Limits and topologies

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December 12, 2018

Abstract

This article provides the solution to a "Math is broken" problem from the undergraduate mathematics curriculum as I experienced it when I was a student. The paradox is that continuous functions are supposed to be the morphisms in the category of topological spaces and are defined via limits but that limits are not defined in the context of topological spaces.

0 Introduction

0.1 Memories

It was in the second semester of my undergraduate education at MIT that I first met pure mathematics, open and closed sets, the book "Baby Rudin" [BRu], and Warren Ambrose. The course was '18.100 Mathematical Analysis'. Warren Ambrose had a great effect on me. Somehow we had a one-to-one conversation where we both confessed that our true love was music and that we were doing math only as a backup. At the time, I was still far from being a professional mathematician and he was a famous geometer nearing the end of his career and his life (it was 1984 and he died in 1995 at the age of 81). He told me that he had been a jazz trumpet player but an accident had made him unable to play properly and so he had pursued mathematics for a profession. His exams (two midterm exams and a final) were all 24 hour open-book closed-friend take-home tests: 10 questions, true or false, graded 1 if correct, -1 if incorrect, and 0 if not answered. The average score across the class (about 20 students) was often around 0. But this mechanism taught you better than any other what proof meant – if you were unable to provide a proof you believed in then you risked getting -1 for that question. The questions were always very interesting. I carried these questions around for years until sometime in 2012 when I accidentally left them in a classroom and, when I came back to find them an hour later, they were gone.

0.2 Math is not broken: continuity and limits

In my first Topology course we covered basic point set topology, open and closed sets *again*; and of course we discussed continuous functions. There was no mention of limits or convergence in this course. Something was wrong. I had always been taught to believe that continuity had something to do with limits. By that time I was already doing plenty of tutoring and teaching

AMS Subject Classifications: Primary 54A20??????; Secondary ??????.

of calculus classes myself and I knew that one could define continuity, at least in a calculus class, by limits. In the Topology course there were no limits and I began to understand that the primary role of continuous functions was for comparing topological spaces. There was a major conflict here: limits or topological spaces? Mathematics was broken, and I very seriously considered quitting and becoming a violin teacher.

Many years later I delved into Bourbaki, learned about filters, and began to understand that there is a definition of limits for topological spaces and that the world of continuous functions is, in fact, not broken. Once you get the definitions right, the definitions are easy (and don't need to be said with filters), the theorem is (not surprisingly)

Theorem. *Let X and Y be topological spaces and let $a \in X$. A function*

$$f\colon X \to Y \text{ is continuous at } a \qquad \text{if and only if} \qquad \lim_{x\to a} f(x) = f(a).$$

and the proof is not difficult.

0.3 Notes and references for this exposition

This general material is found in some form in most textbooks on introductory topology or mathematical analysis such as [Wil] or [BRu]. One comprehensive reference is Bourbaki – see [Bou, Ch. I §1 nos. 1,2,4] for open sets, closed sets and neighborhoods and [Bou, Ch. I §2 no. 1] for continuous functions. The formulation of limits not depending on metric space structure is motivated by [Bou, Ch. I, §6 and 7].

I have made a number of improvements to the standard exposition which I hope will become more widely used:

(a) the formulation of the definition of *neighborhood*;
(b) the presentation of the interior and closure as universal properties;
(c) the formulation of the relation between close points and closure, and between limit points and closure;
(d) the recognition of *sequences as functions* and organisation of the different kinds of limits to highlight the differences and similarities;
(e) the central role of limits in the context of topological spaces;
(f) the use of the term *strict metric space*;
(g) the formulation of the definition of the *metric space topology*;
(h) the *accuracy set* $\mathbb{E}$, and the explicit form of first countability in metric spaces and its role in motivating the precise formulation of limits;
(i) the set $\mathbb{Z}_{\geq \ell}$;
(j) The use of the notations $\mathbb{Z}_{>0}$, $\mathbb{R}_{>0}$, $\mathbb{R}_{[a,b)}$ etc.;
(k) Proof sketches and proofs without skipping steps;
(l) Math, English and Cartoons.

Further explanation:

(a) Although it is traditional to define topological spaces via axioms for **open sets**, there are equivalent (and useful!) definitions of topological spaces by axioms for the **closed sets**, and via axioms for **neighborhoods**. The definition of **neighborhood** varies from author to author (for example, Bourbaki [Bou, Ch. I §2 no. 2] and Munkres [Mun, Ch. 2 §17] use different definitions). I have not chosen to define filters and develop their theory in this exposition but, to make the theory of limits in topological space go through smoothly, it is *crucial* that the neighborhoods of a point form a filter. Therefore, even when filters are not mentioned, it is important that the definition of neighborhood is formulated correctly. This definition of neighborhood is also the right one for an elegant discussion of **close points**, **interior points**, **closures** and **interiors**.

(b) The definition of the **interior** of A is the mathematically precise formulation of "A° is the largest open set contained in A". The definition of the **closure** of A is the mathematically precise formulation of "$\overline{A}$ is the smallest closed set containing A". My students are enticed by the mystery of *categories* and so I find it very productive to explain that these are simple and fundamental examples of *universal properties*. Of course, as with any universal property, it is important to prove that the *universal object* exists (in these cases, the interior and the closure) and is unique. Proposition 2.1, besides being extremely handy for many upcoming proofs, achieves this goal.

(c) The colloquial similarities between the phrases "close point" and "limit point" are alleviated by careful definition of the mathematical terms, careful usage, and Proposition 2.1 and Theorem 3.3, which make precise the relationship between close points and closed sets in topological spaces and between limit points and closed sets in metric spaces, respectively.

(d) An important and useful point of view on topological spaces is to view the topological spaces as a category with the **continuous functions** as morphisms. My students enjoy learning that this is one of their first examples of a category and that hook seems to make them more willing to swallow the axiomatics of the definition of a topology. From the category point of view the notion of *topological space* and the notion of *continuous function* are "equivalent data". Theorems 3.1 and 3.3 tightly connect limits of sequences to closure and continuity and, hence, to the core structure of topological spaces.

(e) When I teach this subject I find it useful to stress that there are *different* limits: limits of sequences, limits of functions, and limits of functions with respect to punctured neighborhoods. The similiarity between the definitions (when sequences are presented as functions from $\mathbb{Z}_{>0}$ to X) provide a good framework for the students (and, myself, as the teacher) to parse carefully the different parts of the definition and confirm how well this all works in the context of topological spaces.

(e) The definition of the metric space topology is carefully chosen to achieve several goals at once: to be the most useful one for doing proofs with metric spaces; to be parallel to the definition of the uniform space topology on a uniform space (see [Ra1]); to inherently incorporate the fact that the open balls form a basis of the topology without having to introduce and study the notion of base of a topology. Proposition 1.1 provides the equivalence between this definition and another commonly used definition where the open sets in the metric space topology are defined as unions of open balls.

(f) I choose to replace the terms *pseudometric space* and *metric space*, by *metric space* and *strict metric space*, respectively, so that the term *metric space* allows infinite distances and allows distinct points to have distance 0. The advantage is that then every uniform space is a metric space (up to supremums of uniformities) and the metric spaces with uniformly continuous functions form a category with good properties (products, limits, colimits, etc). This very slight loosening of definitions allows for the exploitation of powerful category theoretic tools and structure and elucidates some of the quirks of *strict metric spaces* (see [Ra1]).

(g) A primary property of metric spaces is that they are first countable (have **countably generated neighborhood filters**). The introduction of the **accuracy set** $\mathbb{E}$ makes this property of metric spaces evident. The accuracy set $\mathbb{E}$ is countable and defining the metric space topology by using open balls $B_\epsilon(a)$ indexed by the countable set $\mathbb{E}$ displays metric spaces, by their very definition, as first countable topological spaces. The role of this property becomes vividly evident in the metric space results Theorems 3.3 and 3.4, which have the first countable space generalizations given in Theorems 3.5 and 3.6 (and exactly the same proof).

(h) The accuracy set $\mathbb{E}$ has a powerful advantage in the classroom. In class, when I introduce the construction "if $\epsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that if $d(x,a) < \delta$ then $d(f(x), y) < \epsilon$", I also introduce a scenario of a "limit point business" where the clients come in with their machine (the function f) and ask us to calibrate their dials with appropriate precision (the number δ) to achieve a desired number of decimal places of accuracy (the number ϵ) for the output of the machine (the output $f(x)$). My business looks at the client's machine and (for a fee proportional to the desired number of decimal places accuracy) carefully positions the dials to achieve the right output up to the desired accuracy. Using this conceptual model the students are able to parse the large number of quantifiers in the logical statement, and setting up the definitions using the set $\mathbb{E}$ is helpful both for motivation and for execution of the proofs involving limits.

