

c -functions and Macdonald polynomials

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In memory of Georgia Benkart

Abstract

This is a paper about c -functions and Macdonald polynomials. There are c -function formulas for E -expansions of P_λ and $A_{\lambda+\rho}$, principal specializations of P_λ and E_μ , for Macdonald's constant term formulas, and for the norms of Macdonald polynomials. Most of these follow from the creation formulas for Macdonald polynomials, providing alternative proofs to several results from [Mac03]. In addition, we prove the Boson-Fermion correspondence in the Macdonald polynomial setting and the Weyl character formula for Macdonald polynomials.

Key words— Macdonald polynomials, symmetric functions, Hecke algebras

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Introduction

There is a wonderful article by S. Helgason entitled *Harish-Chandra's c -function. A Mathematical Jewel* [Hel94]. Helgason describes how the c -function is at the core of spherical functions, eigenfunctions of invariant differential operators, hypergeometric functions, Gamma functions, Plancherel measures, Fourier transforms, Radon transforms, orbital integrals, and symmetric spaces. In Macdonald's monograph on spherical functions on p -adic groups he pointed to an analogue of the Harish-Chandra's c -function which plays a similar role for spherical functions on p -adic groups and provides for q -analogues of the Gamma functions and the other topics in Helgason's article (see [Mac71, Ch. IV (4.1)]). In Macdonald's 2003 monograph *Double Affine Hecke algebras and Orthogonal polynomials* [Mac03] these analogues of Harish-Chandra's c -function appear everywhere, without explicit mention. In this paper we wish to make these appearances of the c -function more visible and explain how they provide an understanding of combinatorial formulas in Macdonald polynomial theory.

We begin this paper with some philosophical ruminations. In Section 1, we present four broad perspectives indicating why Macdonald polynomials are fascinating objects for continued research:

- (a) the Macdonald polynomials are eigenfunctions (“wave functions”) for a class of operators which play a role analogous to the role played by the Laplacian and other Hamiltonians in classical harmonic analysis and mathematical physics;
- (b) the principal specializations of Macdonald polynomials point to ‘elliptic generalizations’ of Weyl’s dimension formula for irreducible representations of compact Lie groups;
- (c) there is a (q, t) -generalization of a Boson-Fermion type correspondence which hints at an “elliptic” generalization of “geometric Satake”;
- (d) the recursive construction of Macdonald polynomials by intertwining operators has relations to the construction of Schubert polynomials and Grothendieck polynomials by divided-difference operators and Demazure operators.

Section 2 introduces the main objects of study:

- (a) the electronic Macdonald polynomials (known as *nonsymmetric* in the literature);
- (b) the bosonic Macdonald polynomials P_λ (known as *symmetric* in the literature);
- (c) the fermionic Macdonald polynomials $A_{\lambda+\rho}$.

Remark. The first author gave a series of lectures on Macdonald polynomials at the University of Melbourne. After explaining the (q, t) -version of the Boson-Fermion correspondence and how it relates to the polynomials P_λ and $A_{\lambda+\rho}$ it started to feel natural to call the P_λ bosonic Macdonald polynomials and the $A_{\lambda+\rho}$ fermionic Macdonald polynomials. Soon we began exploring the pleasing analogies between the Cherednik-Dunkl operators and the Hamiltonian for the quantum harmonic oscillator and there was a suggestion to call the E_μ electronic Macdonald polynomials. It was fun and helpful for keeping these three families straight in one’s head, and we’ve decided to adopt it here. In previous literature, the E_μ have been called the “nonsymmetric Macdonald polynomials” and the P_λ have been called the “symmetric Macdonald polynomials”. \square

In Section 3 we develop the powerful operator calculus for handling Macdonald polynomials. There are four primary families of operators that are employed:

- (a) the Hecke algebra operators T_1, \dots, T_{n-1} , and the promotion operator T_π ;
- (b) the Cherednik-Dunkl operators Y_1, \dots, Y_n ;
- (c) the intertwiner operators τ_π^\vee and $\tau_1^\vee, \dots, \tau_{n-1}^\vee$;
- (d) the symmetrizers $\mathbf{1}_0$ and ε_0 .

This material is expository in the books of Macdonald [Mac03] and Cherednik [Che05]. We have tried to make an efficient and accessible treatment of these results, in the type GL_n case, in a continuing effort to make these amazing and powerful methods more and more broadly available. In particular, we have found the symmetrizer expressions in Propositions 3.6 and 3.7 to be of great utility and, although they have their roots in the seminal work of Harish-Chandra and Macdonald and many others (see, in particular, [Mac03, (5.5.14)]), we hope our treatment might help others find further uses for these identities.

In Section 4, we study the action of various operators on polynomials and prove important results. In particular, we look at:

- (a) the creation formulas for Macdonald polynomials via intertwiners and symmetrizers;
- (b) the (q, t) Boson-Fermion correspondence and its relation to the symmetrizers;

- (c) the Poincaré polynomial and its relation to the symmetrizers;
- (d) the *E*-expansions of P_λ and $A_{\lambda+\rho}$.

The Boson-Fermion correspondence identifies two different avatars of the ring of symmetric polynomials by relating the P_λ and $A_{\lambda+\rho}$. Moreover, it provides a point of view that connects mathematical physics, geometric representation theory, and the Langlands program. The *E*-expansions are wonderfully explicit combinatorial expressions that generalize the expressions for monomial symmetric polynomials as sums over permutations. Understanding the coefficients in these expansions in terms of *c*-functions makes more explicit the relation between these expansions and formulas like the Gindikin-Karpelevič formula in representation theory (see, for example, [Kn03] and [BN10, (1)]).

Section 5 is devoted to principal specializations of Macdonald polynomials. We give two kinds of formulas:

- (a) *c*-function formulas for principal specializations of E_μ , P_λ and $A_{\lambda+\rho}$;
- (b) hook formulas for principal specializations of P_λ and E_μ .

The *c*-function formulas are reformulations of [Mac03, (5.3.9) and (5.2.14)] which put the focus on type GL_n and the corresponding *c*-functions. The hook formula (Theorem 5.3) for the principal specialization of P_λ is exactly that of [Mac, Ch. VI (6.11')]. The proof we give is different – it uses the intertwiner operators to derive the *c*-functions and then the combinatorial argument of [AGY22].

Section 6 introduces the inner product $(\cdot, \cdot)_{q,t}$ with respect to which the Macdonald polynomials are orthogonal polynomials. We prove that the inner product is sesquilinear, nondegenerate, and normalized Hermitian, that the characterization of the Macdonald polynomials in terms of the inner product, and two amazing formulas:

- (a) the “going up a level” formula relating $(f, g)_{q,t}$ to $(A_\rho f, A_\rho g)_{q,t}$;
- (b) the Weyl character formula for Macdonald polynomials.

This section follows closely the exposition in [Mac03, Ch. 5] except for the changes of notation to focus on type GL_n . In particular, the “going up a level” formula (Proposition 6.4) and the Weyl character formula for Macdonald polynomials (Theorem 6.5) are [Mac03, (5.8.6)] and [Mac03, (5.8.12)], respectively.

Section 7 gives an exposition (with proofs) of

- (a) the *c*-function formulas for $(E_\mu, E_\mu)_{q,t}$, $(P_\lambda, P_\lambda)_{q,t}$ and $(A_{\lambda+\rho}, A_{\lambda+\rho})_{q,t}$; and
- (b) the formulas for “Macdonald’s constant term”.

Our exposition is for type GL_n , although the proof follows the same ideas and pattern of the general type proof exposited in [Mac03, §5.8] (based on the amazing tools developed by Heckman, Opdam, Cherednik, and Macdonald). We have made a special effort to streamline the proof and make it accessible.

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1 Why these things are so incredibly interesting

As mentioned in the introduction, this section aims to give a broad perspective of why Macdonald polynomials are interesting. Therefore, we warn the reader that this section is not too *formal* and that the formal details are presented in the rest of the article.

1.1 Eigenvalues and eigenvectors

Let $n \in \mathbb{Z}_{>0}$ and let $q, t^{\frac{1}{2}} \in \mathbb{C}^\times := \mathbb{C} \setminus \{0\}$. The symmetric group S_n acts on $\mathbb{C}[X] := \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ by permuting x_1, \dots, x_n so that

$$(s_i f)(x_1, \dots, x_n) = f(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n),$$

where s_i denotes the transposition in S_n that switches i and $i+1$. Define operators T_1, \dots, T_{n-1} and T_π on $\mathbb{C}[X]$ by

$$T_i f = -t^{-\frac{1}{2}} f + (1 + s_i) \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} x_i^{-1} x_{i+1}}{1 - x_i^{-1} x_{i+1}} f \quad \text{and} \quad (T_\pi f)(x_1, \dots, x_n) = f(q^{-1} x_n, x_1, \dots, x_{n-1}).$$

The *Cherednik-Dunkl operators* are

$$Y_1 = T_\pi T_{n-1} \cdots T_1, \quad Y_2 = T_1^{-1} Y_1 T_1^{-1}, \quad Y_3 = T_2^{-1} Y_2 T_2^{-1}, \quad \dots, \quad Y_n = T_{n-1}^{-1} Y_{n-1} T_{n-1}^{-1}.$$

If $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ then the minimal length (with respect to the Bruhat order) permutation v_μ such that $v_\mu \mu$ is weakly increasing is given by

$$v_\mu(r) = 1 + \#\{r' \in \{1, \dots, r-1\} \mid \mu_{r'} \leq \mu_r\} + \#\{r' \in \{r+1, \dots, n\} \mid \mu_{r'} < \mu_r\}, \quad \text{for } r \in \{1, \dots, n\}.$$

Theorem 1.1. *There is a unique basis $\{E_\mu \mid \mu \in \mathbb{Z}^n\}$ of $\mathbb{C}[X]$ such that*

$$Y_i E_\mu = q^{-\mu_i} t^{-(v_\mu(i)-1) + \frac{1}{2}(n-1)} E_\mu, \quad \text{for } i \in \{1, \dots, n\},$$

and the coefficient of $x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$ in E_μ is 1.

Remark. In order to make the notation lighter and easier to read, we do not include the variables x_1, \dots, x_n and the parameters q, t in the polynomials.

This is an incredible statement! It says that the operators Y_1, \dots, Y_n all commute, and that their simultaneous eigenvectors form an orthogonal basis with respect to an appropriate inner product and that the eigenvalues are all explicitly determined. This special basis of simultaneous eigenvectors, the electronic Macdonald polynomials E_μ , is the primary object of study in this paper. The inner product $(\cdot, \cdot)_{q,t}$ with respect to which they are orthogonal will be studied in Sections 6 and 7.

1.2 Elliptic, quantum, and ordinary dimension formulas

Let $(\mathbb{Z}^n)^+ = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \dots \geq \lambda_n\}$, and consider $\lambda \in (\mathbb{Z}^n)^+$. The *bosonic Macdonald polynomial* P_λ is a symmetric version of the electronic Macdonald polynomial,

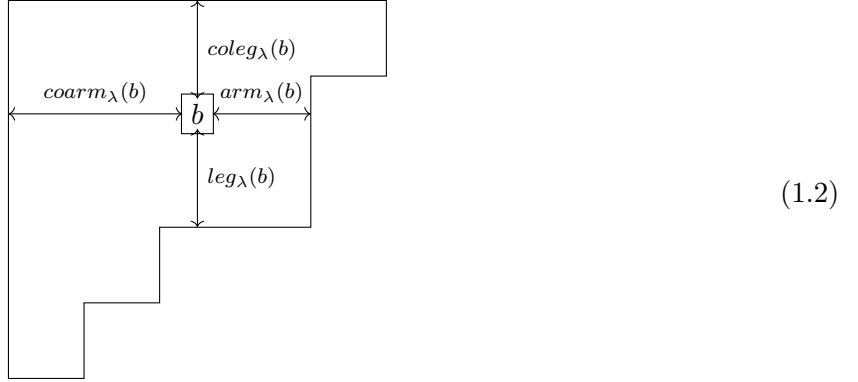
$$P_\lambda = P_\lambda(x; q, t) = \frac{1}{W_\lambda(t)} \sum_{w \in S_n} w \left(E_\lambda \prod_{i < j} \frac{x_i - tx_j}{x_i - x_j} \right),$$

where $W_\lambda(t)$ is the appropriate constant which makes the coefficient of x^λ equal to 1 in $P_\lambda(q, t)$.

Theorem 5.3 and Corollary 5.2 say that

$$\begin{aligned}
 P_\lambda(1, t, t^2, \dots, t^{n-1}; q, t) &= t^{n(\lambda)} \prod_{1 \leq i < j \leq n} \prod_{\ell=0}^{\lambda_i - \lambda_j - 1} \frac{1 - q^\ell t^{j-i+1}}{1 - q^\ell t^{j-i}} \\
 &= t^{n(\lambda)} \prod_{b \in \lambda} \frac{1 - q^{\text{coarm}_\lambda(b)} t^{n - \text{coleg}_\lambda(b)}}{1 - q^{\text{arm}_\lambda(b)} t^{\text{leg}_\lambda(b)+1}}
 \end{aligned} \tag{1.1}$$

where



The Schur polynomial s_λ is the specialization of P_λ at $q = t$, $s_\lambda(x_1, \dots, x_n) = P_\lambda(x_1, \dots, x_n; t, t)$. Note that specializing (1.2) at $q = t$ gives

$$s_\lambda(1, t, t^2, \dots, t^{n-1}) = t^{n(\lambda)} \prod_{1 \leq i < j \leq n} \frac{1 - t^{\lambda_i - \lambda_j + (j-i)}}{1 - t^{j-i}} = t^{n(\lambda)} \prod_{b \in \lambda} \frac{1 - t^{n+c(b)}}{1 - t^{h(b)}}, \tag{1.3}$$

where $h(b)$ is the hook length of the box b and $c(b)$ is the content of the box b . Moreover, setting $t = 1$ in (1.3) gives

$$s_\lambda(1, 1, \dots, 1) = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} = \prod_{b \in \lambda} \frac{n + c(b)}{h(b)}. \tag{1.4}$$

These are special cases of Weyl's integral formula and Weyl's dimension formula (see [BrtD, Ch. VI (1.7)]).

There is also a connection with the representation theory of $GL_n(\mathbb{C})$. Let $\text{char}(L(\lambda))$ denote the character of the irreducible polynomial representation of $GL_n(\mathbb{C})$ indexed by λ (see [Mac, Ch. I App. A (8.4)]). Letting $e^x = \text{diag}(x_1, \dots, x_n)$ in $GL_n(\mathbb{C})$, the Weyl character formula (see [Kac, Theorem 10.4]) says that

$$s_\lambda(x_1, \dots, x_n) = \text{char}(L(\lambda)) = \text{Tr}(L(\lambda), e^x),$$

where $\text{Tr}(L(\lambda), g)$ denotes the trace of the action of g on the vector space $L(\lambda)$. Letting $e^{\rho^\vee} = \text{diag}(1, t, \dots, t^{n-1})$, the specialization

$$s_\lambda(1, t, t^2, \dots, t^{n-1}) = \text{Tr}(L(\lambda), e^{\rho^\vee}) = \text{qdim}(L(\lambda))$$

is the *quantum dimension* of $L(\lambda)$ (see [Kac, Prop, 10.10]) and the specialization

$$s_\lambda(1, 1, \dots, 1) = \text{Tr}(L(\lambda), 1) = \text{dim}(L(\lambda))$$

is the dimension of $L(\lambda)$ (see [Kac, Cor. 10.10]). It would be interesting to give an interpretation of the formula from (1.1) as an ‘‘elliptic dimension’’ formula for $L(\lambda)$.

1.3 Geometric Satake

Let us outline the context of the Boson-Fermion correspondence for symmetric polynomials and the Weyl character formula, and describe some amazing relations between these structural features and the geometric and representation-theoretic settings.

The case $q = 0$ and $t = 0$. Consider the simple reflections in S_n , $s_i = (i, i + 1)$ for $1 \leq i \leq n - 1$, and the spaces

$$\begin{aligned} \mathbb{C}[X]^{S_n} &= \{f \in \mathbb{C}[X] \mid \text{if } i \in \{1, \dots, n-1\} \text{ then } s_i f = f\} \quad \text{and} \\ \mathbb{C}[X]^{\det} &= \{f \in \mathbb{C}[X] \mid \text{if } i \in \{1, \dots, n-1\} \text{ then } s_i f = -f\}. \end{aligned}$$

Let $w_0 = (n, n - 1, \dots, 1) \in S_n$, with $\ell(w_0) = \frac{1}{2}n(n - 1)$, and define the following operators

$$p_0 = \sum_{w \in S_n} w \quad \text{and} \quad e_0 = \sum_{w \in S_n} (-1)^{\ell(w) - \ell(w_0)} w,$$

For $\mu \in \mathbb{Z}^n$, let $x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$. The *monomial symmetric polynomial* is

$$m_\mu = \frac{1}{W_\mu(1)} p_0 x^\mu = \frac{1}{W_\mu(1)} \sum_{w \in S_n} w x^\mu,$$

where the coefficient $\frac{1}{W_\mu(1)}$ makes the coefficient of x^μ in m_μ equal to 1. The *skew orbit sum* is

$$a_\mu = e_0 x^\mu = \sum_{w \in S_n} (-1)^{\ell(w_0) - \ell(w)} x^{w\mu} = \det(x_i^{\mu_j}).$$

The special case where $\rho = (n - 1, n - 2, \dots, 2, 1, 0)$ gives the *Vandermonde determinant*,

$$a_\rho = (-1)^{\ell(w_0)} \det(x_i^{n-j}) = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

Let $(\mathbb{Z}^n)^{++} = \{(\mu_1, \dots, \mu_n) \in \mathbb{Z}^n \mid \mu_1 > \cdots > \mu_n\}$, and recall that $(\mathbb{Z}^n)^+$ is the set of weakly decreasing sequences of integers. Note that the following map gives a bijection between these two sets:

$$\begin{array}{ccc} (\mathbb{Z}^n)^+ & \xrightarrow{\sim} & (\mathbb{Z}^n)^{++} \\ \lambda & \mapsto & \lambda + \rho \end{array}$$

Given $\lambda \in (\mathbb{Z}^n)^+$ and $\mu \in (\mathbb{Z}^n)^{++}$, we have that for $i \in \{1, \dots, n - 1\}$, $m_{s_i \lambda} = m_\lambda$ and $a_\mu = -a_{s_i \mu}$. Thus,

$$\begin{aligned} \{m_\lambda \mid \lambda \in (\mathbb{Z}^n)^+\} & \quad \text{is a basis of} \quad \mathbb{C}[X]^{S_n} = p_0 \mathbb{C}[X], \\ \{a_\mu \mid \mu \in (\mathbb{Z}^n)^{++}\} & \quad \text{is a basis of} \quad \mathbb{C}[X]^{\det} = e_0 \mathbb{C}[X], \end{aligned}$$

Moreover, for $\lambda \in (\mathbb{Z}^n)^+$, the *Schur polynomial* is

$$s_\lambda = \frac{a_{\lambda + \rho}}{a_\rho}. \tag{1.5}$$

Schur definitively recognized the polynomial s_λ as the character of a finite-dimensional irreducible representation of the group $GL_n(\mathbb{C})$. A way of making the Schur polynomial very natural is to

recognize that the following diagram of vector space isomorphisms tells us that $\mathbb{C}[X]^{\det}$ is a free (rank 1) $\mathbb{C}[X]^{S_n}$ -module with basis vector a_ρ .

$$\begin{array}{ccc}
 \mathbb{C}[X]^{S_n} & \xrightarrow{\sim} & \mathbb{C}[X]^{\det} = a_\rho \mathbb{C}[X]^{S_n} \\
 f & \mapsto & a_\rho f \\
 s_\lambda & \mapsto & a_{\lambda+\rho} = e_0 x^{\lambda+\rho} \\
 m_\lambda = p_0 x^\lambda & \mapsto & a_\rho m_\lambda
 \end{array} \tag{1.6}$$

This isomorphism can be thought of as a version of the Boson-Fermion correspondence for symmetric polynomials. Hermann Weyl used this isomorphism in his generalization of Schur's result which recognized that the analogues of the s_λ for crystallographic reflection groups (Weyl groups) provide the characters of the finite-dimensional irreducible representations of compact Lie groups.

The case of $q = 0$ and general t . In view of the operators T_1, \dots, T_{n-1} from Section 1.1, the t -analogues of the elements p_0 and e_0 are given by

$$\mathbf{1}_0 = \sum_{z \in S_n} t^{\frac{1}{2}(\ell(z) - \ell(w_0))} T_z \quad \text{and} \quad \varepsilon_0 = \sum_{w \in S_n} (-t^{-\frac{1}{2}})^{\ell(z) - \ell(w_0)} T_z.$$

It is fruitful to think of the polynomial ring $\mathbb{C}[X] = \mathbb{C}[X]$ as generated by a single element $\mathbf{1}_0$ via multiplication by the variables/operators $x_1^{\pm 1}, \dots, x_n^{\pm 1}$. With this point of view, the polynomial ring $\mathbb{C}[X] = H\mathbf{1}_0$ is an induced representation of the affine Hecke algebra H , where H is the algebra generated by T_1, \dots, T_{n-1} and $x_1^{\pm 1}, \dots, x_n^{\pm 1}$:

$$\mathbb{C}[X] \cong H\mathbf{1}_0 = \text{span}\{x^\mu \mathbf{1}_0 \mid \mu \in \mathbb{Z}^n\}.$$

For $\mu \in \mathbb{Z}^n$, the *Whittaker function*

$$A_\mu(0, t)\mathbf{1}_0 \in \varepsilon_0 H\mathbf{1}_0 \quad \text{is defined by} \quad A_\mu(0, t) = \varepsilon_0 x^\mu \mathbf{1}_0.$$

See, for example, [HKP, §6] for the connection between p -adic groups and the affine Hecke algebra and the explanation of why A_μ is equivalent to the data of a (spherical) Whittaker function for a p -adic group. As proved carefully in [NR04, Theorem 2.7],

$$\varepsilon_0 H\mathbf{1}_0 \quad \text{has } \mathbb{C}\text{-basis} \quad \{A_{\lambda+\rho}(0, t) \mid \lambda \in (\mathbb{Z}^n)^+\}.$$

Following [Lu83] (see [NR04, Theorem 2.4] for another exposition),

$$\begin{array}{l}
 \text{the Satake isomorphism,} \\
 \text{the Casselman-Shalika formula,}
 \end{array}
 \quad
 \begin{array}{l}
 \mathbb{C}[X]^{S_n} \cong \mathbf{1}_0 H\mathbf{1}_0, \\
 A_{\lambda+\rho}(0, t) = s_\lambda A_\rho,
 \end{array}
 \quad \text{and}$$

can be formulated by the following diagram of vector space isomorphisms:

$$\begin{array}{ccccc}
 Z(H) = \mathbb{C}[X]^{S_n} & \xrightarrow{\sim} & \mathbf{1}_0 H\mathbf{1}_0 & \xrightarrow{\sim} & \varepsilon_0 H\mathbf{1}_0 \\
 f & \mapsto & f\mathbf{1}_0 & \mapsto & A_\rho f\mathbf{1}_0 \\
 s_\lambda & \mapsto & s_\lambda \mathbf{1}_0 & \mapsto & A_{\lambda+\rho}(0, t) = \varepsilon_0 x^{\lambda+\rho} \mathbf{1}_0 \\
 P_\lambda(0, t) & \mapsto & P_\lambda(0, t)\mathbf{1}_0 = \mathbf{1}_0 x^\lambda \mathbf{1}_0 & &
 \end{array} \tag{1.7}$$

where $A_\rho = A_\rho(0, t)$. As explained by Lusztig [Lu83], in this diagram

$\mathbf{1}_0 H \mathbf{1}_0$ is the *spherical Hecke algebra*,

s_λ is the *Schur polynomial*,

$P_\lambda(0, t)$ is the *Hall-Littlewood polynomial*, and

$\{P_\lambda(0, t)\mathbf{1}_0 \mid \lambda \in (\mathbb{Z}^n)^+\}$ is the *Kazhdan-Lusztig basis* of $\mathbf{1}_0 H \mathbf{1}_0$.

The spherical Hecke algebra $\mathbf{1}_0 H \mathbf{1}_0$ is the Iwahori-Hecke algebra corresponding to the *loop Grassmannian* $GL_n(\mathbb{C}((\epsilon)))/GL_n(\mathbb{C}[[t]])$. The statement that $P_\lambda(0, t)\mathbf{1}_0$ is a Kazhdan-Lusztig basis element in $\mathbf{1}_0 H \mathbf{1}_0$ indicates that $P_\lambda(0, t)\mathbf{1}_0$ corresponds to the intersection homology of a Schubert variety in the loop Grassmannian (amazing!).

The diagram (1.7) has particular importance due to the fact that $\mathbb{C}[X]^{S_n} = \mathbb{C}[X]^{W_0}$ (where W_0 is the Weyl group) is an avatar of the Grothendieck group of the category $\text{Rep}(G)$ of finite dimensional representations of G , the spherical Hecke algebra $\mathbf{1}_0 H \mathbf{1}_0$ is a form of the Grothendieck group of K -equivariant perverse sheaves on the loop Grassmannian Gr for the Langlands dual group G^\vee , and $\varepsilon_0 H \mathbf{1}_0$ is isomorphic to the Grothendieck group of Whittaker sheaves (appropriately formulated N -equivariant sheaves on Gr); see [FGV].

An analogous picture for general q and general t . The results in Theorem 4.4 and Theorem 6.5 provide an analogous diagram for Macdonald polynomials. Letting \tilde{H} be the affine Hecke algebra and writing the polynomial representation of \tilde{H} as $\mathbb{C}[X] \cong \tilde{H}\mathbf{1}_Y$ as in (4.3), then we have the following diagram:

$$\begin{array}{ccccc}
 \mathbb{C}[X]^{S_n} & \longrightarrow & \mathbb{C}[X]^{S_n}\mathbf{1}_Y = \mathbf{1}_0\tilde{H}\mathbf{1}_Y & \longrightarrow & A_\rho\mathbb{C}[X]^{S_n} = \varepsilon_0\tilde{H}\mathbf{1}_Y \\
 f & \longmapsto & f\mathbf{1}_Y & \longmapsto & A_\rho f\mathbf{1}_Y \\
 P_\lambda(q, qt) & \longmapsto & P_\lambda(q, qt)\mathbf{1}_Y & \longmapsto & A_{\lambda+\rho}(q, t)\mathbf{1}_Y = \varepsilon_0 E_{\lambda+\rho}(q, t)\mathbf{1}_Y \\
 P_\lambda(q, t) & \longmapsto & P_\lambda(q, t)\mathbf{1}_Y = \mathbf{1}_0 E_\lambda(q, t)\mathbf{1}_Y & &
 \end{array} \tag{1.8}$$

In this diagram $A_\rho = A_{0+\rho}(q, t) = A_\rho(0, t)$ and the statement that $P_\lambda(q, qt)$ on the left maps to $A_{\lambda+\rho}(q, t)$ on the right is the *Weyl character formula for Macdonald polynomials*,

$$P_\lambda(q, qt) = \frac{A_{\lambda+\rho}(q, t)}{A_\rho}. \tag{1.9}$$

It would be interesting to understand this diagram in terms of geometric contexts analogous to those which exist for the $q = 0$ case. Some progress in this direction is found, for example, in Ginzburg-Kapranov-Vasserot [GKV95] and Oblomkov-Yun [OY14].

