A Fock space model for decomposition numbers for quantum groups at roots of unity

Martina Lanini  email: lanini@mat.uniroma2.it
Arun Ram  email: aram@unimelb.edu.au
Paul Sobaje  email: sobaje@uga.edu

February 22, 2017

Abstract

In this paper we construct an “abstract Fock space” for general Lie types that serves as a generalisation of the infinite wedge $q$-Fock space familiar in type $A$. Specifically, for each positive integer $\ell$, we define a $\mathbb{Z}[q, q^{-1}]$-module $F_\ell$ with bar involution by specifying generators and “straightening relations” adapted from those appearing in the Kashiwara-Miwa-Stern formulation of the $q$-Fock space. By relating $F_\ell$ to the corresponding affine Hecke algebra we show that the abstract Fock space has standard and canonical bases for which the transition matrix produces parabolic affine Kazhdan-Lusztig polynomials. This property and the convenient combinatorial labeling of bases of $F_\ell$ by dominant integral weights makes $F_\ell$ a useful combinatorial tool for determining decomposition numbers of Weyl modules for quantum groups at roots of unity.

Introduction

The classical Fock space arises in the context of mathematical physics, where one would like to describe the behaviour of certain configurations with an unknown number of identical, non-interacting particles. It is a (non-irreducible) representation of the affine Lie algebra $\widehat{\mathfrak{sl}}_n$. The book [MJD], for example, is an inspiring and friendly tour of applications and connections between this representation, integrable systems, hierarchies of differential equations and infinite dimensional Grassmannians.

Combinatorial models have proven to be incredibly useful in studying the representations of various algebraic objects, such as affine Lie algebras, algebraic groups, Lie algebras, quantum groups and symmetric groups. Often the goal is to express simple modules in terms of “standard” modules (modules whose dimensions and formal characters are computable).

In a wonderful confluence of these two points of view, Lascoux-Leclerc-Thibon [LLT] predicted a connection between Hayashi’s $q$-Fock space [Ha] and decomposition numbers for representations of type A Iwahori-Hecke algebras at roots of unity. The LLT conjecture was proved in work of Ariki [Ar] and Grojnowski [Gr]. The book of Kleshchev [K] shows how successful these methods have been in the study of the modular representation theory of symmetric groups.

This paper arose from an effort to produce an object analogous to the $q$-Fock space that will play the same role in other Lie types, in particular which will be related to the decomposition
numbers for representations of cyclotomic BMW algebras in the same way that the type A case is related to representations of cyclotomic Hecke algebras.

In this paper, we provide a construction of an “abstract” Fock space $\mathcal{F}_\ell$ in a general Lie type setting. Our construction is given by simple combinatorial “straightening relations” which generalize the Kashiwara-Miwa-Stern [KMS] formulation of the $q$-Fock space from the type A case. Adapting the methods used by Leclerc-Thibon [LT] for the type A case, we prove that our abstract Fock space picks up the parabolic affine Kazhdan-Lusztig polynomials for the corresponding affine Hecke algebra of the affine Weyl group (thus generalizing type A results of Varagnolo-Vasserot [VV]). By a combination of the results of Kashiwara-Tanisaki [KT95] and Kazhdan-Lusztig [KL94] and Shan [Sh], these parabolic affine Kazhdan-Lusztig polynomials are graded decomposition numbers of Weyl modules for the corresponding affine Lie algebra at negative level and for the quantum group at a root of unity.

A combinatorial study of the same parabolic affine Kazhdan-Lusztig polynomials was carried out also in [GW], where the authors provided an efficient algorithm which generalizes the algorithm appearing to [LLT] to arbitrary Lie type. The focus of [GW] was the combinatorial understanding of such polynomials rather than the construction of a tool that can play the same role for other Lie types that the infinite wedge space takes in the type A case.

