The Steinberg-Lusztig tensor product theorem, Casselman-Shalika and LLT polynomials

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Dedicated to Friedrich Knop and Peter Littelmann on the occasion of their 60th birthdays

Abstract

In this paper we establish a Steinberg-Lusztig tensor product theorem for abstract Fock space. This is a generalization of the type A result of Leclerc-Thibon and a Grothendieck group version of the Steinberg-Lusztig tensor product theorem for representations of quantum groups at roots of unity. Although the statement can be phrased in terms of parabolic affine Kazhdan-Lusztig polynomials and thus has geometric content, our proof is combinatorial, using the theory of crystals (Littelmann paths). We derive the Casselman-Shalika formula as a consequence of the Steinberg-Lusztig tensor product theorem for abstract Fock space.

Key words— quantum groups, affine Lie algebras, Hecke algebras, symmetric functions

0 Introduction

In our previous paper [LRS] we provided a construction of an “abstract” Fock space $\mathcal{F}_\ell$ in a general Lie type setting. The construction is given by simple combinatorial “straightening relations” which generalize the Kashiwara-Miwa-Stern [KMS] formulation of the q-Fock space from the type A case. We showed that the abstract Fock space is a combinatorial realization of the graded Grothendieck group of finite dimensional representations of the quantum group at a root of unity, where the standard basis elements $|\lambda\rangle$ correspond to the Weyl modules $\Delta_q(\lambda)$ and the KL-basis C_λ corresponds to the simple modules $L_q(\lambda)$.

In Section 1 we prove a product theorem (Theorem 1.3) in abstract Fock space which generalizes the type A theorem of Leclerc and Thibon [LT, Theorem 6.9]. Our proof follows the same pattern as the proof for Type A given in [LT, Theorem 6.9] except that, in order to deal with general Lie type, we have replaced the use of ribbon tableaux with the crystal basis and Littelmann paths. The basic philosophy of our technique is similar to the main idea of a paper of Guilhot [Gu] but we also make use of the elegant cancellation technique of Littelmann [Li, proof of Theorem 9.1] to complete the proof. This technique provides a combinatorial control of the Demazure operator used in the proof of [Knp, Lemma 4.4]. We have not considered the unequal parameter case in this paper but the close relation between our context and that of [Knp] cries out for an extension of the tensor product theorem for abstract Fock space to unequal parameters.

The Casselman-Shalika formula is important in the representation theory of p-adic groups (see [CS]), in its relation to the affine Hecke algebra (see for example [BBF]) and in the geometric

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Langlands program (see [FGV] and [NP]). In Section 2 we show that the Casselman-Shalika formula can be derived as a special case of the Steinberg tensor product theorem for abstract Fock space. This derivation is done by using the relationship between the abstract Fock space and the affine Hecke algebra as detailed in [LRS, Theorem 4.7].

In Section 3, we review the connection between the abstract Fock space and the representations of quantum groups at roots of unity (Theorem 3.1) and the Steinberg-Lusztig tensor product theorem (Theorem 3.2). The Steinberg-Lusztig tensor product theorem is the primary motivation for the product theorem in abstract Fock space. Our approach does provide an alternative proof of the Steinberg-Lusztig tensor product theorem for representations of quantum groups at roots of unity (though hardly elementary since proving the Steinberg-Lusztig tensor product theorem this way relies on deep results of Kazhdan-Lusztig [KL94] and Kashiwara-Tanisaki [KT95]).

As explained in [LT], the Steinberg-Lustzig tensor product theorem and the abstract Fock space are intimately related to the LLT polynomials defined in type A by Lascoux, Leclerc and Thibon [LLT]. Fundamentally, the LLT polynomials are taking the role of the characters of the Frobenius twisted Weyl modules which, by the Steinberg-Lusztig tensor product theorem, are simple modules for the quantum group at a root of unity. General Lie type definitions of LLT polynomials have been given by Grojnowski-Haiman [GH] and Lecouvey [Lcy]. In the second half of Section 3, we summarize a 2008 letter from C. Lecouvey to A. Ram which explains that a consequence of the tensor product theorem for abstract Fock space is that the definition from [GH] and the definition from [Lcy] coincide up to a power of $t^{\frac{1}{2}}$.

Kazhdan and Lusztig [KL94] established an equivalence of categories between an appropriate category of representations of the affine Lie algebra (of negative level) and the finite dimensional representations of the quantum group (of the finite dimensional Lie algebra) at a root of unity. In Section 4 we review this correspondence and make explicit the tensor product theorem in terms of representations of the affine Lie algebra. This produces a character formula for certain negative level irreducible highest weight representations of the affine Lie algebra. From the point of view of this paper this character formula is an easy consequence of [KL94] and [Lu89]. We find it difficult to believe that this formula has not been noticed before but we have not yet been able to locate a suitable specific reference.

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1 A product theorem in abstract Fock space $\mathcal{F}_\ell$

Let W_0 be a finite Weyl group, generated by simple reflections $s_1, \ldots, s_n$, and acting on a lattice of weights $\mathfrak{a}_\mathbb{Z}^*$. For example, this situation arises when T is a maximal torus of a reductive algebraic group G,

$$\mathfrak{a}_\mathbb{Z}^* = \mathrm{Hom}(T, \mathbb{C}^\times) \qquad \text{and} \qquad W_0 = N(T)/T, \tag{1.1}$$

where $N(T)$ is the normalizer of T in G. The simple reflections in W_0 correspond to a choice of Borel subgroup B of G which contains T. Let R^+ denote the positive roots. Let $\alpha_1, \ldots, \alpha_n$ be the simple roots and let $\alpha_1^\vee, \ldots, \alpha_n^\vee$ be the simple coroots. The *dot action* of W_0 on $\mathfrak{a}_\mathbb{Z}^*$ is given by

$$w \circ \lambda = w(\lambda+\rho) - \rho, \qquad \text{where} \quad \rho = \frac{1}{2}\sum_{\alpha\in R^+} \alpha \tag{1.2}$$

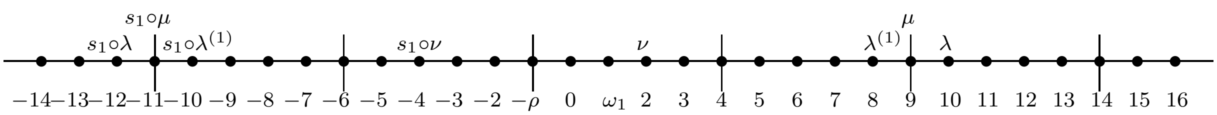
is the half sum of the positive roots for G (with respect to B).

Fix $\ell \in \mathbb{Z}_{>0}$. The *abstract Fock space* $\mathcal{F}_\ell$ is the $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$-module generated by $\{|\lambda\rangle \mid \lambda \in \mathfrak{a}^*_{\mathbb{Z}}\}$ with relations

$$|s_i \circ \lambda\rangle = \begin{cases} -|\lambda\rangle, & \text{if } \langle \lambda+\rho, \alpha_i^\vee\rangle \in \ell\mathbb{Z}_{\geq 0}, \\ -t^{\frac{1}{2}}|\lambda\rangle, & \text{if } 0 < \langle \lambda+\rho, \alpha_i^\vee\rangle < \ell, \\ -t^{\frac{1}{2}}|s_i\circ\lambda^{(1)}\rangle - |\lambda^{(1)}\rangle - t^{\frac{1}{2}}|\lambda\rangle, & \text{if } \langle \lambda+\rho, \alpha_i^\vee\rangle > \ell \text{ and } \langle \lambda+\rho, \alpha_i^\vee\rangle \notin \ell\mathbb{Z}, \end{cases} \tag{1.3}$$

where $\lambda^{(1)} = \lambda - j\alpha_i$ if $\langle \lambda+\rho, \alpha_i^\vee\rangle = k\ell + j$ with $k \in \mathbb{Z}_{>0}$ and $j \in \{1, \ldots, \ell-1\}$.

The following picture illustrates the terms in (1.3). This is the case $G = SL_2$ with $\ell = 5$, $\langle \omega_1, \alpha_1^\vee\rangle = 1$ and $\alpha_1 = 2\omega_1$ and, in the picture, λ corresponds to the third case of (1.3), μ to the first case and ν to the second case.



Define a $\mathbb{Z}$-linear involution $\overline{}\colon \mathcal{F}_\ell \to \mathcal{F}_\ell$ by

$$\overline{t^{\frac{1}{2}}} = t^{-\frac{1}{2}} \qquad \text{and} \qquad \overline{|\lambda\rangle} = (-1)^{\ell(w_0)}(t^{-\frac{1}{2}})^{\ell(w_0)-N_\lambda}\,|w_0\circ\lambda\rangle. \tag{1.4}$$

where w_0 is the longest element of W_0, $\ell(w_0) = \mathrm{Card}(R^+)$ is the length of w_0, and $N_\lambda = \mathrm{Card}\{\alpha \in R^+ \mid \langle \lambda+\rho, \alpha^\vee\rangle \in \ell\mathbb{Z}\}$.