1.4 Demazure operators, Hecke operators, and intertwiners

We start defining several operators in $\mathbb{C}[X]$.

For $i \in \{1, \dots, n\}$,

$$(y_i f)(x_1, \dots, x_n) = f(x_1, \dots, x_{i-1}, q^{-1}x_i, x_{i+1}, \dots, x_n).$$

For $i \in \{1, \dots, n-1\}$,

$$\partial_i = (1 + s_i) \frac{1}{x_i - x_{i+1}}, \quad C_{s_i} = (1 + s_i) \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} x_i^{-1} x_{i+1}}{1 - x_i^{-1} x_{i+1}}, \tag{1.10}$$

$$D_{i, i+1} = (1 + s_i) \frac{1}{1 - x_i x_{i+1}^{-1}}, \quad D_{i+1, i} = (1 + s_i) \frac{1}{1 - x_i^{-1} x_{i+1}}. \tag{1.11}$$

For $i \in \{1, \dots, n-1\}$, $C_{s_i} = T_i + t^{-\frac{1}{2}} = T_i^{-1} + t^{\frac{1}{2}}$.

The *intertwiners or creation operators* $\tau_\pi^\vee, \tau_1^\vee, \dots, \tau_{n-1}^\vee$ are given by

$$\tau_\pi^\vee = X_1 T_1 \cdots T_{n-1}, \quad \text{and} \quad \tau_i^\vee = C_{s_i} - \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}} Y_i^{-1} Y_{i+1}}{1 - Y_i^{-1} Y_{i+1}}. \quad (1.12)$$

These interwiners are used to construct the electronic Macdonald polynomials E_μ .

In the study of Schubert calculus and Demazure characters one encounters the Demazure operators given in (1.11). The relation between the interwiners τ_i^\vee and Demazure operators $D_{i,i+1}$ and $D_{i+1,i}$ is

$$\begin{aligned} t^{-\frac{1}{2}} \tau_i^\vee &= D_{i,i+1} + t^{-1} (D_{i+1,i} - 1) + \frac{(1-t^{-1}) Y_{i+1}^{-1} Y_i}{1 - Y_{i+1}^{-1} Y_i} \quad \text{and} \\ t^{\frac{1}{2}} \tau_i^\vee &= t (D_{i,i+1} - 1) + D_{i+1,i} + \frac{(1-t) Y_i^{-1} Y_{i+1}}{1 - Y_i^{-1} Y_{i+1}}. \end{aligned}$$

The explicit eigenvalues in Theorem 1.1, and the above formulas for τ_i^\vee then give

$$\begin{aligned} t^{\frac{1}{2}} \tau_i^\vee E_\mu(q, 0) &= D_{i+1,i} E_\mu(q, 0), \quad \text{when } \mu_i > \mu_{i+1}, \text{ and} \\ t^{-\frac{1}{2}} \tau_i^\vee E_\nu(q, \infty) &= D_{i,i+1} E_\nu(q, \infty), \quad \text{when } \nu_i < \nu_{i+1}. \end{aligned}$$

This is the reason why the specializations of Macdonald polynomials at $t = 0$ and $t = \infty$ contain information about Demazure characters (see [Ion01] and [MRY19, Theorem 1.3]).

In the study of (principal series) representations of $GL(\mathbb{Q}_p)$ one encounters the Iwahori-Hecke algebra operators T_1, \dots, T_{n-1} and T_π given by

$$T_i = C_{s_i} - t^{-\frac{1}{2}} = t^{\frac{1}{2}} s_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X_i X_{i+1}^{-1}} (1 - s_i), \quad \text{for } i \in \{1, \dots, n-1\}. \quad (1.13)$$

The relation between the interwiners τ_i^\vee and the Iwahori-Hecke algebra operators T_i is

$$t^{-\frac{1}{2}} \tau_i^\vee = t^{-\frac{1}{2}} T_i + \frac{(1-t^{-1}) Y_i Y_{i+1}^{-1}}{1 - Y_i Y_{i+1}^{-1}} \quad \text{and} \quad t^{\frac{1}{2}} \tau_i^\vee = t^{\frac{1}{2}} T_i^{-1} + \frac{(1-t) Y_i^{-1} Y_{i+1}}{1 - Y_i^{-1} Y_{i+1}}. \quad (1.14)$$

The explicit eigenvalues in Theorem 1.1, and the above formulas for τ_i^\vee then give

$$\begin{aligned} t^{-\frac{1}{2}} \tau_i^\vee E_\mu(0, t) &= t^{-\frac{1}{2}} T_i^{-1} E_\mu(0, t), \quad \text{when } q = 0 \text{ and } \mu_i > \mu_{i+1}, \text{ and} \\ t^{\frac{1}{2}} \tau_i^\vee E_\nu(\infty, t) &= t^{\frac{1}{2}} T_i E_\nu(\infty, t), \quad \text{when } q = \infty \text{ and } \nu_i < \nu_{i+1}. \end{aligned}$$

If λ is weakly decreasing then $E_\lambda(0, t) = x^\lambda$ and $E_{w_0 \lambda}(\infty, t) = x^{w_0 \lambda}$. It is because of these relationships that the specializations of Macdonald polynomials at $q = 0$ and $q = \infty$ contain information about principal series representations of $GL_n(\mathbb{Q}_p)$ (see [Ion04]).

2 Macdonald polynomials

In this section, we introduce the three types of Macdonald polynomials that are the focus of this paper, the electronic Macdonald polynomials, the bosonic Macdonald polynomials, and the fermionic Macdonald polynomials.

2.1 Electronic Macdonald polynomials

The *electronic Macdonald polynomial* E_μ , for $\mu \in \mathbb{Z}^n$, is determined by the following recursive process:

(E0) $E_{(0,\dots,0)} = 1$,

(E1) $E_{(\mu_n+1,\mu_1,\dots,\mu_{n-1})} = q^{\mu_n} x_1 E_\mu(x_2, \dots, x_n, q^{-1}x_1)$,

(E2) If $(\mu_1, \dots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$ and $\mu_i > \mu_{i+1}$ then

$$E_{s_i \mu} = \left(\partial_i x_i - t x_i \partial_i + \frac{(1-t)q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}}{1 - q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}} \right) E_\mu,$$

where $v_\mu \in S_n$ is the minimal length permutation such that $v_\mu \mu$ is weakly increasing,

(E3) $E_{(\mu_1-1,\dots,\mu_n-1)} = x_1^{-1} \cdots x_n^{-1} E_{(\mu_1,\dots,\mu_n)}$.

Proposition 2.1 says that E_μ is a homogeneous polynomial and that E_μ is x^μ plus a linear combination of lower terms. The appropriate ordering determining “lower terms” is the DB-lex ordering, defined as follows.

The *dominance partial order* on weakly decreasing elements of \mathbb{Z}^n is given by $\lambda^+ \leq \mu^+$ if $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for $i \in \{1, \dots, n\}$ (where $\lambda^+ = (\lambda_1, \dots, \lambda_n)$ and $\mu^+ = (\mu_1, \dots, \mu_n)$ with $\lambda \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_n$). The *Bruhat order* on the symmetric group S_n is determined by setting $v < vs_{ij}$ if $\ell(v) < \ell(vs_{ij})$ (where s_{ij} denotes the transposition switching i and j). If $\mu \in \mathbb{Z}^n$ let μ^+ be the weakly decreasing rearrangement of μ and let $z_\mu \in S_n$ be minimal length such that $\mu = z_\mu \mu^+$. The *DBlex order* (*dominance-Bruhat-lexicographic order*) is the order on \mathbb{Z}^n given by

$$\lambda^+ < \mu^+ \text{ in dominance order} \\ \lambda \leq \mu \text{ if } \quad \text{or} \\ \lambda^+ = \mu^+ \text{ and } z_\lambda < z_\mu \text{ in Bruhat order.}$$

Proposition 2.1. *Let $\mu \in \mathbb{Z}^n$. Then*

$$E_\mu = x^\mu + \sum_{\substack{\nu < \mu \\ |\nu| = |\mu|}} a_{\mu\nu}(q, t) x^\nu, \quad \text{with } a_{\mu\nu}(q, t) \in \mathbb{C}(q, t).$$

This result follows by verifying that if this property holds before applying one of the operations (E1), (E2), or (E3), then it also holds after the application of (E1), (E2), or (E3). It follows from Proposition 2.1 that $\{E_\mu \mid \mu \in \mathbb{Z}^n\}$ is basis of $\mathbb{C}[X]$.

2.2 Bosonic Macdonald polynomials

The *bosonic Macdonald polynomial* P_λ , for $\lambda \in (\mathbb{Z}^n)^+$, is defined by

$$P_\lambda = P_\lambda(x; q, t) = \frac{1}{W_\lambda(t)} \sum_{w \in S_n} w \left(E_\lambda \prod_{i < j} \frac{x_i - t x_j}{x_i - x_j} \right), \quad (2.1)$$

where $W_\lambda(t)$ is the appropriate constant which makes the coefficient of x^λ equal to 1 in $P_\lambda(q, t)$. The constant $W_\lambda(t)$ is determined explicitly in Proposition 4.6.

Various specializations of the $P_\lambda(x; q, t)$ have their own names.

$$\begin{aligned} m_\lambda &= P_\lambda(x; 0, 1) && \text{are the } \textit{monomial symmetric polynomials}, \\ s_\lambda &= P_\lambda(x; 0, 0) && \text{are the } \textit{Schur polynomials}, \\ P_\lambda(x; 0, t) &&& \text{are the } \textit{Hall-Littlewood polynomials}. \end{aligned}$$

Proposition 2.2. *If $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \dots \geq \lambda_n$ then $E_\lambda(0, t) = x^\lambda$.*

Using Proposition 2.2 gives formulas

$$m_\lambda = \sum_{\gamma \in S_n \lambda} x^\gamma, \quad s_\lambda = \sum_{w \in S_n} w \left(x^\lambda \prod_{i < j} \frac{x_i}{x_i - x_j} \right)$$

and

$$P_\lambda(0, t) = \frac{1}{W_\lambda(t)} \sum_{w \in S_n} w \left(x^\lambda \prod_{i < j} \frac{x_i - tx_j}{x_i - x_j} \right),$$

which are familiar formulas for the monomial symmetric polynomials, the Schur polynomials and the Hall-Littlewood polynomials (see [Mac, Ch. 1 (2.1)], [Mac, Ch. I (3.1)] and [Mac, Ch. III (2.1)]). We do not provide a proof of Proposition 2.2 in this paper. It is an easy consequence of the alcove walks formula for Macdonald polynomials given in [RY08, Theorem 2.2].

2.3 Fermionic Macdonald polynomials

The *fermionic Macdonald polynomial* $A_{\lambda+\rho}$, for $\lambda \in (\mathbb{Z}^n)^+$, is

$$A_{\lambda+\rho}(q, t) = \left(\prod_{i < j} \frac{x_j - tx_i}{x_i - x_j} \right) \sum_{w \in S_n} (-1)^{\ell(w)} w E_{\lambda+\rho}(q, t). \quad (2.2)$$

Theorem 2.3. (*Weyl character formula*) *Let $\lambda \in \mathbb{Z}^n$ with $\lambda_1 \geq \dots \geq \lambda_n$. Then*

$$A_\rho(q, t) = \prod_{i < j} (x_j - tx_i) \quad \text{and} \quad P_\lambda(q, qt) = \frac{A_{\lambda+\rho}(q, t)}{A_\rho(q, t)}.$$

Note that even though $E_\rho(q, t)$ depends on q , the denominator $A_\rho(q, t) = A_\rho(t)$ depends only on t .

3 The DAHA, symmetrizers and c-functions

This section sets up the *operator calculus* which will be exploited in later sections to obtain explicit combinatorial results for Macdonald polynomials. We have defined so far six kinds of operators:

- the multiplication by x_i operators X_1, \dots, X_n ,
- the divided difference operators $\partial_1, \dots, \partial_{n-1}$,
- the Hecke operators T_1, \dots, T_{n-1} , and the promotion operator T_π ,
- the Cherednik-Dunkl operators Y_1, \dots, Y_n ,
- the creation/intertwiner operators $\tau_1^\vee, \dots, \tau_{n-1}^\vee$ and τ_π^\vee , and
- the symmetrizers $\mathbf{1}_0$ and ε_0 .

We also define

$$T_\pi^\vee = X_1 T_1 \cdots T_{n-1}. \quad (3.1)$$

In this section we establish the relations that these operators satisfy and develop the handy calculus for working with these tools. It is in this section that we first see the *c-functions* appear, particularly in the notable formulas for the symmetrizers $\mathbf{1}_0$ and ε_0 which are given in Proposition 3.6 and Proposition 3.7. At this point, we consider the operators as *symbols*, and Section 4 is dedicated to the study of these as operators on polynomials.

3.1 The double affine Hecke algebra (DAHA)

Let $q, t^{\frac{1}{2}} \in \mathbb{C}^\times$. The *double affine Hecke algebra (of type GL_n)* is the algebra \tilde{H} generated by symbols T_π and X_k and T_i for $i, k \in \mathbb{Z}$ with relations

$$T_{i+n} = T_i, \quad X_{i+n} = q^{-1}X_i, \quad X_k X_\ell = X_\ell X_k, \quad \text{for } i, k, \ell \in \mathbb{Z}; \quad (3.2)$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i, \quad T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_i + 1, \quad (3.3)$$

for $i, j \in \mathbb{Z}$ with $j \notin \{i \pm 1\}$;

$$X_{i+1} = T_i X_i T_i, \quad \text{and} \quad T_i X_j = X_j T_i, \quad (3.4)$$

for $i \in \{1, \dots, n-1\}$ and $j \in \{1, \dots, n\}$ with $j \notin \{i, i+1\}$; and

$$T_\pi X_i = X_{i+1} T_\pi \quad \text{and} \quad T_\pi T_i = T_{i+1} T_\pi \quad \text{for } i \in \mathbb{Z}. \quad (3.5)$$

It follows from the last relation in (3.3) and the relations in (3.4) that for $i \in \{1, \dots, n-1\}$,

$$T_i X_i = X_{i+1} T_i - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})X_{i+1} \quad \text{and} \quad T_i X_{i+1} = X_i T_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})X_{i+1}. \quad (3.6)$$

Proposition 3.1. *(The glue relations) The operators X_1, T_i , and T_π^\vee satisfy the relations*

$$T_1^{-1} T_\pi T_\pi^\vee = T_\pi^\vee T_\pi T_{n-1} \quad \text{and} \quad T_{n-1}^{-1} \cdots T_1^{-1} T_\pi (T_\pi^\vee)^{-1} = q (T_\pi^\vee)^{-1} T_\pi T_{n-1} \cdots T_1. \quad (3.7)$$

Proof. We prove the first relation using (3.4) and (3.5) in the following way

$$\begin{aligned} T_\pi^\vee T_\pi T_{n-1} &= X_1 T_1 \cdots T_{n-1} T_\pi T_{n-1} = T_1^{-1} \cdots T_{n-1}^{-1} T_{n-1} \cdots T_1 X_1 T_1 \cdots T_{n-1} T_\pi T_{n-1} \\ &= T_1^{-1} \cdots T_{n-1}^{-1} X_n T_\pi T_{n-1} = T_1^{-1} \cdots T_{n-1}^{-1} T_\pi X_{n-1} T_{n-1} \\ &= T_1^{-1} T_\pi T_1^{-1} \cdots T_{n-2}^{-1} X_{n-1} T_{n-1} = T_1^{-1} T_\pi T_1^{-1} \cdots T_{n-2}^{-1} T_{n-2} \cdots T_1 X_1 T_1 \cdots T_{n-2} T_{n-1} \\ &= T_1^{-1} T_\pi X_1 T_1 \cdots T_{n-1} = T_1^{-1} T_\pi T_\pi^\vee. \end{aligned}$$

We prove the second relation by showing the equivalent relation obtained by taking inverses on both sides. That is, we need to show that

$$q^{-1} T_1^{-1} \cdots T_{n-1}^{-1} T_\pi^{-1} T_\pi^\vee = T_\pi^\vee T_\pi^{-1} T_1 \cdots T_{n-1}.$$

In this case we use (3.2) and (3.4), so that

$$\begin{aligned} q^{-1} T_1^{-1} \cdots T_{n-1}^{-1} T_\pi^{-1} T_\pi^\vee &= T_1^{-1} \cdots T_{n-1}^{-1} q^{-1} T_\pi^{-1} X_1 T_1 \cdots T_{n-1} \\ &= T_1^{-1} \cdots T_{n-1}^{-1} X_n T_\pi^{-1} T_1 \cdots T_{n-1} \\ &= T_1^{-1} \cdots T_{n-1}^{-1} T_{n-1} \cdots T_1 X_1 T_1 \cdots T_{n-1} T_\pi^{-1} T_1 \cdots T_{n-1} \\ &= X_1 T_1 \cdots T_{n-1} T_\pi^{-1} T_1 \cdots T_{n-1} = T_\pi^\vee T_\pi^{-1} T_1 \cdots T_{n-1}. \end{aligned}$$

□

Proof. The statements in (3.10) follow from the relations (3.9) and the fact that the Y_i 's all commute with each other.

For $i \in \{1, \dots, n-2\}$,

$$\begin{aligned} T_\pi^\vee T_i &= X_1 T_1 \cdots T_{n-1} T_i = X_1 T_1 \cdots T_i T_{i+1} T_i T_{i+2} \cdots T_{n-1} = X_1 T_1 \cdots T_{i-1} T_{i+1} T_i T_{i+1} T_{i+2} \cdots T_{n-1} \\ &= X_1 T_{i+1} T_1 \cdots T_{i-1} T_i T_{i+1} \cdots T_{n-1} = T_{i+1} X_1 T_1 \cdots T_{n-1} = T_{i+1} T_\pi^\vee. \end{aligned}$$

Using these relations and the relations from (3.7),

$$\begin{aligned} \tau_\pi^\vee Y_1 &= T_\pi^\vee T_\pi T_{n-1} \cdots T_1 = T_1^{-1} T_\pi T_\pi^\vee T_{n-2} \cdots T_1 = T_1^{-1} T_\pi T_{n-1} \cdots T_2 T_\pi^\vee \\ &= T_1^{-1} T_\pi T_{n-1} \cdots T_2 T_1 T_1^{-1} T_\pi^\vee = T_1^{-1} Y_1 T_1^{-1} T_\pi^\vee = Y_2 T_\pi^\vee = Y_2 \tau_\pi^\vee, \quad \text{and} \\ \tau_\pi^\vee Y_n &= T_\pi^\vee T_{n-1}^{-1} \cdots T_1^{-1} Y_1 T_1^{-1} \cdots T_{n-1}^{-1} \\ &= T_\pi^\vee T_{n-1}^{-1} \cdots T_1^{-1} T_\pi T_{n-1} \cdots T_1 T_1^{-1} \cdots T_{n-1}^{-1} = T_\pi^\vee T_{n-1}^{-1} \cdots T_1^{-1} T_\pi (T_\pi^\vee)^{-1} T_\pi^\vee \\ &= T_\pi^\vee q (T_\pi^\vee)^{-1} T_\pi T_{n-1} \cdots T_1 T_\pi^\vee = q Y_1 T_\pi^\vee = q Y_1 \tau_\pi^\vee. \end{aligned}$$

Now, for $i \in \{2, \dots, n-1\}$, we use induction, so that

$$\tau_\pi^\vee Y_i = T_\pi^\vee T_{i-1}^{-1} \cdots T_1^{-1} Y_1 T_1^{-1} \cdots T_{i-1}^{-1} = T_i^{-1} \cdots T_2^{-1} Y_2 T_2^{-1} \cdots T_i^{-1} T_\pi^\vee = Y_{i+1} T_\pi^\vee = Y_{i+1} \tau_\pi^\vee.$$

□

3.4 XY-parallelism

In this section, we highlight the parallelism between the X_i operators and the Y_i operators. To do so, we define the normalized intertwiners, which are defined in terms of the c -functions. For $i, j \in \mathbb{Z}$ with $i \neq j$, define the c -fuctions

$$c_{ij}^X = \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} X_i X_j^{-1}}{1 - X_i X_j^{-1}} \quad \text{and} \quad c_{ij}^Y = \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} Y_i Y_j^{-1}}{1 - Y_i Y_j^{-1}}.$$

For $i \in \{1, \dots, n-1\}$ define the normalized intertwiners η_{s_i} and ξ_{s_i} by

$$\eta_{s_i} = \frac{1}{c_{i+1,i}^Y} (C_{s_i} - c_{i+1,i}^Y) \quad \text{and} \quad \xi_{s_i} = \frac{1}{c_{i,i+1}^X} (C_{s_i} - c_{i+1,i}^X).$$

Proposition 3.4. For $i, j, k \in \{1, \dots, n-1\}$ with $k \notin \{j \pm 1\}$,

$$\begin{aligned} \eta_{s_i} \eta_{s_{i+1}} \eta_{s_i} &= \eta_{s_i} \eta_{s_{i+1}} \eta_{s_i}, & \eta_{s_j} \eta_{s_k} &= \eta_{s_k} \eta_{s_j}, & \eta_{s_i} C_{s_i} &= C_{s_i}, \\ \xi_{s_i} \xi_{s_{i+1}} \xi_{s_i} &= \xi_{s_i} \xi_{s_{i+1}} \xi_{s_i}, & \xi_{s_j} \xi_{s_k} &= \xi_{s_k} \xi_{s_j}, & \xi_{s_i} C_{s_i} &= C_{s_i}. \end{aligned} \quad (3.12)$$

Moreover, for $i \in \{1, \dots, n-1\}$ and $j \in \{1, \dots, n\}$,

$$\begin{aligned} \eta_{s_i} Y_i &= Y_{i+1} \eta_{s_i}, & \xi_{s_i} X_i &= X_{i+1} \xi_{s_i}, \\ \eta_{s_i} Y_{i+1} &= Y_i \eta_{s_i}, & \xi_{s_i} X_{i+1} &= X_i \xi_{s_i}, \\ \eta_{s_i} Y_j &= Y_j \eta_{s_i}, \quad \text{if } j \notin \{i, i+1\}, & \xi_{s_i} X_j &= X_j \xi_{s_i}, \quad \text{if } j \notin \{i, i+1\}. \end{aligned} \quad (3.13)$$

Proof. The first set of relations in (3.13) follow from the relations in (3.10).

Using the relations

$$C_{s_i}^2 = (t^{\frac{1}{2}} + t^{-\frac{1}{2}}) C_{s_i}, \quad C_{s_i} = \tau_i^\vee + c_{i,i+1}^Y, \quad c_{i,i+1}^Y + c_{i+1,i}^Y = t^{\frac{1}{2}} + t^{-\frac{1}{2}},$$

and the relations in (3.10) gives

$$(\tau_i^\vee)^2 = c_{i,i+1}^Y c_{i+1,i}^Y, \quad (\eta_{s_i} + 1)c_{i,i+1}^Y = C_{s_i} \quad \text{and} \quad \eta_{s_i} C_{s_i} = C_{s_i}. \quad (3.14)$$

A direct computation together with the relations in (3.10) produces

$$\begin{aligned} C_{s_1 s_2 s_1} &= C_{s_1} C_{s_2} C_{s_1} - C_{s_1} = C_{s_2} C_{s_1} C_{s_2} - C_{s_2} \\ &= T_1 T_2 T_1 + t^{-\frac{1}{2}} T_1 T_2 + t^{-\frac{1}{2}} T_2 T_1 + t^{-\frac{2}{2}} T_1 + t^{-\frac{2}{2}} T_2 + t^{-\frac{3}{2}}, \\ &= T_2 T_1 T_2 + t^{-\frac{1}{2}} T_1 T_2 + t^{-\frac{1}{2}} T_2 T_1 + t^{-\frac{2}{2}} T_1 + t^{-\frac{2}{2}} T_2 + t^{-\frac{3}{2}}, \\ &= (\eta_{s_1} \eta_{s_2} \eta_{s_1} + \eta_{s_1} \eta_{s_2} + \eta_{s_2} \eta_{s_1} + \eta_{s_2} + \eta_{s_1} + 1) c_{12}^Y c_{13}^Y c_{23}^Y \\ &= (\eta_{s_2} \eta_{s_1} \eta_{s_2} + \eta_{s_1} \eta_{s_2} + \eta_{s_2} \eta_{s_1} + \eta_{s_2} + \eta_{s_1} + 1) c_{12}^Y c_{13}^Y c_{23}^Y. \end{aligned} \quad (3.15)$$

This establishes that $\eta_{s_1} \eta_{s_2} \eta_{s_1} = \eta_{s_2} \eta_{s_1} \eta_{s_2}$ is equivalent to $T_1 T_2 T_1 = T_2 T_1 T_2$ which is equivalent to $C_{s_1} C_{s_2} C_{s_1} - C_{s_1} = C_{s_2} C_{s_1} C_{s_2} - C_{s_2}$.

The relations for ξ_{s_i} are the same as the relations for η_{s_i} because the relations in (3.6) are the same as the relations (3.9) except for a replacing Y_i with X_{i+1} and replacing Y_{i+1} with X_i . Thus, the same proof also works for the relations for ξ_{s_i} . \square

The XY-parallelism in this section is the core of the duality in double affine Artin groups and double affine Hecke algebras (see [Mac03, §3.5]).