Historically, the mathematical community became infatuated by limits, partly because of the many applications of "calculus" and the ideas of infinitesimals, but also partly because they weren't very well understood. The focus on the epsilon-delta definition of limits has advantages and disadvantages, and the formulation of the definition of limits in topological spaces provides healthy insights, even in an elementary course.

(i) The accuracy set $\mathbb{E}$ has another important role, though it is not visible in this paper. The paper [Ra1] explains that the theory of metric spaces is completely parallel to the theory of uniform spaces and that the accuracy set $\mathbb{E}$ is naturally replaced by the set $\mathcal{E}$ of entourages (or fatdiagonals) in the theory of uniform spaces.

(j) In the definition of the limit of a sequence in a metric space, "if $\epsilon \in \mathbb{E}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $d(x_n, y) < \epsilon$, the set $\mathbb{Z}_{\geq \ell}$ is the inherent appearance of the "Fréchet filter" necessary for the definition of the limit of a sequence (see [Bou, Ch I §6 no. 1 Example 3]). It is pleasant that this fundamental construct has such a simple and unintrusive way of embedding itself into the structure, without having to introduce any tangential remarks about the theory of filters. For an exposition of limits in the context of filters and the relations between filters, nets and sequences see [Ra3].

(j) I have found that the notations $\mathbb{Z}_{>0}$, $\mathbb{Q}_{>0}$, $x \in \mathbb{R}_{>0}$, $\mathbb{Z}_{[1,n]}$, $x \in \mathbb{R}_{(a,b]}$ are usually self evident even to students seeing this material for the first time, and these notations alleviate some of the confusions that arise when writing $\mathbb{N}$, $\mathbb{P}$, $x > 0$, $1 \leq x \leq n$ and $a < x \leq b$.

(k) On the whole our mathematics textbooks provide very few proofs that don't skip steps. A consequence is that my students very often are uncertain what a proof without skipped steps would be. There are both advantages and disadvantages to each style, and I have found it most useful to have both styles simultaneously available. For this reason I have included brief proof skteches in the main text and complete proofs without skipping steps in the Appendix.

(l) I have found it very helpful in my teaching, to loudly recognize that the language of Mathematics is a language of its own different from English (although professional mathematicians usually meld the two intricately in actual conversation). This language of Mathematics is an important tool of the trade as it is what enables us to achieve the standards of proof and correctness that we need. On the other hand the language of English is different (as my wife tells me with a glare), and also important to understanding. Finally, as our handheld devices have made so clear in the 21st century, the power of image is very important to conceptual facility and understanding by humans. Thus, as I tell my students, the **Math** and the **English** and the **Cartoon** are all important at every step and we should strive to present mathematics in all three languages in tandem, but still beautifully and elegantly. Historically, cartoons were discouraged from printed mathematics as the cost of preparing, printing and typsetting figures and pictures was exhorbitant, but this is less true with modern technology and so we can strive towards a new presentation style with more cartoons (though it is still important to stress that cartoons are just cartoons and cannot replace accurate, properly formulated definitions and theorems in the careful language of Mathematics). I have made a concerted effort to include all three languages in this article and, at the same time, keep their roles in clear distinction.

0.4 Acknowledgments

First I would like to thank Hyam Rubinstein, my wonderful colleague at University of Melbourne, who passed on the course "Metric and Hilbert spaces" to me and generously provided me with his lecture notes and materials [Rub]. My point of view has been deeply shaped by his important work in developing this course and his own notes for it. Second I thank my students, who have, mostly patiently, dealt with my quirks and misunderstandings as I have slowly progressed to better understanding of this material. Third I thank Gil Azaria for typesetting of mounds of handwritten notes and pictures in mathml and svg as I worked through version after version and tweak after tweak. I am very grateful to Miaohan Long for doing very careful and helpful proofreading.

1 Spaces

The point of this section is to introduce topological spaces and metric spaces and to explain how to make a metric space into a topological space.

1.1 Topological spaces

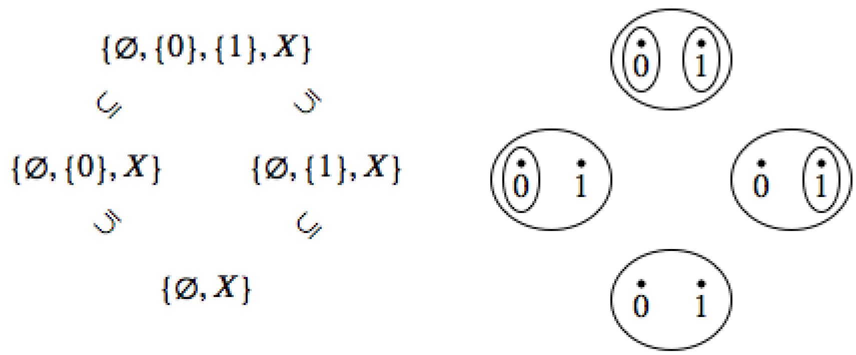
A *topological space* is a set X with a specification of the *open* subsets of X where it is required that

(a) $\emptyset$ is open in X and X is open in X,
(b) Unions of open sets in X are open in X,
(c) Finite intersections of open sets in X are open in X.

In other words, a *topology* on X is a set $\mathcal{T}$ of subsets of X such that

(a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
(b) If $\mathcal{S} \subseteq \mathcal{T}$ then $\left(\bigcup_{U\in\mathcal{S}} U\right) \in \mathcal{T}$,
(c) If $\ell \in \mathbb{Z}_{>0}$ and $U_1, U_2, \ldots, U_\ell \in \mathcal{T}$ then $U_1 \cap U_2 \cap \cdots \cap U_\ell \in \mathcal{T}$.

A *topological space* $(X, \mathcal{T})$ is a set X with a topology $\mathcal{T}$ on X. An *open set in* X is a set in $\mathcal{T}$.

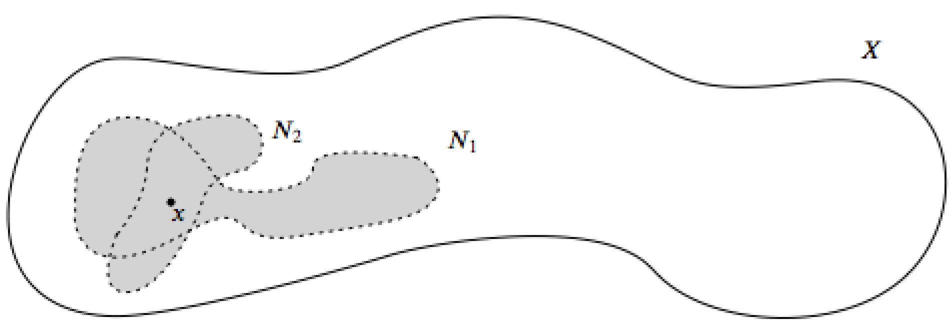


The four possible topologies on $X = \{0, 1\}$.

In a topological space, perhaps even more important than the open sets are the neighborhoods. Let $(X, \mathcal{T})$ be a topological space. Let $x \in X$. The *neighborhood filter of* x is

$$\mathcal{N}(x) = \{N \subseteq X \mid \text{there exists } U \in \mathcal{T} \text{ such that } x \in U \text{ and } U \subseteq N\}. \tag{1.1}$$

A *neighborhood of* x is a set in $\mathcal{N}(x)$.



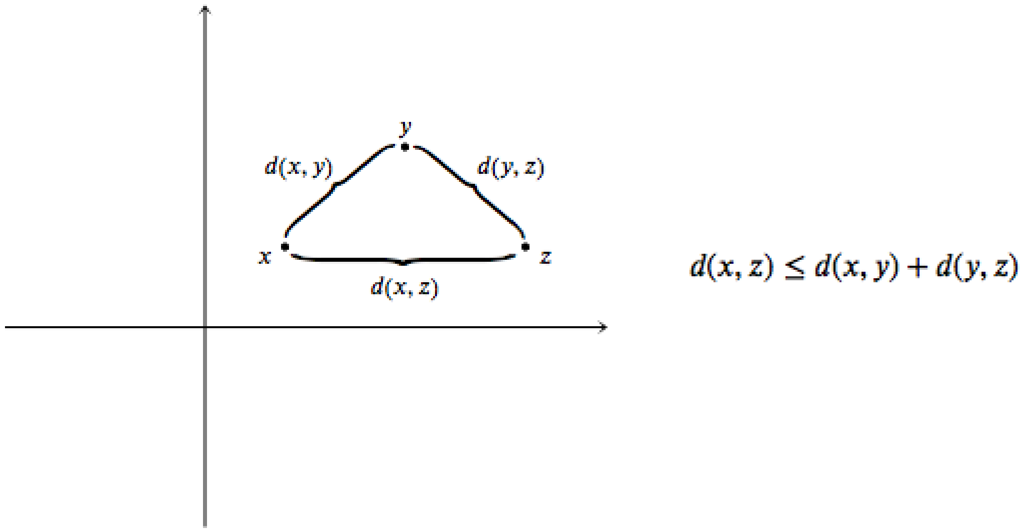
Neighborhoods of x.

