..-1})}\Big)P_{m-3}
\end{aligned}$$

7.12 Proof of the q-t-binomial formulas for E_ℓ and P_ℓ

Proposition 7.1. *The electronic Macdonald polynomials are given by*

$$E_{-\ell}(x) = \sum_{j=0}^{\ell} \begin{bmatrix} \ell \\ j \end{bmatrix}_{q,t} \frac{(1-tq^j)}{(1-tq^\ell)} x^{\ell-2j} \qquad \text{and} \qquad E_\ell(x) = \sum_{j=0}^{\ell-1} \begin{bmatrix} \ell-1 \\ j \end{bmatrix}_{q,t} \frac{q^{\ell-1-j}(1-tq^j)}{(1-tq^{\ell-1})} x^{-\ell+2j+2},$$

and the bosonic Macdonald polynomials are given by

$$P_\ell(x) = \sum_{j=0}^{\ell} \begin{bmatrix} \ell \\ j \end{bmatrix}_{q,t} x^{\ell-2j}.$$

Proof. Following [Mac03, §6.2], where $q^k = t$,

$$\begin{aligned} \begin{bmatrix} k+a \\ b \end{bmatrix} &= \frac{(q;q)_{k+a}}{(q;q)_b (q;q)_{k+a-b}} = \frac{(1-q^{k+a-b+1})\cdots(1-q^{k+a})}{(q;q)_b} \\ &= \frac{(1-tq^{a-b+1})\cdots(1-tq^a)}{(q;q)_b} = \frac{(tq^{a-(b-1)};q)_b}{(q;q)_b}. \end{aligned}$$

Then, from [Mac03, (6.2.7)],

$$\begin{aligned} E_{-m} &= \begin{bmatrix} k+m \\ m \end{bmatrix}^{-1} \sum_{i+j=m} \begin{bmatrix} k+i-1 \\ i \end{bmatrix} \begin{bmatrix} k+j \\ j \end{bmatrix} x^{i-j} \\ &= \frac{(q;q)_m}{(tq^{m-(m-1)};q)_m} \sum_{i+j=m} \frac{(tq^{i-1-(i-1)};q)_i}{(q;q)_i} \frac{(tq^{j-(j-1)};q)_j}{(q;q)_j} x^{i-j} \\ &= \frac{(q;q)_m}{(tq;q)_m} \sum_{j=0}^{m} \frac{(t;q)_{m-j}}{(q;q)_{m-j}} \frac{(tq;q)_j}{(q;q)_j} x^{m-j-j} \\ &= \frac{(1-t)}{(1-tq^m)} \frac{(q;q)_m}{(t;q)_m} \sum_{j=0}^{m} \frac{(t;q)_{m-j}}{(q;q)_{m-j}} \frac{(1-tq^j)}{(1-t)} \frac{(t;q)_j}{(q;q)_j} x^{m-2j} \\ &= \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_{q,t} \frac{(1-tq^j)}{(1-tq^m)} x^{m-2j} \end{aligned}$$

and, from [Mac03, (6.2.8)],

$$E_{m+1} = \begin{bmatrix} k+m \\ m \end{bmatrix}^{-1} \sum_{i+j=m} \begin{bmatrix} k+i-1 \\ i \end{bmatrix} \begin{bmatrix} k+j \\ j \end{bmatrix} q^i x^{-i+j+1} = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_{q,t} \frac{(1-tq^j)}{(1-tq^m)} q^{m-j} x^{-m+2j+1}$$

so that

$$E_m = \sum_{j=0}^{m-1} \begin{bmatrix} m-1 \\ j \end{bmatrix}_{q,t} \frac{(1-tq^j)}{(1-tq^{m-1})} q^{m-1-j} x^{-m+2j+2}.$$