The *dominant integral weights* with the *dominance partial order* $\leq$ are the elements of

$$\begin{gathered} (\mathfrak{a}^*_{\mathbb{Z}})^+ = \{\lambda \in \mathfrak{a}^*_{\mathbb{Z}} \mid \langle \lambda+\rho, \alpha_i^\vee\rangle > 0 \text{ for } i = 1,2,\ldots,n\} \\ \text{with} \qquad \mu \leq \lambda \qquad \text{if} \quad \mu \in \lambda - \textstyle\sum_{\alpha\in R^+}\mathbb{Z}_{\geq 0}\alpha. \end{gathered} \tag{1.5}$$

In [LRS, Theorem 1.1 and Proposition 2.1] we showed that $\mathcal{F}_\ell$ has bases

$$\{|\lambda\rangle \mid \lambda \in (\mathfrak{a}^*_{\mathbb{Z}})^+\} \qquad \text{and} \qquad \{C_\lambda \mid \lambda \in (\mathfrak{a}^*_{\mathbb{Z}})^+\} \tag{1.6}$$

where C_λ are determined by

$$\overline{C_\lambda} = C_\lambda \qquad \text{and} \qquad C_\lambda = |\lambda\rangle + \sum_{\mu\neq\lambda} p_{\mu\lambda}|\mu\rangle, \qquad \text{with } p_{\mu\lambda} \in t^{\frac{1}{2}}\mathbb{Z}[t^{\frac{1}{2}}]. \tag{1.7}$$

1.1 The action of $\mathbb{K}[X]^{W_0}$ on $\mathcal{F}_\ell$

Letting $\mathbb{K} = \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$, the group algebra of $\mathfrak{a}^*_{\mathbb{Z}}$ is

$$\mathbb{K}[X] = \mathbb{K}\text{-span}\{X^\mu \mid \mu \in \mathfrak{a}^*_{\mathbb{Z}}\} \quad \text{with} \quad X^\mu X^\nu = X^{\mu+\nu}. \tag{1.8}$$

The Weyl group W_0 acts $\mathbb{K}$-linearly on $\mathbb{K}[X]$ by

$$wX^\mu = X^{w\mu}, \text{ for } w \in W_0 \text{ and } \mu \in \mathfrak{a}^*_{\mathbb{Z}}, \quad \text{and} \quad \mathbb{K}[X]^{W_0} = \{f \in \mathbb{K}[X] \mid wf = f\} \tag{1.9}$$

is the ring of *symmetric functions*.

Let V be the free $\mathbb{K}$-module generated by $\{|\lambda\rangle \mid \lambda \in \mathfrak{a}_{\mathbb{Z}}^*\}$ so that

$$\mathcal{F}_\ell \cong V/I, \tag{1.10}$$

where I is the subspace of V consisting of $\mathbb{K}$-linear combinations of the elements

$$\begin{aligned} a_\lambda &= |s_i \circ \lambda\rangle + |\lambda\rangle, && \text{with } \langle \lambda+\rho, \alpha_i^\vee\rangle \in \ell\mathbb{Z}_{\geq 0}, \\ b_\lambda &= |s_i \circ \lambda\rangle + t^{\frac{1}{2}}|\lambda\rangle, && \text{with } 0 < \langle \lambda+\rho, \alpha_i^\vee\rangle < \ell, \text{ and} \\ c_\lambda = |s_i \circ \lambda\rangle + t^{\frac{1}{2}}&|s_i \circ \lambda^{(1)}\rangle + |\lambda^{(1)}\rangle + t^{\frac{1}{2}}|\lambda\rangle, && \text{with } \langle \lambda+\rho, \alpha_i^\vee\rangle > \ell \text{ and } \langle \lambda+\rho, \alpha_i^\vee\rangle \notin \ell\mathbb{Z}. \end{aligned}$$

Let $\mathring{\mathfrak{g}}$ be the Lie algebra of the reductive group G alluded to in (1.1). Let φ be the highest weight of the adjoint representation and let $\varphi^\vee \in [\mathring{\mathfrak{g}}_\varphi, \mathring{\mathfrak{g}}_{-\varphi}]$ such that $\langle \varphi, \varphi^\vee\rangle = 2$ (so that $\varphi^\vee$ is an appropriate normalized highest short coroot of $\mathring{\mathfrak{g}}$).

$$\text{The } \textit{dual Coxeter number} \text{ is} \qquad h = \langle \rho, \varphi^\vee\rangle + 1. \tag{1.11}$$

The *level* $(-\ell - h)$ *action* of $\mathbb{K}[X]$ on V is the $\mathbb{K}$-linear extension of

$$X^\mu \cdot |\gamma\rangle = |-\ell w_0\mu + \gamma\rangle, \qquad \text{for } \mu, \gamma \in \mathfrak{a}_{\mathbb{Z}}^*. \tag{1.12}$$

Letting w_0 be the longest element of W_0, define

$$\mu^* = -w_0\mu \qquad \text{and} \qquad w^* = w_0 w w_0, \qquad \text{for } \mu \in \mathfrak{a}_{\mathbb{Z}}^* \text{ and } w \in W.$$

(This notation is such that if $\mu \in \mathfrak{a}_{\mathbb{Z}}^*$ and $L_{\mathring{\mathfrak{g}}}(\mu)$ denotes the irreducible $\mathring{\mathfrak{g}}$-module of highest weight μ then the dual $L_{\mathring{\mathfrak{g}}}(\mu)^* \cong L_{\mathring{\mathfrak{g}}}(\mu^*)$ and ${}^*\colon W_0 \to W_0$ is the involutive automorphism of W_0 induced by the automorphism of the Dynkin diagram specified by $s_i^* = s_{i^*}$.) Then

$$X^\mu \cdot |\gamma\rangle = |\ell\mu^* + \gamma\rangle \qquad \text{and} \qquad (w\mu)^* = w^*\mu^*. \tag{1.13}$$

The following proposition establishes an action of the ring of symmetric functions $\mathbb{K}[X]^{W_0}$ on the abstract Fock space $\mathcal{F}_\ell$. From the point of view of Theorem 2.2 below, this action is coming from an action of the center of the affine Hecke algebra which, by an important result of Bernstein, is the ring of symmetric functions (inside the affine Hecke algebra). Our proof of Proposition 1.1 provides an independent proof of the existence of the action of $\mathbb{K}[X]^{W_0}$ without referring to the affine Hecke algebra and the characterization of its center.

Proposition 1.1. *The action of* $\mathbb{K}[X]$ *on* V *given in* (1.12) *induces a* $\mathbb{K}$*-linear action of the ring* $\mathbb{K}[X]^{W_0}$ *of symmetric functions on* $\mathcal{F}_\ell$ *by*

$$\left(\sum_{w \in W_0} X^{w\mu}\right) \cdot |\gamma\rangle = \sum_{w \in W_0} |\ell(w\mu)^* + \gamma\rangle, \quad \textit{for } \mu \in \mathfrak{a}_{\mathbb{Z}}^* \textit{ and } \gamma \in \mathfrak{a}_{\mathbb{Z}}^*.$$

Proof. Let f be an element of the subspace I defined in (1.10), let $\mu \in \mathfrak{a}_{\mathbb{Z}}^*$ and let $i \in \{1, \ldots, n\}$. Summing over a set of representatives of the cosets in $\{1, s_i^*\}\backslash W_0$,

$$\left(\sum_{w \in W_0} X^{w\mu}\right) \cdot f = \left(\sum_{v \in \{1, s_i^*\}\backslash W_0} (X^{v\mu} + X^{s_i^* v\mu})\right) \cdot f,$$

where the representatives $v \in \{1, s_i^*\}\backslash W_0$ are chosen such that $\langle v\mu, \alpha_{i^*}^\vee\rangle \in \mathbb{Z}_{\geq 0}$.

Case 1: $f = |s_i \circ \lambda\rangle + |\lambda\rangle$ with $\langle \lambda+\rho, \alpha_i^\vee\rangle \in \ell\mathbb{Z}_{\geq 0}$. Then

$$\begin{aligned}
&(X^{s_i^* v\mu} + X^{v\mu}) \cdot (|s_i \circ \lambda\rangle + |\lambda\rangle) \\
&\quad = |\ell(s_i^* v\mu)^* + s_i \circ \lambda\rangle + |\ell(v\mu)^* + s_i \circ \lambda\rangle + |\ell(s_i^* v\mu)^* + \lambda\rangle + |\ell(v\mu)^* + \lambda\rangle \\
&\quad = |\ell s_i v^*\mu^* + s_i \circ \lambda\rangle + |\ell v^*\mu^* + s_i \circ \lambda\rangle + |\ell s_i v^*\mu^* + \lambda\rangle + |\ell v^*\mu^* + \lambda\rangle \\
&\quad = |s_i \circ (\ell v^*\mu^* + \lambda)\rangle + |s_i \circ (\ell s_i v^*\mu^* + \lambda)\rangle + |\ell s_i v^*\mu^* + \lambda\rangle + |\ell v^*\mu^* + \lambda\rangle \\
&\quad = \begin{cases} a_{\ell v^*\mu^* + s_i \circ \lambda} + a_{\ell v^*\mu^* + \lambda}, & \text{if } \langle \ell v^*\mu^*, \alpha_i^\vee\rangle > \langle \lambda+\rho, \alpha_i^\vee\rangle, \\ a_{\ell s_i v^*\mu^* + \lambda} + a_{\ell v^*\mu^* + \lambda}, & \text{if } \langle \ell v^*\mu^*, \alpha_i^\vee\rangle \leq \langle \lambda+\rho, \alpha_i^\vee\rangle. \end{cases}
\end{aligned}$$

Thus the right hand side is an element of I.