1.2 Metric spaces

A *strict metric space* is a set X with a function $d\colon X \times X \to \mathbb{R}_{\geq 0}$ such that

(a) (diagonal condition) If $x \in X$ then $d(x, x) = 0$,
(b) (diagonal condition) If $x, y \in X$ and $d(x, y) = 0$ then $x = y$,
(c) (symmetry condition) If $x, y \in X$ then $d(x, y) = d(y, x)$,
(d) (the triangle inequality) If $x, y, z \in X$ then $d(x, y) \leq d(x, z) + d(z, y)$.

Conditions (a) and (b) are equivalent to $d^{-1}(0) = \Delta(X)$, where the *diagonal of* X is $\Delta(X) = \{(x, x) \mid x \in X\}$ and $d^{-1}(0) = \{(x, y) \in X \times X \mid d(x, y) = 0\}$.



Distances between points in the metric space $\mathbb{R}^2$.

1.3 Making metric spaces into topological spaces

Let $\mathbb{E} = \{10^{-k} \mid k \in \mathbb{Z}_{>0}\}$. The set $\mathbb{E}$ is the *accuracy set*. Specifying an element of $\mathbb{E}$ specifies the desired number of decimal places of accuracy.

Let (X, d) be a strict metric space. Let $x \in X$ and let $\epsilon \in \mathbb{E}$. The *open ball of radius ϵ at x* is

$$B_\epsilon(x) = \{y \in X \mid d(x, y) < \epsilon\}.$$

The *neighborhood filter of an element $x \in X$* is

$$\mathcal{N}(x) = \{N \subseteq X \mid \text{there exists } \epsilon \in \mathbb{E} \text{ such that } B_\epsilon(x) \subseteq N\}.$$

The *metric space topology on X* is

$$\mathcal{T} = \{U \subseteq X \mid \text{if } x \in U \text{ then there exists } \epsilon \in \mathbb{E} \text{ such that } B_\epsilon(x) \subseteq U\}.$$

The following characterization of the metric space topology is frequently used as the definition of the metric space topology.

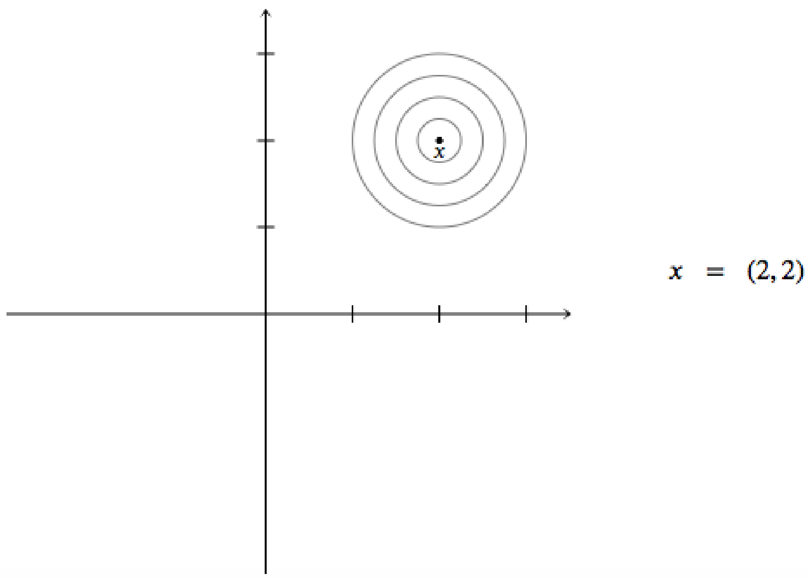
Proposition 1.1. *Let (X, d) be a strict metric space.*

$$\textit{Let} \quad \mathbb{E} = \{10^{-k} \mid k \in \mathbb{Z}_{>0}\} \quad \textit{and let} \quad \mathcal{B} = \{B_\epsilon(x) \mid \epsilon \in \mathbb{E} \textit{ and } x \in X\}.$$

Let $\mathcal{T}$ be the metric space topology on X. Let $U \subseteq X$. Then $U \in \mathcal{T}$ if and only if

$$\textit{there exists } \mathcal{S} \subseteq \mathcal{B} \textit{ such that} \quad U = \bigcup_{B \in \mathcal{S}} B.$$

Proof. (Sketch) If $U = \bigcup_{B \in \mathcal{S}} B$ and $x \in U$ then there exists $B_\delta(y) \in \mathcal{S}$ with $x \in B_\delta(y)$. Letting $\epsilon < \delta - d(x, y)$ then $B_\epsilon(x) \subseteq U$. So $U \in \mathcal{T}$. □



Generators of the neighborhood filter of $x = (2, 2)$ in the metric space $\mathbb{R}^2$.

2 Continuous functions, interiors and closures

2.1 Interiors and closures

Let $(X, \mathcal{T})$ be a topological space. An *open set in* X is a subset U of X such that $U \in \mathcal{T}$. A *closed set in* X is a subset C of X such that the complement of C is an open set in X, i.e.

$$C \text{ is closed if} \qquad X - C = \{x \in X \mid x \notin C\} \quad \text{is an open set in } X.$$

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$.
The *interior* of A is the subset A° of X such that

(a) A° is open in X and $A^\circ \subseteq A$,
(b) If U is open X and $U \subseteq A$ then $U \subseteq A^\circ$.

The *closure of* A is the subset $\overline{A}$ of X such that

(a) $\overline{A}$ is closed in X and $\overline{A} \supseteq A$,
(b) If C is closed in X and $C \supseteq A$ then $C \supseteq \overline{A}$.

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$.
An *interior point of* A is a element $x \in X$ such that

$$\text{there exists } N \in \mathcal{N}(x) \text{ such that } N \subseteq A.$$

A *close point* to A is an element $x \in X$ such that

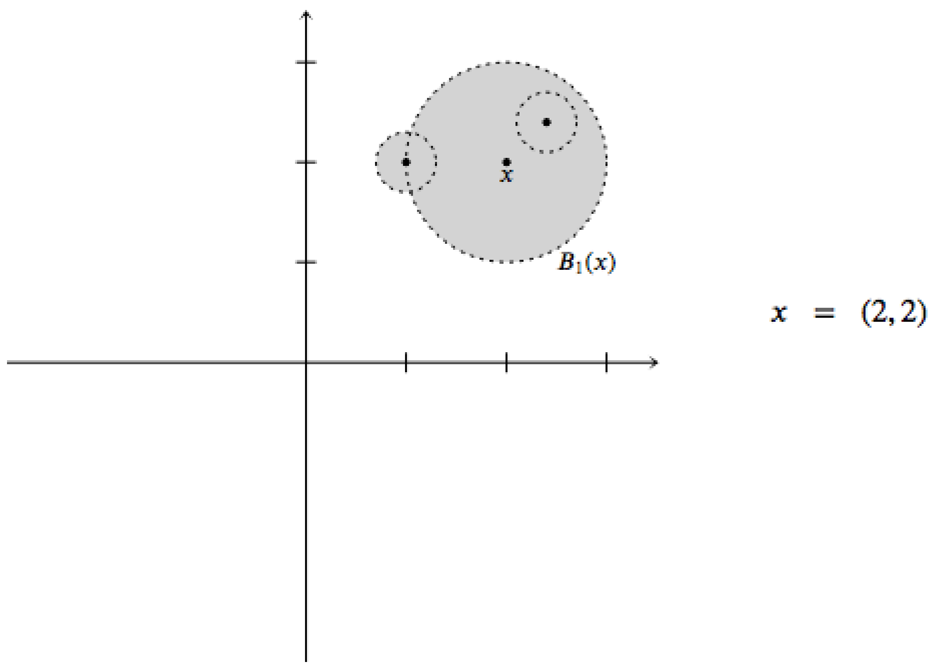
$$\text{if } N \in \mathcal{N}(x) \text{ then } N \cap A \neq \emptyset.$$

Proposition 2.1. *Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$.*

(a) *The interior of A is the set of interior points of A.*

(b) *The closure of A is the set of close points of A.*

Proof. (Sketch) For part (a): Let $I = \{\text{interior points of } A\}$ and use the definitions to show that $I \subseteq A^\circ$ and $A^\circ \subseteq I$. Part (b) is obtained from part(a) by carefully taking complements. $\square$



An interior point and a close point of $B_1(x)$ where $x = (2, 2)$ in $\mathbb{R}^2$.