From [Mac03, (6.3.7)],

$$
\begin{aligned}
P_m &= \begin{bmatrix} k+m-1 \\ m \end{bmatrix}^{-1} \sum_{i+j=m} \begin{bmatrix} k+i-1 \\ i \end{bmatrix} \begin{bmatrix} k+j-1 \\ j \end{bmatrix} x^{i-j} \\
&= \frac{(q;q)_m}{(tq^{m-1-(m-1)};q)_m} \sum_{i+j=m} \frac{(tq^{i-1-(i-1)};q)_i}{(q;q)_i} \frac{(tq^{j-1-(j-1)};q)_j}{(q;q)_j} x^{i-j} \\
&= \frac{(q;q)_m}{(t;q)_m} \sum_{j=0}^{m} \frac{(t;q)_{m-j}}{(q;q)_{m-j}} \frac{(t;q)_j}{(q;q)_j} x^{m-j-j} = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_{q,t} x^{m-2j}.
\end{aligned}
$$

$\square$

7.13 Examples of $E_\ell(X)\mathbf{1}_0$

This page provides examples of the identities in (3.29). Since $E_1(X) = X$ and $E_{-1}(X) = X^{-1}+\frac{(1-t)}{(1-qt)}X$ then

$$\begin{aligned}
E_1(X)\mathbf{1}_0 &= X\mathbf{1}_0 = \tau_\pi^\vee T_1^{-1}\mathbf{1}_0 = t^{-\frac{1}{2}}\tau_\pi^\vee\mathbf{1}_0 = t^{-\frac{1}{2}}\tau_\pi^\vee E_0(X)\mathbf{1}_0 = t^{-\frac{1}{2}}\eta_\pi\mathbf{1}_0,\\
E_{-1}(X)\mathbf{1}_0 &= \Big(X^{-1} + \frac{(1-t)}{(1-qt)}X\Big)\mathbf{1}_0 = \Big(T_1\tau_\pi^\vee + \frac{(1-t)}{(1-tq)}\tau_\pi^\vee T_1^{-1}\Big)\mathbf{1}_0\\
&= \Big(T_1 + t^{-\frac{1}{2}}\frac{(1-t)}{(1-tq)}\Big)\tau_\pi^\vee\mathbf{1}_0 = \Big(T_1 + t^{-\frac{1}{2}}\frac{(1-t)}{(1-tq)}\Big)t^{\frac{1}{2}}E_1(X)\mathbf{1}_0
\end{aligned}$$

Since $E_3(X) = X^3 + \Big(\frac{(1-t)q}{(1-tq)} + \frac{(1-t)q^2}{(1-tq^2)}\frac{(1-t)}{(1-tq)}\Big)X + \frac{(1-t)q^2}{(1-tq^2)}X^{-1}$ then