Remark 3.5. The reader should be warned to be careful with denominators when working with the normalized interwiners. When using the relations in Proposition 3.4 it is wise to always write any denominators as the leftmost factors so that when acting on (left) \tilde{H} -modules the expressions are 0 as appropriate. As a concrete example, from the first identity in (3.14) it follows that

$$\eta_{s_i}^2 = \frac{1}{c_{i,i+1}^Y c_{i+1,i}^Y} c_{i,i+1}^Y c_{i+1,i}^Y, \quad \text{which does not always reduce to } \eta_{s_i}^2 = 1. \quad (3.16)$$

Even on the polynomial representation considered in Section 4, the operator η_{s_i} can act by 0 on some vectors (for example $\eta_{s_i} \cdot 1 = 0$). However the identity for $\eta_{s_i}^2$ in (3.16) is still valid since $\eta_{s_i}^2 \cdot 1 = 0$ and

$$\frac{1}{c_{i,i+1}^Y c_{i+1,i}^Y} c_{i,i+1}^Y c_{i+1,i}^Y \cdot 1 = 0 \quad (\text{because } c_{i,i+1}^Y \cdot 1 = 0),$$

In the language of localizations of rings, the elements η_{s_i} and η_w (defined in (3.17)) do not live in the double affine Hecke algebra \tilde{H} proper but in a localization at the multiplicative set generated by the elements $(1 - Y_i Y_j^{-1})$ and $(1 - t Y_i Y_j^{-1})$.

3.5 Symmetrizers

Given $w \in S_n$ with reduced word $w = s_{i_1} \cdots s_{i_\ell}$, we define

$$\xi_w = \xi_{s_{i_1}} \cdots \xi_{s_{i_\ell}}, \quad \eta_w = \eta_{s_{i_1}} \cdots \eta_{s_{i_\ell}}, \quad \text{and} \quad T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}, \quad (3.17)$$

together with the following symmetrizers

$$\begin{aligned} X\text{-symmetrizer} \quad p_0^X &= \sum_{w \in S_n} \xi_w, & X\text{-antisymmetrizer} \quad e_0^X &= \sum_{w \in S_n} (-1)^{\ell(w) - \ell(w_0)} \xi_w, \\ Y\text{-symmetrizer} \quad p_0^Y &= \sum_{w \in S_n} \eta_w, & Y\text{-antisymmetrizer} \quad e_0^Y &= \sum_{w \in S_n} (-1)^{\ell(w) - \ell(w_0)} \eta_w. \end{aligned}$$

The *bosonic symmetrizer* and the *fermionic symmetrizer* are defined by

$$\mathbf{1}_0 = \sum_{w \in S_n} t^{\frac{1}{2}(\ell(w) - \ell(w_0))} T_w \quad \text{and} \quad \varepsilon_0 = \sum_{w \in S_n} (-t^{-\frac{1}{2}})^{\ell(w) - \ell(w_0)} T_w. \quad (3.18)$$

The bosonic symmetrizer $\mathbf{1}_0$ is a t -analogue of p_0^X and p_0^Y and the fermionic symmetrizer ε_0 is a t -analogue of e_0^X and e_0^Y . Using the relation $T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_i + 1$ and the last relation in (3.14) gives

$$T_i \mathbf{1}_0 = t^{\frac{1}{2}} \mathbf{1}_0, \quad C_{s_i} \mathbf{1}_0 = (t^{\frac{1}{2}} + t^{-\frac{1}{2}}) \mathbf{1}_0 \quad \text{and} \quad \eta_{s_i} \mathbf{1}_0 = \mathbf{1}_0,$$

since $\eta_{s_i} \mathbf{1}_0 = \eta_{s_i} (t^{\frac{1}{2}} + t^{-\frac{1}{2}})^{-1} C_{s_i} \mathbf{1}_0 = (t^{\frac{1}{2}} + t^{-\frac{1}{2}})^{-1} C_{s_i} \mathbf{1}_0 = \mathbf{1}_0$.

Let us denote by w_0 the longest element in S_n . That is, $w_0(i) = n - i + 1$, for $i \in \{1, \dots, n\}$. It has maximal length, $\ell(w_0) = \frac{n(n-1)}{2} = \binom{n}{2}$, since its set of inversions is

$$\text{Inv}(w_0) = \{(i, j) \mid i, j \in \{1, \dots, n\} \text{ and } i < j\}.$$

Then, we define

$$c_{w_0}^X = \prod_{1 \leq i < j \leq n} c_{ij}^X = \prod_{1 \leq i < j \leq n} \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} X_i X_j^{-1}}{1 - X_i X_j^{-1}} \quad \text{and} \quad c_{w_0}^{X^{-1}} = \prod_{1 \leq i < j \leq n} \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} X_i^{-1} X_j}{1 - X_i^{-1} X_j}.$$

The following result rewrites the bosonic and fermionic symmetrizers in terms of the X -symmetrizers and the Y -symmetrizers.

Proposition 3.6.

$$\mathbf{1}_0 = p_0^X c_{w_0}^{X^{-1}} = p_0^Y c_{w_0}^Y \quad \text{and} \quad \varepsilon_0 = c_{w_0}^X e_0^X = c_{w_0}^{Y^{-1}} e_0^Y. \quad (3.19)$$

Proof. Let $w \in S_n$ with reduced word $w = s_{i_1} \cdots s_{i_\ell}$. Using that $T_i = \xi_{s_i} c_{i, i+1}^{X^{-1}} + (t^{\frac{1}{2}} - c_{i, i+1}^X)$ and expanding the result, we have that

$$T_w = T_{i_1} \cdots T_{i_\ell} = T_w = \xi_w c_w^{X^{-1}} + \sum_{v < w} \xi_v b_v(X), \quad \text{with } b_v(X) \in \mathbb{C}(X_1, \dots, X_n).$$

Thus there are $a_v(X) \in \mathbb{C}(X_1, \dots, X_n)$ such that

$$\mathbf{1}_0 = \sum_{w \in S_n} t^{-\frac{1}{2}(\ell(w) - \ell(w_0))} T_w = \xi_{w_0} c_{w_0}^{X^{-1}} + \sum_{w < w_0} \xi_w a_w(X), \quad (3.20)$$

Since $p_0^X = \sum_{w \in S_n} \xi_w$ and $\xi_{s_i} p_0^X c_{w_0}^{X^{-1}} = p_0^X c_{w_0}^{X^{-1}}$ then

$$\begin{aligned} T_i(p_0^X c_{w_0}^{X^{-1}}) &= (c_{i, i+1}^X \xi_{s_i} + (t^{\frac{1}{2}} - c_{i, i+1}^X)) p_0^X c_{w_0}^{X^{-1}} \\ &= (c_{i, i+1}^X + (t^{\frac{1}{2}} - c_{i, i+1}^X)) p_0^X c_{w_0}^{X^{-1}} = t^{\frac{1}{2}} (p_0^X c_{w_0}^{X^{-1}}). \end{aligned}$$

Now, $\mathbf{1}_0$ is determined, up to multiplication by a constant, by the fact that $T_i \mathbf{1}_0 = t^{\frac{1}{2}} \mathbf{1}_0$ for $i \in \{1, \dots, n-1\}$, and so it follows from (3.20) that $\mathbf{1}_0 = p_0^X c_{w_0}^{X^{-1}}$ (see the example for $n = 3$ in (3.15), where $\mathbf{1}_0 = C_{s_1 s_2 s_1}$). The other relation in (3.19) is proven similarly. \square

3.5.1 Symmetrizers and stabilizers

Let $\lambda \in (\mathbb{Z}^n)^+$. The stabilizer of λ under the action of S_n is $W_\lambda = \{v \in S_n \mid v\lambda = \lambda\}$, and denote by w_λ its longest element. Let W^λ be the set of minimal length representatives of the cosets in S_n/W_λ , and denote by v_λ its longest element. Then, $w_0 = v_\lambda w_\lambda$ with $\ell(w_0) = \ell(v_\lambda) + \ell(w_\lambda)$. Consider the following symmetrizers and antisymmetrizers for W^λ and W_λ .

$$\begin{aligned}
 p_X^\lambda &= \sum_{u \in W^\lambda} \xi_u & \text{and} & & p_\lambda^X &= \sum_{v \in W_\lambda} \xi_v & \text{so that} & & p_0^X &= p_X^\lambda p_\lambda^X, \\
 e_X^\lambda &= \sum_{u \in W^\lambda} \det(v_\lambda u) \xi_u & \text{and} & & e_\lambda^X &= \sum_{v \in W_\lambda} \det(w_\lambda v) \xi_v & \text{so that} & & e_0^X &= e_X^\lambda e_\lambda^X, \\
 p_Y^\lambda &= \sum_{u \in W^\lambda} \eta_u & \text{and} & & p_\lambda^Y &= \sum_{v \in W_\lambda} \eta_v, & \text{so that} & & p_0^Y &= p_Y^\lambda p_\lambda^Y, \\
 e_Y^\lambda &= \sum_{u \in W^\lambda} \det(v_\lambda u) \eta_u & \text{and} & & e_\lambda^Y &= \sum_{v \in W_\lambda} \det(w_\lambda v) \eta_v & \text{so that} & & e_0^Y &= e_Y^\lambda e_\lambda^Y.
 \end{aligned} \tag{3.21}$$

We also define the t -analogues of the elements in (3.21):

$$\begin{aligned}
 \mathbf{1}^\lambda &= \sum_{u \in W^\lambda} (t^{\frac{1}{2}})^{\ell(u) - \ell(v_\lambda)} T_u & \text{and} & & \mathbf{1}_\lambda &= \sum_{v \in W_\lambda} (t^{\frac{1}{2}})^{\ell(v) - \ell(w_\lambda)} T_v, \\
 \varepsilon^\lambda &= \sum_{u \in W^\lambda} (-t^{-\frac{1}{2}})^{\ell(u) - \ell(v_\lambda)} T_u & \text{and} & & \varepsilon_\lambda &= \sum_{v \in W_\lambda} (-t^{-\frac{1}{2}})^{\ell(v) - \ell(w_\lambda)} T_v.
 \end{aligned} \tag{3.22}$$

Then $\mathbf{1}_0 = \mathbf{1}^\lambda \mathbf{1}_\lambda$ and $\varepsilon_0 = \varepsilon^\lambda \varepsilon_\lambda$.

For $\lambda \in (\mathbb{Z}^n)^+$, consider the sets of inversions of the elements w_λ and v_λ .

$$\text{Inv}(w_\lambda) = \{(i, j) \mid i < j \text{ and } \lambda_i = \lambda_j\} \quad \text{and} \quad \text{Inv}(v_\lambda) = \{(i, j) \mid i < j \text{ and } \lambda_i > \lambda_j\}.$$

Thus, the associated c -functions are defined by

$$c_{w_\lambda}^{X^{-1}} = \prod_{\substack{1 \leq i < j \leq n \\ \lambda_i = \lambda_j}} c_{ij}^{X^{-1}} \quad \text{and} \quad c_{v_\lambda}^{X^{-1}} = \prod_{\substack{1 \leq i < j \leq n \\ \lambda_i > \lambda_j}} c_{ij}^{X^{-1}}.$$

The following is a generalization of Proposition 3.6.

Proposition 3.7. For $\lambda \in (\mathbb{Z}^n)^+$,

$$\mathbf{1}_0 = p_X^\lambda c_{v_\lambda}^{X^{-1}} \mathbf{1}_\lambda = p_Y^\lambda c_{v_\lambda}^Y \mathbf{1}_\lambda \quad \text{and} \quad \varepsilon_0 = c_{v_\lambda}^X e_X^\lambda \varepsilon_\lambda = c_{v_\lambda}^{Y^{-1}} e_Y^\lambda \varepsilon_\lambda.$$

Proof. If $u \in W_\lambda$ and $\lambda_i > \lambda_j$ then $\lambda_{u(i)} > \lambda_{u(j)}$. Therefore,

$$u \cdot \text{Inv}(v_\lambda) = \{(u(i), u(j)) \mid i < j \text{ and } \lambda_i > \lambda_j\} = \text{Inv}(v_\lambda),$$

and so, $w_\lambda^{-1} c_{v_\lambda}^X = u c_{v_\lambda}^X = c_{v_\lambda}^X$ for $u \in W_\lambda$. This shows that

$$c_{w_0}^{X^{-1}} = c_{v_\lambda w_\lambda}^{X^{-1}} = (w_\lambda^{-1} c_{v_\lambda}^{X^{-1}}) c_{w_\lambda}^{X^{-1}} = c_{v_\lambda}^{X^{-1}} c_{w_\lambda}^{X^{-1}} \quad \text{and} \quad p_\lambda^X c_{v_\lambda}^{X^{-1}} = c_{v_\lambda}^{X^{-1}} p_\lambda^X. \tag{3.23}$$

Replacing S_n by the group W_λ in the proof of Proposition 3.6 gives $\mathbf{1}_\lambda = p_\lambda^X c_{w_\lambda}^{X^{-1}}$. Using the relations in (3.23) and the identity $\mathbf{1}_\lambda = p_\lambda^X c_{w_\lambda}^{X^{-1}}$ gives

$$\mathbf{1}_0 = p_0^X c_{w_0}^{X^{-1}} = p_X^\lambda p_\lambda^X (w_\lambda^{-1} c_{v_\lambda}^{X^{-1}}) c_{w_\lambda}^{X^{-1}} = p_X^\lambda c_{v_\lambda}^{X^{-1}} p_\lambda^X c_{w_\lambda}^{X^{-1}} = p_X^\lambda c_{v_\lambda}^{X^{-1}} \mathbf{1}_\lambda.$$

The proof for ε_0 is similar. □

4 The action on polynomials

In this section we see the operators of the last section acting on polynomials. Using this action we can reinterpret the constructions of the Macdonald polynomials given in Section 2 as the result of operators acting on the initial polynomial 1. These are the ‘creation formulas’ for Macdonald polynomials. The symmetrizer formulas from Proposition 3.6 and Proposition 3.7 are then used to set up and prove the Boson-Fermion correspondence in the Macdonald polynomial setting and to provide explicit combinatorial formulas for the expansion of bosonic and fermionic Macdonald polynomials in terms of the basis of electronic Macdonald polynomials. These symmetrizer formulas are also the key to product formulas for the Poincaré polynomials which arise as the normalizing constants for the bosonic Macdonald polynomials P_λ .

We have structured this section so that the proofs are postponed to the last subsection. This allows us to first focus on the results and the “big picture” structure that relates the bosonic, fermionic, and electronic Macdonald polynomials and leaves the necessary computations to the last subsection.

4.1 DAHA acts on polynomials

Recall the definition of the *divided difference operators* $\partial_i: \mathbb{C}[X] \rightarrow \mathbb{C}[X]$, the *Hecke algebra operators* $T_i: \mathbb{C}[X] \rightarrow \mathbb{C}[X]$ and the *promotion operator* $T_\pi: \mathbb{C}[X] \rightarrow \mathbb{C}[X]$ by

$$\partial_i f = \frac{f - s_i f}{x_i - x_{i+1}}, \quad T_i = t^{-\frac{1}{2}} x_{i+1} \partial_i - t^{\frac{1}{2}} \partial_i x_{i+1} \quad \text{and} \quad T_\pi = s_1 \cdots s_{n-1} y_n. \quad (4.1)$$

Letting

$$s_0 = T_\pi s_{n-1} T_\pi^{-1} = y_1 y_n^{-1} s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1},$$

then

$$T_\pi s_0 T_\pi^{-1} = s_1, \quad T_\pi X_n T_\pi^{-1} = q^{-1} X_1, \quad T_\pi s_i T_\pi^{-1} = s_{i+1}, \quad \text{and} \quad T_\pi X_i T_\pi^{-1} = X_{i+1},$$

for $i \in \{1, \dots, n-1\}$.

Theorem 4.1. *Let \tilde{H} be the double affine Hecke algebra as defined by generators and relations in Section 3.1. The formulas (4.1) define an action of \tilde{H} on $\mathbb{C}[X]$.*

A way of deriving the formulas in (4.1) is to consider the induced representation

$$\mathbb{C}[X] \cong \text{Ind}_{\tilde{H}_Y}^{\tilde{H}}(\mathbf{1}_Y) = \tilde{H}\mathbf{1}_Y = \mathbb{C}\text{-span}\{X_1^{\mu_1} \cdots X_n^{\mu_n} \mathbf{1}_Y \mid \mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n\}$$

determined by

$$T_\pi \mathbf{1}_Y = \mathbf{1}_Y \quad \text{and} \quad T_i \mathbf{1}_Y = t^{\frac{1}{2}} \mathbf{1}_Y, \quad \text{for } i \in \{1, \dots, n\}. \quad (4.2)$$

Then the formulas in (4.1) are consequences of the relations in (3.4) and (3.5). In other words, the map

$$\begin{array}{ccc} \mathbb{C}[X] & \longrightarrow & \tilde{H}\mathbf{1}_Y \\ x^\mu & \longmapsto & X^\mu \mathbf{1}_Y \end{array} \quad \text{is a } \tilde{H}\text{-module isomorphism.} \quad (4.3)$$

4.2 Creation formulas

Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$. The *minimal length permutation* v_μ such that $v_\mu \mu$ is weakly increasing is given by

$$v_\mu(r) = 1 + \#\{r' \in \{1, \dots, r-1\} \mid \mu_{r'} \leq \mu_r\} + \#\{r' \in \{r+1, \dots, n\} \mid \mu_{r'} < \mu_r\},$$

for $r \in \{1, \dots, n\}$. A *box* in μ is a pair (r, c) with $r \in \{1, \dots, n\}$ and $c \in \{1, \dots, \mu_r\}$. If $b = (r, c)$ is a box in μ then define

$$u_\mu(r, c) = \#\{r' \in \{1, \dots, r-1\} \mid \mu_{r'} \leq c-1\} + \#\{r' \in \{r+1, \dots, n\} \mid \mu_{r'} < c-1\}.$$

Let us take the following creation formula for E_μ from [GR21, Proposition 5.5 and Proposition 2.2(a)], where a complete proof is included.

Theorem 4.2. *Let $\mu \in \mathbb{Z}_{\geq 0}^n$. Letting*

$$\tau_{u_\mu}^\vee = \prod_{(r,c) \in \mu} (\tau_{u_\mu(r,c)}^\vee \cdots \tau_2^\vee \tau_1^\vee \tau_\pi^\vee), \quad \text{then} \quad E_\mu = t^{-\frac{1}{2}\ell(v_\mu^{-1})} \tau_{u_\mu}^\vee 1,$$

where 1 is the polynomial $1 \in \mathbb{C}[X]$.

The creation formulas for P_λ and $A_{\lambda+\rho}$ are given in terms of the bosonic and fermionic symmetrizers defined in (3.18).

Theorem 4.3. *Let $\lambda \in (\mathbb{Z}^n)^{++}$, and recall that $\rho = (n-1, n-2, \dots, 1)$. Then*

$$P_\lambda = \frac{t^{\frac{1}{2}\ell(w_0)}}{W_\lambda(t)} \mathbf{1}_0 E_\lambda \quad \text{and} \quad A_{\lambda+\rho} = t^{\frac{1}{2}\ell(w_0)} \varepsilon_0 E_{\lambda+\rho}, \quad (4.4)$$

where $W_\lambda(t)$ is a normalizing constant which makes the coefficient of x^λ in P_λ equal to 1.

By (3.19), the formulas in (4.4) match the formulas in (2.1) and (2.2). The constant $W_\lambda(t)$ is determined explicitly in Proposition 4.6.

4.3 The Boson-Fermion correspondence

Define the following vector subspaces of $\mathbb{C}[X]$

$$\begin{aligned} \mathbb{C}[X]^{S_n} &= \{f \in \mathbb{C}[X] \mid s_i f = f \text{ for all } i \in \{1, \dots, n-1\}\}, \\ \mathbb{C}[X]^{\det} &= \{f \in \mathbb{C}[X] \mid s_i f = -f \text{ for all } i \in \{1, \dots, n-1\}\}, \\ \mathbb{C}[X]^{\text{Bos}} &= \{f \in \mathbb{C}[X] \mid T_{s_i} f = t^{\frac{1}{2}} f \text{ for all } i \in \{1, \dots, n-1\}\}, \\ \mathbb{C}[X]^{\text{Fer}} &= \{f \in \mathbb{C}[X] \mid T_{s_i} f = -t^{-\frac{1}{2}} f \text{ for all } i \in \{1, \dots, n-1\}\}. \end{aligned}$$

The Boson-Fermion correspondence in Theorem 4.4 puts the formulas in (4.4) into a structural context.

Theorem 4.4. *Let $\mathbf{1}_0$ and ε_0 be as defined in (3.18). Then*

$$\mathbf{1}_0 \mathbb{C}[X] = \mathbb{C}[X]^{\text{Bos}} = \mathbb{C}[X]^{S_n} \quad \text{and} \quad \varepsilon_0 \mathbb{C}[X] = \mathbb{C}[X]^{\text{Fer}} = A_\rho \mathbb{C}[X]^{S_n}.$$

Moreover, there are $\mathbb{C}[X]^{S_n}$ -module isomorphisms

$$\begin{array}{ccc} \mathbb{C}[X]^{S_n} & \rightarrow & \mathbb{C}[X]^{\det} \\ f & \mapsto & a_\rho f \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{C}[X]^{\text{Bos}} & \rightarrow & \mathbb{C}[X]^{\text{Fer}} \\ f & \mapsto & A_\rho f \end{array} \quad (4.5)$$

where

$$a_\rho = \prod_{1 \leq i < j \leq n} (x_j - x_i) \quad \text{and} \quad A_\rho = \prod_{1 \leq i < j \leq n} (x_j - tx_i), \quad (4.6)$$

4.4 The Poincaré polynomial $W_0(t)$

The *Poincaré polynomial* for the symmetric group S_n is

$$W_0(t) = \sum_{w \in S_n} t^{\ell(w)}.$$

The following result writes the Poincaré polynomial in terms of the c -functions $c_{w_0}^Y$ and $c_{w_0}^{X^{-1}}$.

Proposition 4.5. *We have that*

$$t^{-\frac{1}{2}\ell(w_0)}W_0(t) = \text{ev}_0^t(c_{w_0}^Y) = \sum_{w \in S_n} wc_{w_0}^{X^{-1}} \quad (4.7)$$

and

$$W_0(t) = \prod_{1 \leq i < j \leq n} \frac{1 - t^{j-i+1}}{1 - t^{j-i}} = [n]!, \quad \text{where} \quad [n]! = \prod_{d=1}^n \frac{1 - t^d}{1 - t}.$$

By Proposition 4.5,

$$W_0(t) = \sum_{w \in S_n} wt^{\frac{1}{2}\ell(w_0)}c_{w_0}^{X^{-1}} = \sum_{w \in S_n} w \left(\prod_{i < j} \frac{x_i - tx_j}{x_i - x_j} \right).$$

This coincides with the definition of $W_0(t)$ (see (2.1)) as the appropriate constant that makes the coefficient of $1 = x^0 = x_1^0 \cdots x_n^0$ in

$$P_0 = \frac{1}{W_0(t)} \sum_{w \in S_n} w \left(\prod_{i < j} \frac{x_i - tx_j}{x_i - x_j} \right).$$

The following result is the analog for the subgroup W_λ , and its proof is the same as for Proposition 4.5, except restricted to the subgroup W_λ . It provides explicit formulas for the normalizing factor $W_\lambda(t)$ which makes the coefficient of x^λ in the bosonic Macdonald polynomial $P_\lambda(x_1, \dots, x_n; q, t)$ equal to 1.

Proposition 4.6. *Let $\lambda \in (\mathbb{Z}^n)^+$. Let $W_\lambda = \{v \in S_n \mid v\lambda = \lambda\}$ be the stabilizer of λ in S_n and $W_\lambda(t)$ the length generating function for W_λ . Then*

$$t^{-\frac{1}{2}\ell(w_\lambda)}W_\lambda(t) = \text{ev}_0^t(c_{w_\lambda}^Y) = \sum_{w \in W_\lambda} wc_{w_\lambda}^{X^{-1}},$$

where w_λ is the longest element of W_λ . Alternatively,

$$W_\lambda(t) = \sum_{w \in W_\lambda} w \left(\prod_{\substack{1 \leq i < j \leq n \\ \lambda_i = \lambda_j}} \frac{1 - tx_i^{-1}x_j}{1 - x_i^{-1}x_j} \right) = \prod_{\substack{1 \leq i < j \leq n \\ \lambda_i = \lambda_j}} \frac{1 - t^{j-i+1}}{1 - t^{j-i}} = \prod_{i \in \mathbb{Z}} [m_i]!,$$

where $m_i = \#\{k \in \{1, \dots, n\} \mid \lambda_k = i\}$.

Example 4.1. *For $\lambda = (5, 5, 3, 2, 2, 2, 2, -1, -1, -1)$, we have that $m_5 = 2$, $m_3 = 1$, $m_2 = 5$, $m_{-1} = 3$. By Proposition 4.6,*

$$W_\lambda = S_5 \times S_1 \times S_5 \times S_3 \subseteq S_{11} \quad \text{and} \quad W_\lambda(t) = [2]! \cdot [1]! \cdot [5]! \cdot [3]!.$$

4.5 H_Y -decomposition of the polynomial representation

Let H_Y be the algebra generated by the operators T_1, \dots, T_{n-1} and Y_1, \dots, Y_n (so that H_Y is an affine Hecke algebra). As H_Y -modules

$$\mathbb{C}[X] = \bigoplus_{\lambda} \mathbb{C}[X]^{\lambda} \quad \text{where} \quad \mathbb{C}[X]^{\lambda} = \text{span}\{E_{\mu} \mid \mu \in S_n \lambda\},$$

where the direct sum is over decreasing $\lambda = (\lambda_1 \geq \dots \geq \lambda_n) \in \mathbb{Z}^n$, and $S_n \lambda$ denotes the set of distinct rearrangements of λ .

A description of the action of H_Y on $\mathbb{C}[X]^{\lambda}$ is given by the following. Let $\mu \in \mathbb{Z}^n$ and $v_{\mu} \in S_n$ be the minimal length permutation such that $v_{\mu} \mu$ is weakly increasing. Fix $i \in \{1, \dots, n-1\}$, and define

$$a_{\mu} = q^{\mu_i - \mu_{i+1}} t^{v_{\mu}(i) - v_{\mu}(i+1)}, \quad \text{and} \quad D_{\mu} = \frac{(1 - ta_{\mu})(1 - ta_{s_i \mu})}{(1 - a_{\mu})(1 - a_{s_i \mu})}.$$

We have two cases depending on the relationship between μ_i and μ_{i+1} .