2.2 Continuous functions

Continuous functions are for comparing topological spaces.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A *continuous function from X to Y* is a function $f : X \to Y$ such that

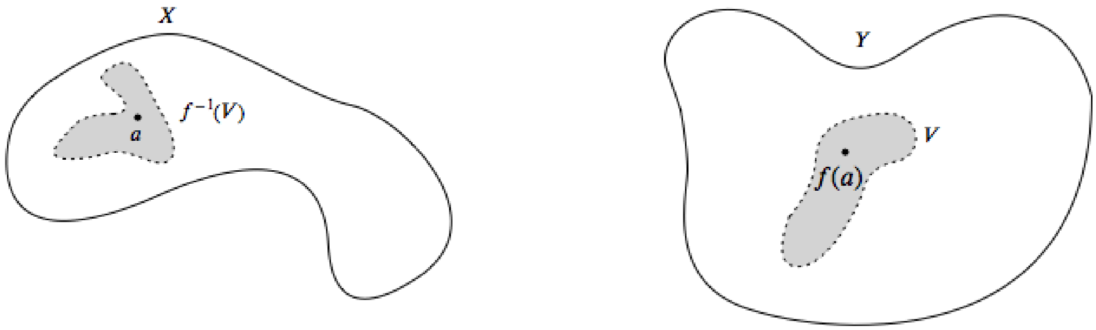
$$\text{if } V \text{ is an open set of } Y \quad \text{then} \quad f^{-1}(V) \text{ is an open set of } X,$$

where $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$. An *isomorphism of topological spaces*, or *homeomorphism*, is a continuous function $f : X \to Y$ such that the inverse function $f^{-1} : Y \to X$ exists and is continuous.

Let X and Y be topological spaces and let $a \in X$. A function $f : X \to Y$ is *continuous at a* if f satisfies the condition

$$\text{if } V \text{ is a neighborhood of } f(a) \text{ in } Y \quad \text{then} \quad f^{-1}(V) \text{ is a neighborhood of } a \text{ in } X,$$

i.e. if $V \in \mathcal{N}(f(a))$ then $f^{-1}(V) \in \mathcal{N}(a)$.



Proposition 2.2. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $f\colon X \to Y$ be a function. Then f is continuous if and only if f satisfies*

$$\text{if } a \in X \qquad \text{then} \quad f \text{ is continuous at } a.$$

Proof. (Sketch) This is a combination of the definitions of continuous, continuous at a, and the definition of $\mathcal{N}(a)$ as in (1.1). ☐

3 Limits in topological spaces

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f\colon X \to Y$ be a function and let $a \in X$ and $y \in Y$. Write

$$y = \lim_{x \to a} f(x) \qquad \text{if } f \text{ satisfies:} \quad \begin{array}{l} \text{if } N \in \mathcal{N}(y) \quad \text{then} \\ \quad \text{there exists } P \in \mathcal{N}(a) \text{ such that } N \supseteq f(P). \end{array}$$

Assume $a \in X$ such that $a \in \overline{X - \{a\}}$ (in English: a is in the closure of the complement of $\{a\}$ so that a is not an isolated point). Write

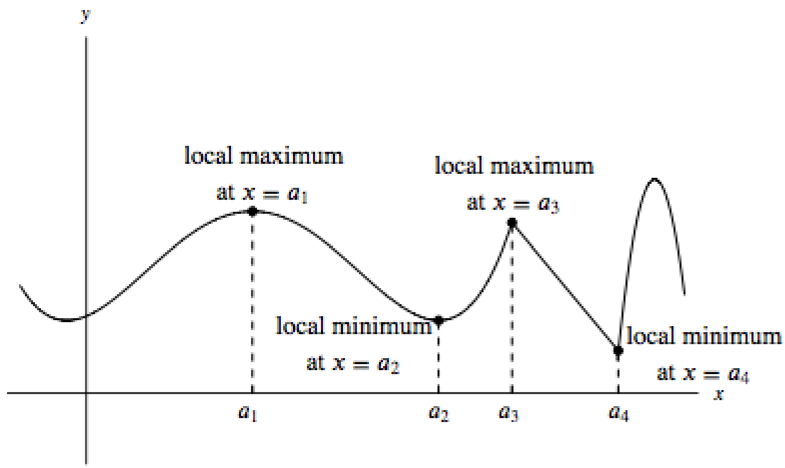
$$y = \lim_{\substack{x \to a \\ x \neq a}} f(x) \qquad \text{if } f \text{ satisfies:} \quad \begin{array}{l} \text{if } N \in \mathcal{N}(y) \quad \text{then} \\ \quad \text{there exists } P \in \mathcal{N}(a) \text{ such that } N \supseteq f(P - \{a\}). \end{array}$$

For example, using the standard topology on $\mathbb{R}$, the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

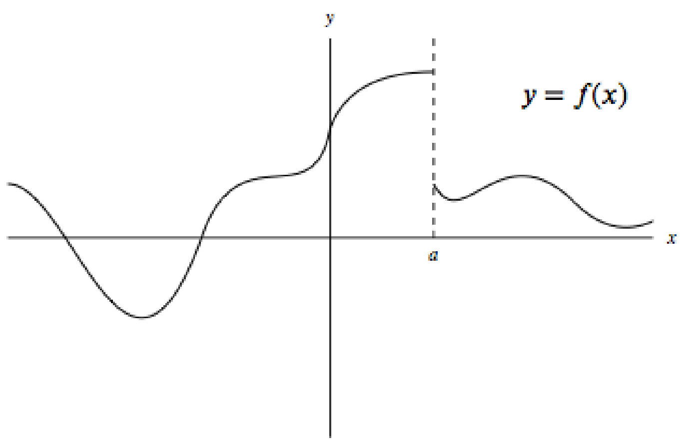
$$f(x) = \begin{cases} 2, & \text{if } x \neq 0, \\ 4, & \text{if } x = 0, \end{cases} \qquad \text{has} \qquad \lim_{\substack{x \to 0 \\ x \neq 0}} f(x) = 2 \quad \text{and} \quad \lim_{x \to 0} f(x) \text{ does not exist,}$$

and, using the subspace topology on $\{0, 1\}$ (a subspace of $\mathbb{R}$), the function $g\colon \{0,1\} \to \mathbb{R}$ given by

$$g(x) = 2, \qquad \text{has} \qquad \lim_{x \to 0} f(x) = 2 \quad \text{and} \quad \lim_{\substack{x \to 0 \\ x \neq 0}} f(x) \text{ is not defined.}$$



$f\colon \mathbb{R} \to \mathbb{R}$ is continuous



$f\colon \mathbb{R} \to \mathbb{R}$ is not continuous at a

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

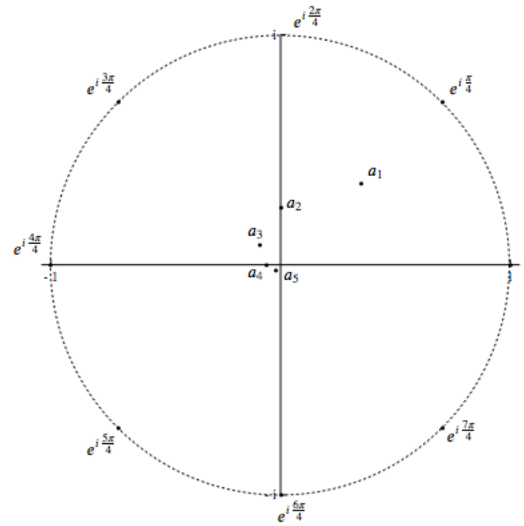
$$\text{A } \textit{sequence in } X \text{ is} \qquad \text{a function} \quad \begin{matrix} \vec{x}\colon & \mathbb{Z}_{>0} & \to & X \\ & n & \mapsto & x_n \end{matrix}$$

Let $(X, \mathcal{T})$ be a topological space. Let $(x_1, x_2, \ldots)$ be a sequence in X and let $z \in X$. Write