Case 2: $f = |s_i \circ \lambda\rangle + t^{\frac{1}{2}}|\lambda\rangle$ with $0 < \langle \lambda+\rho, \alpha_i^\vee\rangle < \ell$. Then $\ell v^*\mu^* + s_i \circ \lambda = (\ell v^*\mu^* + \lambda)^{(1)}$ so that

$$\begin{aligned}
&(X^{s_i^* v\mu} + X^{v\mu}) \cdot (|s_i \circ \lambda\rangle + t^{\frac{1}{2}}|\lambda\rangle) \\
&\quad = |\ell(s_i^* v\mu)^* + s_i \circ \lambda\rangle + |\ell(v\mu)^* + s_i \circ \lambda\rangle + t^{\frac{1}{2}}|\ell(s_i^* v\mu)^* + \lambda\rangle + t^{\frac{1}{2}}|\ell(v\mu)^* + \lambda\rangle \\
&\quad = |\ell s_i v^*\mu^* + s_i \circ \lambda\rangle + |\ell v^*\mu^* + s_i \circ \lambda\rangle + t^{\frac{1}{2}}|\ell s_i v^*\mu^* + \lambda\rangle + t^{\frac{1}{2}}|\ell v^*\mu^* + \lambda\rangle \\
&\quad = |s_i \circ (\ell v^*\mu^* + \lambda)\rangle + |(\ell v^*\mu^* + \lambda)^{(1)}\rangle + t^{\frac{1}{2}}|s_i \circ (\ell v^*\mu^* + s_i \circ \lambda)\rangle + t^{\frac{1}{2}}|\ell v^*\mu^* + \lambda\rangle \\
&\quad = |s_i \circ (\ell v^*\mu^* + \lambda)\rangle + t^{\frac{1}{2}}|s_i \circ (\ell v^*\mu^* + \lambda)^{(1)}\rangle + |(\ell v^*\mu^* + \lambda)^{(1)}\rangle + t^{\frac{1}{2}}|\ell v^*\mu^* + \lambda\rangle \\
&\quad = \begin{cases} c_{\ell v^*\mu^* + \lambda}, & \text{if } \langle v^*\mu^*, \alpha_i^\vee\rangle \in \mathbb{Z}_{>0}, \\ 2b_{\ell v^*\mu^* + \lambda}, & \text{if } \langle v^*\mu^*, \alpha_i^\vee\rangle = 0, \end{cases}
\end{aligned}$$

since if $s_i^* v\mu \neq v\mu$ then $s_i v^*\mu^* \neq v^*\mu^*$ and $\langle v^*\mu^*, \alpha_i^\vee\rangle \in \mathbb{Z}_{>0}$ then $\langle \ell v^*\mu^* + \lambda + \rho, \alpha_i^\vee\rangle > \ell$ and $\langle \ell v^*\mu^* + \lambda + \rho, \alpha_i^\vee\rangle \notin \ell\mathbb{Z}$. Thus the right hand side is an element of I.

Case 3: Assume $\lambda \in \mathfrak{a}_\mathbb{Z}^*$ with $\langle \lambda+\rho, \alpha_i^\vee\rangle > \ell$ and $\langle \lambda+\rho, \alpha_i^\vee\rangle \notin \ell\mathbb{Z}$. If $\mu \in \mathfrak{a}_\mathbb{Z}^*$ and $\langle \nu, \alpha_i^*\rangle \in \mathbb{Z}_{\geq 0}$ then

$$s_i \circ (v^*\mu^* + \nu) = s_i(v^*\mu^* + \nu + \rho) - \rho = s_i v^*\mu^* + s_i \circ \nu \quad \text{and} \quad (\ell\nu^* + \lambda)^{(1)} = \ell\nu^* + \lambda^{(1)},$$

so that, with $\langle \lambda+\rho, \alpha_i^\vee\rangle = k\ell + j$ with $k > 0$ and $0 \leq j < \ell$,

$$\begin{aligned}
&(X^{s_i^* v\mu} + X^{v\mu}) \cdot (|s_i \circ \lambda\rangle + t^{\frac{1}{2}}|s_i \circ \lambda^{(1)}\rangle + |\lambda^{(1)}\rangle + t^{\frac{1}{2}}|\lambda\rangle) \\
&= |\ell(s_i^* v\mu)^* + s_i \circ \lambda\rangle + |\ell(v\mu)^* + s_i \circ \lambda\rangle + t^{\frac{1}{2}}|\ell(s_i^* v\mu)^* + s_i \circ \lambda^{(1)}\rangle + t^{\frac{1}{2}}|\ell(v\mu)^* + s_i \circ \lambda^{(1)}\rangle \\
&\quad + |\ell(s_i^* v\mu)^* + \lambda^{(1)}\rangle + |\ell(v\mu)^* + \lambda^{(1)}\rangle + t^{\frac{1}{2}}|\ell(s_i^* v\mu)^* + \lambda\rangle + t^{\frac{1}{2}}|\ell(v\mu)^* + \lambda\rangle \\
&= |\ell s_i v^*\mu^* + s_i \circ \lambda\rangle + |\ell v^*\mu^* + s_i \circ \lambda\rangle + t^{\frac{1}{2}}|\ell s_i v^*\mu^* + s_i \circ \lambda^{(1)}\rangle + t^{\frac{1}{2}}|\ell v^*\mu^* + s_i \circ \lambda^{(1)}\rangle \\
&\quad + |\ell s_i v^*\mu^* + \lambda^{(1)}\rangle + |\ell v^*\mu^* + \lambda^{(1)}\rangle + t^{\frac{1}{2}}|\ell s_i v^*\mu^* + \lambda\rangle + t^{\frac{1}{2}}|\ell v^*\mu^* + \lambda\rangle \\
&= |s_i \circ (\ell v^*\mu^* + \lambda)\rangle + |s_i \circ (\ell s_i v^*\mu^* + \lambda)\rangle + t^{\frac{1}{2}}|s_i \circ (\ell v^*\mu^* + \lambda^{(1)})\rangle + t^{\frac{1}{2}}|s_i \circ (\ell s_i v^*\mu^* + \lambda^{(1)})\rangle \\
&\quad + |\ell s_i v^*\mu^* + \lambda^{(1)}\rangle + |(\ell v^*\mu^* + \lambda)^{(1)}\rangle + t^{\frac{1}{2}}|\ell s_i v^*\mu^* + \lambda\rangle + t^{\frac{1}{2}}|\ell v^*\mu^* + \lambda\rangle \\
&= \begin{cases} c_{\lambda + \ell v^*\mu^*} + c_{s_i \circ \lambda^{(1)} + v^*\mu^*}, & \text{if } \langle \ell v^*\mu^*, \alpha_i^\vee\rangle > \ell k > 0, \\ c_{\lambda + \ell v^*\mu^*} + c_{\lambda + s_i v^*\mu^*}, & \text{if } 0 < \langle \ell v^*\mu^*, \alpha_i^\vee\rangle < \ell k, \\ c_{\lambda + \ell v^*\mu^*} + b_{\lambda + s_i v^*\mu^*} + b_{s_i \circ \lambda^{(1)} + v^*\mu^*}, & \text{if } \langle \ell v^*\mu^*, \alpha_i^\vee\rangle = \ell k. \end{cases}
\end{aligned}$$

Thus the right hand side is an element of I.

These computations show that I is stable under the action of $\mathbb{K}[X]^{W_0}$. Thus the action of $\mathbb{K}[X]^{W_0}$ on $\mathcal{F}_\ell = V/I$ is well defined. □

Remark 1.2. One might be tempted to try to define an action of $\mathbb{K}[X]$ on $\mathcal{F}_\ell$ by $X^\mu \cdot |\gamma\rangle = |\gamma + \ell\mu\rangle$ for $\mu, \gamma \in \mathfrak{a}_\mathbb{Z}^*$ but this action is not well defined. For example in the $G = SL_2$ case with $\ell = 5$ pictured after (1.3), one would have $0 = X^{\omega_1} \cdot |-1\rangle = |5-1\rangle = |4\rangle$, which is a contradiction to (1.6). On the other hand $0 = (X^{-\omega_1} + X^{\omega_1}) \cdot |-1\rangle = |-5-1\rangle + |4\rangle = 0$, as it should be.