Case 1: $\mu_i > \mu_{i+1}$. Using the identity $E_{s_i \mu} = t^{\frac{1}{2}} \tau_i^{\vee} E_{\mu}$ if $\mu_i > \mu_{i+1}$ from (E2), the eigenvalue from Theorem 1.1 and the formulas in (1.14) for τ_i^{\vee} ,

$$\begin{aligned} Y_i^{-1} Y_{i+1} E_{\mu} &= a_{\mu} E_{\mu}, & t^{\frac{1}{2}} \tau_i^{\vee} E_{\mu} &= E_{s_i \mu}, & \text{and} & & t^{\frac{1}{2}} T_i E_{\mu} &= -\frac{1-t}{1-a_{\mu}} E_{\mu} + E_{s_i \mu}, \\ Y_i^{-1} Y_{i+1} E_{s_i \mu} &= a_{s_i \mu} E_{s_i \mu}, & t^{\frac{1}{2}} \tau_i^{\vee} E_{s_i \mu} &= D_{\mu} E_{\mu}, & & & t^{\frac{1}{2}} T_i E_{s_i \mu} &= D_{\mu} E_{\mu} + \frac{1-t}{1-a_{s_i \mu}} E_{s_i \mu}. \end{aligned} \quad (4.8)$$

Case 2: If $\mu_i = \mu_{i+1}$ then $v_{\mu}(i+1) = v_{\mu}(i) + 1$ and $a_{\mu} = t^{-1}$, so that

$$Y_i^{-1} Y_{i+1} E_{\mu} = t^{-1} E_{\mu}, \quad (t^{\frac{1}{2}} \tau_i^{\vee}) E_{\mu} = 0, \quad \text{and} \quad (t^{\frac{1}{2}} T_i) E_{\mu} = t E_{\mu}. \quad (4.9)$$

These formulas make explicit the action of H_Y on $\mathbb{C}[X]^{\lambda}$ in the basis $\{E_{\mu} \mid \mu \in S_n \lambda\}$.

4.6 E-expansions

Let $\mathbb{C}(Y)$ be the field of fractions of $\mathbb{C}[Y]$. For $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ define homomorphisms $\text{ev}_{\mu}^t: \mathbb{C}[Y] \rightarrow \mathbb{C}[Y]$ by

$$\text{ev}_{\mu}^t(Y_i) = q^{-\mu_i} t^{-(v_{\mu}(i)-1) + \frac{1}{2}(n-1)}, \quad \text{for } i \in \{1, \dots, n\}.$$

Extend ev_{μ}^t to those elements of the field $\mathbb{C}(Y)$ for which the evaluated denominator is not 0. We refer to these homomorphisms as *evaluations*. Then Theorem 1.1 gives that

$$f E_{\mu} = \text{ev}_{\mu}^t(f) E_{\mu}, \quad \text{for } f \in \mathbb{C}[Y] \text{ and } \mu \in \mathbb{Z}^n. \quad (4.10)$$

The following result expressed the bosonic and fermionic Macdonald polynomials in terms of the electronic Macdonald polynomials using the evaluations and c -functions.

Proposition 4.7. *Let $\lambda \in (\mathbb{Z}^n)^{\dagger}$. Then*

$$\begin{aligned} P_{\lambda} &= \sum_{z \in W^{\lambda}} t^{\frac{1}{2} \ell(v_{\lambda} z)} \text{ev}_{z\lambda}^t(c_{v_{\lambda} z}^Y) E_{z\lambda} \quad \text{and} \\ A_{\lambda+\rho} &= \sum_{z \in W_0} (-t^{\frac{1}{2}})^{\ell(w_0 z)} \text{ev}_{z(\lambda+\rho)}^t(c_{w_0 z}^{Y^{-1}}) E_{z(\lambda+\rho)}. \end{aligned}$$

Alternatively,

$$P_\lambda = \sum_{\mu \in S_n \lambda} \left(\prod_{\substack{1 \leq i < j \leq n \\ \mu_i > \mu_j}} t \left(\frac{1 - q^{\mu_i - \mu_j} t^{v_\mu(i) - v_\mu(j) - 1}}{1 - q^{\mu_i - \mu_j} t^{v_\mu(i) - v_\mu(j)}} \right) \right) E_\mu \quad \text{and}$$

$$A_{\lambda + \rho} = \sum_{\mu \in S_n(\lambda + \rho)} \left(\prod_{\substack{1 \leq i < j \leq n \\ \mu_i > \mu_j}} (-1) \left(\frac{1 - q^{\mu_i - \mu_j} t^{v_\mu(i) - v_\mu(j) + 1}}{1 - q^{\mu_i - \mu_j} t^{v_\mu(i) - v_\mu(j)}} \right) \right) E_\mu.$$

Example 4.2. For $n = 2$ and $m \in \mathbb{Z}_{>0}$,

$$P_{(m,0)} = E_{(0,m)} + t^{\frac{1}{2}} \text{ev}_{(m,0)}^t(c_{12}^Y) E_{(m,0)} = E_{(0,m)} + t \left(\frac{1 - q^m}{1 - q^m t} \right) E_{(m,0)},$$

$$A_{(m,0)} = E_{(0,m)} - t^{\frac{1}{2}} \text{ev}_{(m,0)}^t(c_{21}^Y) E_{(m,0)} = E_{(0,m)} - \frac{1 - q^m t^2}{1 - q^m t} E_{(m,0)}.$$

Note that these expressions relate to c-functions since $t \left(\frac{1 - q^m}{1 - q^m t} \right) = t^{\frac{1}{2}} \left(\frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} q^{-m} t^{-1}}{1 - q^{-m} t^{-1}} \right)$.

For $n = 3$,

$$P_{(1,0,0)} = E_{(0,0,1)} + t \left(\frac{1 - q}{1 - qt} \right) E_{(0,1,0)} + t^2 \left(\frac{1 - q}{1 - qt} \right) \left(\frac{1 - qt}{1 - qt^2} \right) E_{(1,0,0)}.$$

For general n and $m \in \mathbb{Z}_{>0}$, denote $\varepsilon_i = (0, \dots, 0, 1, 0, \dots, 0)$, that is the sequence of length n with 1 in the i th spot and 0 elsewhere. Then

$$P_{(m,0,\dots,0)} = \sum_{i=1}^n t^{n-i} \left(\frac{1 - q^m}{1 - q^m t} \right) \left(\frac{1 - q^m t}{1 - q^m t^2} \right) \cdots \left(\frac{1 - q^m t^{n-i-1}}{1 - q^m t^{n-i}} \right) E_{m\varepsilon_i} = \sum_{i=1}^n t^{n-i} \left(\frac{1 - q^m}{1 - q^m t^{n-i}} \right) E_{m\varepsilon_i}.$$

Example 4.3.

$$P_{(2,1,0)} = E_{(0,1,2)} + t \left(\frac{1 - q}{1 - qt} \right) E_{(1,0,2)} + t \left(\frac{1 - q}{1 - qt} \right) E_{(0,2,1)} + t^2 \left(\frac{1 - qt}{1 - qt^2} \right) \left(\frac{1 - q^2}{1 - q^2 t} \right) E_{(2,0,1)}$$

$$+ t^2 \left(\frac{1 - qt}{1 - qt^2} \right) \left(\frac{1 - q^2}{1 - q^2 t} \right) E_{(1,2,0)} + t^3 \left(\frac{1 - q}{1 - qt} \right) \left(\frac{1 - q^2 t}{1 - q^2 t^2} \right) \left(\frac{1 - q}{1 - qt} \right) E_{(2,1,0)}.$$

4.7 Proofs

4.7.1 Proof that the action on polynomials is a representation of \tilde{H}

Theorem 4.1. Let \tilde{H} be the double affine Hecke algebra as defined by generators and relations in Section 3.1. The formulas (4.1) define an action of \tilde{H} on $\mathbb{C}[X]$.

Proof. Let $i \in \{1, \dots, n-1\}$. Since $y_n X_i = X_i y_n$ and $s_1 \cdots s_{n-1} X_i = X_{i+1} s_1 \cdots s_{n-1}$ then

$$T_\pi X_i = s_1 \cdots s_{n-1} y_n X_i = X_{i+1} s_1 \cdots s_{n-1} y_n = X_{i+1} T_\pi.$$

As operators on $\mathbb{C}[X]$,

$$T_\pi X_n = s_1 \cdots s_{n-1} y_n X_n = s_1 \cdots s_{n-1} q^{-1} X_n y_n = q^{-1} X_1 s_1 \cdots s_{n-1} y_n = q^{-1} X_1 T_\pi = X_{n+1} T_\pi.$$

Recall that we define $s_0 = T_\pi s_{n-1} T_\pi^{-1}$ so that

$$s_0 = T_\pi s_{n-1} T_\pi^{-1} = (y_1 s_1 \cdots s_{n-1}) s_{n-1} (s_{n-1} \cdots s_1 y_1^{-1}) = y_1 s_1 \cdots s_{n-2} s_{n-1} s_{n-2} \cdots s_1 y_1^{-1}$$

$$= y_1 y_n^{-1} s_1 \cdots s_{n-2} s_{n-1} s_{n-2} \cdots s_1 = y_1 y_n^{-1} s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1}.$$

Then

$$T_\pi s_0 T_\pi^{-1} = (s_1 \cdots s_{n-1} y_n)(y_1 y_n^{-1} s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1})(s_{n-1} \cdots s_1 y_1^{-1}) = s_1 y_2 y_1 y_2^{-1} y_1^{-1} = s_1.$$

Define $\partial_0 = T_\pi \partial_{n-1} T_\pi^{-1}$, so that

$$\partial_0 = T_\pi \partial_{n-1} T_\pi^{-1} = T_\pi \frac{1}{x_{n-1} - x_n} (1 - s_{n-1}) T_\pi^{-1} = \frac{1}{x_n - q^{-1} x_1} (1 - s_0).$$

Then

$$T_\pi \partial_0 T_\pi^{-1} = T_\pi \frac{1}{x_n - q^{-1} x_1} (1 - s_0) T_\pi^{-1} = \frac{1}{q^{-1} x_1 - q^{-1} x_2} (1 - s_1) = q \partial_1.$$

Finally,

$$T_0 = T_\pi T_{n-1} T_\pi^{-1} = T_\pi (t^{-\frac{1}{2}} x_n \partial_{n-1} - t^{\frac{1}{2}} \partial_{n-1} x_n) T_\pi^{-1} = t^{-\frac{1}{2}} q^{-1} x_1 \partial_0 - t^{\frac{1}{2}} \partial_0 q^{-1} x_n,$$

and

$$T_\pi T_0 T_\pi^{-1} = T_\pi (t^{-\frac{1}{2}} q^{-1} x_1 \partial_0 - t^{\frac{1}{2}} \partial_0 q^{-1} x_1) T_\pi^{-1} = t^{-\frac{1}{2}} q^{-1} x_2 q \partial_1 - t^{\frac{1}{2}} q \partial_1 q^{-1} x_2 = T_1.$$

These computations show that the relations in (3.5) hold.

Let $i \in \{1, \dots, n-1\}$. Then, as operators on $\mathbb{C}[X]$,

$$\begin{aligned} \xi_{s_i} &= \frac{1}{c_{i,i+1}^X} (C_{s_i} - c_{i+1,i}^X) = \frac{1}{c_{i,i+1}^X} (T_i + t^{-\frac{1}{2}} - c_{i+1,i}^X) = \frac{1}{c_{i,i+1}^X} (t^{-\frac{1}{2}} x_{i+1} \partial_i - t^{\frac{1}{2}} \partial_i x_{i+1} + t^{-\frac{1}{2}} - c_{i+1,i}^X) \\ &= \frac{1}{c_{i,i+1}^X} \left(t^{-\frac{1}{2}} \frac{x_{i+1}}{x_i - x_{i+1}} (1 - s_i) - t^{\frac{1}{2}} \frac{1}{x_i - x_{i+1}} (1 - s_i) x_{i+1} + \frac{t^{-\frac{1}{2}} x_i - t^{-\frac{1}{2}} x_{i+1}}{x_i - x_{i+1}} - \frac{t^{-\frac{1}{2}} x_i - t^{\frac{1}{2}} x_{i+1}}{x_i - x_{i+1}} \right) \\ &= \frac{1}{c_{i,i+1}^X} \left(\frac{t^{-\frac{1}{2}} x_{i+1} - t^{\frac{1}{2}} x_{i+1} + t^{-\frac{1}{2}} x_i - t^{-\frac{1}{2}} x_{i+1} - t^{-\frac{1}{2}} x_i + t^{\frac{1}{2}} x_{i+1}}{x_i - x_{i+1}} + \frac{-t^{-\frac{1}{2}} x_{i+1} + t^{\frac{1}{2}} x_i}{x_i - x_{i+1}} s_i \right) \\ &= \frac{1}{c_{i,i+1}^X} \left(0 + \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} x_i x_{i+1}^{-1}}{1 - x_i x_{i+1}^{-1}} s_i \right) = s_i. \end{aligned}$$

Then, as in the proof of Proposition 3.4, the relations

$$s_i^2 = 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_j s_k = s_k s_j,$$

for $i, j, k \in \{1, \dots, n-1\}$ with $k \notin \{j-1, j+1\}$, imply the relations in (3.3). Similarly, as in the proof of Proposition 3.4, the relations

$$s_i X_i = X_{i+1} s_i, \quad s_i X_{i+1} = x_i s_i, \quad s_i X_j = X_j s_i,$$

for $i \in \{1, \dots, n-1\}$ with $j \notin \{i, i+1\}$, imply the relations in (3.6), which, in the presence of the last relation in (3.3), are equivalent to the relations in (3.4). \square

4.7.2 Proof of the Boson-Fermion equalities

The following result is a restatement of Theorem 4.4.

Proposition 4.8.

$$p_0\mathbb{C}[X] = \mathbb{C}[X]^{S_n} = \mathbb{C}[X]^{W_0}, \quad e_0\mathbb{C}[X] = \mathbb{C}[X]^{\det} = a_\rho\mathbb{C}[X]^{W_0} \quad \text{and} \quad a_\rho = e_0x^\rho,$$

$$\mathbf{1}_0\mathbb{C}[X] = \mathbb{C}[X]^{\text{Bos}} = \mathbb{C}[X]^{W_0}, \quad \varepsilon_0\mathbb{C}[X] = \mathbb{C}[X]^{\text{Fer}} = A_\rho\mathbb{C}[X]^{W_0} \quad \text{and} \quad A_\rho = \varepsilon_0x^\rho.$$

Proof.

◦ $p_0\mathbb{C}[X] = \mathbb{C}[X]^{S_n}$

– If $f \in \mathbb{C}[X]^{S_n}$ then $f = p_0\left(\frac{1}{n!}f\right)$ so that $f \in p_0\mathbb{C}[X]$. Thus $\mathbb{C}[X]^{S_n} \subseteq p_0\mathbb{C}[X]$.

– If $f \in p_0\mathbb{C}[X]$ then $f = p_0g$. We also know that if $w \in S_n$ then $wf = wp_0g = p_0g$, and so $f \in \mathbb{C}[X]^{S_n}$. Thus $p_0\mathbb{C}[X] \subseteq \mathbb{C}[X]^{S_n}$.

◦ $a_\rho\mathbb{C}[X]^{W_0} = \mathbb{C}[X]^{\det}$

– If $f \in \mathbb{C}[X]^{\det}$ then $(1 - s_{ij})f = 0$, where s_{ij} denotes the transposition in S_n that switches i and j . Thus f is divisible by each $x_j - x_i$, which implies that

$$f \text{ is divisible by } a_\rho = \prod_{1 \leq i < j \leq n} x_j - x_i.$$

Then $\frac{1}{a_\rho}f \in \mathbb{C}[X]^{S_n}$ and $f \in a_\rho\mathbb{C}[X]^{W_0}$, so that $\mathbb{C}[X]^{\det} \subseteq a_\rho\mathbb{C}[X]^{W_0}$.

◦ $e_0\mathbb{C}[X] = \mathbb{C}[X]^{\det}$

– If $f \in e_0\mathbb{C}[X]$ then $s_\alpha f = s_\alpha e_0g = -e_0g = -f$. Thus $f \in \mathbb{C}[X]^{\det}$ and $e_0\mathbb{C}[X] \subseteq \mathbb{C}[X]^{\det}$.

– If $f \in \mathbb{C}[X]^{\det}$ then $e_0f = \text{Card}(W_0)f$. Thus $f \in e_0\mathbb{C}[X]$ and $\mathbb{C}[X]^{\det} \subseteq e_0\mathbb{C}[X]$.

◦ $\mathbf{1}_0\mathbb{C}[X] = \mathbb{C}[X]^{\text{Bos}} = \mathbb{C}[X]^{S_n}$

– Let $h \in \mathbf{1}_0\mathbb{C}[X]$, we can write $h = \mathbf{1}_0f$ with $f \in \mathbb{C}[X]$ and then

$$T_{s_i}h = T_{s_i}\mathbf{1}_0f = t^{\frac{1}{2}}\mathbf{1}_0f = t^{\frac{1}{2}}h. \quad \text{Thus } h \in \mathbb{C}[X]^{\text{Bos}} \text{ and } \mathbf{1}_0\mathbb{C}[X] \subseteq \mathbb{C}[X]^{\text{Bos}}.$$

– Let $f \in \mathbb{C}[X]^{\text{Bos}}$. Then, by Proposition 3.6 and (4.7),

$$f = \frac{t^{\frac{1}{2}\ell(w_0)}}{W_0(t)}\mathbf{1}_0f = \frac{1}{[n]!} \sum_{w \in W_0} w\left(f \prod_{1 \leq i < j \leq n} \frac{x_i - tx_j}{x_i - x_j}\right) \in \mathbb{C}[X]^{W_0}.$$

Thus $\mathbb{C}[X]^{\text{Bos}} \subseteq \mathbb{C}[X]^{W_0}$.

– Let $f \in \mathbb{C}[X]^{W_0}$. Then, by Proposition 3.6 and (4.7),

$$\mathbf{1}_0 \frac{t^{\frac{1}{2}\ell(w_0)}}{W_0(t)}f = \frac{1}{[n]!} \sum_{w \in W_0} w\left(f \prod_{1 \leq i < j \leq n} \frac{x_i - tx_j}{x_i - x_j}\right) = f \frac{1}{[n]!} \sum_{w \in W_0} w\left(\prod_{1 \leq i < j \leq n} \frac{x_i - tx_j}{x_i - x_j}\right) = f.$$

Thus $f \in \mathbf{1}_0\mathbb{C}[X]$ and $\mathbb{C}[X]^{W_0} \subseteq \mathbf{1}_0\mathbb{C}[X]$.

◦ $\varepsilon_0\mathbb{C}[X] = \mathbb{C}[X]^{\text{Fer}} = A_\rho\mathbb{C}[X]^{W_0}$

- Let $h = \varepsilon_0 \mathbb{C}[X]$ and let $f \in \mathbb{C}[X]$ such that $h = \varepsilon_0 f$. Then

$$T_{s_i} h = T_{s_i} \varepsilon_0 f = -t^{-\frac{1}{2}} \varepsilon_0 f = -t^{-\frac{1}{2}} h.$$

Thus $h \in \mathbb{C}[X]^{\text{Fer}}$ and $\varepsilon_0 \mathbb{C}[X] \subseteq \mathbb{C}[X]^{\text{Fer}}$.

- Let $f \in \mathbb{C}[X]^{\text{Fer}}$. Then $T_i f = -t^{-\frac{1}{2}} f$ gives

$$f = \frac{t^{\frac{1}{2}\ell(w_0)}}{W_0(t)} \varepsilon_0 f = \frac{1}{W_0(t)} \frac{A_\rho}{a_\rho} \sum_{w \in W_0} \det(w) w f \in A_\rho \mathbb{C}[X]^{W_0}.$$

Thus $\mathbb{C}[X]^{\text{Fer}} \subseteq A_\rho \mathbb{C}[X]^{W_0}$.

- Let $f \in A_\rho \mathbb{C}[X]^{W_0}$ and let $g \in \mathbb{C}[X]^{S_n}$ be such that $f = A_\rho g$. Write g as a linear combination, $g = \sum c_\lambda s_\lambda$, where s_λ are Schur polynomials. Then

$$f = A_\rho g = \sum_\lambda c_\lambda A_\rho s_\lambda = \sum_\lambda c_\lambda \frac{A_\rho}{a_\rho} \sum_{w \in W_0} \det(w_0 w) w x^{\lambda+\rho} = \varepsilon_0 \left(\sum_\lambda c_\lambda x^{\lambda+\rho} \right).$$

By (3.19) and the fact that

$$t^{-\frac{1}{2}\ell(w_0)} \frac{A_\rho}{a_\rho} = t^{-\frac{1}{2}\ell(w_0)} \prod_{1 \leq i < j \leq n} \frac{x_j - t x_i}{x_j - x_i} = \prod_{1 \leq i < j \leq n} \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} x_i x_j^{-1}}{1 - x_i x_j^{-1}} = c_{w_0}(x)$$

conclude that $f \in \varepsilon_0 \mathbb{C}[X]$. Thus $A_\rho \mathbb{C}[X]^{W_0} \subseteq \varepsilon_0 \mathbb{C}[X]$.

□

4.7.3 Proof of the formulas for Poincaré polynomials

Proposition 4.5. *We have that*

$$t^{-\frac{1}{2}\ell(w_0)} W_0(t) = \text{ev}_0^t(c_{w_0}^Y) = \sum_{w \in S_n} w c_{w_0}^{X^{-1}}$$

and

$$W_0(t) = \prod_{1 \leq i < j \leq n} \frac{1 - t^{j-i+1}}{1 - t^{j-i}} = [n]!, \quad \text{where} \quad [n]! = \prod_{d=1}^n \frac{1 - t^d}{1 - t}.$$

Proof. Identify $\mathbb{C}[X]$ and $\tilde{H}\mathbf{1}_Y$ via the isomorphism in (4.3). Applying the identity $\mathbf{1}_0 = p_0^X c_{w_0}^{X^{-1}}$ from Proposition 3.19 to the polynomial $1 = x^0 = X^0 \mathbf{1}_Y = \mathbf{1}_Y$,

$$\begin{aligned} t^{-\frac{1}{2}\ell(w_0)} \sum_{w \in S_n} w \left(\prod_{i < j} \frac{x_i - t x_j}{x_i - x_j} \right) &= \left(\sum_{w \in S_n} w \right) c_{w_0}^{X^{-1}} \mathbf{1}_Y = p_0^X c_{w_0}^{X^{-1}} \mathbf{1}_Y = \mathbf{1}_0 \mathbf{1}_Y \quad \text{and} \\ \mathbf{1}_0 \mathbf{1}_Y &= \sum_{w \in S_n} (t^{\frac{1}{2}})^{\ell(w) - \ell(w_0)} T_w \mathbf{1}_Y = t^{-\frac{1}{2}\ell(w_0)} \left(\sum_{w \in S_n} t^{\ell(w)} \right) \mathbf{1}_Y = t^{-\frac{1}{2}\ell(w_0)} W_0(t) \mathbf{1}_Y. \end{aligned}$$

Applying the identity $\mathbf{1}_0 = p_0^Y c_{w_0}^Y$ from Proposition 3.19 to $\mathbf{1}_Y$ and using that $\eta_w \mathbf{1}_Y = 0$ if $w \in S_n$ and $w \neq 1$ (see the last identity in (3.12)), gives

$$\begin{aligned} \mathbf{1}_0 \mathbf{1}_Y &= p_0^Y c_{w_0}^Y = \left(\sum_{w \in W_0} \eta_w \right) c_{w_0}^Y \mathbf{1}_Y = \text{ev}_0^t(c_{w_0}^Y) \left(1 + \sum_{w \in W_0, w \neq 1} \eta_w \right) \mathbf{1}_Y \\ &= \text{ev}_0^t(c_{w_0}^Y) (1 + 0) \mathbf{1}_Y = \text{ev}_0^t(c_{w_0}^Y) \mathbf{1}_Y = \text{ev}_0^t \left(\prod_{i < j} \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} Y_i Y_j^{-1}}{1 - Y_i Y_j^{-1}} \right) \mathbf{1}_Y \\ &= \left(\prod_{i < j} \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} Y_i Y_j^{-1}}{1 - Y_i Y_j^{-1}} \right) \mathbf{1}_Y = t^{-\frac{1}{2} \ell(w_0)} \left(\prod_{i < j} \frac{1 - t^{j-i+1}}{1 - t^{j-i}} \right) \mathbf{1}_Y. \end{aligned}$$

Finally,

$$\begin{aligned} \prod_{1 \leq i < j \leq n} \frac{1 - t^{j-i+1}}{1 - t^{j-i}} &= \prod_{d=1}^{n-1} \prod_{\substack{i < j \\ j-i=d}} \frac{1 - t^{d+1}}{1 - t^d} = \prod_{d=1}^{n-1} \left(\frac{1 - t^{d+1}}{1 - t^d} \right)^{n-d} \\ &= \left(\frac{1 - t^n}{1 - t^{n-1}} \right) \left(\frac{1 - t^{n-1}}{1 - t^{n-2}} \right)^2 \cdots \left(\frac{1 - t^2}{1 - t} \right)^{n-1} = \frac{(1 - t^n)(1 - t^{n-1}) \cdots (1 - t^2)}{(1 - t)^{n-1}} = [n]!. \end{aligned}$$

□

Proposition 4.6. *Let $\lambda \in (\mathbb{Z}^n)^+$. Let $W_\lambda = \{v \in S_n \mid v\lambda = \lambda\}$ be the stabilizer of λ in S_n and $W_\lambda(t)$ is the length generating function for W_λ . Then*

$$t^{-\frac{1}{2} \ell(w_\lambda)} W_\lambda(t) = \text{ev}_0^t(c_{w_\lambda}^Y) = \sum_{w \in W_\lambda} w c_{w_\lambda}^{X^{-1}},$$

where w_λ is the longest element of W_λ . Alternatively,

$$W_\lambda(t) = \sum_{w \in W_\lambda} w \left(\prod_{\substack{1 \leq i < j \leq n \\ \lambda_i = \lambda_j}} \frac{1 - t x_i^{-1} x_j}{1 - x_i^{-1} x_j} \right) = \prod_{\substack{1 \leq i < j \leq n \\ \lambda_i = \lambda_j}} \frac{1 - t^{j-i+1}}{1 - t^{j-i}} = \prod_{i \in \mathbb{Z}} [m_i]!,$$

where $m_i = \#\{k \in \{1, \dots, n\} \mid \lambda_k = i\}$.