$$\begin{aligned}
E_3(X)\mathbf{1}_0 &= \Big(X^3 + \Big(\frac{(1-t)q}{(1-tq)} + \frac{(1-t)q^2}{(1-tq^2)}\frac{(1-t)}{(1-tq)}\Big)X + \frac{(1-t)q^2}{(1-tq^2)}X^{-1}\Big)\mathbf{1}_0\\
&= \Big(\tau_\pi^\vee T_1^{-1}\tau_\pi^\vee T_1^{-1}\tau_\pi^\vee T_1^{-1} + t^{-\frac{1}{2}}\frac{t^{-\frac{1}{2}}(1-t)tq}{(1-tq)}\tau_\pi^\vee T_1^{-1}\\
&\qquad + \frac{t^{-\frac{1}{2}}(1-t)tq^2}{(1-tq^2)}\frac{t^{-\frac{1}{2}}(1-t)}{(1-tq)}\tau_\pi^\vee T_1^{-1} + t^{-\frac{1}{2}}\frac{t^{-\frac{1}{2}}(1-t)tq^2}{(1-tq^2)}T_1\tau_\pi^\vee\Big)\mathbf{1}_0\\
&= \Big(\tau_\pi^\vee T_1^{-1}\tau_\pi^\vee T_1^{-1}\tau_\pi^\vee T_1^{-1} + t^{-\frac{1}{2}}\frac{t^{-\frac{1}{2}}(1-t)tq}{(1-tq)}\tau_\pi^\vee T_1^{-1}\\
&\qquad + \frac{t^{-\frac{1}{2}}(1-t)tq^2}{(1-tq^2)}\frac{t^{-\frac{1}{2}}(1-t)}{(1-tq)}\tau_\pi^\vee T_1^{-1} + t^{-\frac{1}{2}}\frac{t^{-\frac{1}{2}}(1-t)tq^2}{(1-tq^2)}(T_1^{-1} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}))\tau_\pi^\vee\Big)\mathbf{1}_0\\
&= \Big(\tau_\pi^\vee T_1^{-1}\tau_\pi^\vee T_1^{-1}\tau_\pi^\vee t^{-\frac{1}{2}} + t^{-\frac{1}{2}}\frac{t^{-\frac{1}{2}}(1-t)tq}{(1-tq)}\tau_\pi^\vee T_1^{-1}\tau_\pi^\vee\tau_\pi^\vee\\
&\qquad + t^{-\frac{1}{2}}\frac{t^{-\frac{1}{2}}(1-t)tq^2}{(1-tq^2)}\frac{t^{-\frac{1}{2}}(1-t)}{(1-tq)}\tau_\pi^\vee + t^{-\frac{1}{2}}\frac{t^{-\frac{1}{2}}(1-t)tq^2}{(1-tq^2)}T_1^{-1}\tau_\pi^\vee\\
&\qquad - t^{-\frac{1}{2}}\frac{t^{-\frac{1}{2}}(1-t)tq^2}{(1-tq^2)}\frac{t^{-\frac{1}{2}}(1-t)}{(1-tq)}(1-tq)\tau_\pi^\vee\Big)\mathbf{1}_0\\
&= \Big(\tau_\pi^\vee T_1^{-1}\tau_\pi^\vee T_1^{-1}\tau_\pi^\vee t^{-\frac{1}{2}} + t^{-\frac{1}{2}}\frac{t^{-\frac{1}{2}}(1-t)tq}{(1-tq)}\tau_\pi^\vee T_1^{-1}\tau_\pi^\vee\tau_\pi^\vee\\
&\qquad + t^{-\frac{1}{2}}\frac{t^{-\frac{1}{2}}(1-t)tq^2}{(1-tq^2)}T_1^{-1}\tau_\pi^\vee\\
&\qquad + t^{-\frac{1}{2}}\frac{t^{-\frac{1}{2}}(1-t)tq^2}{(1-tq^2)}\frac{t^{-\frac{1}{2}}(1-t)}{(1-tq)}tq\tau_\pi^\vee\Big)\mathbf{1}_0\\
&= \Big(t^{-\frac{1}{2}}\tau_\pi^\vee T_1^{-1}\tau_\pi^\vee T_1^{-1}\tau_\pi^\vee + t^{-\frac{1}{2}}\frac{t^{-\frac{1}{2}}(1-t)tq}{(1-tq)}\tau_\pi^\vee T_1^{-1}\tau_\pi^\vee\tau_\pi^\vee\\
&\qquad + t^{-\frac{1}{2}}\frac{t^{-\frac{1}{2}}(1-t)tq^2}{(1-tq^2)}\tau_\pi^\vee\tau_\pi^\vee T_1^{-1}\tau_\pi^\vee + t^{-\frac{1}{2}}\frac{t^{-\frac{1}{2}}(1-t)tq^2}{(1-tq^2)}\frac{t^{-\frac{1}{2}}(1-t)tq}{(1-tq)}\tau_\pi^\vee\tau_\pi^\vee\tau_\pi^\vee\Big)\mathbf{1}_0\\
&= t^{-\frac{1}{2}}\tau_\pi^\vee\Big(T_1^{-1} + \frac{t^{-\frac{1}{2}}(1-t)tq^2}{(1-tq^2)}\Big)\tau_\pi^\vee\Big(T_1^{-1} + \frac{t^{-\frac{1}{2}}(1-t)tq}{(1-tq)}\Big)\tau_\pi^\vee\mathbf{1}_0.
\end{aligned}$$

7.14 Examples of the $E_\ell(X)\mathbf{1}_0$ expansion

Let

$$c(Y) = \frac{(1-tY^2)}{(1-Y^2)} \qquad \text{and} \qquad F_\ell(Y) = \frac{(1-t)(1-tY^2q^\ell)}{(1-tq^\ell)(1-Y^2)}.$$