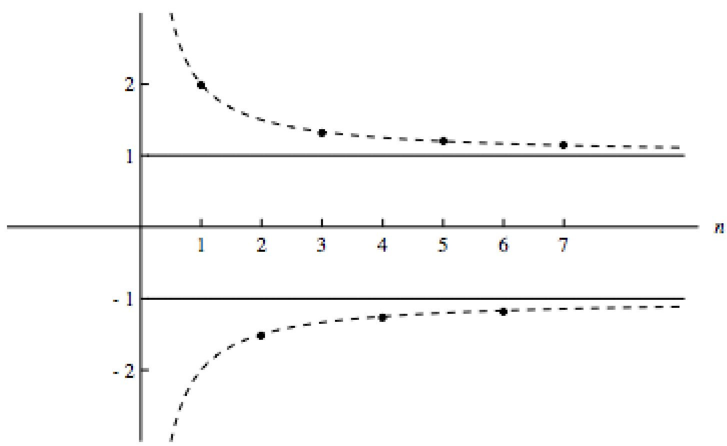
$$z = \lim_{n\to\infty} x_n \quad \text{if } (x_1, x_2, \ldots) \text{ satisfies:} \quad \begin{array}{l} \text{if } N \in \mathcal{N}(z) \quad \text{then } N \text{ contains all but} \\ \quad \text{a finite number of elements of } \{x_1, x_2, \ldots\}. \end{array}$$

More precisely,

$$z = \lim_{n\to\infty} x_n \quad \text{if } (x_1, x_2, \ldots) \text{ satisfies:} \quad \begin{array}{l} \text{if } N \in \mathcal{N}(z) \quad \text{then there exists } \ell \in \mathbb{Z}_{>0} \\ \quad \text{such that } N \supseteq \{x_\ell, x_{\ell+1}, \ldots\}. \end{array}$$



The spiral sequence $a_n = \left(\frac{1}{2}e^{i\pi/4}\right)^n$ in $\mathbb{C}$ has limit point 0



The sequence $a_n = (-1)^{n-1}(1 + \frac{1}{n})$ in $\mathbb{R}$ has cluster points at 1 and at -1

3.1 Limits and continuity

Proposition 3.1. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f\colon X \to Y$ be a function.*

(a) Let $a \in X$. Then

$$f \text{ is continuous at } a \qquad \text{if and only if} \qquad \lim_{x\to a} f(x) = f(a).$$

(b) Let $a \in X$ such that $a \in \overline{X - \{a\}}$. Then

$$f \text{ is continuous at } a \qquad \text{if and only if} \qquad \lim_{\substack{x\to a\\ x\neq a}} f(x) = f(a).$$

Proof. (Sketch) The notation $\lim_{x\to a} f(x) = f(a)$ means that if $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \supseteq P$, where $P \in \mathcal{N}(a)$. But then $f^{-1}(N) \in \mathcal{N}(a)$. □

3.2 Limits in metric spaces

Let $\mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\}$.

Proposition 3.2. *Let (X, d_X) and (Y, d_Y) be strict metric spaces. Let $f\colon X \to Y$ be a function and let $y \in Y$.*

(a) Let $a \in X$. Then

$$\lim_{x \to a} f(x) = y \quad \text{if and only if} \quad f \text{ satisfies}$$

$$\begin{gathered}\text{if } \epsilon \in \mathbb{E} \text{ then there exists } \delta \in \mathbb{E} \text{ such that} \\ \text{if } x \in X \text{ and } d_X(x, a) < \delta \quad \text{then} \quad d_Y(f(x), y) < \epsilon.\end{gathered}$$

(b) Let $a \in X$ be such that $a \in \overline{X - \{a\}}$. Then

$$\lim_{\substack{x \to a \\ x \neq a}} f(x) = y \quad \text{if and only if} \quad f \text{ satisfies}$$

$$\begin{gathered}\text{if } \epsilon \in \mathbb{E} \text{ then there exists } \delta \in \mathbb{E} \text{ such that} \\ \text{if } x \in X \text{ and } 0 < d_X(x, a) < \delta \quad \text{then} \quad d_Y(f(x), y) < \epsilon.\end{gathered}$$

(c) Let $(x_1, x_2, \ldots)$ be a sequence in X and let $z \in X$. Then

$$\lim_{n \to \infty} x_n = z \quad \text{if and only if} \quad (x_1, x_2, \ldots) \text{ satisfies}$$

$$\begin{gathered}\text{if } \epsilon \in \mathbb{E} \text{ then there exists } \ell \in \mathbb{Z}_{>0} \text{ such that} \\ \text{if } n \in \mathbb{Z}_{\geq \ell} \quad \text{then} \quad d(x_n, z) < \epsilon.\end{gathered}$$

Proof. (Sketch) The proof is accomplished by a careful conversion of the definitions of the limits using the definition of the metric space topology and the definition of the open ball $B_\epsilon(y)$ of radius ϵ centered at y. □

3.3 Limits of sequences capture closure and continuity in metric spaces

Theorem 3.3. *(Closure in metric spaces) Let (X, d) be a strict metric space and let $\mathcal{T}_X$ be the metric space topology on X. Let $A \subseteq X$. Then*

$$\overline{A} = \left\{ z \in X \mid \text{there exists a sequence } (a_1, a_2, \ldots) \text{ in } A \text{ such that } z = \lim_{n \to \infty} a_n \right\},$$

where $\overline{A}$ is the closure of A in X.

Proof. (Sketch) If z is a close point to A then a sequence $(a_1, a_2, \ldots)$ such that

$$a_1 \in B_{0.1}(z) \cap A, \quad a_2 \in B_{0.01}(z) \cap A, \quad a_3 \in B_{0.001}(z) \cap A, \quad \ldots,$$

will have $z = \lim_{n \to \infty} a_n$. □

Theorem 3.4. *(Continuity for metric spaces) Let (X, d_X) and (Y, d_Y) be strict metric spaces. Let $\mathcal{T}_X$ be the metric space topology on X and let $\mathcal{T}_Y$ be the metric space topology on Y. Let $f\colon X \to Y$ be a function. Then f is continuous if and only if f satisfies*

$$\text{if } (x_1, x_2, \ldots) \text{ is a sequence in } X \text{ and } \lim_{n \to \infty} x_n \text{ exists} \quad \text{then} \quad f\left(\lim_{n \to \infty} x_n\right) = \lim_{n \to \infty} f(x_n).$$

Proof. (Sketch) The $\Rightarrow$ implication is similar to the proof of Theorem 3.1. For the $\Leftarrow$ implication prove the contrapositive: If f is not continuous at a then there exists $N \in \mathcal{N}(f(a))$ such that $f^{-1}(N) \notin \mathcal{N}(a)$ and letting

$$x_1 \in B_{0.1}(a) \cap f^{-1}(N)^c, \quad x_2 \in B_{0.01}(a) \cap f^{-1}(N)^c, \quad \ldots$$

produces a sequence such that $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} f(x_n) \neq f(a)$. □

3.4 Limits of sequences capture closure and continuity in topological spaces with countably generated neighborhood filters

A topological space $(X, \mathcal{T})$ has *countably generated neighborhood filters*, or is *first countable*, if $(X, \mathcal{T})$ satisfies:

$$\begin{aligned}&\text{if } x \in X \text{ then there exist subsets } B_1, B_2, \ldots \text{ of } X \text{ such that}\\ &\qquad \mathcal{N}(x) = \{N \subseteq X \mid \text{there exists } k \in \mathbb{Z}_{>0} \text{ such that } N \supseteq B_k\}.\end{aligned}$$

Theorem 3.5. *(Closure in topological spaces with countably generated neighborhood filters) Let $(X, \mathcal{T})$ be a topological space with countably generated neighborhood filters. Let $A \subseteq X$. Then*

$$\overline{A} = \left\{z \in X \mid \text{there exists a sequence } (a_1, a_2, \ldots) \text{ in } A \text{ such that } z = \lim_{n\to\infty} a_n\right\},$$

Proof. (Sketch) If z is a close point to A and $B_1, B_2, \ldots$ are generators of $\mathcal{N}(z)$ then a sequence $(a_1, a_2, \ldots)$ such that

$$a_1 \in B_1(z) \cap A, \quad a_2 \in B_2(z) \cap A, \quad a_3 \in B_3(z) \cap A, \quad \ldots,$$

will have $z = \lim_{n\to\infty} a_n$. □

Theorem 3.6. *(Continuity for topological spaces with countably generated neighborhood filters) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and assume that $(X, \mathcal{T}_X)$ has countably generated neighborhood filters. Let $f\colon X \to Y$ be a function. Then f is continuous if and only if f satisfies*

$$\text{if } (x_1, x_2, \ldots) \text{ is a sequence in } X \text{ and } \lim_{n\to\infty} x_n \text{ exists} \quad \text{then} \quad f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n).$$

Proof. (Sketch) The proof is similar to the proof of Theorem 3.4 except with generators $B_1, B_2, \ldots$ of $\mathcal{N}(a)$ replacing the open balls $B_{0.1}(a), B_{0.01}(a), \ldots$. □

4 Appendix: Proofs without skipping any steps

Years of practice are what enables a professional mathematician to perfect the skill of taking a proof sketch and expanding it properly to fill in all the steps. For students learning this skill (and for teachers such as myself that need to save time preparing their class) expanded proofs of the proof sketches given in previous sections are included below.