Proof. Identify $\mathbb{C}[X]$ and $\tilde{H}\mathbf{1}_Y$ via the isomorphism in (4.3). By the definition of $\mathbf{1}_\lambda$ and $\mathbf{1}_Y$ from (3.22) and (4.2),

$$\mathbf{1}_\lambda \mathbf{1}_Y = \sum_{w \in W_\lambda} (t^{\frac{1}{2}})^{\ell(w) - \ell(w_\lambda)} T_w \mathbf{1}_Y = t^{-\frac{1}{2} \ell(w_\lambda)} \left(\sum_{w \in W_\lambda} t^{\ell(w)} \right) \mathbf{1}_Y = t^{-\frac{1}{2} \ell(w_\lambda)} W_\lambda(t) \mathbf{1}_Y.$$

Let $v_\lambda = w_0 w_\lambda^{-1}$ so that $v_\lambda \in S_n$ is minimal length such that $v_\lambda \lambda$ is weakly increasing. Applying the identities $\mathbf{1}_0 = p_0^X c_{w_0}^{X^{-1}}$ and $\mathbf{1}_0 = p_\lambda^X c_{v_\lambda}^{X^{-1}} \mathbf{1}_\lambda$ from (3.7), and using the facts that $\ell(v_\lambda) + \ell(w_\lambda) = \ell(w_0)$ and the coefficient of 1 in $c_{v_\lambda}^{X^{-1}}$ is $t^{-\frac{1}{2} \ell(v_\lambda)}$, then the expression

$$\left(\sum_{w \in S_n} w \right) c_{w_0}^{X^{-1}} E_\lambda = \mathbf{1}_0 E_\lambda \mathbf{1}_Y = p_\lambda^X c_{v_\lambda}^{X^{-1}} \mathbf{1}_\lambda E_\lambda \mathbf{1}_Y = p_\lambda^X c_{v_\lambda}^{X^{-1}} E_\lambda \mathbf{1}_\lambda \mathbf{1}_Y = t^{-\frac{1}{2} \ell(w_\lambda)} W_\lambda(t) p_\lambda^X c_{v_\lambda}^{X^{-1}} E_\lambda \mathbf{1}_Y$$

has coefficient of x^λ equal to $t^{-\frac{1}{2} \ell(w_0)} W_\lambda(t)$. Since $\lambda_1 \geq \dots \geq \lambda_n$ and the coefficient of $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ in P_λ is 1 then

$$P_\lambda = \frac{t^{\frac{1}{2} \ell(w_0)}}{W_\lambda(t)} \left(\sum_{w \in S_n} w \right) c_{w_0}^{X^{-1}} E_\lambda = \frac{1}{W_\lambda(t)} \sum_{w \in S_n} w \left(E_\lambda \prod_{i < j} \frac{x_i - t x_j}{x_i - x_j} \right).$$

Applying the identity $\mathbf{1}_\lambda = p_\lambda^Y c_{w_\lambda}^Y$ from Proposition (3.7) to $\mathbf{1}_Y$ and using that $\eta_w \mathbf{1}_Y = 0$ if $w \in S_n$ and $w \neq 1$ (see the last identity in (3.12)), gives

$$\mathbf{1}_\lambda \mathbf{1}_Y = p_\lambda^Y c_{w_\lambda}^Y = \left(\sum_{w \in W_\lambda} \eta_w \right) c_{w_\lambda}^Y \mathbf{1}_Y = \text{ev}_0^t(c_{w_\lambda}^Y) \left(1 + \sum_{w \in W_\lambda, w \neq 1} \eta_w \right) \mathbf{1}_Y = \text{ev}_0^t(c_{w_\lambda}^Y) (1 + 0) \mathbf{1}_Y.$$

so that $t^{-\frac{1}{2}\ell(w_\lambda)} W_\lambda(t) = \text{ev}_0^t(c_{w_\lambda}^Y)$. Explicitly,

$$\text{ev}_0^t(c_{w_\lambda}^Y) = \text{ev}_0^t \left(\prod_{\substack{i < j \\ \lambda_i = \lambda_j}} \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} Y_i Y_j^{-1}}{1 - Y_i Y_j^{-1}} \right) \mathbf{1}_Y = t^{-\frac{1}{2}\ell(w_\lambda)} \left(\prod_{\substack{i < j \\ \lambda_i = \lambda_j}} \frac{1 - t^{j-i+1}}{1 - t^{j-i}} \right) \mathbf{1}_Y.$$

Finally, letting $m_i = \#\{k \in \{1, \dots, n\} \mid \lambda_k = i\}$,

$$\begin{aligned} \prod_{\substack{1 \leq i < j \leq n \\ \lambda_i = \lambda_j}} \frac{1 - t^{j-i+1}}{1 - t^{j-i}} &= \prod_{i \in \mathbb{Z}} \prod_{d_i=1}^{m_i-1} \prod_{\substack{i < j \\ j-i=d_i}} \frac{1 - t^{d_i+1}}{1 - t^{d_i}} = \prod_{i \in \mathbb{Z}} \prod_{d_i=1}^{m_i-1} \left(\frac{1 - t^{d_i+1}}{1 - t^{d_i}} \right)^{n-d_i} \\ &= \prod_{i \in \mathbb{Z}} \left(\frac{1 - t^{m_i}}{1 - t^{m_i-1}} \right) \left(\frac{1 - t^{m_i-1}}{1 - t^{m_i-2}} \right)^2 \cdots \left(\frac{1 - t^2}{1 - t} \right)^{m_i-1} \\ &= \prod_{i \in \mathbb{Z}} \frac{(1 - t^{m_i})(1 - t^{m_i-1}) \cdots (1 - t^2)}{(1 - t)^{m_i-1}} = \prod_{i \in \mathbb{Z}} [m_i]!. \end{aligned}$$

□

4.7.4 Proof of the eigenvalue formula

The following result establishes Theorem 1.1.

Proposition 4.9. *Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$. Then*

$$Y_i E_\mu = q^{-\mu_i} t^{-(v_\mu(i)-1) + \frac{1}{2}(n-1)} E_\mu.$$

Proof. First do the base case $\mu = 0$ when $E_\mu = 1$. Using $T_\pi 1 = 1$ and $T_i 1 = t^{\frac{1}{2}} 1$ and $T_i^{-1} 1 = t^{-\frac{1}{2}} 1$ gives

$$Y_i 1 = T_{i-1}^{-1} \cdots T_{n-1}^{-1} T_\pi T_{n-1} \cdots T_i \cdot 1 = t^{\frac{1}{2}(-(i-1) + (n-i-1))} \cdot 1 = t^{-(i-1) + \frac{1}{2}(n-1)} \cdot 1.$$

Then, by the creation formula for E_μ in Theorem 4.2 and the relations (3.10) and (3.11) for moving Y_i past τ_j^\vee ,

$$\begin{aligned} Y_i E_\mu &= t^{-\frac{1}{2}\ell(v_\mu^{-1})} \tau_{u_\mu}^\vee 1 = t^{-\frac{1}{2}\ell(v_\mu^{-1})} \tau_{u_\mu}^\vee Y_{u_\mu^{-1}(i)} 1 = t^{-\frac{1}{2}\ell(v_\mu^{-1})} \tau_{u_\mu}^\vee q^{-\mu_i} Y_{v_\mu(i)} 1 \\ &= t^{-\frac{1}{2}\ell(v_\mu^{-1})} \tau_{u_\mu}^\vee q^{-\mu_i} t^{-(v_\mu(i)-1) + \frac{1}{2}(n-1)} 1 = q^{-\mu_i} t^{-(v_\mu(i)-1) + \frac{1}{2}(n-1)} E_\mu. \end{aligned}$$

□

4.7.5 Proof of the E-expansions

Proposition 4.7. *Let $\lambda \in (\mathbb{Z}^n)^+$. Then*

$$P_\lambda = \sum_{z \in W^\lambda} t^{\frac{1}{2}\ell(v_\lambda z)} \text{ev}_{z\lambda}^t(c_{v_\lambda z}^Y) E_{z\lambda} \quad \text{and}$$

$$A_{\lambda+\rho} = \sum_{z \in W_0} (-t^{\frac{1}{2}})^{\ell(w_0 z)} \text{ev}_{z(\lambda+\rho)}^t(c_{w_0 z}^{Y^{-1}}) E_{z(\lambda+\rho)}.$$

Alternatively,

$$P_\lambda = \sum_{\mu \in S_n \lambda} \left(\prod_{\substack{1 \leq i < j \leq n \\ \mu_i > \mu_j}} t \left(\frac{1 - q^{\mu_i - \mu_j} t^{v_\mu(i) - v_\mu(j) - 1}}{1 - q^{\mu_i - \mu_j} t^{v_\mu(i) - v_\mu(j)}} \right) \right) E_\mu \quad \text{and}$$

$$A_{\lambda+\rho} = \sum_{\mu \in S_n(\lambda+\rho)} \left(\prod_{\substack{1 \leq i < j \leq n \\ \mu_i > \mu_j}} (-1) \left(\frac{1 - q^{\mu_i - \mu_j} t^{v_\mu(i) - v_\mu(j) + 1}}{1 - q^{\mu_i - \mu_j} t^{v_\mu(i) - v_\mu(j)}} \right) \right) E_\mu.$$

Proof. Note that the coefficient of $E_{w_0 \lambda}$ in P_λ is 1 and the coefficient of $E_{w_0(\lambda+\rho)}$ in $A_{\lambda+\rho}$ is 1.

For the first statement, (3.19), (4.4) and (4.7) give

$$\begin{aligned} P_\lambda &= \frac{t^{\frac{1}{2}\ell(w_0)}}{W_\lambda(t)} \mathbf{1}_0 E_\lambda = \frac{t^{\frac{1}{2}(\ell(w_0) - \ell(w_\lambda))}}{t^{-\frac{1}{2}\ell(w_\lambda)} W_\lambda(t)} \left(\sum_{z \in W^\lambda} \eta_z \right) c_{v_\lambda}^Y \mathbf{1}_\lambda E_\lambda = t^{\frac{1}{2}(\ell(w_0) - \ell(w_\lambda))} \sum_{z \in W^\lambda} \tau_z^\vee \frac{c_{v_\lambda}^Y}{c_z^Y} E_\lambda \\ &= t^{\frac{1}{2}\ell(v_\lambda)} \sum_{z \in W^\lambda} \tau_z^\vee (z^{-1} c_{v_\lambda z}^Y) E_\lambda = t^{\frac{1}{2}\ell(v_\lambda)} \sum_{z \in W^\lambda} c_{v_\lambda z}^Y \tau_z^\vee E_\lambda \\ &= t^{\frac{1}{2}\ell(v_\lambda)} \sum_{z \in W^\lambda} \text{ev}_{z\lambda}^t(c_{v_\lambda z}^Y) t^{-\frac{1}{2}\ell(z)} E_{z\lambda} = \sum_{z \in W^\lambda} t^{\frac{1}{2}\ell(v_\lambda z)} \text{ev}_{z\lambda}^t(c_{v_\lambda z}^Y) E_{z\lambda}. \end{aligned}$$

If $z \in W^\lambda$ and $\mu = z\lambda$ then $\text{Inv}(v_\lambda z) = \{(i, j) \mid 1 \leq i < j \leq n \text{ and } \mu_i > \mu_j\}$, so that

$$\begin{aligned} t^{\frac{1}{2}\ell(v_\lambda z)} \text{ev}_\mu^t(c_{v_\lambda z}^Y) &= \text{ev}_\mu^t \left(\prod_{\substack{1 \leq i < j \leq n \\ \mu_i > \mu_j}} t^{\frac{1}{2}} \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} Y_i Y_j^{-1}}{1 - Y_i Y_j^{-1}} \right) = \text{ev}_\mu^t \left(\prod_{\substack{1 \leq i < j \leq n \\ \mu_i > \mu_j}} t \frac{t^{-1} Y_i^{-1} Y_j - 1}{Y_i^{-1} Y_j - 1} \right) \\ &= \prod_{\substack{1 \leq i < j \leq n \\ \mu_i > \mu_j}} t \left(\frac{1 - q^{\mu_i - \mu_j} t^{v_\mu(i) - v_\mu(j) - 1}}{1 - q^{\mu_i - \mu_j} t^{v_\mu(i) - v_\mu(j)}} \right), \end{aligned} \tag{4.11}$$

where we have used that $\text{ev}_\mu^t(Y_i) = q^{-\mu_i} t^{-(v_\mu(i) - 1) + \frac{1}{2}(n-1)}$. Thus

$$\text{ev}_\mu^t(Y_i^{-1} Y_j) = q^{\mu_i} t^{(v_\mu(i) - 1) - \frac{1}{2}(n-1)} q^{-\mu_j} t^{-(v_\mu(j) - 1) + \frac{1}{2}(n-1)} = q^{\mu_i - \mu_j} t^{v_\mu(i) - v_\mu(j)}.$$

For the second statement,

$$\begin{aligned} A_{\lambda+\rho}(q, t) &= t^{\frac{1}{2}\ell(w_0)} \varepsilon_0 E_{\lambda+\rho}(q, t) = t^{\frac{1}{2}\ell(w_0)} c_{w_0}^{Y^{-1}} \sum_{z \in S_n} \det(w_0 z) \eta_z E_{\lambda+\rho}(q, t) \\ &= t^{\frac{1}{2}\ell(w_0)} \sum_{z \in S_n} \det(w_0 z) c_{w_0 z}^{Y^{-1}} t^{-\frac{1}{2}\ell(z)} t^{\frac{1}{2}\ell(z)} \tau_z^\vee E_{\lambda+\rho}(q, t) \\ &= \sum_{z \in S_n} \det(w_0 z) c_{w_0 z}^{Y^{-1}} t^{\frac{1}{2}\ell(w_0 z)} E_{z(\lambda+\rho)}(q, t) \\ &= \sum_{z \in S_n} (-1)^{\ell(w_0 z)} t^{\frac{1}{2}\ell(w_0 z)} \text{ev}_{z(\lambda+\rho)}^t(c_{w_0 z}^{Y^{-1}}) E_{z(\lambda+\rho)}(q, t). \end{aligned}$$

If $\mu = z(\lambda + \rho)$ then, in a manner similar to the computation in (4.11),

$$\begin{aligned} (-t^{\frac{1}{2}})^{\ell(w_0 z)} \text{ev}_\mu^t(c_{w_0 z}^{Y^{-1}}) &= \prod_{\substack{1 \leq i < j \leq n \\ \mu_i > \mu_j}} (-t^{\frac{1}{2}}) \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} Y_i^{-1} Y_j}{1 - Y_i^{-1} Y_j} \\ &= \prod_{\substack{1 \leq i < j \leq n \\ \mu_i > \mu_j}} (-1) \frac{1 - tq^{\mu_i - \mu_j} t^{v_\mu(i) - v_\mu(j)}}{1 - q^{\mu_i - \mu_j} t^{v_\mu(i) - v_\mu(j)}}. \end{aligned} \quad (4.12)$$

□

5 Principal specializations and hook formulas

In Section 4 we saw that c -functions provide the explicit constants for normalization and the E -expansion of bosonic and fermionic Macdonald polynomials. In this section we see the role of c -functions in the amazing product formulas for principal specializations. These principal specializations capture many of the hook type formulas that appear in formulas for dimensions of irreducible representations of the general linear group and the symmetric group. As discussed in Section 1.2, the representation theoretic interpretation of the hook formulas that arise from the principal specializations of Macdonald polynomials is still rather mysterious, but we hope that viewing these results as being sourced from evaluations of c -functions will help to provide insight. Our exposition follows [Mac03, (5.2.14) and (5.3.9)] (with some streamlining) for the c -function formulas, and then follows [AGY22] for rewriting the c -function formulas into hook formulas. This provides an alternative route to the proof of the principal specialization formulas for P_λ which are given in [Mac, (6.11')].

5.1 c -function formulas

An n -periodic permutation is a bijection $w: \mathbb{Z} \rightarrow \mathbb{Z}$ such that $w(i+n) = w(i) + n$ for all $i \in \mathbb{Z}$. Given an n -periodic permutation w , define its *set of inversions* and *length* by

$$\text{Inv}(w) = \left\{ (i, k) \mid \begin{array}{l} i \in \{1, \dots, n\}, k \in \mathbb{Z} \\ i < k \text{ and } w(i) > w(k) \end{array} \right\} \quad \text{and} \quad \ell(w) = \#\text{Inv}(w).$$

Define an action of the n -periodic permutations on \mathbb{Z}^n by setting

$$w(\mu_1, \dots, \mu_n) = (\mu_{v(1)} + \ell_1, \dots, \mu_{v(n)} + \ell_n),$$

where $v(i) \in \{1, \dots, n\}$ and $\ell_i \in \mathbb{Z}$ are determined by $w(i) = v(i) + \ell_i n$. Given $\mu \in \mathbb{Z}^n$, we consider the following n -periodic permutations associated to μ

u_μ be the minimal length n -periodic permutation such that $u_\mu(0, \dots, 0) = (\mu_1, \dots, \mu_n)$ and t_μ be the n -periodic permutation given by $t_\mu(i) = i + n\mu_i$.

Recall that $v_\mu \in S_n$ denotes the minimal length permutation such that $v_\mu \mu$ is weakly increasing. The three permutations are related by $u_\mu = t_\mu v_\mu^{-1}$.

We extend the definition of c -functions to n -periodic permutations. For $i, j \in \{1, \dots, n\}$ and $\ell \in \mathbb{Z}$ define

$$c_{(i, j + \ell n)}^{Y^{-1}} = t^{-\frac{1}{2}} \frac{1 - q^\ell t Y_i^{-1} Y_j}{1 - q^\ell Y_i^{-1} Y_j} \quad \text{and} \quad c_w^{Y^{-1}} = \prod_{(i, k) \in \text{Inv}(w)} c_{(i, k)}^{Y^{-1}}.$$

Define ring homomorphisms $\text{ev}_0^t: \mathbb{C}[Y] \rightarrow \mathbb{C}$ and $\text{ev}_0^{t^{-1}}: \mathbb{C}[Y] \rightarrow \mathbb{C}$ by

$$\text{ev}_0^t(Y_i) = t^{-(i-1) + \frac{1}{2}(n-1)} \quad \text{and} \quad \text{ev}_0^{t^{-1}}(Y_i) = t^{(i-1) - \frac{1}{2}(n-1)}, \quad \text{for } i \in \{1, \dots, n\}.$$

Theorem 5.1. *Let $\mu, \lambda \in \mathbb{Z}^n$ with $\lambda_1 \geq \dots \geq \lambda_n$. Then*

$$\begin{aligned} E_\mu(1, t, t^2, \dots, t^{n-1}; q, t) &= t^{\frac{(n-1)}{2}|\lambda|} t^{-\frac{1}{2}\ell(v_\mu^{-1})} \text{ev}_0^t(c_{u_\mu}^{Y^{-1}}), \\ P_\lambda(1, t, t^2, \dots, t^{n-1}; q, t) &= t^{\frac{(n-1)}{2}|\lambda|} \text{ev}_0^{t^{-1}}(c_{t_\lambda}^{Y^{-1}}) \quad \text{and} \\ A_{\lambda+\rho}(1, t, t^2, \dots, t^{n-1}; q, t) &= 0. \end{aligned}$$

Let $\mu \in \mathbb{Z}_{\geq 0}^n$. Using the formulas (see [GR21, Proposition 2.1(d) and Proposition 2.2(b)])

$$\text{Inv}(t_\lambda) = \{(i, j + \ell n) \mid i, j \in \{1, \dots, n\}, i < j \text{ and } \ell \in \{0, 1, \dots, \lambda_j - \lambda_i - 1\}\} \quad (5.1)$$

and

$$\text{Inv}(u_\mu) = \{(v_\mu(r), i + (\mu_r - c + 1)n) \mid (r, c) \in \mu \text{ and } i \in \{1, \dots, u_\mu(r, c)\}\}. \quad (5.2)$$

gives the following corollary.

Corollary 5.2. *Let $\mu \in \mathbb{Z}_{\geq 0}^n$. Denote by λ the decreasing rearrangement of μ and $n(\lambda) = \sum_{i=1}^n (i-1)\lambda_i$. Then*

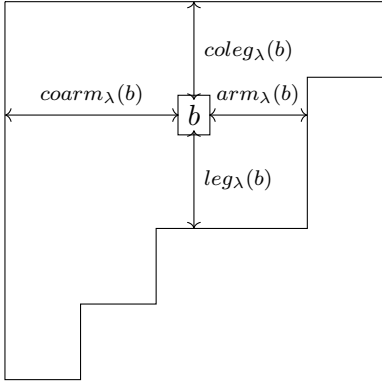
$$P_\lambda(1, t, t^2, \dots, t^{n-1}; q, t) = t^{n(\lambda)} \prod_{1 \leq i < j \leq n} \prod_{\ell=0}^{\lambda_i - \lambda_j - 1} \frac{1 - q^\ell t^{j-i+1}}{1 - q^\ell t^{j-i}}$$

and

$$E_\mu(1, t, t^2, \dots, t^{n-1}; q, t) = t^{-\frac{1}{2}\ell(v_\mu^{-1})} \prod_{(r,c) \in \mu} \prod_{i=1}^{u_\mu(r,c)} \frac{1 - q^{\mu_r - c + 1} t^{v_\mu(r) - i + 1}}{1 - q^{\mu_r - c + 1} t^{v_\mu(r) - i}}.$$

5.2 Hook formulas for the bosonic and electronic cases

Let $\lambda \in \mathbb{Z}_{\geq 0}^n$ with $\lambda_1 \geq \dots \geq \lambda_n$. Let λ' denote the conjugate partition to λ (i.e. for $c \in \mathbb{Z}_{>0}$ let $\lambda'_c = \#\{j \in \mathbb{Z}_{>0} \mid \lambda_j \geq c\}$). A *box* in λ is a pair $b = (r, c)$ with $r \in \{1, \dots, n\}$ and $c \in \{1, \dots, \lambda_j\}$. For a box $b = (r, c)$ in λ define



$$coleg_\lambda(b) = r - 1,$$

$$coarm_\lambda(b) = c - 1, \quad b = (r, c), \quad arm_\lambda(b) = \lambda_r - c,$$

$$leg_\lambda(b) = \lambda'_c - r.$$

The *hook length* $h(b)$ and the *content* $c(b)$ of the box b are defined by

$$h(b) = arm_\lambda(b) + leg_\lambda(b) + 1 \quad \text{and} \quad c(b) = coarm_\lambda(b) - coleg_\lambda(b).$$

Theorem 5.3. *Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n$ with $\lambda_1 \geq \dots \geq \lambda_n$. Then*

$$P_\lambda(1, t, t^2, \dots, t^{n-1}; q, t) = t^{n(\lambda)} \prod_{b \in \lambda} \frac{1 - q^{coarm_\lambda(b)} t^{n - coleg_\lambda(b)}}{1 - q^{arm_\lambda(b)} t^{leg_\lambda(b) + 1}}.$$

Theorem 5.4. *Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$ and let λ be the weakly decreasing rearrangement of μ . Then*

$$E_\mu(1, t, t^2, \dots, t^{n-1}; q, t) = t^{n(\lambda)} \prod_{(r,c) \in \mu} \frac{1 - q^c t^{v_\mu(r)}}{1 - q^{\mu_r - c + 1} t^{v_\mu(r) - u_\mu(r,c)}}.$$

5.3 Proofs

5.3.1 Proof of the c-function formula

Theorem 5.1. *Let $\mu, \lambda \in \mathbb{Z}^n$ with $\lambda_1 \geq \dots \geq \lambda_n$. Then*

$$\begin{aligned} E_\mu(1, t, t^2, \dots, t^{n-1}; q, t) &= t^{\frac{(n-1)}{2}|\lambda|} t^{-\frac{1}{2}\ell(v_\mu^{-1})} \text{ev}_0^t(c_{u_\mu}^{Y^{-1}}), \\ P_\lambda(1, t, t^2, \dots, t^{n-1}; q, t) &= t^{\frac{(n-1)}{2}|\lambda|} \text{ev}_0^{t^{-1}}(c_{t_\lambda}^{Y^{-1}}) \quad \text{and} \\ A_{\lambda+\rho}(1, t, t^2, \dots, t^{n-1}; q, t) &= 0. \end{aligned}$$

Proof. For this proof use the realization of the polynomial representation $\mathbb{C}[X]$ as an induced module $\widetilde{H}\mathbf{1}_Y$ via the \widetilde{H} -module isomorphism of (4.3). Then the creation formulas for E_μ , P_λ and $A_{\lambda+\rho}$ are

$$E_\mu = t^{-\frac{1}{2}\ell(v_\mu^{-1})} \tau_{u_\mu}^\vee \mathbf{1}_0, \quad P_\lambda = \frac{t^{\frac{1}{2}\ell(w_0)}}{W_\lambda(t)} \mathbf{1}_0 E_\lambda, \quad \text{and} \quad A_{\lambda+\rho} = t^{\frac{1}{2}\ell(w_0)} \varepsilon_0 E_{\lambda+\rho}$$

(see Theorem 4.2 and (4.4)).

Let $\mathbf{1}_X$ be a formal symbol which satisfies $\mathbf{1}_X T_j = t^{\frac{1}{2}} \mathbf{1}_X$ and $\mathbf{1}_X g^\vee = \mathbf{1}_X$, for $j \in \{1, \dots, n-1\}$. Since $g^\vee = x_1 T_1 \cdots T_{n-1}$ and $x_{i+1} = T_i x_i T_i$ then $x_1 = g^\vee T_{n-1}^{-1} \cdots T_1^{-1}$ and

$$\mathbf{1}_X x_i = t^{-\frac{1}{2}(n-1)} t^{i-1} \mathbf{1}_X, \quad \text{for } i \in \{1, \dots, n\}.$$

Thus, if $\mu \in \mathbb{Z}^n$ then

$$\mathbf{1}_X E_\mu(x_1, \dots, x_n; q, t) = \mathbf{1}_X t^{-\frac{1}{2}(n-1)|\mu|} E_\mu(1, t, t^2, \dots, t^{n-1}; q, t).$$

For $i \in \{1, \dots, n-1\}$,

$$\mathbf{1}_X \tau_i^\vee = \mathbf{1}_X \left(T_i + \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}}}{1 - Y_i^{-1} Y_{i+1}} \right) = \mathbf{1}_X \left(t^{\frac{1}{2}} + \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}}}{1 - Y_i^{-1} Y_{i+1}} \right) = \mathbf{1}_X \left(\frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} Y_i^{-1} Y_{i+1}}{1 - Y_i^{-1} Y_{i+1}} \right) = \mathbf{1}_X c_{i,i+1}^{Y^{-1}}.$$

By (4.10),

$$c_{i,i+1}^{Y^{-1}} \mathbf{1}_Y = \text{ev}_0^t(c_{i,i+1}^{Y^{-1}}) \mathbf{1}_Y.$$

If $w \in W$ and $\ell(s_i w) > \ell(w)$ then

$$\mathbf{1}_X \tau_i^\vee \tau_w^\vee \mathbf{1}_Y = \mathbf{1}_X c_{i,i+1}^{Y^{-1}} \tau_w^\vee \mathbf{1}_Y = \mathbf{1}_X \tau_w^\vee c_{w^{-1}(i), w^{-1}(i+1)}^{Y^{-1}} \mathbf{1}_Y = \text{ev}_0^t(c_{w^{-1}(i), w^{-1}(i+1)}^{Y^{-1}}) \mathbf{1}_X \tau_w^\vee \mathbf{1}_Y.$$

By induction, conclude that if $w \in W$ and $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced word for w then

$$\mathbf{1}_X \tau_w^\vee \mathbf{1}_Y = \mathbf{1}_X \tau_{i_1}^\vee \cdots \tau_{i_\ell}^\vee \mathbf{1}_Y = \mathbf{1}_X \text{ev}_0^t(c_w^{Y^{-1}}) \mathbf{1}_Y = \text{ev}_0^t(c_w^{Y^{-1}}) \mathbf{1}_X \mathbf{1}_Y.$$

Thus, by the creation formula $E_\mu = t^{-\frac{1}{2}\ell(v_\mu^{-1})} \tau_{u_\mu}^\vee \mathbf{1}_Y$,

$$t^{-\frac{1}{2}(n-1)|\mu|} E_\mu(1, t, \dots, t^{n-1}; q, t) \mathbf{1}_X \mathbf{1}_Y = \mathbf{1}_X E_\mu \mathbf{1}_Y = \mathbf{1}_X t^{-\frac{1}{2}\ell(v_\mu^{-1})} \tau_{u_\mu}^\vee \mathbf{1}_Y = t^{-\frac{1}{2}\ell(v_\mu^{-1})} \text{ev}_0^t(c_{u_\mu}^{Y^{-1}}) \mathbf{1}_X \mathbf{1}_Y.$$

which completes the proof of the first statement.