1.2 The product theorem

Let $\mathring{\mathfrak{g}}$ be the Lie algebra of the reductive group G alluded to in (1.1). For $\lambda \in (\mathfrak{a}_\mathbb{Z}^*)^+$ let $L_{\mathring{\mathfrak{g}}}(\lambda)$ be the irreducible $U\mathring{\mathfrak{g}}$-module of highest weight λ and let $B(\lambda)$ be the crystal of $L_{\mathring{\mathfrak{g}}}(\lambda)$,

$$B(\lambda) = \{\text{LS paths } p \text{ of type } \lambda\} \quad \text{and} \quad \mathrm{wt}(p) \text{ denotes the endpoint of } p,$$

see [Ra, §5]. The *Weyl character* corresponding to λ is the element of $\mathbb{K}[X]^{W_0}$ given by

$$s_\lambda = \mathrm{char}(L_{\mathring{\mathfrak{g}}}(\lambda)) = \frac{\sum_{w \in W_0} \det(w) X^{w \circ \lambda}}{\sum_{w \in W_0} \det(w) X^{w \circ 0}} = \sum_{p \in B(\lambda)} X^{\mathrm{wt}(p)}. \tag{1.14}$$

An *ℓ-restricted* dominant integral weight is $\lambda_0 \in (\mathfrak{a}_\mathbb{Z}^*)^+$ such that $\langle \lambda_0, \alpha_i^\vee \rangle < \ell$ for $i \in \{1, \ldots, n\}$. In other words, if $\omega_1, \ldots, \omega_n$ are the fundamental weights for $\mathring{\mathfrak{g}}$ then a weight $\lambda_0 \in (\mathfrak{a}_\mathbb{Z}^*)^+$ is ℓ-restricted if λ_0 is an element of

$$\Pi_\ell = \{a_1\omega_1 + \cdots + a_n\omega_n \mid a_1, \ldots, a_n \in \{0, 1, \ldots, \ell - 1\}\}. \tag{1.15}$$

Theorem 1.3. *Let $\lambda \in (\mathfrak{a}_\mathbb{Z}^*)^+$ be a dominant integral weight and write*

$$\lambda = \ell\lambda_1 + \lambda_0, \qquad \textit{with } \lambda_0 \in \Pi_\ell \textit{ and } \lambda_1 \in (\mathfrak{a}_\mathbb{Z}^*)^+.$$

Then, with $C_\lambda \in \mathcal{F}_\ell$ as in (1.7) and the $\mathbb{K}[X]^{W_0}$-action on $\mathcal{F}_\ell$ as in Proposition 1.1,

$$C_\lambda = s_{\lambda_1^*} \cdot C_{\lambda_0}.$$

Proof. The proof is accomplished in two steps:

(a) Show that $s_{\lambda_1^*} \cdot C_{\lambda_0}$ is bar invariant.

(b) Show that $s_{\lambda_1^*} \cdot C_{\lambda_0} = |\lambda\rangle + \sum_{\mu \neq \lambda} c_\mu |\mu\rangle$ with $c_\mu \in t^{\frac{1}{2}}\mathbb{Z}[t^{\frac{1}{2}}]$.

Proof of (a): The bar involution and N_γ are defined in (1.4). Since $\langle -\ell w_0\mu, \alpha^\vee \rangle \in \ell\mathbb{Z}$ then

$$\begin{aligned} N_{\gamma - \ell w_0 \mu} &= \mathrm{Card}\{\alpha \in R^+ \mid \langle \gamma - \ell w_0\mu + \rho, \alpha^\vee \rangle \in \ell\mathbb{Z}\} \\ &= \mathrm{Card}\{\alpha \in R^+ \mid \langle \gamma + \rho, \alpha^\vee \rangle \in \ell\mathbb{Z}\} = N_\gamma. \end{aligned}$$

Thus

$$\begin{aligned} \overline{X^\mu \cdot |\gamma\rangle} = \overline{|\gamma - \ell w_0 \mu\rangle} &= (-1)^{\ell(w_0)} (t^{-\frac{1}{2}})^{\ell(w_0) - N_{\gamma - \ell w_0 \mu}} |w_0 \circ (\gamma - \ell w_0 \mu)\rangle \\ &= (-1)^{\ell(w_0)} (t^{-\frac{1}{2}})^{\ell(w_0) - N_{\gamma - \ell w_0 \mu}} |w_0(\gamma + \rho) - \rho - \ell\mu\rangle \\ &= (-1)^{\ell(w_0)} (t^{-\frac{1}{2}})^{\ell(w_0) - N_{\gamma - \ell w_0 \mu}} |w_0 \circ \gamma - \ell\mu\rangle \\ &= (-1)^{\ell(w_0)} (t^{-\frac{1}{2}})^{\ell(w_0) - N_\gamma} |w_0 \circ \gamma - \ell\mu\rangle \\ &= (-1)^{\ell(w_0)} (t^{-\frac{1}{2}})^{\ell(w_0) - N_\gamma} X^{w_0\mu} \cdot |w_0 \circ \gamma\rangle = X^{w_0\mu} \cdot \overline{|\gamma\rangle}, \end{aligned}$$

and since $s_{\lambda_1^*}$ is W_0-invariant,

$$\overline{s_{\lambda_1^*}\cdot C_{\lambda_0}} = (w_0 s_{\lambda_1^*})\cdot \overline{C_{\lambda_0}} = s_{\lambda_1^*}\cdot C_{\lambda_0}.$$

(b) Let $\lambda = \lambda_0 + \ell\lambda_1$ as in the statement of the Theorem and let $a \equiv b$ mean $a = b \bmod t^{\frac{1}{2}}$. By the second formula in (1.7),

$$s_{\lambda_1^*}\cdot C_{\lambda_0} \equiv s_{\lambda_1^*}\cdot|\lambda_0\rangle = \sum_{p\in B(\lambda_1^*)} X^{\mathrm{wt}(p)}|\lambda_0\rangle = \sum_{p\in B(\lambda_1^*)} |\ell\mathrm{wt}(p)^* + \lambda_0\rangle. \tag{1.16}$$

By (1.3), if $\lambda\in\mathfrak{a}_{\mathbb{Z}}^*$ and $\langle\lambda+\rho,\alpha_i^\vee\rangle \geq 0$ then

$$|s_i\circ\lambda\rangle \equiv \begin{cases} -|\lambda\rangle, & \text{if } \langle\lambda+\rho,\alpha_i^\vee\rangle\in\ell\mathbb{Z}_{\geq 0},\\ 0, & \text{if } 0<\langle\lambda+\rho,\alpha_i^\vee\rangle<\ell,\\ -|\lambda^{(1)}\rangle, & \text{otherwise,}\end{cases}$$

where $\lambda^{(1)} = \lambda - j\alpha_i$ if $\langle\lambda+\rho,\alpha_i^\vee\rangle = k\ell + j$ with $j\in\{0,1,\ldots,\ell-1\}$. Since $\lambda^{(1)} = \lambda$ if $\langle\lambda+\rho,\alpha_i^\vee\rangle\in\ell\mathbb{Z}_{\geq 0}$, the first case can be viewed as a special case of the last case to read

$$|s_i\circ\lambda\rangle \equiv \begin{cases} 0, & \text{if } 0<\langle\lambda+\rho,\alpha_i^\vee\rangle<\ell,\\ -|\lambda^{(1)}\rangle, & \text{otherwise.}\end{cases}$$

Assume $\langle\nu+\rho,\alpha_i^\vee\rangle\in\mathbb{Z}_{\leq 0}$ and let $\lambda = s_i\circ(\lambda_0+\ell\nu)$. Since $\langle\rho,\alpha_i^\vee\rangle = 1$ then

$$\begin{aligned}\langle\lambda+\rho,\alpha_i^\vee\rangle &= \langle s_i\circ(\lambda_0+\ell\nu)+\rho,\alpha_i^\vee\rangle = \langle s_i(\lambda_0+\ell\nu+\rho),\alpha_i^\vee\rangle = \langle\lambda_0+\ell\nu+\rho, s_i\alpha_i^\vee\rangle\\ &= -\langle\lambda_0+\ell\nu+\rho,\alpha_i^\vee\rangle = \ell(-\langle\nu+\rho,\alpha_i^\vee\rangle) + (\ell-1-\langle\lambda_0,\alpha_i^\vee\rangle).\end{aligned}$$

Since $\lambda_0\in\Pi_\ell$ then $0\leq \ell-1-\langle\lambda_0,\alpha_i^\vee\rangle<\ell$ and so

$$\begin{aligned}\lambda^{(1)} &= \lambda - (\ell-1-\langle\lambda_0,\alpha_i^\vee\rangle)\alpha_i = s_i\circ(\lambda_0+\ell\nu) - (\ell-1-\langle\lambda_0,\alpha_i^\vee\rangle)\alpha_i\\ &= s_i\lambda_0 + \ell s_i\nu + s_i\rho - \rho - (\ell-1-\langle\lambda_0,\alpha_i^\vee\rangle)\alpha_i\\ &= (\lambda_0 - \langle\lambda_0,\alpha_i^\vee\rangle\alpha_i) + \ell(s_i\nu + s_i\rho-\rho) + (\ell-1)\alpha_i - (\ell-1-\langle\lambda_0,\alpha_i^\vee\rangle)\alpha_i\\ &= \lambda_0 + \ell(s_i\circ\nu).\end{aligned}$$

Thus, since $s_i\circ\lambda = \lambda_0+\ell\nu$,

$$|\lambda_0+\ell\nu\rangle \equiv \begin{cases} 0, & \text{if } \langle\nu+\rho,\alpha_i^\vee\rangle = 0,\\ -|\lambda_0+\ell(s_i\circ\nu)\rangle, & \text{if } \langle\nu+\rho,\alpha_i^\vee\rangle<0.\end{cases}$$

Since $s_i\circ\nu=\nu$ when $\langle\nu+\rho,\alpha_i^\vee\rangle = 0$, then $|\lambda_0+\ell\nu\rangle \equiv -|\lambda_0+\ell(s_i\circ\nu)\rangle$ when $\langle\nu+\rho,\alpha_i^\vee\rangle\in\mathbb{Z}_{\leq 0}$ and, replacing ν by $s_i\circ\nu$, gives $|\lambda_0+\ell\nu\rangle\equiv -|\lambda_0+\ell(s_i\circ\nu)\rangle$ for $\langle\nu+\rho,\alpha_i^\vee\rangle\in\mathbb{Z}_{\geq 0}$. Thus

$$|\lambda_0+\ell\nu\rangle \equiv -|\lambda_0+\ell(s_i\circ\nu)\rangle, \qquad \text{for } \nu\in\mathfrak{a}_{\mathbb{Z}}. \tag{1.17}$$

With formula (1.17) established, follow [Ra, proof of Theorem 5.5] (see also [Li, proof of Theorem 9.1]) to define an involution ι on the set $B(\lambda_1^*)\setminus\{p^+_{\lambda_1^*}\}$, where $p^+_{\lambda_1^*}$ is the unique highest weight path in $B(\lambda_1^*)$.