4.1 Alternative characterization of the metric space topology

Proposition 4.1. *Let (X, d) be a strict metric space. Let*

$$\mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\} \quad \textit{and let} \quad \mathcal{B} = \{B_\epsilon(x) \mid \epsilon \in \mathbb{E} \textit{ and } x \in X\},$$

the set of open balls in X. Let $\mathcal{T}$ be the metric space topology on X. Let $U \subseteq X$. Then $U \in \mathcal{T}$ if and only if

$$\textit{there exists } \mathcal{S} \subseteq \mathcal{B} \textit{ such that} \quad U = \bigcup_{B \in \mathcal{S}} B.$$

Proof.
$\Leftarrow$: Assume $U = \bigcup_{B\in\mathcal{S}} B$.
To show: $U \in \mathcal{T}$.
To show: If $x \in U$ then there exists $\epsilon \in \mathbb{E}$ such that $B_\epsilon(x) \subseteq U$.
Assume $x \in U$.
Since $U = \bigcup_{B\in\mathcal{S}} B$ then there exists $B \in \mathcal{S}$ such that $x \in B$.
By definition of $\mathcal{B}$ there exists $\delta \in \mathbb{E}$ and $y \in X$ such that $B = B_\delta(y)$.
Since $x \in B = B_\delta(y)$ then $d(x, y) < \delta$.
Let $\epsilon = 10^{-k}$, where $k \in \mathbb{Z}_{>0}$ is such that $0 < 10^{-k} < \delta - d(x, y)$.
To show: $B_\epsilon(x) \subseteq B_\delta(y)$.
To show: If $p \in B_\epsilon(x)$ then $p \in B_\delta(y)$.
Assume $p \in B_\epsilon(x)$.
Since $d(p, y) \le d(p, x) + d(x, y) < \epsilon + d(x, y) < \delta$ then $p \in B_\delta(y)$.
So $B_\epsilon(x) \subseteq B_\delta(y) \subseteq U$.
Since $B_\delta(y) = B$ and $B \in \mathcal{S}$ then $B_\epsilon(x) \subseteq U$.
So $U \in \mathcal{T}$.

$\Rightarrow$: Assume $U \in \mathcal{T}$.
If $x \in U$ then there exists $\epsilon_x \in \mathbb{E}$ such that $B_{\epsilon_x}(x) \subseteq U$.
To show: There exists $\mathcal{S} \subseteq \mathcal{B}$ such that $U = \bigcup_{B\in\mathcal{S}} B$.
Let $\mathcal{S} = \{B_{\epsilon_x}(x) \mid x \in U\}$.
To show: $U = \bigcup_{B\in\mathcal{S}} B$.
To show: (a) $U \supseteq \bigcup_{B\in\mathcal{S}} B$.
(b) $U \subseteq \bigcup_{B\in\mathcal{S}} B$.

(a) If $B \in \mathcal{S}$ then $B = B_{\epsilon_x}(x) \subseteq U$.
So $U \supseteq \bigcup_{B\in\mathcal{S}} B$.

(b) To show: If $x \in U$ then $x \in \left(\bigcup_{B\in\mathcal{S}} B\right)$.
Assume $x \in U$.
Since $x \in B_{\epsilon_x}(x)$ and $B_{\epsilon_x}(x) \in \mathcal{S}$ then $x \in \bigcup_{B\in\mathcal{S}} B$.
So $U \subseteq \left(\bigcup_{B\in\mathcal{S}} B\right)$.

So $U = \bigcup_{B\in\mathcal{S}} B$. □

4.2 Interiors and closures

Proposition 4.2. *Let X be a topological space. Let $A \subseteq X$.*

(a) The interior of A is the set of interior points of A.

(b) The closure of A is the set of close points of A.

Proof.

(a) Let $I = \{x \in A \mid x \text{ is an interior point of } A\}$.
To show: $A^\circ = I$.
To show: (aa) $I \subseteq A^\circ$.
(ab) $A^\circ \subseteq I$.

(aa) Let $x \in I$.
Then there exists a neighborhood N of x with $N \subseteq A$.
So there exists an open set U with $x \in U \subseteq N \subseteq A$.
Since $U \subseteq A$ and U is open $U \subseteq A^\circ$.
So $x \in A^\circ$.
So $I \subseteq A^\circ$.

(ab) Assume $x \in A^\circ$.
Then A° is open and $x \in A^\circ \subseteq A$.
So x is a interior point of A.
So $x \in I$.
So $A^\circ \subseteq I$.

So $I = A^\circ$.

(b) Let $C = \{x \in X \mid \text{if } N \in \mathcal{N}(x) \text{ then } N \cap A \neq \emptyset\}$ be the set of close points of A.
Then

$$\begin{aligned} C^c &= \{x \in X \mid \text{there exists } N \in \mathcal{N}(x) \text{ such that } N \cap A = \emptyset\} \\ &= \{x \in X \mid \text{there exists } N \in \mathcal{N}(x) \text{ such that } N \subseteq A^c\}. \end{aligned}$$

which is the set of interior points of A^c.
Thus, by part (a), $C^c = (A^c)^\circ$.
So $C = ((A^c)^\circ)^c$.
To show: $C = \overline{A}$.
To show: $((A^c)^\circ)^c = \overline{A}$.

Claim: If $F \subseteq X$ then $(F^\circ)^c = \overline{F^c}$.
Let $F \subseteq X$.
Then F° is open and $(F^\circ)^c$ is closed.
Since $F^\circ \subseteq F$, then $(F^\circ)^c \supseteq F^c$.
So $(F^\circ)^c \supseteq \overline{F^c}$.
If V is closed and $V \supseteq F^c$ then V^c is open and $V^c \subseteq F$.
Thus, if V is closed and $V \supseteq F^c$ then $V^c \subseteq F^\circ$.
Thus, if V is closed and $V \supseteq F^c$ then $V \supseteq (F^\circ)^c$.
So $(F^\circ)^c = \overline{F^c}$.

Thus $((A^c)^\circ)^c = \overline{(A^c)^c}$.
Thus $C = ((A^c)^\circ)^c = \overline{(A^c)^c} = \overline{A}$.

□

4.3 Limits and continuity

Theorem 4.3. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f: X \to Y$ be a function.*

(a) *[Bou, Ch. 1 §2 Theorem 1(d)] f is continuous if and only if f satisfies:*

$$\text{if } a \in X \qquad \text{then} \qquad f \text{ is continuous at } a.$$

(b) *[Bou, Ch. 1 §7 Prop. 9] Let $a \in X$. Then*

$$f \text{ is continuous at } a \qquad \text{if and only if} \qquad \lim_{x \to a} f(x) = f(a).$$

(c) *[Bou, Ch. 1 §7 no. 5] Let $a \in X$ such that $a \in \overline{X - \{a\}}$. Then*

$$f \text{ is continuous at } a \qquad \text{if and only if} \qquad \lim_{\substack{x \to a \\ x \neq a}} f(x) = f(a).$$

(d) *[Bou, Ch. IX §2 no. 7 Proposition 10 and the remark following] Let (X, d) be a strict metric space and let $\mathcal{T}_X$ be the metric space topology on X. Then f is continuous if and only if f satisfies:*

if $(x_1, x_2, \ldots)$ is a sequence in X and

$$\text{if } \lim_{n \to \infty} x_n \text{ exists} \qquad \text{then} \qquad \lim_{n \to \infty} f(x_n) = f\Big(\lim_{n \to \infty} x_n\Big).$$

Proof.

(a) $\Rightarrow$: To show: If f is continuous then f satisfies: if $a \in X$ then f is continuous at a.
Assume f is continuous.
To show: If $a \in X$ then f is continuous at a.
Assume $a \in X$.
To show: If $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \in \mathcal{N}(a)$.
Assume $N \in \mathcal{N}(f(a))$.
Then there exists $V \in \mathcal{T}_Y$ such that $f(a) \in V \subseteq N$.
To show: $f^{-1}(N) \in \mathcal{N}(a)$.
To show: There exists $U \in \mathcal{T}_X$ such that $a \in U \subseteq f^{-1}(N)$.
Let $U = f^{-1}(V)$.
Since f is continuous then U is open in X.
Since $f(a) \in V \subseteq N$ then $a \in f^{-1}(V) = U \subseteq f^{-1}(N)$.
So $f^{-1}(N) \in \mathcal{N}(a)$.
So f is continuous at a.