Using the creation formula $P_\lambda = \frac{t^{\frac{1}{2}\ell(w_0)}}{W_\lambda(t)} \mathbf{1}_0 E_\lambda$ gives

$$\begin{aligned} t^{-\frac{1}{2}(n-1)|\lambda|} P_\lambda(1, t, \dots, t^{n-1}; q, t) \mathbf{1}_X \mathbf{1}_Y &= \mathbf{1}_X P_\lambda \mathbf{1}_Y = \mathbf{1}_X \frac{t^{\frac{1}{2}\ell(w_0)}}{W_\lambda(t)} \mathbf{1}_0 E_\lambda \mathbf{1}_Y \\ &= \mathbf{1}_X \frac{t^{\frac{1}{2}\ell(w_0)} W_0(t)}{W_\lambda(t)} E_\lambda \mathbf{1}_Y = t^{\frac{1}{2}\ell(w_0)} t^{-\frac{1}{2}\ell(v_\lambda^{-1})} \frac{W_0(t)}{W_\lambda(t)} \text{ev}_0^t(c_{u_\lambda}^{Y^{-1}}) \mathbf{1}_X \mathbf{1}_Y. \end{aligned}$$

Since $v_\lambda^{-1} = (w_0 w_\lambda)^{-1} = w_\lambda w_0$ and $t_\lambda = u_\lambda v_\lambda$ then

$$\begin{aligned} \frac{t^{-\frac{1}{2}\ell(w_0)} W_0(t)}{t^{-\frac{1}{2}\ell(w_\lambda)} W_\lambda(t)} \text{ev}_0^t(c_{u_\lambda}^{Y^{-1}}) &= \text{ev}_0^{t^{-1}} \left(\frac{c_{w_0}^{Y^{-1}}}{c_{w_\lambda}^{Y^{-1}}} \right) \text{ev}_0^t(c_{u_\lambda}^{Y^{-1}}) = \text{ev}_0^{t^{-1}} \left(\frac{c_{w_0}^{Y^{-1}}}{c_{w_\lambda}^{Y^{-1}}} \right) \text{ev}_0^{t^{-1}}(v_\lambda^{-1} c_{u_\lambda}^{Y^{-1}}) \\ &= \text{ev}_0^{t^{-1}}(c_{v_\lambda}^{Y^{-1}}) \text{ev}_0^{t^{-1}}(v_\lambda^{-1} c_{u_\lambda}^{Y^{-1}}) = \text{ev}_0^{t^{-1}}(c_{u_\lambda v_\lambda}^{Y^{-1}}) = \text{ev}_0^{t^{-1}}(c_{t_\lambda}^{Y^{-1}}). \end{aligned}$$

The first equality comes from Proposition 4.6 which gives

$$t^{-\frac{1}{2}\ell(w_\lambda)} W_\lambda(t) = \text{ev}_0^t(c_{w_\lambda}^Y) = \text{ev}_0^{t^{-1}}(c_{w_\lambda}^{Y^{-1}}),$$

and the second equality results from the fact that

$$\text{if } i, j \in \{1, \dots, n\} \text{ and } i < j \text{ then } t^{j-i} = (t^{-1})^{(n-j)-(n-i)} = (t^{-1})^{w_0(j)-w_0(i)},$$

which gives

$$\text{ev}_0^t(c_{u_\lambda}^{Y^{-1}}) = \text{ev}_0^{t^{-1}}(v_\lambda^{-1} c_{u_\lambda}^{Y^{-1}}).$$

Using the creation formula $A_{\lambda+\rho} = t^{\frac{1}{2}\ell(w_0)} \varepsilon_0 E_{\lambda+\rho}$ gives

$$t^{-\frac{1}{2}(n-1)|\lambda|} A_{\lambda+\rho}(1, t, \dots, t^{n-1}; q, t) \mathbf{1}_X \mathbf{1}_Y = \mathbf{1}_X A_{\lambda+\rho} \mathbf{1}_Y = t^{\frac{1}{2}\ell(w_0)} \mathbf{1}_X \varepsilon_0 E_{\lambda+\rho} \mathbf{1}_Y = 0,$$

since $\mathbf{1}_X \varepsilon_0 = 0$. Thus $A_{\lambda+\rho}(1, t, \dots, t^{n-1}; q, t) = 0$. □

5.3.2 Proof of the root product formula

Corollary 5.2. *Let $\mu \in \mathbb{Z}_{\geq 0}^n$. Denote by λ the decreasing rearrangement of μ and $n(\lambda) = \sum_{i=1}^n (i-1)\lambda_i$. Then*

$$P_\lambda(1, t, t^2, \dots, t^{n-1}; q, t) = t^{n(\lambda)} \prod_{1 \leq i < j \leq n} \prod_{\ell=0}^{\lambda_i - \lambda_j - 1} \frac{1 - q^\ell t^{j-i+1}}{1 - q^\ell t^{j-i}}$$

and

$$E_\mu(1, t, t^2, \dots, t^{n-1}; q, t) = t^{-\frac{1}{2}\ell(v_\mu^{-1})} \prod_{(r,c) \in \mu} \prod_{i=1}^{u_\mu(r,c)} \frac{1 - q^{\mu_r - c + 1} t^{v_\mu(r) - i + 1}}{1 - q^{\mu_r - c + 1} t^{v_\mu(r) - i}}.$$

Proof. Let $\rho^\vee = \frac{1}{2}(n-1, n-3, \dots, -(n-3), -(n-1))$. Then

$$n(\lambda) = \langle (\lambda_1, \dots, \lambda_n), (0, 1, \dots, n-1) \rangle = \langle \lambda, -\rho^\vee + \frac{(n-1)}{2}(1, 1, \dots, 1) \rangle = \frac{(n-1)}{2}|\lambda| - \langle \lambda, \rho^\vee \rangle$$

and, since $\ell(t_\lambda) = \sum_{i < j} (\lambda_i - \lambda_j)$ then

$$\begin{aligned} \ell(t_\lambda) &= (n-1)\lambda_1 + (n-2)\lambda_2 + \dots + (n-n)\lambda_n - ((n-1)\lambda_n + (n-2)\lambda_{n-1} + \dots + (n-n)\lambda_1) \\ &= (n-1)\lambda_1 + (n-3)\lambda_2 + (n-5)\lambda_3 + \dots - (n-3)\lambda_{n-1} - (n-1)\lambda_n = \langle \lambda, 2\rho^\vee \rangle, \end{aligned}$$

so that

$$n(\lambda) = \frac{1}{2}(n-1)|\lambda| - \frac{1}{2}\ell(t_\lambda). \quad (5.3)$$

Since

$$\begin{aligned} \text{ev}_0^{t^{-1}}(c_{i,j+rn}^{Y^{-1}}) &= \text{ev}_0^{t^{-1}}\left(t^{-\frac{1}{2}}\frac{1-tq^rY_i^{-1}Y_j}{1-q^rY_i^{-1}Y_j}\right) \\ &= t^{-\frac{1}{2}}\frac{1-tq^r t^{-(i-1)+\frac{1}{2}(n-1)}t^{(j-1)-\frac{1}{2}(n-1)}}{1-q^r t^{-(i-1)+\frac{1}{2}(n-1)}t^{(j-1)-\frac{1}{2}(n-1)}} = t^{-\frac{1}{2}}\frac{1-q^r t^{j-i+1}}{1-q^r t^{j-i}} \end{aligned}$$

then, using (5.1) and (5.3),

$$t^{\frac{(n-1)}{2}|\lambda|}\text{ev}_0^{t^{-1}}(c_{t_\lambda}^{Y^{-1}}) = t^{\frac{(n-1)}{2}|\lambda|}t^{-\frac{1}{2}\ell(t_\lambda)}\prod_{i<j}\prod_{r=0}^{\lambda_i-\lambda_j-1}\frac{1-q^r t^{j-i+1}}{1-q^r t^{j-i}} = t^{n(\lambda)}\prod_{i<j}\prod_{r=0}^{\lambda_i-\lambda_j-1}\frac{1-q^r t^{j-i+1}}{1-q^r t^{j-i}}.$$

The formula in the first statement follows then from Theorem 5.1.

Again by Theorem 5.1,

$$E_\mu(1, t, t^2, \dots, t^{n-1}; q, t) = t^{\frac{(n-1)}{2}|\mu|}t^{-\frac{1}{2}\ell(v_\mu^{-1})}\text{ev}_{k\rho}(c_{u_\mu}^{Y^{-1}}) = t^{\frac{(n-1)}{2}|\mu|-\frac{1}{2}\ell(u_\mu)-\frac{1}{2}\ell(v_\mu)}\left(t^{\frac{1}{2}\ell(u_\mu)}\text{ev}_{k\rho}(c_{u_\mu}^{Y^{-1}})\right)$$

and, if λ is the weakly decreasing rearrangement of μ then, by (5.3),

$$\frac{(n-1)}{2}|\mu| - \frac{1}{2}\ell(u_\mu) - \frac{1}{2}\ell(v_\mu) = \frac{(n-1)}{2}|\lambda| - \frac{1}{2}\ell(t_\mu) = \frac{(n-1)}{2}|\lambda| - \frac{1}{2}\ell(t_\lambda) = n(\lambda).$$

Then (5.2) gives that $t^{\frac{1}{2}\ell(u_\mu)}\text{ev}_0^t(c_{u_\mu}^{Y^{-1}})$ is equal to

$$\text{ev}_0^t\left(\prod_{(r,c)\in\mu}\prod_{j=1}^{u_\mu(r,c)}\frac{1-q^{\mu_r-c+1}tY_{v_\mu(r)}^{-1}Y_j}{1-q^{\mu_r-c+1}Y_{v_\mu(r)}^{-1}Y_j}\right) = \prod_{(r,c)\in\mu}\prod_{j=1}^{u_\mu(r,c)}\frac{1-tq^{\mu_r-c+1}t^{v_\mu(r)-j}}{1-q^{\mu_r-c+1}t^{v_\mu(r)-j}}. \quad (5.4)$$

□

5.3.3 Proof of the bosonic hook formula

Theorem 5.3. *Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n$ with $\lambda_1 \geq \dots \geq \lambda_n$. Then*

$$P_\lambda(1, t, t^2, \dots, t^{n-1}; q, t) = t^{n(\lambda)}\prod_{b\in\lambda}\frac{1-q^{\text{coarm}_\lambda(b)}t^{n-\text{coleg}_\lambda(b)}}{1-q^{\text{arm}_\lambda(b)}t^{\text{leg}_\lambda(b)+1}}.$$

Proof. In view of Corollary 5.2 the result will follow if we prove that

$$\prod_{i<j}\prod_{\ell=0}^{\lambda_i-\lambda_j-1}\frac{1-q^\ell t^{j-i+1}}{1-q^\ell t^{j-i}} = \prod_{b\in\lambda}\frac{1-q^{\text{coarm}_\lambda(b)}t^{n-\text{coleg}_\lambda(b)}}{1-q^{\text{arm}_\lambda(b)}t^{\text{leg}_\lambda(b)+1}}.$$

The left-hand side is

$$LHS = \prod_{i<j}\prod_{\ell=0}^{\lambda_i-\lambda_j-1}\frac{1-q^\ell t^{j-i+1}}{1-q^\ell t^{j-i}} = \prod_{r=1}^n\prod_{j=r+1}^n\prod_{\ell=0}^{\lambda_r-\lambda_j-1}\frac{1-tq^\ell t^{j-r}}{1-q^\ell t^{j-r}}.$$

Let m be the number of columns of length n in λ . Let $r \in \{1, \dots, n\}$. Switching the products over j and ℓ gives

$$\prod_{j=r+1}^n \prod_{\ell=0}^{\lambda_r - \lambda_j - 1} \frac{1 - tq^\ell t^{j-r}}{1 - q^\ell t^{j-r}} = \prod_{c=m+1}^{\lambda_r} \prod_{j=\lambda'_c+1}^n \frac{1 - tq^{\lambda_r - c} t^{j-r}}{1 - q^{\lambda_r - c} t^{j-r}} = \prod_{c=m+1}^{\lambda_r} \frac{1 - tq^{\lambda_r - c} t^{n-r}}{1 - q^{\lambda_r - c} t^{\lambda'_c + 1 - r}} \quad (5.5)$$

The definitions of arms, legs, coarms and colegs of boxes give that

$$RHS = t^{n(\lambda)} \prod_{b=(r,c) \in \lambda} \frac{1 - q^{\text{coarm}_\lambda(b)} t^{n - \text{coleg}_\lambda(b)}}{1 - q^{\text{arm}_\lambda(b)} t^{\text{leg}_\lambda(b)+1}} = t^{n(\lambda)} \prod_{r=1}^n \prod_{c=1}^{\lambda_r} \frac{1 - q^{c-1} t^{n-(r-1)}}{1 - q^{\lambda_r - c} t^{\lambda'_c - r + 1}}$$

For $r \in \{1, \dots, n\}$ let $\ell = \lambda_r$ and write

$$\prod_{c=1}^{\lambda_r} \frac{1 - q^{c-1} t^{n-(r-1)}}{1 - q^{\lambda_r - c} t^{\lambda'_c - r + 1}} = \frac{(1 - q^0 t^{n-(r-1)})(1 - q^1 t^{n-(r-1)}) \dots (1 - q^{\lambda_r - 1} t^{n-(r-1)})}{(1 - q^{\lambda_r - 1} t^{\lambda'_1 - r + 1}) \dots (1 - q^1 t^{\lambda'_{\ell-1} - r - 1 + 1})(1 - q^0 t^{\lambda'_\ell - r + 1})}$$

to observe that the last m factors in the numerator cancel with the first m terms in the denominator. Thus

$$\begin{aligned} \prod_{c=1}^{\lambda_r} \frac{1 - q^{c-1} t^{n-(r-1)}}{1 - q^{\lambda_r - c} t^{\lambda'_c - r + 1}} &= \frac{(1 - q^0 t^{n-(r-1)})(1 - q^1 t^{n-(r-1)}) \dots (1 - q^{\lambda_r - m} t^{n-(r-1)})}{(1 - q^{\lambda_r - m} t^{\lambda'_{\ell-m} - r + 1}) \dots (1 - q^1 t^{\lambda'_{\ell-1} - r - 1 + 1})(1 - q^0 t^{\lambda'_\ell - r + 1})} \\ &= \prod_{c=m+1}^{\lambda_r} \frac{1 - q^{\lambda_r - c} t^{n-(r-1)}}{1 - q^{\lambda_r - c} t^{\lambda'_c - r + 1}} = \prod_{c=m+1}^{\lambda_r} \frac{1 - tq^{\lambda_r - c} t^{n-r}}{1 - q^{\lambda_r - c} t^{\lambda'_c - r + 1}}. \end{aligned}$$

Since this is equal to the expression in (5.5), the result follows. \square

5.3.4 Proof of the electronic hook formula

Theorem 5.4. *Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$ and let λ be the weakly decreasing rearrangement of μ . Then*

$$E_\mu(1, t, t^2, \dots, t^{n-1}; q, t) = t^{n(\lambda)} \prod_{(r,c) \in \mu} \frac{1 - q^c t^{v_\mu(r)}}{1 - q^{\mu_r - c + 1} t^{v_\mu(r) - u_\mu(r,c)}}.$$

Proof. In view of Corollary 5.2 the result will follow if we prove that

$$\prod_{b=(r,c) \in \mu} \prod_{j=1}^{u_\mu(r,c)} \frac{1 - q^{\mu_r - c + 1} t^{v_\mu(r) - j + 1}}{1 - q^{\mu_r - c + 1} t^{v_\mu(r) - j}} = \prod_{(r,c) \in \mu} \frac{1 - q^c t^{v_\mu(r)}}{1 - q^{\mu_r - c + 1} t^{v_\mu(r) - u_\mu(r,c)}}.$$

For a single box $b = (r, c)$, the product $\prod_{j=1}^{u_\mu(r,c)} \frac{1 - tq^{\mu_r - c + 1} t^{v_\mu(r) - j}}{1 - q^{\mu_r - c + 1} t^{v_\mu(r) - j}}$ is equal to

$$\begin{aligned} &\frac{(1 - q^{\mu_r - c + 1} t^{v_\mu(r)})}{(1 - q^{\mu_r - c + 1} t^{v_\mu(r) - 1})} \frac{(1 - q^{\mu_r - c + 1} t^{v_\mu(r) - 1})}{(1 - q^{\mu_r - c + 1} t^{v_\mu(r) - 2})} \dots \frac{(1 - q^{\mu_r - c + 1} t^{v_\mu(r) - (u_\mu(r,c) - 1)})}{(1 - q^{\mu_r - c + 1} t^{v_\mu(r) - u_\mu(r,c)})} \\ &= \frac{1 - q^{\mu_r - c + 1} t^{v_\mu(r)}}{1 - q^{\mu_r - c + 1} t^{v_\mu(r) - u_\mu(r,c)}}. \end{aligned}$$

For a single fixed row r , the product $\prod_{c=1}^{\mu_r} (1 - q^{\mu_r - c + 1} t^{v_\mu(r)}) = \prod_{c=1}^{\mu_r} 1 - q^c t^{v_\mu(r)}$, and so

$$\prod_{(r,c) \in \mu} \prod_{j=1}^{u_\mu(r,c)} \frac{1 - q^{\mu_r - c + 1} t^{v_\mu(r) - j + 1}}{1 - q^{\mu_r - c + 1} t^{v_\mu(r) - j}} = \prod_{(r,c) \in \mu} \frac{1 - q^{\mu_r - c + 1} t^{v_\mu(r)}}{1 - q^{\mu_r - c + 1} t^{v_\mu(r) - u_\mu(r,c)}} = \prod_{(r,c) \in \mu} \frac{1 - q^c t^{v_\mu(r)}}{1 - q^{\mu_r - c + 1} t^{v_\mu(r) - u_\mu(r,c)}}.$$

□

6 The inner product and the Weyl character formula

One of the amazing formulas of mathematics is the Weyl character formula which, in the type GL_n case says that the Schur polynomial is a quotient of two determinants (the denominator is a Vandermonde determinant). Miraculously, the Weyl character formula generalizes to a similar quotient formula for the bosonic Macdonald polynomial $P_\lambda(q, qt)$. In this section we prove this quotient formula for $P_\lambda(q, qt)$, following the same proof as in [Mac03, §5] but with some simplifications in notation.

6.1 The inner product $(\cdot, \cdot)_{q,t}$

Define an involution $\bar{\cdot} : \mathbb{C}[X] \rightarrow \mathbb{C}[X]$ by

$$\bar{f}(x_1, \dots, x_n; q, t) = f(x_1^{-1}, \dots, x_n^{-1}; q^{-1}, t^{-1}),$$

Define

$$\Delta_\infty^X = \Delta_\infty^X(t) = \prod_{1 \leq i < j \leq n} \prod_{r \in \mathbb{Z}_{\geq 0}} \frac{1 - tq^r x_i x_j^{-1}}{1 - q^r x_i x_j^{-1}}, \quad \Delta_\infty^{X^{-1}} = \Delta_\infty^{X^{-1}}(t) = \prod_{1 \leq i < j \leq n} \prod_{r \in \mathbb{Z}_{\geq 0}} \frac{1 - tq^r x_i^{-1} x_j}{1 - q^r x_i^{-1} x_j},$$

$$\Delta_0^X = \Delta_0^X(t) = \prod_{1 \leq i < j \leq n} \frac{1 - tx_i x_j^{-1}}{1 - x_i x_j^{-1}} \quad \text{and} \quad \Delta_0^{X^{-1}} = \Delta_0^{X^{-1}}(t) = \prod_{1 \leq i < j \leq n} \frac{1 - tx_i^{-1} x_j}{1 - x_i^{-1} x_j}.$$

Define a scalar product $(\cdot, \cdot)_{q,t} : \mathbb{C}[X] \times \mathbb{C}[X] \rightarrow \mathbb{C}(q, t)$ by

$$(f_1, f_2)_{q,t} = \text{ct} \left(\frac{f_1 \bar{f}_2}{\Delta_\infty^X \Delta_0^X \Delta_\infty^{X^{-1}}} \right), \quad \text{where } \text{ct}(f) = (\text{constant term in } f), \quad (6.1)$$

for $f \in \mathbb{C}[X]$. Proposition 6.1 shows that, in a suitable sense, the inner product $(\cdot, \cdot)_{q,t}$ is nondegenerate and normalized Hermitian.

Proposition 6.1.

(a) (sesquilinear) If $f, g \in \mathbb{C}[X]$ and $c \in \mathbb{C}[q^{\pm 1}]$ then

$$(cf, g)_{q,t} = c(f, g)_{q,t}, \quad \text{and} \quad (f, cg)_{q,t} = \bar{c}(f, g)_{q,t}.$$

(b) (nonisotropy) If $f \in \mathbb{C}[X]$ and $f \neq 0$ then $(f, f)_{q,t} \neq 0$.

(c) (nondegeneracy) If F is a subspace of $\mathbb{C}[X]$ and $(\cdot, \cdot)_F : F \times F \rightarrow \mathbb{C}$ is the restriction of $(\cdot, \cdot)_{q,t}$ to F , then $(\cdot, \cdot)_F$ is nondegenerate.

(d) (normalized Hermitian) If $f_1, f_2 \in \mathbb{C}[X]$ then

$$\frac{(f_2, f_1)_{q,t}}{(1, 1)_{q,t}} = \overline{\frac{(f_1, f_2)_{q,t}}{(1, 1)_{q,t}}}.$$

Proof. (a) Let $f_1, f_2 \in \mathbb{C}[X]$ and $c \in \mathbb{C}[q^{\pm 1}, t^{\pm 1}]$. Then $(cf_1, f_2)_{q,t} = \text{ct}(cf_1 \overline{f_2}) = c \cdot \text{ct}(f_1 \overline{f_2}) = c(f, g)_{q,t}$ and

$$(f_1, cf_2)_{q,t} = \text{ct}(f_1 \overline{cf_2}) = \text{ct}(f_1 \overline{c} \overline{f_2}) = \overline{c} \cdot \text{ct}(f_1 \overline{f_2}) = \overline{c}(f, g)_{q,t}.$$

(b) Let $f \in \mathbb{C}[X]$ with $f \neq 0$. By clearing denominators appropriately, renormalize f so that f specializes to something nonzero at $q = 1$. If

$$f = \sum_{\mu} f_{\mu} x^{\mu} \quad \text{then} \quad (f, f)_{1,1} = (f, f)_{1,1^k} = \sum_{\mu} |f_{\mu}|^2 \in \mathbb{R}_{>0}.$$

Thus $(f, f)_t \neq 0$.

(c) Let $f \in F$ with $f \neq 0$. Since $(f, f)_{q,t} \neq 0$ then there exists $p \in F$ such that $(f, p)_{q,t} \neq 0$. Thus the restriction of $(\cdot, \cdot)_{q,t}$ to F is nondegenerate.

(d) Let $f_1 = \sum_{\lambda} a_{\lambda} x^{\lambda}$ and $f_2 = \sum_{\mu} b_{\mu} x^{\mu}$ and

$$\frac{\Delta_{\infty}^X \Delta_0^X \Delta_{\infty}^{X^{-1}}}{(1, 1)_{q,t}} = \frac{\Delta_{\infty}^X \Delta_0^X \Delta_{\infty}^{X^{-1}}}{\text{ct}(\Delta_{\infty}^X \Delta_0^X \Delta_{\infty}^{X^{-1}})} = \sum_{\mu \in \mathbb{Z}^n} d_{\mu}(q, t) x^{\mu}.$$

Then

$$\frac{(f_2, f_1)_{q,t}}{(1, 1)_{q,t}} = \sum_{\lambda, \mu \in \mathbb{Z}^n} \overline{a_{\lambda}} b_{\mu} d_{\lambda - \mu} = \sum_{\lambda, \mu \in \mathbb{Z}^n} \overline{a_{\lambda} b_{\mu} d_{\mu - \lambda}} = \overline{\left(\frac{(f_1, f_2)_{q,t}}{(1, 1)_{q,t}} \right)}.$$

□

6.2 The inner product characterization of E_{μ} and P_{λ}

Recall that the elements of \mathbb{Z}^n are partially ordered with the DBlex order given by

$$\lambda \leq \mu \quad \text{if} \quad \begin{array}{l} \lambda^+ < \mu^+ \text{ in dominance order} \\ \text{or} \\ \lambda^+ = \mu^+ \text{ and } z_{\lambda} < z_{\mu} \text{ in Bruhat order.} \end{array}$$

Proposition 6.2. *Let $\mu \in \mathbb{Z}^n$. The nonsymmetric Macdonald polynomial $E_{\mu}(q, t)$ is the unique element of $\mathbb{C}[X]$ such that*

(a) $E_{\mu}(q, t) = x^{\mu} + (\text{lower terms in DBlex order})$;

(b) If $\nu \in \mathbb{Z}^n$ and $\nu < \mu$ then $(E_{\mu}(q, t), x^{\nu})_{q,t} = 0$.