Let $p\in B(\lambda_1^*)$ and $p\neq p^+_{\lambda_1^*}$. Since $p\neq p^+_{\lambda_1^*}$ the path p crosses a wall out of the fundamental chamber at some point during its trajectory. Let r be such that the first time p leaves the dominant chamber is by crossing the hyperplane $\{x \in \mathfrak{a}_{\mathbb{R}}^* \mid \langle x, \alpha_r^\vee \rangle = 0\}$. Letting $\tilde{e}_r$ and $\tilde{f}_r$ denote the root operators on $B(\lambda_1^*)$, the r-string containing p is

$$S_r(p) = \{q \in B(\lambda_1^*) \mid q = \tilde{e}_r^k p \text{ or } q = \tilde{f}_r^k p \text{ where } k \in \mathbb{Z}_{\geq 0}\}.$$

Let

$$\iota(p) \text{ be the element of } S_r(p) \text{ such that } \mathrm{wt}(\iota(p)) = s_r \circ \mathrm{wt}(p).$$

By (1.17),

$$|\lambda_0 + \ell \mathrm{wt}(p)^*\rangle \equiv -|\lambda_0 + \ell(s_r^* \circ \mathrm{wt}(p)^*)\rangle = -|\lambda_0 + \ell(s_r \circ \mathrm{wt}(p))^*\rangle = -|\lambda_0 + \ell \mathrm{wt}(\iota(p))^*\rangle,$$

and so the map ι partitions the set $B(\lambda_1^*) \setminus \{p^+_{\lambda_1^*}\}$ into pairs $\{p, \iota(p)\}$ which cancel each other in the mod $t^{\frac{1}{2}}$ straightening of the terms of $s_{\lambda_1^*} \cdot C_{\lambda_0}$ in (1.16). Thus

$$s_{\lambda_1^*} \cdot C_{\lambda_0} \equiv |\ell(\lambda_1^*)^* + \lambda_0\rangle = |\lambda_0 + \ell\lambda_1\rangle, \qquad \text{which proves (b).}$$

□

2 The Casselman-Shalika formula

In order to establish the Casselman-Shalika formula it is necessary to use the connection between the abstract Fock space $\mathcal{F}_\ell$ and the affine Hecke algebra H. Let us recall this relationship from [LRS].

2.1 The affine Hecke algebra H

Keep the notation for the finite Weyl group W_0, the simple reflections $s_1, \ldots, s_n$ and the weight lattice $\mathfrak{a}_{\mathbb{Z}}^*$ as in (1.1). For $i, j \in \{1, \ldots, n\}$ with $i \neq j$, let m_{ij} denote the order of $s_i s_j$ in W_0 so that $s_i^2 = 1$ and $(s_i s_j)^{m_{ij}} = 1$ are the relations for the Coxeter presentation of W_0. Let $\mathbb{K} = \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$. The *affine Hecke algebra* is

$$H = \mathbb{K}\text{-span}\{X^\mu T_w \mid \mu \in \mathfrak{a}_{\mathbb{Z}}^*, w \in W_0\}, \tag{2.1}$$

with $\mathbb{K}$-basis $\{X^\mu T_w \mid \mu \in \mathfrak{a}_{\mathbb{Z}}^*, w \in W_0\}$ and relations

$$(T_{s_i} - t^{\frac{1}{2}})(T_{s_i} + t^{-\frac{1}{2}}) = 0, \qquad \underbrace{T_{s_i} T_{s_j} T_{s_i} \ldots}_{m_{ij} \text{ factors}} = \underbrace{T_{s_j} T_{s_i} T_{s_j} \ldots}_{m_{ij} \text{ factors}}, \tag{2.2}$$

$$X^{\lambda+\mu} = X^\lambda X^\mu, \quad \text{and} \quad T_{s_i} X^\lambda - X^{s_i\lambda} T_{s_i} = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \left(\frac{X^\lambda - X^{s_i\lambda}}{1 - X^{-\alpha_i}} \right), \tag{2.3}$$

for $i, j \in \{1, \ldots, n\}$ with $i \neq j$ and $\lambda, \mu \in \mathfrak{a}_{\mathbb{Z}}^*$. The *bar involution on* H is the $\mathbb{Z}$-linear automorphism $\overline{} : H \to H$ given by

$$\overline{t^{\frac{1}{2}}} = t^{-\frac{1}{2}}, \qquad \overline{T_{s_i}} = T_{s_i}^{-1}, \qquad \text{and} \qquad \overline{X^\lambda} = T_{w_0} X^{w_0\lambda} T_{w_0}^{-1}. \tag{2.4}$$

for $i = 1, \ldots, n$ and $\lambda, \mu \in \mathfrak{a}_{\mathbb{Z}}^*$. For $\mu \in \mathfrak{a}_{\mathbb{Z}}^*$ and $w \in W_0$ define

$$X^{t_\mu w} = X^\mu (T_{w^{-1}})^{-1} \qquad \text{and} \qquad T_{t_\mu w} = T_x X^{\mu^+} T_{w_{\mu^+}} (T_{w^{-1} x w_{\mu^+}})^{-1}, \tag{2.5}$$

where μ^+ is the dominant representative of $W_0\mu$, $x \in W_0$ is minimal length such that $\mu = x\mu^+$ and w_{μ^+} is the longest element of the stabilizer $W_{\mu^+} = \mathrm{Stab}_{W_0}(\mu+)$. Define

$$\varepsilon_0 = (-t^{\frac{1}{2}})^{\ell(w_0)} \sum_{z\in W_0} (-t^{-\frac{1}{2}})^{\ell(z)} T_z \qquad \text{and} \qquad \mathbf{1}_0 = (t^{-\frac{1}{2}})^{\ell(w_\nu)} \sum_{z\in W_0} (t^{\frac{1}{2}})^{\ell(z)} T_z,$$

so that

$$\overline{\varepsilon_0} = \varepsilon_0, \quad \overline{\mathbf{1}_0} = \mathbf{1}_0, \qquad \text{and} \qquad \varepsilon_0 T_{s_i} = -t^{-\frac{1}{2}}\varepsilon_0, \quad \text{and} \quad T_{s_i}\mathbf{1}_0 = t^{\frac{1}{2}}\mathbf{1}_0, \tag{2.6}$$

for $i \in \{1, \ldots, n\}$. The algebra $\mathbb{K}[X]$ defined in (1.8) is a subalgebra of H and, by a theorem of Bernstein (see [NR, Theorem 1.4]), the center of H is the ring of symmetric functions,

$$Z(H) = \mathbb{K}[X]^{W_0}. \tag{2.7}$$

Remark 2.1. Formulas (2.4) and (2.5) are just a reformulation of the usual bar involution and the conversion between the Bernstein and Coxeter presentations of the affine Hecke algebra (see for example [NR, Lemma 2.8 and (1.22)]).

2.2 The relation between H and the abstract Fock space $\mathcal{F}_\ell$

In this subsection we follow [LRS, §4.2]. The *affine Weyl group* is

$$W = \{t_\mu w \mid \mu \in \mathfrak{a}^*_\mathbb{Z}, w \in W_0\}, \qquad \text{with} \qquad t_\mu t_\nu = t_{\mu+\nu}, \quad \text{and} \quad w t_\mu = t_{w\mu} w, \tag{2.8}$$

for $\mu, \nu \in \mathfrak{a}^*_\mathbb{Z}$ and $w \in W_0$. Let $\varphi^\vee$ and h be as in (1.11). For $\ell \in \mathbb{Z}_{>0}$, the *level* $(-\ell - h)$ *dot action of* W *on* $\mathfrak{a}^*_\mathbb{Z}$ is given by

$$(t_\mu w) \circ \lambda = (w \circ \lambda) - \ell\mu = w(\lambda + \rho) - \rho - \ell\mu, \tag{2.9}$$

for $\mu \in \mathfrak{a}^*_\mathbb{Z}$, $w \in W_0$ and $\lambda \in \mathfrak{a}^*_\mathbb{Z}$. Note that this is an extension of the dot action of W_0 given in (1.2). Define

$$A_{-\ell-h} = \{\nu \in \mathfrak{a}^*_\mathbb{Z} \mid \langle \nu, \varphi^\vee \rangle \geq -\ell - 1 \text{ and } \langle \nu, \alpha_i^\vee \rangle \leq -1 \text{ for } i \in \{1, \ldots, n\}\}. \tag{2.10}$$

and

$$\mathcal{P}^+_{-\ell-h} = \bigoplus_{\nu \in A_{-\ell-h}} \varepsilon_0 H \mathbf{p}_\nu, \tag{2.11}$$

where ε_0 is as in (2.6) and $\mathbf{p}_\nu$ are formal symbols indexed by $\nu \in A_{-\ell-h}$ satisfying

$$\overline{\mathbf{p}_\nu} = \mathbf{p}_\nu \qquad \text{and} \qquad T_y \mathbf{p}_\nu = (t^{\frac{1}{2}})^{\ell(y)} \mathbf{p}_\nu \text{ for } y \in W_\nu,$$

where $W_\nu = \mathrm{Stab}_W(\nu)$ is the stabilizer of ν under the level $(-\ell - h)$ dot action of W on $\mathfrak{a}^*_\mathbb{Z}$. Define a bar involution

$$\overline{} \colon \mathcal{P}^+_{-\ell-h} \to \mathcal{P}^+_{-\ell-h} \qquad \text{by} \qquad \overline{\varepsilon_0 f \mathbf{p}_\nu} = \varepsilon_0 \bar{f} \mathbf{p}_\nu, \quad \text{for } \nu \in A_{-\ell-h} \text{ and } f \in H. \tag{2.12}$$