(a) $\Leftarrow$: Assume that if $a \in X$ then f is continuous at a.
To show: f is continuous.
To show: If $V \in \mathcal{T}_Y$ then $f^{-1}(V) \in \mathcal{T}_X$.
Assume $V \in \mathcal{T}_Y$.
To show: $f^{-1}(V)$ is open in X.
To show: If $a \in f^{-1}(V)$ then a is an interior point of $f^{-1}(V)$.
Assume $a \in f^{-1}(V)$.
To show: There exists $U \in \mathcal{N}(a)$ such that $a \in U \subseteq f^{-1}(V)$.
Since $V \in \mathcal{T}_Y$ and $f(a) \in V$ then $V \in \mathcal{N}(f(a))$.

Since f is continuous at a then $f^{-1}(V) \in \mathcal{N}(a)$.
Let $U = f^{-1}(V)$.
Then $a \in U \subseteq f^{-1}(V)$.
So a is an interior point of $f^{-1}(V)$.
So $f^{-1}(V)$ is open in X.
So f is continuous.

(b) $\Rightarrow$: To show: If f is continuous at a then $\lim_{x\to a} f(x) = f(a)$.
Assume f is continuous at a.
To show: $\lim_{x\to a} f(x) = f(a)$.
To show: If $N \in \mathcal{N}(f(a))$ then there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P)$.
Assume $N \in \mathcal{N}(f(a))$.
To show: There exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P)$.
Since f is continuous at a and $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \in \mathcal{N}(a)$.
Let $P = f^{-1}(N)$.
Then $f(P) = f(f^{-1}(N)) \subseteq N$.
So $\lim_{x\to a} f(x) = f(a)$.

(b) $\Leftarrow$: To show: If $\lim_{x\to a} f(x) = f(a)$ then f is continuous at a.
Assume $\lim_{x\to a} f(x) = f(a)$.
To show: f is continuous at a.
To show: If $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \in \mathcal{N}(a)$.
Assume $N \in \mathcal{N}(f(a))$.
To show: $f^{-1}(N) \in \mathcal{N}(a)$.
To show: There exists $U \in \mathcal{T}_X$ such that $a \in U \subseteq f^{-1}(N)$.
Since $\lim_{x\to a} f(x) = f(a)$ then there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P)$.
So $f^{-1}(N) \supseteq P$.
Since $P \in \mathcal{N}(a)$, there exists $U \in \mathcal{T}_X$ such that $a \in U \subseteq P$.
So there exists $U \in \mathcal{T}_X$ such that $a \in U \subseteq P \subseteq f^{-1}(N)$.
So $f^{-1}(N) \in \mathcal{N}(a)$.
So f is continuous at a.

(c) $\Rightarrow$: Assume $a \in \overline{X - \{a\}}$.
To show: If f is continuous at a then $\lim\limits_{\substack{x\to a\\ x\neq a}} f(x) = f(a)$.
Assume f is continuous at a.
To show: $\lim\limits_{\substack{x\to a\\ x\neq a}} f(x) = f(a)$.
To show: If $N \in \mathcal{N}(f(a))$ then there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P - \{a\})$.
Assume $N \in \mathcal{N}(f(a))$.
To show: There exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P - \{a\})$.
Since f is continuous at a and $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \in \mathcal{N}(a)$.
Let $P = f^{-1}(N)$.
Then $f(P - \{a\}) \subseteq f(P) = f(f^{-1}(N)) \subseteq N$.
So $\lim\limits_{\substack{x\to a\\ x\neq a}} f(x) = f(a)$.

(c) $\Leftarrow$: Assume $a \in \overline{X - \{a\}}$.
To show: If $\lim\limits_{\substack{x\to a\\ x\neq a}} f(x) = f(a)$ then f is continuous at a.
Assume $\lim\limits_{\substack{x\to a\\ x\neq a}} f(x) = f(a)$.
To show: f is continuous at a.
To show: If $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \in \mathcal{N}(a)$.
Assume $N \in \mathcal{N}(f(a))$.

To show: $f^{-1}(N) \in \mathcal{N}(a)$.
To show: There exists $U \in \mathcal{T}_X$ such that $a \in U \subseteq f^{-1}(N)$.
Since $\lim_{\substack{x\to a\\ x\neq a}} f(x) = f(a)$ there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P - \{a\})$.
So $f^{-1}(N) \supseteq P - \{a\}$.
Since $N \in \mathcal{N}(f(a))$ then $f(a) \in N$ and $a \in f^{-1}(N)$.
So $f^{-1}(N) \supseteq P$.
Since $P \in \mathcal{N}(a)$, there exists $U \in \mathcal{T}_X$ such that $a \in U \subseteq P$.
So there exists $U \in \mathcal{T}_X$ such that $a \in U \subseteq P \subseteq f^{-1}(N)$.
So $f^{-1}(N) \in \mathcal{N}(a)$.
So f is continuous at a.

(d) $\Rightarrow$: Assume f is continuous.
To show: f satisfies

$$\begin{array}{c}\text{if } (x_1, x_2, \ldots) \text{ is a sequence in } X \text{ and } \lim_{n\to\infty} x_n \text{ exists}\\ \text{then}\quad f\Big(\lim_{n\to\infty} x_n\Big) = \lim_{n\to\infty} f(x_n).\end{array} \qquad (*)$$

Assume $(x_1, x_2, \ldots)$ is a sequence in X and $\lim_{n\to\infty} x_n = a$.
To show: $f(a) = \lim_{n\to\infty} f(x_n)$.
To show: If $N \in \mathcal{N}(f(a))$ then there exists $t \in \mathbb{Z}_{>0}$ such that $N \supseteq (f(x_t), f(x_{t+1}), \ldots)$.
Assume $N \in \mathcal{N}(f(a))$.
Since f is continuous then $f^{-1}(N) \in \mathcal{N}(a)$.
Since $\lim_{n\to\infty} x_n = a$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that $f^{-1}(N) \supseteq \{x_\ell, x_{\ell+1}, \ldots\}$.
Let $t = \ell$.
Then $f^{-1}(N) \supseteq \{x_t, x_{t+1}, \ldots\}$.
So $N \supseteq \{f(x_t), f(x_{t+1}), \ldots\}$.
So f satisfies (*).

(d) $\Leftarrow$: To show: If f is not continuous then f does not satisfy (*).
Assume f is not continuous.
Then there exists a such that f is not continuous at a.
So there exists $N \in \mathcal{N}(f(a))$ such that $f^{-1}(N) \notin \mathcal{N}(a)$.
To show: There exists a sequence $(x_1, x_2, \ldots)$ such that $\lim_{n\to\infty} x_n$ exists and $\lim_{n\to\infty} f(x_n) \neq f\big(\lim_{n\to\infty} x_n\big)$.
Since $f^{-1}(N) \notin \mathcal{N}(a)$ then $f^{-1}(N) \not\supseteq B_{10^{-\ell}}(a)$, for $\ell \in \mathbb{Z}_{>0}$. Let

$$x_1 \in B_{10^{-1}}(a) \cap f^{-1}(N)^c, \quad x_2 \in B_{10^{-2}}(a) \cap f^{-1}(N)^c, \quad \ldots.$$

To show: (da) $\lim_{n\to\infty} x_n = a$.
(db) $\lim_{n\to\infty} f(x_n) \neq f(a)$.

(da) To show: If $P \in \mathcal{N}(a)$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $x_n \in P$.
Assume $P \in \mathcal{N}(a)$.
To show: There exists $\ell \in \mathbb{Z}_{>0}$ such that $P \supseteq \{x_\ell, x_{\ell+1}, \ldots\}$.
Since $P \in \mathcal{N}(a)$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that $P \supseteq B_{10^{-\ell}}(a)$.
To show: $P \supseteq \{x_\ell, x_{\ell+1}, \ldots\}$.
To show: If $n \in \mathbb{Z}_{\geq \ell}$ then $x_n \in P$.
Assume $n \in \mathbb{Z}_{\geq \ell}$.
Since $n \geq \ell$ then $10^{-\ell} \leq 10^{-n}$ and $x_n \in B_{10^{-n}}(a) \subseteq B_{10^{-\ell}}(a) \subseteq P$.
So $P \supseteq \{x_\ell, x_{\ell+1}, \ldots\}$.
So $\lim_{n\to\infty} x_n = a$.

(db) To show: $\lim_{n\to\infty} f(x_n) \neq f(a)$.
To show: There exists $M \in \mathcal{N}(f(a))$ such that $\{j \in \mathbb{Z}_{>0} \mid f(x_j) \in M^c\}$ is infinite.