Proof. Let $V = \text{span}\{x^{\mu} \mid \mu \in \mathbb{Z}^n \text{ and } \nu \leq \mu\}$,

$$S = \text{span}\{x^{\nu} \mid \nu \in \mathbb{Z}^n \text{ and } \nu < \mu\} \quad \text{and} \quad S^{\perp} = \{f \in \mathbb{C}[X] \mid \text{if } p \in S \text{ then } (f, p)_{q,t} = 0\}.$$

Since the inner product $(\cdot, \cdot)_{q,t}$ is nonisotropic then the restriction of $(\cdot, \cdot)_{q,t}$ to V is nondegenerate and so $\dim(S^{\perp}) = 1$. Then the normalization of $E_{\mu} \in S^{\perp}$ is determined by condition (a). □

For $\gamma \in (\mathbb{Z}^n)^+$, define the *monomial symmetric polynomial* m_{γ} by

$$m_{\gamma} = \sum_{\mu \in S_n \gamma} x^{\mu}, \quad \text{where the sum is over all distinct rearrangements of } \gamma.$$

Proposition 6.3. *Let $\lambda \in (\mathbb{Z}^n)^+$. The symmetric Macdonald polynomial P_λ is the unique element of $\mathbb{C}[X]^{S_n}$ such that*

- (a) $P_\lambda(q, t) = m_\lambda + (\text{lower terms in dominance order});$
- (b) *If $\gamma \in (\mathbb{Z}^n)^+$ and $\gamma < \lambda$ then $(P_\lambda(q, t), m_\gamma)_{q,t} = 0.$*

Proof. The proof is completed in the same manner as the proof of Proposition 6.2. □

6.3 Going up a level from t to qt

As in (4.6) and (3.19), let

$$A_\rho = A_\rho(t) = \prod_{1 \leq i < j \leq n} (x_j - tx_i) \quad \text{and} \quad W_0(t) = \sum_{w \in S_n} t^{\ell(w)}.$$

Proposition 6.4. *Let $f, g \in \mathbb{C}[X]^{S_n}$ so that f and g are symmetric polynomials. Then*

$$(f, g)_{q,qt} = \frac{W_0(qt)}{W_0(t^{-1})} (A_\rho f, A_\rho g)_{q,t}.$$

Proof. Letting

$$a_\rho(x) = \prod_{i < j} (x_j - x_i) \quad \text{and} \quad a_\rho(x^{-1}) = \prod_{i < j} (x_i - x_j), \quad \text{then} \quad \Delta_\infty^X(qt) = \frac{\Delta_\infty^X(t)}{\Delta_0^X(qt)a_\rho(x)}.$$

Then

$$\begin{aligned} (f, g)_{q,qt} &= \text{ct} \left(\frac{f\bar{g}}{\Delta_\infty^X(qt)\Delta_\infty^{X^{-1}}(qt)\Delta_0^X(qt)} \right) = \text{ct} \left(f\bar{g} \frac{\Delta_0^X(qt)a_\rho(x)\Delta_0^{X^{-1}}(qt)a_\rho(x^{-1})}{\Delta_\infty^X(t)\Delta_\infty^{X^{-1}}(t)\Delta_0^X(qt)} \right) \\ &= \text{ct} \left(f\bar{g} \frac{a_\rho(x)a_\rho(x^{-1})}{\Delta_\infty^X(t)\Delta_\infty^{X^{-1}}(t)} \Delta_0^{X^{-1}}(qt) \right) = \frac{W_0(qt)}{n!} \text{ct} \left(f\bar{g} \frac{a_\rho(x)a_\rho(x^{-1})}{\Delta_\infty^X(t)\Delta_\infty^{X^{-1}}(t)} \right), \end{aligned}$$

where the last equality follows from the fact that if H is symmetric then

$$\begin{aligned} \text{ct}(H\Delta_0^{X^{-1}}(t)) &= \frac{1}{n!} \text{ct} \left(\sum_{w \in S_n} w(H\Delta_0^{X^{-1}}(t)) \right) = \frac{1}{n!} \text{ct} \left(H \sum_{w \in S_n} w(\Delta_0^{X^{-1}}(t)) \right) \\ &= \frac{1}{n!} \text{ct} \left(HW_0(t) \right) = \frac{W_0(t)}{n!} \text{ct}(H). \end{aligned}$$

Similarly, using $A_\rho = \Delta_0^X(t)a_\rho(x)$ gives

$$\begin{aligned} (A_\rho f, A_\rho g)_{q,t} &= \text{ct} \left(f\bar{g} \frac{\Delta_0^X(t)a_\rho(x)\Delta_0^{X^{-1}}(t)a_\rho(x^{-1})}{\Delta_\infty^X(t)\Delta_\infty^{X^{-1}}(t)\Delta_0^X(t)} \right) \\ &= \text{ct} \left(f\bar{g} \frac{a_\rho(x)a_\rho(x^{-1})}{\Delta_\infty^X(t)\Delta_\infty^{X^{-1}}(t)} \Delta_0^{X^{-1}}(t) \right) = \frac{W_0(t)}{n!} \text{ct} \left(f\bar{g} \frac{a_\rho(x)a_\rho(x^{-1})}{\Delta_\infty^X(t)\Delta_\infty^{X^{-1}}(t)} \right). \end{aligned}$$

Comparing these expressions for $(f, g)_{q,qt}$ and $(A_\rho f, A_\rho g)_{q,t}$ gives the statement. □

6.4 Weyl character formula for Macdonald polynomials

Theorem 6.5. *Let $\lambda \in \mathbb{Z}^n$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. Then*

$$P_\lambda(q, qt) = \frac{A_{\lambda+\rho}(q, t)}{A_\rho(t)}.$$

Proof. Since $A_{\lambda+\rho} = t^{\frac{1}{2}\ell(w_0)} \varepsilon_0 E_{\lambda+\rho}$ then $A_{\lambda+\rho} \in \mathbb{C}[X]^{\text{Fer}}$. Thus, by Proposition 4.8,

$$\text{there exists } f \in \mathbb{C}[X]^{S_n} \quad \text{such that} \quad A_{\lambda+\rho} = A_\rho f.$$

If $\mu \in \mathbb{Z}^n$ is such that the coefficient of x^μ in $A_{\lambda+\rho}$ is nonzero then $\mu \leq w_0(\lambda + \rho)$. Thus

$$f = m_\lambda + (\text{lower terms}).$$

The E -expansion for $A_{\lambda+\rho}$ in Proposition 4.7 gives that

$$A_{\lambda+\rho} = \sum_{\mu \in S_n(\lambda+\rho)} d_{\lambda+\rho}^\mu E_\mu = E_{w_0(\lambda+\rho)} + (\text{lower terms})$$

and, from the definitions of A_ρ and m_ν ,

$$A_\rho m_\nu = x^{w_0(\nu+\rho)} + (\text{lower terms}).$$

Since $(E_{w_0(\lambda+\rho)}, x^\gamma)_{q,t} = 0$ for $\gamma \in \mathbb{Z}^n$ with $\gamma < w_0(\lambda + \rho)$, then

$$(A_\rho f, A_\rho m_\nu)_{q,t} = (A_{\lambda+\rho}, A_\rho m_\nu)_{q,t} = 0, \quad \text{for } \nu \in (\mathbb{Z}^n)^+ \text{ with } \nu < \lambda.$$

Thus, by (6.4), since $f \in \mathbb{C}[X]^{S_n}$ and $m_\nu \in \mathbb{C}[X]^{S_n}$ then

$$(f, m_\nu)_{q,qt} = \frac{W_0(qt)}{W_0(t^{-1})} (A_\rho f, A_\rho m_\nu)_{q,t} = 0, \quad \text{for } \nu \in (\mathbb{Z}^n)^+ \text{ with } \nu < \lambda.$$

Thus, by Proposition 6.3, $f = P_\lambda(q, qt)$. □

7 Norms and c -functions

The Macdonald polynomials form an incredible family of orthogonal polynomials with respect to the inner product $(\cdot, \cdot)_{q,t}$ defined in (6.1). These orthogonal polynomials generalize the Askey-Wilson polynomials to the multivariate case and to all affine root systems. In this section we see how the c -functions enter in the formulas for the norms of the Macdonald polynomials. The special case of the norm of the polynomial 1 is the celebrated Macdonald constant term conjecture [Mac82]. The generalization [Mac87] of the Macdonald constant term conjectures to conjectures for the norms of the P_λ was astounding. After much work on special cases, the methods and tools of Heckman, Opdam, Macdonald, and Cherednik yielded a proof of the norm conjecture (about 1994, see the Notes and References at the end of [Mac03, §5]). This proof is exposited beautifully in [Mac03, §5.8 and §5.9]. In this section we provide a type GL_n specific exposition, highlighting the role of the c -functions, and using them to simplify some steps. The general framework of our exposition follows the proof found in [Mac03, §5.8].

7.1 Adjoints and orthogonality

For a linear operator $M: \mathbb{C}[X] \rightarrow \mathbb{C}[X]$, the *adjoint of M* is the linear operator $M^*: \mathbb{C}[X] \rightarrow \mathbb{C}[X]$ determined by

$$(Mf_1, f_2)_{q,t} = (f_1, M^*f_2)_{q,t}, \quad \text{for } f_1, f_2 \in \mathbb{C}[X],$$

where the inner product on $\mathbb{C}[X]$ is as defined in (6.1). Also, simplify the notation by letting

$$J_{q,t} = \frac{1}{\Delta_\infty^X \Delta_0^X \Delta_\infty^{X^{-1}}} = \prod_{1 \leq i < j \leq n} \frac{(x_i x_j^{-1}; q)_\infty}{(t x_i x_j^{-1}; q)_\infty} \cdot \frac{(q x_i^{-1} x_j; q)_\infty}{(q t x_i^{-1} x_j; q)_\infty}, \quad \text{and} \quad \Delta_{ij} = \frac{1 - t x_i x_j^{-1}}{1 - x_i x_j^{-1}}.$$

Proposition 7.1. *Let $i \in \{1, \dots, n\}$ and $k \in \{1, \dots, n-1\}$. Then, as operators on $\mathbb{C}[X]$,*

$$x_i^* = x_i^{-1}, \quad s_k^* = \frac{\Delta_{k,k+1}}{\Delta_{k+1,k}} s_k, \quad T_\pi^* = T_\pi^{-1}, \quad T_k^* = T_k^{-1}, \quad Y_i^* = Y_i^{-1}.$$

Proof.

- Adjoint of multiplication by x_i :

$$(x_i f, g)_{q,t} = \text{ct}(x_i f \bar{g} J_{q,t}) = \text{ct}(f \cdot \overline{x_i^{-1} g} \cdot J_{q,t}) = (f, x_i^{-1} g)_{q,t}.$$

- Adjoint of s_k : For $i, j \in \{1, \dots, n\}$ with $i \neq j$, note that

$$\overline{\Delta_{ij}} = \frac{1 - t^{-1} x_i^{-1} x_j}{1 - x_i^{-1} x_j} = \frac{t x_i x_j^{-1} - 1}{t(x_i x_j^{-1} - 1)} = t^{-1} \Delta_{ij}.$$

Therefore

$$\begin{aligned} s_k J_{q,t} &= s_k \left(\frac{1}{\Delta_\infty^X \Delta_0^X \Delta_\infty^{X^{-1}}} \right) = s_k \left(\frac{1}{\Delta_\infty^X \Delta_0^X \Delta_\infty^{X^{-1}} \Delta_\infty^{X^{-1}}} \cdot \Delta_0^{X^{-1}} \right) = \frac{1}{\Delta_\infty^X \Delta_0^X \Delta_\infty^{X^{-1}} \Delta_\infty^{X^{-1}}} \cdot s_k(\Delta_0^{X^{-1}}) \\ &= \frac{1}{\Delta_\infty^X \Delta_0^X \Delta_\infty^{X^{-1}} \Delta_\infty^{X^{-1}}} \cdot \Delta_0^{X^{-1}} \frac{\Delta_{k,k+1}}{\Delta_{k+1,k}} = \left(\frac{1}{\Delta_\infty^X \Delta_0^X \Delta_\infty^{X^{-1}}} \right) \frac{\Delta_{k,k+1}}{\Delta_{k+1,k}} = J_{q,t} \frac{\Delta_{k,k+1}}{\Delta_{k+1,k}}. \end{aligned}$$

Then

$$\begin{aligned} (s_k f_1, f_2)_{q,t} &= \text{ct}((s_k f_1) \bar{f}_2 J_{q,t}) = \text{ct}(s_k(f_1(s_k(\bar{f}_2 J_{q,t})))) = \text{ct}(f_1(s_k(f_2 J_{q,t}))) \\ &= \text{ct}\left(f_1(s_k \bar{f}_2) J_{q,t} \frac{\Delta_{k,k+1}}{\Delta_{k+1,k}}\right) = \text{ct}\left(f_1 \frac{\Delta_{k,k+1}}{\Delta_{k+1,k}} (s_k f_2) J_{q,t}\right) = (f_1, \frac{\Delta_{k,k+1}}{\Delta_{k+1,k}} (s_k f_2))_{q,t}. \end{aligned}$$

- Adjoint of T_π : Recall that the action of T_π is given by $T_\pi = s_1 s_2 \cdots s_{n-1} y_n$ and so

$$T_\pi x_n = q^{-1} x_1, \quad \text{and} \quad T_\pi x_i = x_{i+1} \text{ for } i \in \{1, \dots, n-1\}.$$

Thus, if $i \in \{1, \dots, n-1\}$ then $T_\pi x_i x_n^{-1} = q x_{i+1} x_1^{-1}$ and $T_\pi q x_i^{-1} x_n = q q^{-1} x_{i+1}^{-1} x_1$. Therefore

$$\begin{aligned} T_\pi J_{q,t} &= T_\pi \left(\prod_{1 \leq i < j \leq n-1} \frac{(x_i x_j^{-1}; q)_\infty}{(t x_i x_j^{-1}; q)_\infty} \cdot \frac{(q x_i^{-1} x_j; q)_\infty}{(q t x_i^{-1} x_j; q)_\infty} \right) \left(\prod_{i=1}^{n-1} \frac{(x_i x_n^{-1}; q)_\infty}{(t x_i x_n^{-1}; q)_\infty} \cdot \frac{(q x_i^{-1} x_n; q)_\infty}{(q t x_i^{-1} x_n; q)_\infty} \right) \\ &= \left(\prod_{2 \leq i < j \leq n} \frac{(x_i x_j^{-1}; q)_\infty}{(t x_i x_j^{-1}; q)_\infty} \cdot \frac{(q x_i^{-1} x_j; q)_\infty}{(q t x_i^{-1} x_j; q)_\infty} \right) \left(\prod_{i=2}^n \frac{(q x_i x_1^{-1}; q)_\infty}{(q t x_i x_1^{-1}; q)_\infty} \cdot \frac{(q q^{-1} x_i^{-1} x_1; q)_\infty}{(q t q^{-1} x_i^{-1} x_1; q)_\infty} \right) \\ &= \left(\prod_{2 \leq i < j \leq n} \frac{(x_i x_j^{-1}; q)_\infty}{(t x_i x_j^{-1}; q)_\infty} \cdot \frac{(q x_i^{-1} x_j; q)_\infty}{(q t x_i^{-1} x_j; q)_\infty} \right) \left(\prod_{i=2}^n \frac{(q x_i^{-1} x_1; q)_\infty}{(q t x_i^{-1} x_1; q)_\infty} \cdot \frac{(x_1 x_i^{-1}; q)_\infty}{(t x_1 x_i^{-1}; q)_\infty} \right) = J_{q,t}. \end{aligned}$$

Thus

$$\begin{aligned} (T_\pi f_1, f_2)_{q,t} &= \text{ct}((T_\pi f_1) \overline{f_2} J_{q,t}) = \text{ct}(T_\pi(f_1 T_{\pi^{-1}}(\overline{f_2} J_{q,t}))) = \text{ct}(f_1 T_{\pi^{-1}}(\overline{f_2} J_{q,t})) \\ &= \text{ct}(f_1 (T_{\pi^{-1}}(\overline{f_2}) J_{q,t})) = \text{ct}(f_1 \cdot \overline{T_{\pi^{-1}} f_2} \cdot J_{q,t}) = (f_1, T_{\pi^{-1}} f_2)_{q,t}. \end{aligned}$$

◦ Adjoint of T_k :

$$\begin{aligned} (T_{s_k})^* &= (-t^{-\frac{1}{2}} + (1 + s_k)t^{-\frac{1}{2}} \Delta_{k+1,k})^* = -t^{\frac{1}{2}} + t^{\frac{1}{2}} \overline{\Delta_{k+1,k}} (1 + s_k^*) \\ &= -t^{\frac{1}{2}} + t^{-\frac{1}{2}} \Delta_{k+1,k} \left(1 + \frac{\Delta_{k,k+1}}{\Delta_{k+1,k}} s_k\right) = -t^{\frac{1}{2}} + (1 + s_k)t^{-\frac{1}{2}} \Delta_{k+1,k} = T_{s_k}^{-1}. \end{aligned}$$

◦ Adjoint of Y_j :

$$Y_1^* = (T_\pi T_{n-1} \cdots T_1)^* = T_1^{-1} \cdots T_{n-1}^{-1} T_\pi^{-1} = (T_\pi T_{n-1} \cdots T_1)^{-1} = Y_1^{-1},$$

and if $j \in \{2, \dots, n\}$ then

$$Y_j^* = (T_{j-1}^{-1} Y_{j-1} T_{j-1})^* = T_{j-1} Y_{j-1}^{-1} T_{j-1} = (T_{j-1}^{-1} Y_{j-1} T_{j-1})^{-1} = Y_j^{-1}.$$

□

Next, we look at the adjoint operator of the bosonic and fermionic symmetrizers. Since $T_i^{-1} \mathbf{1}_0^* = T_i^* \mathbf{1}_0^* = (\mathbf{1}_0 T_i)^* = (t^{\frac{1}{2}} \mathbf{1}_0)^* = t^{-\frac{1}{2}} \mathbf{1}_0$ and

$$\mathbf{1}_0^* = T_{w_0}^{-1} + (\text{lower terms}) = T_{w_0} + (\text{lower terms}) \quad \text{then} \quad \mathbf{1}_0^* = \mathbf{1}_0.$$

Similarly, since $T_i^{-1} \varepsilon_0^* = T_i^* \varepsilon_0^* = (\varepsilon_0 T_i)^* = (-t^{-\frac{1}{2}} \varepsilon_0)^* = -t^{\frac{1}{2}} \varepsilon_0^*$ and

$$\varepsilon_0^* = T_{w_0}^{-1} + (\text{lower terms}) = T_{w_0} + (\text{lower terms}) \quad \text{then} \quad \varepsilon_0^* = \varepsilon_0.$$

The relations $Y_i^* = Y_i^{-1}$ in combination with the knowledge of the eigenvalues for the action of the Y_i on the E_μ give the following orthogonality relations for Macdonald polynomials.

Proposition 7.2.

- (a) Let $\lambda, \mu \in \mathbb{Z}^n$. If $\mu \neq \lambda$ then $(E_\lambda, E_\mu)_{q,t} = 0$.
- (b) Let $\lambda, \mu \in (\mathbb{Z}^n)^+$. If $\mu \neq \lambda$ then $(P_\lambda, P_\mu)_{q,t} = 0$.
- (c) Let $\lambda, \mu \in (\mathbb{Z}^n)^+$. If $\mu \neq \lambda$ then $(A_{\lambda+\rho}, A_{\mu+\rho})_{q,t} = 0$.

Proof. Let $i \in \{1, \dots, n\}$. Then, by Theorem 1.1,

$$\begin{aligned} q^{-\lambda_i} t^{-(v_\lambda(i)-1) + \frac{1}{2}(n-1)} (E_\lambda, E_\mu)_{q,t} &= (Y_i E_\lambda, E_\mu)_{q,t} = (E_\lambda, Y_i^{-1} E_\mu)_{q,t} = (E_\lambda, q^{\mu_i} t^{(v_\mu(i)-1) + \frac{1}{2}(n-1)} E_\mu)_{q,t} \\ &= \overline{q^{\mu_i} t^{(v_\mu(i)-1) - \frac{1}{2}(n-1)}} (E_\lambda, E_\mu)_{q,t} = q^{-\mu_i} t^{-(v_\mu(i)-1) + \frac{1}{2}(n-1)} (E_\lambda, E_\mu)_{q,t}. \end{aligned}$$

If $(E_\lambda, E_\mu)_{q,t} \neq 0$ then $q^{-\lambda_i} = q^{-\mu_i}$ for $i \in \{1, \dots, n\}$. Thus $\lambda_i = \mu_i$ for $i \in \{1, \dots, n\}$ and so $\lambda = \mu$ (and $v_\lambda = v_\mu$).

Parts (b) and (c) follow from (a) and the E -expansions in Proposition 4.7. □

7.2 Reductions for norms

Proposition 7.3.

(a) Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$. Then

$$\frac{(E_\mu, E_\mu)_{q,t}}{(1,1)_{q,t}} = \text{ev}_0^t(c_{u_\mu}^Y c_{u_\mu}^{Y^{-1}}).$$

(b) Let $\lambda \in (\mathbb{Z}^n)^+$ and let $\rho = (n-1, n-2, \dots, 1, 0)$. Then

$$\frac{(P_\lambda, P_\lambda)_{q,t}}{(E_\lambda, E_\lambda)_{q,t}} = \frac{W_0(t)}{W_\lambda(t)} t^{-\frac{1}{2}\ell(v_\lambda)} \text{ev}_\lambda^t(c_{v_\lambda}^Y) \quad \text{and} \quad \frac{(A_{\lambda+\rho}, A_{\lambda+\rho})_{q,t}}{(P_{\lambda+\rho}, P_{\lambda+\rho})_{q,t}} = \text{ev}_{\lambda+\rho}^t\left(\frac{c_{w_0}^{Y^{-1}}}{c_{w_0}^Y}\right).$$

(c) Let $\lambda \in (\mathbb{Z}^n)^+$. Then

$$\frac{(P_\lambda(q, qt), P_\lambda(q, qt))_{q,qt}}{(P_{\lambda+\rho}(q, t), P_{\lambda+\rho}(q, t))_{q,t}} = \frac{W_0(qt)}{W_0(t)} \text{ev}_{\lambda+\rho}^t\left(t^{\ell(w_0)} \frac{c_{w_0}^{Y^{-1}}}{c_{w_0}^Y}\right).$$

Proof. (a) Recalling the intertwiners τ_i^\vee from (1.12) (see also the proof of Proposition 3.4),

$$(\tau_i^\vee)^* = \left(T_i + \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}}}{1 - Y_i^{-1}Y_{i+1}}\right)^* = T_i^* + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - Y_i Y_{i+1}^{-1}} = T_i^{-1} + \frac{(t^{-\frac{1}{2}} - t^{\frac{1}{2}})Y_i^{-1}Y_{i+1}}{1 - Y_i^{-1}Y_{i+1}} = \tau_i^\vee,$$

and

$$(\tau_i^\vee)^2 = \frac{(1 - tY_i Y_{i+1}^{-1})(1 - tY_i^{-1}Y_{i+1})}{(1 - Y_i Y_{i+1}^{-1})(1 - Y_i^{-1}Y_{i+1})} = c_{i,i+1}^Y c_{i,i+1}^{Y^{-1}}.$$

Then, using the creation formula for E_μ (Theorem 4.2),

$$\begin{aligned} (E_\mu, E_\mu)_{q,t} &= (t^{-\frac{1}{2}\ell(v_\mu^{-1})} \tau_{u_\mu}^\vee \mathbf{1}_Y, t^{-\frac{1}{2}\ell(v_\mu^{-1})} \tau_{u_\mu}^\vee \mathbf{1}_Y)_{q,t} = (\tau_{u_\mu^{-1}}^\vee \tau_{u_\mu}^\vee \mathbf{1}_Y, \mathbf{1}_Y)_{q,t} \\ &= (c_{u_\mu}^Y c_{u_\mu}^{Y^{-1}} \mathbf{1}_Y, \mathbf{1}_Y)_{q,t} = \text{ev}_0^t(c_{u_\mu}^Y c_{u_\mu}^{Y^{-1}}) \cdot (1,1)_{q,t}. \end{aligned}$$

(b) Note that

$$\mathbf{1}_0^2 = t^{-\frac{1}{2}\ell(w_0)} W_0(t) \mathbf{1}_0 \quad \text{and} \quad \varepsilon_0^2 = (-1)^{\ell(w_0)} t^{-\frac{1}{2}\ell(w_0)} W_0(t) \varepsilon_0.$$

By Proposition 4.7,

$$\begin{aligned} P_\lambda &= \sum_{\mu \in S_n \lambda} b_\lambda^\mu E_\mu, & \text{with} & \quad b_\lambda^\lambda = t^{\frac{1}{2}\ell(v_\lambda)} \text{ev}_\lambda^t(c_{v_\lambda}^Y), \\ A_{\lambda+\rho} &= \sum_{\mu \in S_n(\lambda+\rho)} d_{\lambda+\rho}^\mu E_\mu, & \text{with} & \quad d_{\lambda+\rho}^{\lambda+\rho} = (-t^{\frac{1}{2}})^{\ell(w_0)} \text{ev}_{\lambda+\rho}^t(c_{w_0}^{Y^{-1}}). \end{aligned}$$

Since $W_\lambda(t^{-1}) = t^{-\ell(w_\lambda)} W_\lambda(t)$ and $\ell(w_0) - \ell(w_\lambda) = \ell(v_\lambda)$ then using $P_\lambda = \frac{t^{\frac{1}{2}\ell(w_0)}}{W_\lambda(t)} \mathbf{1}_0 E_\lambda$ gives

$$\begin{aligned} (P_\lambda, P_\lambda)_{q,t} &= \left(\frac{t^{\frac{1}{2}\ell(w_0)}}{W_\lambda(t)} \mathbf{1}_0 E_\lambda, \frac{t^{\frac{1}{2}\ell(w_0)}}{W_\lambda(t)} \mathbf{1}_0 E_\lambda\right)_{q,t} = \frac{1}{W_\lambda(t)W_\lambda(t^{-1})} (\mathbf{1}_0^2 E_\lambda, E_\lambda)_{q,t} \\ &= \frac{t^{-\frac{1}{2}\ell(w_0)} W_0(t)}{W_\lambda(t)W_\lambda(t^{-1})} (\mathbf{1}_0 E_\lambda, E_\lambda)_{q,t} = \frac{t^{-\ell(w_0)} W_0(t)}{W_\lambda(t^{-1})} (P_\lambda, E_\lambda)_{q,t} \\ &= \frac{t^{-\ell(w_0)} W_0(t)}{t^{-\ell(w_\lambda)} W_\lambda(t)} b_\lambda^\lambda (E_\lambda, E_\lambda)_{q,t} = \frac{t^{-\ell(v_\lambda)} W_0(t)}{W_\lambda(t)} t^{\frac{1}{2}\ell(v_\lambda)} \text{ev}_\lambda^t(c_{v_\lambda}^Y) (E_\lambda, E_\lambda)_{q,t}. \end{aligned}$$