For $\lambda \in \mathfrak{a}^*_\mathbb{Z}$ define

$$[X_\lambda] = [X_{w_0 v \circ \nu}] = \varepsilon_0 X^v \mathbf{p}_\nu, \qquad \text{where} \quad \lambda = w_0 v \circ \nu \quad \text{with } \nu \in A_{-\ell-h}, \tag{2.13}$$

and $v \in W$ is such that $X^{vu} = X^v T_u$ for any $u \in W_\nu$. It is helpful to stress that the $(-\ell - h)$ dot action of (2.9) applies here so that, when $v = t_\mu w$ with $\mu \in \mathfrak{a}^*_\mathbb{Z}$ and $w \in W_0$, then $\lambda = -\ell w_0 \mu + (w_0 w) \circ \nu$ and

$$[X_\lambda] = [X_{-\ell w_0 \mu + (w_0 w)\circ\nu}] = \varepsilon_0 X^{t_\mu w} \mathbf{p}_\nu = \varepsilon_0 X^\mu (T_{w^{-1}})^{-1} \mathbf{p}_\nu. \tag{2.14}$$

With these notations, a main result of [LRS] is

Theorem 2.2. *(see [LRS, Theorem 4.7]) Let $\le$ be the dominance order on the set $(\mathfrak{a}_{\mathbb{Z}}^*)^+$ of dominant integral weights. Then the $\mathbb{K}$-linear map $\Phi\colon \mathcal{F}_\ell \to \mathcal{P}_{-\ell-h}^+$ given by*

$$\Phi(\ |\lambda\rangle\ ) = [X_\lambda], \qquad \text{for } \lambda \in \mathfrak{a}_{\mathbb{Z}}^*, \tag{2.15}$$

is a well defined $\mathbb{K}$-module isomorphism satisfying $\overline{\Phi(f)} = \Phi(\overline{f})$.

Since elements of $Z(H) = \mathbb{K}[X]^{W_0}$ commute with ε_0 there is a $\mathbb{K}[X]^{W_0}$-action on $\mathcal{P}_{-\ell-h}$ by left multiplication. The pullback of this action by the isomorphism Φ is the source of the $\mathbb{K}[X]^{W_0}$ action on $\mathcal{F}_\ell$ given in Proposition 1.1,

$$z\Phi(f) = \Phi(zf), \qquad \text{for } z \in Z(H) = \mathbb{K}[X]^{W_0} \text{ and } f \in \mathcal{F}_\ell. \tag{2.16}$$

2.3 Deducing the Casselman-Shalika formula

For $\mu \in \mathfrak{a}_{\mathbb{Z}}^*$ define the "Whittaker function" $A_\mu \in \varepsilon_0 H\mathbf{1}_0$ by

$$A_\mu = \varepsilon_0 X^\mu \mathbf{1}_0. \tag{2.17}$$

See, for example, [HKP, §6] for the connection between p-adic groups and the affine Hecke algebra and the explanation of why A_μ is equivalent to the data of a (spherical) Whittaker function for a p-adic group. As proved carefully in [NR, Theorem 2.7], it follows from (2.6) and (2.3) that

$$\varepsilon_0 H\mathbf{1}_0 \text{ has } \mathbb{K}\text{-basis} \quad \{A_{\lambda+\rho} \mid \ \langle \lambda+\rho, \alpha_i\rangle \in \mathbb{Z}_{\ge 0} \text{ for } i \in \{1,\ldots,n\}\}\,.$$

Following [NR, Theorem 2.4], the Satake isomorphism, $\mathbb{K}[X]^{W_0} \cong \mathbf{1}_0 H\mathbf{1}_0$, and the Casselman-Shalika formula, $A_{\lambda+\rho} = s_\lambda A_\rho$, can be formulated by the following diagram of vector space (free $\mathbb{K}$-module) isomorphisms:

$$\begin{array}{ccccc} Z(H) = \mathbb{K}[X]^{W_0} & \stackrel{\sim}{\longrightarrow} & \mathbf{1}_0 H\mathbf{1}_0 & \stackrel{\sim}{\longrightarrow} & \varepsilon_0 H\mathbf{1}_0 \\ f & \longmapsto & f\mathbf{1}_0 & \longmapsto & A_\rho f\mathbf{1}_0 \\ s_\lambda & \longmapsto & s_\lambda \mathbf{1}_0 & \longmapsto & A_{\lambda+\rho} \end{array} \tag{2.18}$$

This diagram has particular importance due to the fact that $\mathbb{K}[X]^{W_0}$ is an avatar of the Grothendieck group of the category $\mathrm{Rep}(G)$ of finite dimensional representations of G, the spherical Hecke algebra $\mathbf{1}_0 H\mathbf{1}_0$ is a form of the Grothendieck group of K-equivariant perverse sheaves on the loop Grassmanian Gr, and $\varepsilon_0 H\mathbf{1}_0$ is isomorphic to the Grothendieck group of Whittaker sheaves (appropriately formulated N-equivariant sheaves on Gr), see [FGV].

Our proof of the Casselman-Shalika formula is accomplished by restricting Theorem 1.3 to the summand in (2.11) corresponding to $-\rho \in A_{-\ell-h}$. We shall identify this summand with $\varepsilon_0 H\mathbf{1}_0$ via the $Z(H)$-isomorphism

$$\begin{array}{ccc} \varepsilon_0 H\mathbf{1}_0 & \stackrel{\sim}{\longrightarrow} & \varepsilon_0 H\mathbf{p}_{-\rho} \\ \varepsilon_0 X^\mu \mathbf{1}_0 & \longmapsto & \varepsilon_0 X^\mu \mathbf{p}_{-\rho} \end{array}$$

Using the level $(-\ell-h)$ dot action of W from (2.9), the stabilizer of $-\rho$ is W_0 and

$$W \circ (-\rho) = \{t_{-\lambda} \circ (-\rho) \mid \lambda \in \mathfrak{a}_{\mathbb{Z}}^*\} = \{\ell\lambda - \rho \mid \lambda \in \mathfrak{a}_{\mathbb{Z}}^*\}.$$

Since $\langle (\ell\lambda-\rho)+\rho, \alpha^\vee\rangle \in \ell\mathbb{Z}$ for $\alpha \in R^+$, the straightening law (1.3) for elements of $W \circ (-\rho)$ is

$$|s_i \circ (\ell\lambda - \rho)\rangle = -|\ell\lambda - \rho\rangle. \tag{2.19}$$

Theorem 2.3. *(Casselman-Shalika) For $\lambda \in (\mathfrak{a}_{\mathbb{Z}}^*)^+$ and $\mu \in \mathfrak{a}_{\mathbb{Z}}^*$ let s_λ be the Weyl character as defined in* (1.14) *and let A_μ be the Whittaker function as defined in* (2.17)*. Then*

$$s_\lambda A_\rho = A_{\lambda+\rho}.$$

Proof. Using (2.19),

$$\overline{|\ell\lambda-\rho\rangle} = (-1)^{\ell(w_0)}(t^{-\frac{1}{2}})^{\ell(w_0)-\ell(w_0)}|w_0\circ(\ell\lambda-\rho)\rangle = (-1)^{\ell(w_0)}|w_0\circ(\ell\lambda-\rho)\rangle = |\ell\lambda-\rho\rangle$$

and thus $|\ell\lambda-\rho\rangle$ satisfies the conditions of (1.7) so that

$$C_{\ell\lambda-\rho} = |\ell\lambda-\rho\rangle, \qquad \text{for } \lambda \in (\mathfrak{a}_{\mathbb{Z}}^*)^+. \tag{2.20}$$

By (2.14) and (2.17),

$$[X_{-\ell w_0\mu-\rho}] = \varepsilon_0 X^\mu T_{w_0}^{-1}\mathbf{p}_{-\rho} = t^{-\ell(w_0)/2}\varepsilon_0 X^\mu \mathbf{p}_{-\rho} = t^{-\ell(w_0)/2}A_\mu, \qquad \text{for } \mu \in (\mathfrak{a}_{\mathbb{Z}}^*)^+.$$

Using (2.16), (2.20), (2.15) and that $w_0\rho = -\rho$,

$$\begin{aligned} t^{-\ell(w_0)/2}s_\lambda A_\rho &= s_\lambda[X_{-\ell w_0\rho-\rho}] = s_\lambda[X_{(\ell-1)\rho}] = s_\lambda\Phi(|(\ell-1)\rho\rangle) = \Phi(s_\lambda\,|(\ell-1)\rho\rangle\ ) \\ &= \Phi(s_\lambda C_{(\ell-1)\rho}) = \Phi(C_{-\ell w_0\lambda+(\ell-1)\rho}), \qquad \text{by Theorem 1.3}, \\ &= \Phi(|(-\ell w_0\lambda)+(\ell-1)\rho\rangle) = [X_{-\ell w_0\lambda+(\ell-1)\rho}] = [X_{-\ell w_0(\lambda+\rho)-\rho}] \\ &= t^{-\ell(w_0)/2}A_{\lambda+\rho}. \end{aligned}$$

□

3 Quantum groups and LLT polynomials

In this section we describe the main motivation for Theorem 1.3 namely, the Steinberg-Lusztig tensor product theorem for representations of quantum groups at roots of unity. Then we explain the connection between these results and the theory of LLT polynomials.