Then

$$E_1(X)\mathbf{1}_0 = t^{-\frac{1}{2}}\eta_\pi\mathbf{1}_0 = t^{-\frac{1}{2}}\eta_\pi\eta_{s_1}\mathbf{1}_0 = \eta D_0^{(0)}(Y)\mathbf{1}_0,$$

$$\begin{aligned}
E_2(X)\mathbf{1}_0 &= t^{-\frac{2}{2}}(\eta c(Y) + \eta_\pi F_1(Y))\eta\mathbf{1}_0 \\
&= t^{-\frac{2}{2}}\eta^2 c(q^{-\frac{1}{2}}Y)\mathbf{1}_0 + t^{-\frac{2}{2}}F_1(q^{-\frac{1}{2}}Y^{-1})\mathbf{1}_0 \\
&= \eta^2 t^{-\frac{2}{2}}\frac{(1-tY^2q^{-1})}{(1-Y^2q^{-1})}\mathbf{1}_0 + t^{-\frac{2}{2}}\frac{(1-t)(1-tq^{-1}Y^{-2}q)}{(1-tq)(1-q^{-1}Y^{-2})}\mathbf{1}_0 \\
&= \eta^2 t^{-\frac{2}{2}}\cdot\frac{(1-tY^2q^{-1})}{(1-Y^2q^{-1})}\mathbf{1}_0 + t^{-\frac{2}{2}}\cdot qt\frac{(1-t)(1-t^{-1}Y^2)}{(1-tq)(1-Y^2q)}\mathbf{1}_0 \\
&= \eta^2 D_0^{(1)}(Y)\mathbf{1}_0 + D_1^{(1)}(Y)\mathbf{1}_0
\end{aligned}$$

and

$$\begin{aligned}
E_3(X)\mathbf{1}_0 &= t^{-\frac{1}{2}}(\eta c(Y) + \eta_\pi F_2(Y))(\eta^2 D_0^{(1)}(Y)\mathbf{1}_0 + D_1^{(1)}\mathbf{1}_0) \\
&= \eta^3 t^{-\frac{1}{2}}c(q^{-2}Y)D_0^{(1)}(Y)\mathbf{1}_0 + \eta t^{-\frac{1}{2}}c(Y)D_1^{(1)}(Y)\mathbf{1}_0 \\
&\quad + \eta^{-1}t^{-\frac{1}{2}}F_2(q^{-2}Y^{-1})D_0^{(1)}(Y^{-1})\mathbf{1}_0 + \eta t^{-\frac{1}{2}}F_2(Y^{-1})D_1^{(1)}(Y^{-1})\mathbf{1}_0 \\
&= \eta^3 t^{-\frac{1}{2}}c(q^{-2}Y)D_0^{(1)}(Y)\mathbf{1}_0 \\
&\quad + \eta t^{-\frac{1}{2}}(c(Y)D_1^{(1)}(Y) + F_2(Y^{-1})D_1^{(1)}(Y^{-1}))\mathbf{1}_0 \\
&\quad + \eta^{-1}t^{-\frac{1}{2}}F_2(q^{-2}Y^{-1})D_0^{(1)}(Y^{-1})\mathbf{1}_0 \\
&= \eta^3 t^{-\frac{3}{2}}\frac{(1-tY^2q^{-2})(1-tY^2q^{-1})}{(1-Y^2q^{-2})(1-Y^2q^{-1})}\mathbf{1}_0 \\
&\quad + \eta t^{-\frac{1}{2}}\frac{(1-t)(1-q^2)}{(1-tq^2)(1-q)}\cdot q\frac{(1-tY^2)(1-tqY^{-2})}{(1-Y^2q)(1-Y^{-2}q)}\mathbf{1}_0 \\
&\quad + \eta^{-1}t^{-\frac{3}{2}}\frac{(1-t)}{(1-tq^2)}\cdot\frac{(1-tY^{-2})(1-tY^{-2}q^{-1})}{(1-Y^{-2}q^{-2})(1-Y^{-2}q^{-1})}\mathbf{1}_0 \\
&= \eta^3 D_0^{(2)}(Y)\mathbf{1}_0 + \eta D_1^{(2)}(Y)\mathbf{1}_0 + \eta^{-1}D_2^{(2)}(Y)\mathbf{1}_0.
\end{aligned}$$

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