Similarly, using $A_{\lambda+\rho} = t^{\frac{1}{2}\ell(w_0)}\varepsilon_0 E_{\lambda+\rho}$ gives

$$\begin{aligned} (A_{\lambda+\rho}, A_{\lambda+\rho})_{q,t} &= (t^{\frac{1}{2}\ell(w_0)}\varepsilon_0 E_{\lambda+\rho}, t^{\frac{1}{2}\ell(w_0)}\varepsilon_0 E_{\lambda+\rho})_{q,t} = (\varepsilon_0^2 E_{\lambda+\rho}, E_{\lambda+\rho})_{q,t} \\ &= (-1)^{\ell(w_0)} t^{-\frac{1}{2}\ell(w_0)} W_0(t) (\varepsilon_0 E_{\lambda+\rho}, E_{\lambda+\rho})_{q,t} = (-1)^{\ell(w_0)} t^{-\ell(w_0)} W_0(t) (A_{\lambda+\rho}, E_{\lambda+\rho})_{q,t} \\ &= (-1)^{\ell(w_0)} W_0(t^{-1}) a_{\lambda+\rho}^{\lambda+\rho} (E_{\lambda+\rho}, E_{\lambda+\rho})_{q,t} = W_0(t^{-1}) t^{\frac{1}{2}\ell(w_0)} \text{ev}_{\lambda+\rho}^t (c_{w_0}^{Y^{-1}}) (E_{\lambda+\rho}, E_{\lambda+\rho})_{q,t}. \end{aligned}$$

Using $(P_{\lambda+\rho}, P_{\lambda+\rho})_{q,t} = W_0(t) t^{-\frac{1}{2}\ell(w_0)} \text{ev}_{\lambda+\rho}^t (c_{w_0}^Y) (E_{\lambda+\rho}, E_{\lambda+\rho})_{q,t}$ gives

$$\frac{(A_{\lambda+\rho}, A_{\lambda+\rho})_{q,t}}{(P_{\lambda+\rho}, P_{\lambda+\rho})_{q,t}} = \frac{W_0(t^{-1}) t^{\frac{1}{2}\ell(w_0)} \text{ev}_{\lambda+\rho}^t (c_{w_0}^{Y^{-1}})}{W_0(t) t^{-\frac{1}{2}\ell(w_0)} \text{ev}_{\lambda+\rho}^t (c_{w_0}^Y)} = \text{ev}_{\lambda+\rho}^t \left(\frac{c_{w_0}^{Y^{-1}}}{c_{w_0}^Y} \right).$$

(c) Using Proposition 6.4 and the Weyl character formula (Theorem 6.5),

$$\begin{aligned} (P_\lambda(q, qt), P_\lambda(q, qt))_{q,qt} &= \frac{W_0(qt)}{W_0(t^{-1})} (A_\rho(t) P_\lambda(q, qt), A_\rho(t) P_\lambda(q, qt))_{q,t} \\ &= \frac{W_0(qt)}{t^{-\ell(w_0)} W_0(t)} (A_{\lambda+\rho}(q, t), A_{\lambda+\rho}(q, t))_{q,t} = \frac{W_0(qt)}{W_0(t)} \text{ev}_{\lambda+\rho}^t \left(t^{\ell(w_0)} \frac{c_{w_0}^{Y^{-1}}}{c_{w_0}^Y} \right) (P_{\lambda+\rho}(q, t), P_{\lambda+\rho}(q, t))_{q,t}, \end{aligned}$$

where the last equality follows from the second formula in (b). □

Evaluating the c -functions appearing in Proposition 7.3 gives the following corollary.

Corollary 7.4.

$$\begin{aligned} (E_\mu, E_\mu)_{q,t} &= \left(\prod_{(r,c) \in \mu} \prod_{i=1}^{u_\mu(r,c)} \frac{(1 - q^{\mu_r - c + 1} t^{v_\mu(r) - i + 1}) (1 - q^{\mu_r - c + 1} t^{v_\mu(r) - i - 1})}{(1 - q^{\mu_r - c + 1} t^{v_\mu(r) - i})^2} \right) \cdot (1, 1)_{q,t}, \\ (P_\lambda, P_\lambda)_{q,t} &= \frac{W_0(t)}{W_\lambda(t)} \left(\prod_{i < j} \frac{1 - q^{\lambda_i - \lambda_j} t^{j - i - 1}}{1 - q^{\lambda_i - \lambda_j} t^{j - i}} \right) \cdot (E_\lambda, E_\lambda)_{q,t}, \\ (A_{\lambda+\rho}, A_{\lambda+\rho})_{q,t} &= W_0(t^{-1}) \left(\prod_{i < j} \frac{1 - q^{\lambda_i - \lambda_j + j - i} t^{j - i + 1}}{1 - q^{\lambda_i - \lambda_j + j - i} t^{j - i}} \right) \cdot (E_{\lambda+\rho}, E_{\lambda+\rho})_{q,t}, \\ \frac{(P_\lambda(q, qt), P_\lambda(q, qt))_{q,qt}}{(P_{\lambda+\rho}(q, t), P_{\lambda+\rho}(q, t))_{q,t}} &= \frac{W_0(qt)}{W_0(t)} \left(\prod_{i < j} \frac{1 - q^{\lambda_i - \lambda_j + j - i} t^{j - i + 1}}{1 - q^{\lambda_i - \lambda_j + j - i} t^{j - i - 1}} \right). \end{aligned}$$

Proof. Using that

$$c_{ij}^Y c_{ij}^{Y^{-1}} = \left(\frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} Y_i Y_j^{-1}}{1 - Y_i Y_j^{-1}} \right) \left(\frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} Y_i^{-1} Y_j}{1 - Y_i^{-1} Y_j} \right) = \frac{(1 - t Y_i Y_j^{-1}) (1 - t^{-1} Y_i Y_j^{-1})}{(1 - Y_i Y_j^{-1})^2},$$

the formula for $(E_\mu, E_\mu)_{q,t}$ follows from Proposition 7.3(a) and the computation in (5.4). The formula for $(P_\lambda, P_\lambda)_{q,t}$ follows from Proposition 7.3(b) and the computation in (4.11). The formula for $(P_\lambda, P_\lambda)_{q,t}$ follows from Proposition 7.3(b) and the computation in (4.12).

Using $\text{ev}_\mu^t(Y_i) = q^{-\mu_i} t^{-(v_\mu(i) - 1) + \frac{1}{2}(n-1)}$ gives

$$\text{ev}_\mu^t(Y_i^{-1} Y_j) = q^{\mu_i} t^{(v_\mu(i) - 1) - \frac{1}{2}(n-1)} q^{-\mu_j} t^{-(v_\mu(j) - 1) + \frac{1}{2}(n-1)} = q^{\mu_i - \mu_j} t^{v_\mu(i) - v_\mu(j)},$$

for $i, j \in \{1, \dots, n\}$ with $i \neq j$. Then

$$t \frac{c_{ij}^{Y^{-1}}}{c_{ij}^Y} = t \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} Y_i^{-1} Y_j}{1 - Y_i^{-1} Y_j} = t \frac{1 - t Y_i^{-1} Y_j}{1 - Y_i^{-1} Y_j} = \frac{1 - t Y_i^{-1} Y_j}{t^{-1}(t - Y_i^{-1} Y_j)} = \frac{1 - t Y_i^{-1} Y_j}{1 - t^{-1} Y_i^{-1} Y_j}$$

Using that $w_{\lambda+\rho} = w_0$ and $w_0(i) = n - i + 1$ then

$$\text{ev}_{\lambda+\rho}^t \left(t^{\ell(w_0)} \frac{c_{w_0}^{Y^{-1}}}{c_{w_0}^Y} \right) = \text{ev}_{\lambda+\rho}^t \left(\prod_{i < j} \frac{1 - t Y_i^{-1} Y_j}{1 - t^{-1} Y_i^{-1} Y_j} \right) = \prod_{i < j} \frac{1 - t q^{\lambda_i - \lambda_j + j - i} t^{j-i}}{1 - t^{-1} q^{\lambda_i - \lambda_j + j - i} t^{j-i}}. \quad (7.1)$$

In view of Proposition 7.3(c), this proves the formula for $\frac{(P_\lambda(q, qt), P_\lambda(q, qt))_{q, qt}}{(P_{\lambda+\rho}(q, t), P_{\lambda+\rho}(q, t))_{q, t}}$. \square

7.3 Formulas for norms and the constant term

Theorem 7.5. *Let $\lambda \in (\mathbb{Z}^n)^+$ and let $k \in \mathbb{Z}_{\geq 0}$. Then*

$$(P_\lambda(q, q^k), P_\lambda(q, q^k))_{q, q^k} = W_0(q^k) \prod_{r=1}^{k-1} \text{ev}_{\lambda+(k-r)\rho}^{q^r} \left(q^{r \cdot \ell(w_0)} \frac{c_{w_0}^{Y^{-1}}(q^r)}{c_{w_0}^Y(q^r)} \right).$$

Alternatively,

$$(P_\lambda(q, q^k), P_\lambda(q, q^k))_{q, q^k} = W_0(q^k) \cdot \prod_{i < j} \prod_{r=1}^{k-1} \frac{1 - q^{\lambda_i - \lambda_j + r} q^{k(j-i)}}{1 - q^{\lambda_i - \lambda_j - r} q^{k(j-i)}}.$$

Proof. The proof is by induction on k . The base case is $k = 0$, where

$$(P_\lambda(q, q^0), P_\lambda(q, q^0))_{q, q^0} = (m_\lambda, m_\lambda)_{q, 1} = W_\lambda(1).$$

Using Proposition 7.3(c), the first step of the induction is

$$\begin{aligned} (P_\lambda(q, q), P_\lambda(q, q))_{q, q} &= \frac{(P_\lambda(q, q), P_\lambda(q, q))_{q, q}}{(P_{\lambda+\rho}(q, q^0), P_{\lambda+\rho}(q, q^0))_{q, q^0}} (P_{\lambda+\rho}(q, q^0), P_{\lambda+\rho}(q, q^0))_{q, q^0} \\ &= \frac{W_0(q)}{W_0(1)} \text{ev}_{\lambda+\rho}^1 \left(1^{\ell(w_0)} \frac{c_{w_0}^{Y^{-1}}(q)}{c_{w_0}^Y(q)} \right) (m_{\lambda+\rho}, m_{\lambda+\rho})_{q, 1} = \frac{W_0(q)}{W_0(1)} \cdot 1 \cdot W_0(1) = W_0(q), \end{aligned}$$

and the general induction step is

$$\begin{aligned} (P_\lambda(q, q^k), P_\lambda(q, q^k))_{q, q^k} &= \frac{(P_\lambda(q, q^k), P_\lambda(q, q^k))_{q, q^k}}{(P_{\lambda+\rho}(q, q^{k-1}), P_{\lambda+\rho}(q, q^{k-1}))_{q, q^{k-1}}} (P_{\lambda+\rho}(q, q^{k-1}), P_{\lambda+\rho}(q, q^{k-1}))_{q, q^{k-1}} \\ &= \frac{W_0(q^k)}{W_0(q^{k-1})} \text{ev}_{\lambda+\rho}^{q^{k-1}} \left(q^{(k-1)\ell(w_0)} \frac{c_{w_0}^{Y^{-1}}(q)}{c_{w_0}^Y(q)} \right) W_0(q^{k-1}) \prod_{r=1}^{k-2} \text{ev}_{\lambda+(k-1-r)\rho}^{q^r} \left(q^{r \cdot \ell(w_0)} \frac{c_{w_0}^{Y^{-1}}}{c_{w_0}^Y} \right) \\ &= W_0(q^k) \prod_{r=1}^{k-1} \text{ev}_{\lambda+(k-r)\rho}^{q^r} \left(q^{r \cdot \ell(w_0)} \frac{c_{w_0}^{Y^{-1}}}{c_{w_0}^Y} \right). \end{aligned}$$

In a similar manner to the computation in (7.1),

$$\text{ev}_{\lambda+(k-r)\rho}^{q^r} \left(q^{r \cdot \ell(w_0)} \frac{c_{w_0}^{Y^{-1}}}{c_{w_0}^Y} \right) = \prod_{i < j} \frac{1 - q^r q^{\lambda_i - \lambda_j + (k-r)(j-i)} q^{r(j-i)}}{1 - q^{-r} q^{\lambda_i - \lambda_j + (k-r)(j-i)} q^{r(j-i)}} = \prod_{i < j} \frac{1 - q^{\lambda_i - \lambda_j + r} q^{k(j-i)}}{1 - q^{\lambda_i - \lambda_j - r} q^{k(j-i)}}.$$

\square

Specializing Proposition 7.5 at $\lambda = 0$ provides a proof of Macdonald's constant term conjectures. This proof follows the same framework as the proof exposited in [Mac03, §5.8].

Proposition 7.6. *Let $k \in \mathbb{Z}_{\geq 0}$ and let $t = q^k$. Then*

$$(1, 1)_{q, q^k} = \text{ct} \left(\frac{1}{\Delta_{\infty}^X \Delta_0^X \Delta_{\infty}^{X^{-1}}} \right) = \prod_{i=2}^n \begin{bmatrix} ik \\ k \end{bmatrix}$$

and

$$\frac{1}{W_0(q^k)} (1, 1)_{q, q^k} = \text{ct} \left(\frac{1}{\Delta_{\infty}^X \Delta_0^X \Delta_0^{X^{-1}} \Delta_{\infty}^{X^{-1}}} \right) = \prod_{h=2}^{n-1} \begin{bmatrix} hk - 1 \\ k - 1 \end{bmatrix}.$$

Proof. The first equality is the definition of the inner product $(1, 1)_{q, q^k}$. Then

$$\begin{aligned} \prod_{i < j} \prod_{r=1}^{k-1} \frac{1 - q^r q^{k(j-i)}}{1 - q^{-r} q^{k(j-i)}} &= \prod_{h=1}^{n-1} \prod_{\substack{i < j \\ j-i=h}} \frac{(1 - q^{kh+1}) \dots (1 - q^{kh+k-1})}{(1 - q^{kh-1}) \dots (1 - q^{kh-(k-1)})} \\ &= \prod_{h=1}^{n-1} \frac{((1 - q^{kh+1}) \dots (1 - q^{kh+(k-1)}))^{n-h}}{((1 - q^{k(h-1)+1}) \dots (1 - q^{k(h-1)+(k-1)}))^{n-h}} \\ &= \prod_{h=1}^{n-1} \frac{((1 - q^{kh+1}) \dots (1 - q^{kh+(k-1)}))^{n-h}}{((1 - q^{k(h-1)+1}) \dots (1 - q^{k(h-1)+(k-1)}))^{n-(h-1)}} \cdot (1 - q^{k(h-1)+1}) \dots (1 - q^{k(h-1)+(k-1)}) \\ &= \frac{((1 - q^{k(n-1)+1}) \dots (1 - q^{k(n-1)+(k-1)}))^{n-(n-1)}}{((1 - q) \dots (1 - q^{k-1}))^{n-(1-1)}} \left(\prod_{h=1}^{n-1} (1 - q^{k(h-1)+1}) \dots (1 - q^{k(h-1)+(k-1)}) \right) \\ &= \left(\frac{(1 - q^{kn-(k-1)}) \dots (1 - q^{kn-1})}{(q; q)_{k-1}} \right) \prod_{h=1}^{n-1} \frac{(1 - q^{kh-(k-1)}) \dots (1 - q^{kh-1})}{(q; q)_{k-1}} \\ &= \prod_{h=2}^{n-1} \begin{bmatrix} hk - 1 \\ k - 1 \end{bmatrix}. \end{aligned}$$

Then, using $1 = P_0(q, q^k)$ and Theorem 7.5,

$$(1, 1)_{q, q^k} = W_0(q^k) \prod_{i=2}^n \begin{bmatrix} ik - 1 \\ k - 1 \end{bmatrix} = \left(\prod_{i=2}^n \frac{1 - q^{ik}}{1 - q^k} \right) \prod_{i=2}^n \begin{bmatrix} ik - 1 \\ k - 1 \end{bmatrix} = \prod_{i=2}^n \begin{bmatrix} ik \\ k \end{bmatrix}.$$

□

Remark 7.7. Converting to general q and t . Define

$$\begin{aligned} \Delta_{\infty}^Y(t) &= \prod_{1 \leq i < j \leq n} \prod_{r=1}^{\infty} \frac{1 - tq^r Y_i Y_j^{-1}}{1 - q^r Y_i Y_j^{-1}}, & \Delta_{\infty}^{Y^{-1}}(t^{-1}) &= \prod_{1 \leq i < j \leq n} \prod_{r=1}^{\infty} \frac{1 - t^{-1} q^r Y_i^{-1} Y_j}{1 - q^r Y_i^{-1} Y_j}, \\ \Delta_0^Y(t) &= \prod_{1 \leq i < j \leq n} \frac{1 - t Y_i Y_j^{-1}}{1 - Y_i Y_j^{-1}}. \end{aligned}$$

Using the notation $(z; q)_{\infty} = (1 - z)(1 - qz)(1 - q^2z) \dots$, then

$$\Delta_{\infty}^Y(t) = \prod_{i < j} \frac{(qt Y_i Y_j^{-1}; \infty)}{(q Y_i Y_j^{-1}; \infty)} \quad \text{and} \quad \Delta_{\infty}^{Y^{-1}}(t^{-1}) = \prod_{i < j} \frac{(qt^{-1} Y_i^{-1} Y_j; \infty)}{(q Y_i^{-1} Y_j; \infty)}.$$

Define an evaluation homomorphism $\text{ev}_{q^\lambda t^\rho} : \mathbb{C}[Y] \rightarrow \mathbb{C}$ by

$$\text{ev}_{q^\lambda t^\rho}(Y_i) = q^{\lambda_i} t^{n-j}, \quad \text{so that} \quad \text{ev}_{q^\lambda t^\rho}(Y_i^{-1} Y_j) = q^{\lambda_i - \lambda_j} t^{j-i}.$$

Let $i, j \in \{1, \dots, n\}$ with $i < j$ and let $t = q^k$. Using

$$(x; q)_\infty = (1-x)(qx; q)_\infty, \quad (x; q)_k = \frac{(x; q)_\infty}{(q^k x; q)_\infty},$$

and

$$(x; q^{-1})_k = (q^{-(k-1)}x; q)_k = (q^{-k}qx; q)_k = (t^{-1}qx; q),$$

then

$$\begin{aligned} \prod_{r=1}^{k-1} \frac{1-xq^r}{1-xq^{-r}} &= \prod_{r=0}^{k-1} \frac{1-xq^r}{1-xq^{-r}} = \frac{(x; q)_k}{(x; q^{-1})_k} = \frac{(x; q)_k}{q^{-k}qx; q)_k} = \frac{(x; q)_\infty}{(q^k x; q)_\infty} \frac{(q^k q^{-k}qx; q)_\infty}{(q^{-k}qx; q)_\infty} \\ &= \frac{(x; q)_\infty}{(q^k x; q)_\infty} \frac{(qx; q)_\infty}{(q^{-k}qx; q)_\infty} = \frac{(qx; q)_\infty}{(q^k qx; q)_\infty} \frac{(1-x)}{(1-q^k x)} \frac{(qx; q)_\infty}{(q^{-k}qx; q)_\infty} \\ &= \frac{(x; q)_\infty}{(q^k x; q)_\infty} \frac{(qx; q)_\infty}{(q^{-k}qx; q)_\infty} = \frac{(qx; q)_\infty}{(tqx; q)_\infty} \frac{(1-x)}{(1-tx)} \frac{(qx; q)_\infty}{(t^{-1}qx; q)_\infty}. \end{aligned}$$

Then using $\text{ev}_{q^\lambda t^\rho}(Y_i Y_j^{-1}) = q^{\lambda_i - \lambda_j} t^{j-i} = \text{ev}_{q^{-\lambda} t^{-\rho}}(Y_i^{-1} Y_j)$, gives

$$\begin{aligned} \prod_{i < j} \prod_{r=1}^{k-1} \frac{1-q^{\lambda_i - \lambda_j + r} t^{j-i}}{1-q^{\lambda_i - \lambda_j - r} t^{j-i}} &= \prod_{i < j} \frac{(qq^{\lambda_i - \lambda_j} t^{j-i}; q)_\infty}{(tqq^{\lambda_i - \lambda_j} t^{j-i}; q)_\infty} \frac{(1-q^{\lambda_i - \lambda_j} t^{j-i})}{(1-tq^{\lambda_i - \lambda_j} t^{j-i})} \frac{(qq^{\lambda_i - \lambda_j} t^{j-i}; q)_\infty}{(t^{-1}qq^{\lambda_i - \lambda_j} t^{j-i}; q)_\infty} \\ &= \text{ev}_{q^\lambda t^\rho} \left(\frac{1}{\Delta_\infty^Y(t)} \right) \text{ev}_{q^\lambda t^\rho} \left(\frac{1}{\Delta_0^Y(t)} \right) \text{ev}_{q^{-\lambda} t^{-\rho}} \left(\frac{1}{\Delta_\infty^{Y^{-1}}(t^{-1})} \right). \end{aligned}$$

Thus the last statement of Proposition 7.5 can be written in the form

$$(P_\lambda(q, t), P_\lambda(q, t))_{q, t} = W_0(t) \cdot \text{ev}_{q^\lambda t^\rho} \left(\frac{1}{\Delta_\infty^Y(t)} \right) \text{ev}_{q^\lambda t^\rho} \left(\frac{1}{\Delta_0^Y(t)} \right) \text{ev}_{q^{-\lambda} t^{-\rho}} \left(\frac{1}{\Delta_\infty^{Y^{-1}}(t^{-1})} \right),$$

a formula which, since it holds for all $t = q^k$ with $k \in \mathbb{Z}_{>0}$, holds for general t . □

7.4 The symmetric inner product

Define involutions

$$\begin{aligned} \bar{\cdot} : \mathbb{C}[X] &\rightarrow \mathbb{C}[X] & \text{by} & \quad \bar{f}(x_1, \dots, x_n; q, t) = f(x_1^{-1}, \dots, x_n^{-1}; q^{-1}, t^{-1}), \\ \sigma : \mathbb{C}[X] &\rightarrow \mathbb{C}[X] & \text{by} & \quad f^\sigma(x_1, \dots, x_n; q, t) = f(x_1^{-1}, \dots, x_n^{-1}; q, t), \\ t : \mathbb{C}[X] &\rightarrow \mathbb{C}[X] & \text{by} & \quad f^t(x_1, \dots, x_n; q, t) = f(x_1, \dots, x_n; q^{-1}, t^{-1}). \end{aligned}$$

Define two scalar products $(\ , \) : \mathbb{C}[X] \times \mathbb{C}[X] \rightarrow \mathbb{C}(q, t)$ and $\langle \ , \ \rangle : \mathbb{C}[X] \times \mathbb{C}[X] \rightarrow \mathbb{C}(q, t)$ by

$$(f_1, f_2)_{q, t} = \text{ct} \left(\frac{f_1 \bar{f}_2}{\Delta_\infty^X \Delta_0^X \Delta_\infty^{X^{-1}}} \right) \quad \text{and} \quad \langle f_1, f_2 \rangle_{q, t} = \frac{1}{|W_0|} \text{ct} \left(\frac{f_1 f_2^\sigma}{\Delta_\infty^X \Delta_0^X \Delta_\infty^{X^{-1}} \Delta_\infty^{X^{-1}}} \right).$$

The following result provides a comparison of $(\ , \)_{q, t}$ and $\langle \ , \ \rangle_{q, t}$ as inner products on symmetric polynomials.

Proposition 7.8. *Let $f, g \in \mathbb{C}[X]^{S_n}$ so that f and g are symmetric polynomials. Then*

$$\langle f, g \rangle_{q,t} = \frac{1}{W_0(t)} (f, g^t)_{q,t}.$$

Proof. Let $f, g \in \mathbb{C}[X]^{S_n}$. Equation (4.7) says

$$W_0(t) = \sum_{w \in W_0} w(t^{\frac{1}{2}\ell(w)} c_{w_0}^{X^{-1}}), \quad \text{and using} \quad \Delta_0^{X^{-1}} = t^{\frac{1}{2}\ell(w_0)} c_{w_0}^{X^{-1}},$$

gives

$$\begin{aligned} \langle f, g \rangle_{q,t} &= \frac{1}{|W_0|} \text{ct} \left(\frac{f g^\sigma}{\Delta_\infty^X \Delta_0^X \Delta_0^{X^{-1}} \Delta_\infty^{X^{-1}}} \right) = \frac{1}{W_0(t) |W_0|} \text{ct} \left(\frac{f \bar{g}^t}{\Delta_\infty^X \Delta_0^X \Delta_0^{X^{-1}} \Delta_\infty^{X^{-1}}} W_0(t) \right) \\ &= \frac{1}{W_0(t) |W_0|} \text{ct} \left(\frac{f \bar{g}^t}{\Delta_\infty^X \Delta_0^X \Delta_0^{X^{-1}} \Delta_\infty^{X^{-1}}} \left(\sum_{w \in W_0} w(t^{\frac{1}{2}\ell(w)} c_{w_0}^{X^{-1}}) \right) \right) \\ &= \frac{1}{W_0(t) |W_0|} \text{ct} \left(\sum_{w \in W_0} w \left(\frac{f \bar{g}^t}{\Delta_\infty^X \Delta_0^X \Delta_0^{X^{-1}} \Delta_\infty^{X^{-1}}} t^{\frac{1}{2}\ell(w)} c_{w_0}^{X^{-1}} \right) \right) \\ &= \frac{1}{W_0(t) |W_0|} \text{ct} \left(\sum_{w \in W_0} w \left(\frac{f \bar{g}^t}{\Delta_\infty^X \Delta_0^X \Delta_\infty^{X^{-1}}} \right) \right) = \frac{1}{W_0(t)} \text{ct} \left(\frac{f \bar{g}^t}{\Delta_\infty^X \Delta_0^X \Delta_\infty^{X^{-1}}} \right) = \frac{1}{W_0(t)} (f, g^t)_{q,t}. \end{aligned}$$

□

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