3.1 Representations of quantum groups at a root of unity

Let $\mathring{\mathfrak{g}}$ be the Lie algebra of the group G alluded to in (1.1). Let $q \in \mathbb{C}^\times$ and let $U_q(\mathring{\mathfrak{g}})$ be the Drinfel'd-Jimbo quantum group corresponding to $\mathring{\mathfrak{g}}$. Let

$\Delta_q(\lambda)$	the Weyl module for $U_q(\mathring{\mathfrak{g}})$ of highest weight λ,
$L_q(\lambda)$	the simple module for $U_q(\mathring{\mathfrak{g}})$ of highest weight λ,

Let

$K(\mathrm{fd}U_q(\mathring{\mathfrak{g}})\text{-mod})$ be the free $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$-module generated by symbols $[\Delta_q(\lambda)]$,

for $\lambda \in \mathfrak{a}_{\mathbb{Z}}^*$. For $\mu \in \mathfrak{a}_{\mathbb{Z}}^*$, denote by W^μ, resp. ${}^\mu W$, the set of minimal length coset representatives for W/W_μ, resp. $W_\mu\backslash W$. Define elements $[L_q(w_0y\circ\nu)]$, for $\nu \in A_{-\ell-h}$ and $y \in {}^0W$ such that $w_0y \in W^\nu$, by the equation

$$[\Delta_q(w_0x\circ\nu)] = \sum_{y\le x}\left(\sum_{i\in\mathbb{Z}_{\ge0}}\left[\frac{\Delta_q(w_0x\circ\nu)^{(i)}}{\Delta_q(w_0x\circ\nu)^{(i+1)}} : L_q(w_0y\circ\nu)\right](t^{\frac{1}{2}})^i\right)[L_q(w_0y\circ\nu)],$$

where $[M : L_q(\mu)]$ denotes the multiplicity of the simple $\mathfrak{g}$-module $L_q(\mu)$ of highest weight μ in a composition series of M and

$$\Delta_q(\lambda) = \Delta_q(\lambda)^{(0)} \supseteq \Delta_q(\lambda)^{(1)} \supseteq \cdots \qquad \text{is the Jantzen filtration of } \Delta_q(\lambda)$$

(see, for example, [Sh, §1.4, §2.3 and §2.10 and Cor. 2.14] and [JM, §4] for the Jantzen filtration in this context).

The combination of [LRS, (3.20)] and [LRS, Theorem 4.7]) is the following connection between the representation theory of the quantum group at a root of unity and the abstract Fock space.

Theorem 3.1. *Let $\ell \in \mathbb{Z}_{>0}$ and let $q \in \mathbb{C}^\times$ such that $q^{2\ell} = 1$. Let $\mathbb{K} = \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$. Then the $\mathbb{K}$-linear map given by*

$$\begin{array}{ccc} K(\mathrm{fd}U_q(\mathring{\mathfrak{g}})\text{-mod}) & \xrightarrow{\Psi_2} & \mathcal{F}_\ell \\ [\Delta_q(\lambda)] & \longmapsto & |\lambda\rangle \\ [L_q(\lambda)] & \longmapsto & C_\lambda \end{array} \qquad \textit{is a well defined isomorphism of } \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]\textit{-modules.}$$

The enveloping algebra $U\mathring{\mathfrak{g}}$ has a presentation by generators $e_1, \ldots, e_n$, $f_1, \ldots, f_n$ and $h_1, \ldots, h_n$ and Serre relations and the quantum group $U_q\mathring{\mathfrak{g}}$ has a presentation by generators $E_1, \ldots, E_n$, $F_1, \ldots, F_n$, and $K_1, \ldots, K_n$ and quantum Serre relations such that, at $q = 1$, E_i becomes e_i and F_i becomes f_i. Following [Lu89] and [CP, Theorem 9.3.12], with appropriate restrictions on ℓ as in [CP, just before Proposition 9.3.5 and Theorem 9.3.12], the *Frobenius map* is the Hopf algebra homomorphism

$$\begin{array}{cccc} Fr\colon & U_q\mathring{\mathfrak{g}} & \longrightarrow & U\mathring{\mathfrak{g}} \\ & E_i^{(r)} & \mapsto & \begin{cases} e_i^{(r/\ell)}, & \text{if } \ell \text{ divides } r, \\ 0, & \text{otherwise,} \end{cases} \\ & F_i^{(r)} & \mapsto & \begin{cases} f_i^{(r/\ell)}, & \text{if } \ell \text{ divides } r, \\ 0, & \text{otherwise,} \end{cases} \\ & K_i & \mapsto & 1. \end{array} \tag{3.1}$$

The *Frobenius twist* of a $U\mathring{\mathfrak{g}}$-module M is the $U_q\mathring{\mathfrak{g}}$-module M^{Fr} with underlying vector space M and $U_q\mathring{\mathfrak{g}}$-action given by

$$um = Fr(u)m, \qquad \text{for } u \in U_q\mathring{\mathfrak{g}} \text{ and } m \in M.$$

Theorem 3.2. *([Lu89, Theorem 7.4], see also [CP, 11.2.9]) Let $\ell \in \mathbb{Z}_{>0}$ and let Π_ℓ be as defined in* (1.15)*. Let $\lambda \in (\mathfrak{a}_\mathbb{Z}^*)^+$ and write*

$$\lambda = \ell\lambda_1 + \lambda_0, \qquad \textit{with } \lambda_0 \in \Pi_\ell \textit{ and } \lambda_1 \in (\mathfrak{a}_\mathbb{Z}^*)^+.$$

Let $q \in \mathbb{C}^\times$ be such that $q^{2\ell} = 1$ and let $L_q(\lambda)$ denote the simple $U_q\mathring{\mathfrak{g}}$-module of highest weight λ. Then

$$L_q(\lambda) \cong \Delta(\lambda_1)^{Fr} \otimes L_q(\lambda_0),$$

where $\Delta(\mu)$ denotes the irreducible $U\mathring{\mathfrak{g}}$-module of highest weight μ.

Accepting Theorem 3.1, Theorem 3.2 is equivalent to the product theorem for abstract Fock space, Theorem 1.3.

3.2 LLT polynomials for general Lie type

In [LLT] and [LT, (43)] and [GH, Definition 6.6], the LLT polynomials for type A are defined by

$$G^{(\ell)}_{\mu/\nu}(x,t^{-1}) \sum_{T\in SSRT_\ell(\mu/\nu)} t^{-\mathrm{spin}(T)}x^T, \tag{3.2}$$

where $SSRT_\ell(\mu/\nu)$ is the set of semistandard ℓ ribbon tableaux of shape μ/ν, $\mathrm{spin}(T)$ is the spin of the tableaux T and X^T is the weight of the tableaux T (see [GH, §6.5] for an efficient review of the combinatorial definitions of semistandard ribbon tableaux, spin and weight).

In Lecouvey [Lcy], there is a definition of LLT polynomials for general Lie type generalizing the definition of [LLT] from type A which proceeds as follows. Define a $\mathbb{K}$-algebra homomorphism

$$\begin{matrix} \psi_\ell\colon & \mathbb{K}[X] & \longrightarrow & \mathbb{K}[X] \\ & X^\mu & \longmapsto & X^{\ell\mu} \end{matrix} \quad \text{so that} \quad \psi_\ell(s_\lambda) = \mathrm{char}(\Delta_q(\lambda)^{Fr}),$$

in the framework of Theorem 3.2. Then [Lcy, (57)] defines

$$G^\ell_\mu = \sum_{\lambda\in(\mathfrak{a}^*_\mathbb{Z})^+} p_{\ell\lambda,\mu}s_\lambda, \tag{3.3}$$

where $p_{\ell\lambda,\mu}\in\mathbb{Z}[t^{\frac{1}{2}}]$ are as in (1.7). As pointed out in [Lcy, Cor. 5.1.3], Theorem 3.1 gives

$$\psi_\ell(s_\lambda) = \mathrm{char}(\Delta(\lambda)^{Fr}) = \mathrm{char}(L_q(\ell\lambda)) = \sum_{\mu\in(\mathfrak{a}^*_\mathbb{Z})^+} p_{\ell\lambda,\mu}(1)\mathrm{char}(\Delta_q(\mu)) = \sum_{\mu\in(\mathfrak{a}^*_\mathbb{Z})^+} p_{\ell\lambda,\mu}(1)s_\mu.$$

As explained carefully in [LRS, Theorem 4.8(b)], the polynomials $p_{\ell\lambda,\mu}$ are parabolic singular Kazhdan-Lusztig polynomials.