Let $M = N$.
To show: $\{j \in \mathbb{Z}_{>0} \mid f(x_j) \in N^c\}$ is infinite.
Since $x_j \in f^{-1}(N)^c$ then $f(x_j) \notin N$, for $j \in \mathbb{Z}_{>0}$.
So $\{f(x_1), f(x_2), \ldots\} \subseteq N^c$.
So $\{j \in \mathbb{Z}_{>0} \mid f(x_j) \in N^c\}$ is infinite.
So $\lim_{n\to\infty} f(x_n) \neq f(a)$.

So f does not satisfy (*).

$\square$

To change the proof of (d) above to a proof for first countable topological spaces $(X, \mathcal{T}_X)$, replace the use of the open balls $B_{10^{-1}}(a) \supseteq B_{10^{-2}}(a) \supseteq \cdots$ by generators $B_1 \supseteq B_2 \supseteq \cdots$ of $\mathcal{N}(a)$, the neighborhood filter of a.

4.4 The topology in a metric space is determined by limits of sequences

Theorem 4.4. *Let (X, d) be a strict metric space and let $A \subseteq X$ and let $\overline{A}$ be the closure of A. Then*

$$\overline{A} = \big\{z \in X \mid \text{there exists a sequence } (a_1, a_2, \ldots) \text{ in } A \text{ with } z = \lim_{n\to\infty} a_n\big\}.$$

Proof. Let $R = \big\{z \in X \mid \text{there exists a sequence } (a_1, a_2, \ldots) \text{ in } A \text{ with } z = \lim_{n\to\infty} a_n\big\}$.

To show: (a) $R \subseteq \overline{A}$.
(b) $\overline{A} \subseteq R$.

(a) To show: If $z \in R$ then $z \in \overline{A}$.
Assume $z \in R$.
To show: $z \in \overline{A}$.
We know there exists a sequence $(a_1, a_2, \ldots)$ in A with $z = \lim_{n\to\infty} a_n$.
To show: z is a close point of A.
To show: If N is a neighborhood of z then $N \cap A \neq \emptyset$.
Assume N is a neighborhood of z.
Since $\lim_{n\to\infty} a_n = z$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $a_n \in N$.
So $N \cap A \neq \emptyset$.
So z is a close point of A.
So $R \subseteq \overline{A}$.

(b) To show: $\overline{A} \subseteq R$.
To show: If $z \in \overline{A}$ then $z \in R$.
Let $z \in \overline{A}$.
To show: $z \in R$.
To show: There exists a sequence $(a_1, a_2, \ldots)$ in A with $z = \lim_{n\to\infty} a_n$.
Using that z is a close point of A,

$$\text{let} \quad a_1 \in B_{0.1}(z) \cap A, \quad a_2 \in B_{0.01}(z) \cap A, \quad a_3 \in B_{0.001}(z) \cap A, \quad \ldots.$$

To show: $z = \lim_{n\to\infty} a_n$.
To show: If P is a neighborhood of z then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $a_n \in P$.
Let P be a neighborhood of z.
Then there exists $\ell \in \mathbb{Z}_{>0}$ such that $B_{10^{-\ell}}(z) \subseteq P$.
To show: If $n \in \mathbb{Z}_{\geq \ell}$ then $a_n \in P$.

Assume $n \in \mathbb{Z}_{\geq \ell}$.
Since $n \geq \ell$ then $10^{-n} \leq 10^{-\ell}$ and

$$a_n \in B_{10^{-n}}(z) \subseteq B_{10^{-\ell}}(z) \subseteq P,$$

So $\lim_{n\to\infty} a_n = z$.
So $z \in R$.
So $\overline{A} \subseteq R$.

□

To change the proof of (b) above to a proof for first countable topological spaces $(X, \mathcal{T}_X)$, replace the use of the open balls $B_{10^{-1}}(a) \supseteq B_{10^{-2}}(a) \supseteq \cdots$ by generators $B_1 \supseteq B_2 \supseteq \cdots$ of $\mathcal{N}(a)$, the neighborhood filter of a.

4.5 Limits in metric spaces

Proposition 4.5. *Let (X, d_X) and (Y, d_Y) be strict metric spaces, let $\mathcal{T}_X$ be the metric space topology on X and let $\mathcal{T}_Y$ be the metric space topology on Y. Let $f\colon X \to Y$ be a function and let $y \in Y$.*

(a) Let $a \in X$. Then $\lim_{x\to a} f(x) = y$ if and only if f satisfies

$$\text{if } \epsilon \in \mathbb{E} \text{ then there exists } \delta \in \mathbb{E} \text{ such that} \\ \text{if } x \in X \text{ and } d_X(x,a) < \delta \quad \text{then} \quad d_Y(f(x), y) < \epsilon.$$

(b) Let $a \in X$ such that $a \in \overline{X - \{a\}}$. Then $\lim_{\substack{x\to a\\ x\neq a}} f(x) = y$ if and only if f satisfies

$$\text{if } \epsilon \in \mathbb{E} \text{ then there exists } \delta \in \mathbb{E} \text{ such that} \\ \text{if } x \in X \text{ and } 0 < d_X(x,a) < \delta \quad \text{then} \quad d_Y(f(x), y) < \epsilon.$$

(c) Let $(x_1, x_2, \ldots)$ be a sequence in X and let $z \in X$. Then $\lim_{n\to\infty} x_n = z$ if and only if $(x_1, x_2, \ldots)$ satisfies

$$\text{if } \varepsilon \in \mathbb{E} \text{ then there exists } \ell \in \mathbb{Z}_{>0} \text{ such that} \quad \text{if } n \in \mathbb{Z}_{\geq \ell} \text{ then } d(x_n, z) < \varepsilon.$$

Proof. (a) By definition, $\lim_{x\to a} f(x) = y$ if and only if f satisfies: if $N \in \mathcal{N}(y)$ then there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P)$.
By definition of the metric space topology, $N \in \mathcal{N}(y)$ if and only if there exists $\epsilon \in \mathbb{E}$ such that $B_\epsilon(y) \subseteq N$.
Thus $\lim_{x\to a} f(x) = y$ if and only if f satisfies: if $B_\epsilon(y)$ is an open ball at y then there exists $B_\delta(a)$, an open ball at a such that $B_\epsilon(y) \supseteq f(B_\delta(a))$.
By definition, $B_\delta(a) = \{x \in X \mid d(x,a) < \delta\}$.
Thus, $\lim_{x\to a} f(x) = y$ if and only if f satisfies: if $\varepsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that if $x \in X$ and $d_X(x,a) < \delta$ then $d_Y(f(x), y) < \varepsilon$.

(b) By definition, $\lim_{\substack{x\to a\\ x\neq a}} f(x) = y$ if and only if f satisfies: if $N \in \mathcal{N}(y)$ then there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P - \{a\})$.

By definition of the metric space topology, $N \in \mathcal{N}(y)$ if and only if there exists $\epsilon \in \mathbb{E}$ such that $B_\epsilon(y) \subseteq N$.

Thus $\lim\limits_{\substack{x\to a\\ x\neq a}} f(x) = y$ if and only if f satisfies: if $B_\epsilon(y)$ is an open ball at y then there exists $B_\delta(a)$, an open ball at a such that $B_\epsilon(y) \supseteq f(B_\delta(a) - \{a\})$.

By definition, $B_\epsilon(y) = \{x \in Y \mid d(x,y) < \epsilon\}$ and $B_\delta(a) - \{a\} = \{x \in X \mid 0 < d(x,a) < \delta\}$.

Thus, $\lim\limits_{\substack{x\to a\\ x\neq a}} f(x) = y$ if and only if f satisfies: if $\epsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that if $x \in X$ and $0 < d_X(x,a) < \delta$ then $d_Y(f(x),y) < \epsilon$.

(c) By definition, $\lim\limits_{n\to\infty} x_n = z$ if and only if $(x_1, x_2, \ldots)$ satisfies: if $P \in \mathcal{N}(z)$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that $P \supseteq \{x_\ell, x_{\ell+1}, \ldots\}$.

By definition of the metric space topology, $P \in \mathcal{N}(y)$ if and only if there exists $\epsilon \in \mathbb{E}$ such that $B_\epsilon(y) \subseteq P$.

So $\lim\limits_{n\to\infty} x_n = z$ if and only if $(x_1, x_2, \ldots)$ satisfies: if $\epsilon \in \mathbb{E}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that $B_\epsilon(z) \supseteq \{x_\ell, x_{\ell+1}, \ldots, \}$.

By definition, $B_\epsilon(a) = \{x \in X \mid d(x,a) < \epsilon\}$.

Thus, $\lim\limits_{n\to\infty} x_n = z$ if and only if $(x_1, x_2, \ldots)$ satisfies: if $\epsilon \in \mathbb{E}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $d(x_n, z) < \epsilon$. □

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