In [GH, Definition 5.12 and Corollary 6.4] there is another definition of LLT polynomials for general Lie type:

$$\mathcal{L}^G_{L,\beta,\gamma} = t^{l_{\beta-\gamma}+\ell(w)-\ell(v)} \sum_{\lambda\in(\mathfrak{a}^*_\mathbb{Z})^+} Q^\lambda_{\mu\nu}s_\lambda, \quad \text{where} \quad s_{\lambda^*}\cdot|\nu\rangle = \sum_\mu Q^\lambda_{\mu\nu}|\mu\rangle \tag{3.4}$$

determine the polynomials $Q^\lambda_{\nu\mu}$. Here G is the reductive algebraic group alluded to in (1.1), L is a Levi subgroup of G with Weyl group W_ν, $l_{\beta-\gamma}$ is the nonnegative integer defined in [GH, Remark 5.10], and

$$\mu = v\circ(\eta+\ell\beta) \quad \text{and} \quad \nu = w\circ(\eta+k\gamma), \quad \text{where} \quad v\in W_0t_\beta W_\eta \quad \text{and} \quad w\in W_0t_\gamma W_\eta$$

are minimal representatives. At this point, the reader's discomfort occurring from the transitions between β and γ and v and w and μ and ν is mitigated by recognizing that the relation between these two definitions occurs in the special case $\nu=0$: Theorem 1.3 and (2.20) and the definition of $Q^\lambda_{\mu\nu}$ in (3.4) give

$$C_{\ell\lambda} = s_{\lambda^*}\cdot C_0 = s_{\lambda^*}\cdot|0\rangle = \sum_\mu Q^\lambda_{\mu 0}|\mu\rangle, \quad \text{and comparing with (1.7) gives} \quad p_{\ell\lambda,\mu} = Q^\lambda_{\mu,0}$$

and specifies the close relationship between G^ℓ_μ and $\mathcal{L}^G_{L,\beta,\gamma}$ which occurs at $\nu=0$. They are the same up to a power of t.

4 Tensor product theorem on affine Lie algebra representations

Let $\mathring{\mathfrak{g}}$ be the Lie algebra of G and let $\mathfrak{g} = \mathring{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[\epsilon, \epsilon^{-1}] + \mathbb{C}K + \mathbb{C}d$ be the corresponding affine Kac-Moody Lie algebra (see [Kac, §6.2] – we follow the notation of [LRS, (3.17)]). Let $\ell \in \mathbb{Z}_{>0}$ and let h be the dual Coxeter number. As explained in [LRS, Theorem 3.2], an important result of Kazhdan-Lusztig establishes a relation between level $(-\ell - h)$-representations in parabolic category $\mathcal{O}^{\mathfrak{g}}_{\mathring{\mathfrak{g}}}$ for the affine Lie algebra and the finite dimensional representations of the quantum group $U_q\mathring{\mathfrak{g}}$ with $q^{2\ell} = 1$.

Let

$$\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] = \mathring{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[\epsilon, \epsilon^{-1}] + \mathbb{C}K.$$

By restriction, the modules in $\mathcal{O}^{\mathfrak{g}}_{\mathring{\mathfrak{g}}}$ are $\mathfrak{g}'$-modules. Let Λ_0 be the fundamental weight of the affine Lie algebra so that $L(c\Lambda_0 + \lambda)$ is an irreducible highest weight $\mathfrak{g}$-module of level c (i.e. K acts by the constant c).

Theorem 4.1. *[KL94, Theorem 38.1] There is an equivalence of categories*

$$\begin{array}{ccc}
\left\{\begin{matrix}\text{finite length } \mathfrak{g}'\text{-modules} \\ \text{of level } -\ell-h \text{ in } \mathcal{O}^{\mathfrak{g}}_{\mathring{\mathfrak{g}}}\end{matrix}\right\} & \xrightarrow{\Psi_1} & \left\{\begin{matrix}\text{finite dimensional } U_q(\mathring{\mathfrak{g}})\text{-modules} \\ \text{with } q^{2\ell}=1\end{matrix}\right\} \\
\Delta^{\mathfrak{g}}_{\mathring{\mathfrak{g}}}((-\ell-h)\Lambda_0+\lambda) & \longmapsto & \Delta_q(\lambda) \\
L((-\ell-h)\Lambda_0+\lambda) & \longmapsto & L_q(\lambda)
\end{array}$$

This statement of Theorem 4.1 is for the simply-laced (symmetric) case. With the proper modifications to this statement the result holds for non-simply laced cases as well, see [Lu94, §8.4] and [Lu95].

Let $\lambda \in \mathfrak{a}^*_{\mathbb{Z}}$. Under the composition of the map Ψ_1 in Theorem 4.1 and the map Ψ_2 from Theorem 3.1,

$$\Psi_2(\Psi_1([L((-\ell-h)\Lambda_0 + \ell\lambda)])) = \Psi_2([L_q(\ell\lambda)]) = C_{\ell\lambda} = |\ell\lambda\rangle.$$

Thus it follows from Theorem 4.1, Theorem 3.1 and (2.20) that

$$L((-\ell-h)\Lambda_0 + \ell\lambda) = \Delta^{\mathfrak{g}}_{\mathring{\mathfrak{g}}}((-\ell-h)\Lambda_0 + \ell\lambda) = \mathrm{Ind}^{\mathfrak{g}}_{\mathring{\mathfrak{g}}_0+\mathfrak{b}}(L_{\mathring{\mathfrak{g}}}(\ell\lambda)) \cong U\mathfrak{g} \otimes_{U\mathfrak{k}} L_{\mathring{\mathfrak{g}}}(\ell\lambda), \tag{4.1}$$

where

$$\mathfrak{k} = \bigoplus_{k\in\mathbb{Z}_{\geq 0}} \epsilon^k \Big(\mathfrak{a} \oplus \bigoplus_{\alpha\in R^+} \mathring{\mathfrak{g}}_\alpha + \mathring{\mathfrak{g}}_{-\alpha}\Big) \quad \text{with } R^+ \text{ the set of positive roots of } \mathring{\mathfrak{g}}.$$

As given in (1.14), the Weyl character formula for the $\mathring{\mathfrak{g}}$-module $L_{\mathring{\mathfrak{g}}}(\ell\lambda)$ is

$$\mathrm{char}(L_{\mathring{\mathfrak{g}}}(\ell\lambda)) = s_{\ell\lambda} = \left(\prod_{\alpha\in R^+} \frac{1}{1-X^{-\alpha}}\right) \cdot \sum_{w\in W_0} \det(w) X^{w\circ\ell\lambda}. \tag{4.2}$$

Letting $q = e^\delta$ and using the Poincaré-Birkhoff-Witt theorem, the character of the $\mathfrak{g}$-module in (4.1) is

$$\begin{aligned}
&\mathrm{char}(L((-\ell-h)\Lambda_0 + \ell\lambda)) = \mathrm{char}(\Delta^{\mathfrak{g}}_{\mathring{\mathfrak{g}}}((-\ell-h)\Lambda_0 + \ell\lambda)) = \mathrm{char}(U\mathfrak{g} \otimes_{U\mathfrak{k}} L_{\mathring{\mathfrak{g}}}(\lambda)) \\
&= s_{\ell\lambda} \prod_{k\in\mathbb{Z}_{>0}} \left(\frac{1}{(1-q^{-k})^n} \prod_{\alpha\in R^+} \frac{1}{1-q^{-k}X^\alpha} \cdot \frac{1}{1-q^{-k}X^{-\alpha}}\right) \qquad (4.3) \\
&= \left(\prod_{k\in\mathbb{Z}_{>0}} \frac{1}{(1-q^{-k})^n}\right) \left(\prod_{k\in\mathbb{Z}_{>0}} \prod_{\alpha\in R^+} \frac{1}{1-q^{-k}X^\alpha}\right) \left(\prod_{k\in\mathbb{Z}_{\geq 0}} \prod_{\alpha\in R^+} \frac{1}{1-q^{-k}X^{-\alpha}}\right) \left(\sum_{w\in W_0} \det(w) X^{w\circ\ell\lambda}\right).
\end{aligned}$$

This formula is reminiscent of Weyl-Kac character formula for integrable representations, but we have not yet found a reference for it in the literature. As we have explained in (4.1), this formula is an easy consequence of [KL94] and [Lu89].

The equivalence in Theorem 4.1 is an equivalence of monoidal categories where the product on the left hand side is the fusion tensor product $\hat{\otimes}$ and the product on right hand side is the tensor product coming from the Hopf algebra structure of $U_q\mathring{\mathfrak{g}}$. Thus, in terms of affine Lie algebra representations, the Lusztig-Steinberg tensor product theorem says that

$$\text{if } \lambda \in (\mathfrak{a}_{\mathbb{Z}}^*)^+ \quad \text{and} \quad \lambda = \lambda_0 + \ell\lambda_1 \text{ with } \lambda_0 \in \Pi_\ell$$

where Π_ℓ is as in (1.15), then

$$\begin{aligned} L((-\ell-h)\Lambda_0+\lambda) &\cong L((-\ell-h)\Lambda_0+\lambda_0)\hat{\otimes}L((-\ell-h)\Lambda_0+\ell\lambda_1) \\ &\cong L((-\ell-h)\Lambda_0+\lambda_0)\hat{\otimes}\Delta_{\mathring{\mathfrak{g}}}^{\mathfrak{g}}((-\ell-h)\lambda_0+\ell\lambda_1). \end{aligned} \tag{4.4}$$

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