In Section 1 we give the simple construction of the general Lie type “abstract Fock space” $\mathcal{F}_\ell$. We then explain exactly how this general construction relates to the classical type A setting, the framework of Kashiwara-Miwa-Stern and the familiar formulations in terms of semi-infinite wedges, partitions and Maya diagrams. In Section 2 we give an expository treatment of modules with bar involution, general bar-invariant KL-bases, and the construction of KL-polynomials for Hecke algebras, including the singular, parabolic and parabolic-singular cases. Although this material is well known (see, for example, [Soe97]) it is crucial for us to set this up in a form suitable for connecting to the abstract Fock space so that we can eventually see the parabolic affine KL-polynomials in the abstract Fock space $\mathcal{F}_\ell$. In Section 3 we review the results of Kashiwara-Tanisaki, Kazhdan-Lusztig and Shan and concretely connect the decomposition numbers for Weyl modules of affine Lie algebras at negative level and quantum groups at roots of unity to the parabolic and parabolic-singular KL polynomials that have been treated in Section 2. In Section 4, we prove that a certain module with bar involution which is constructed from the affine Hecke algebra is isomorphic to the abstract Fock space $\mathcal{F}_\ell$. This is the key step for proving that the abstract Fock space picks up the appropriate parabolic and parabolic-singular affine KL-polynomials. Finally, at the end of section 4 we tie together the results of Section 3 and 4 to conclude that the abstract Fock space, a combinatorial construct, computes the decomposition numbers of Weyl modules for quantum groups at roots of unity.

Our construction is an important first step in providing combinatorial tools for general Lie type that are direct analogues of the tools that have been so useful in the Type A case. There is much to be done. In particular, we hope that in the future someone will complete the following:

(a) Development of the combinatorics of $\mathcal{F}_\ell$ in parallel to the way it is used in the type A case (see, for example, Kleshchev’s book [Kl]) to provide a “theory of crystals” for other types which applies to the representation theory of the cyclotomic BMW algebras in the same way that the classical crystal theory applies to the modular representation theory of cyclotomic Hecke algebras.

(b) Provide operators on $\mathcal{F}_\ell$ analogous to the $U_q\widehat{\mathfrak{sl}}(\ell)$ action on $\mathcal{F}_\ell$ in the type A case. Taking the point of view of [RT] these operators are the (graded Grothendieck group) images of translation functors for representations of the quantum group at a root of unity. There is significant evidence (see, for example, [ES13], [BW], [BSWW] and [FLLLW]) leading one
to expect that in the type $B, C$ and $D$ cases these operators will provide actions of coideal quantum groups on $F_\ell$.

(c) Elias-Williamson [EW] introduced the diagrammatic Hecke category $D_{BS}$ over a field, which in characteristic zero provides a generators and relations presentation of the Soergel bimodule category. It is expected [RW Conjecture 5.1] that a regular block Rep$_0(G(\mathbb{F}_p))$ is equipped with an action of the category $D_{BS}$ over $\mathbb{F}_p$. This conjecture can be viewed as a (categorical) extension of the project described in (b). Indeed, our abstract Fock space $F_p$ is designed to be a decategorification of Rep$(G(\mathbb{F}_p))$. For the type A case, Riche-Williamson [RW] have used the $U(\widehat{\mathfrak{g}}_n)$-action on $F_p$ (in its infinite wedge space formulation) to prove their conjecture and hence to show that the $p$-canonical basis corresponds to the indecomposable tilting modules in Rep$_0(G(\mathbb{F}_p))$. It is possible that our abstract Fock space $F_p$ could be a useful tool for generalizing the results of [RW] to other Lie types in a uniform fashion (taking care also of singular blocks).

It is a pleasure to thank all the institutions which have supported our work on this paper, including especially the University of Melbourne, the Australian Research Council (grants DP1201001942 and DP130100674) and ICERM (Institute for Computational and Experimental Research in Mathematics). M.L. would like to thank the University of Edinburgh, which supported her research during the final part of this project.

1 The abstract Fock space

1.1 Fock space $F_\ell$

Let $W_0$ be a finite Weyl group, generated by simple reflections $s_1, \ldots, s_n$, and acting on a lattice of weights $a_\mathbb{Z}^*$. For example, this situation arises when $T$ is a maximal torus of a reductive algebraic group $G$,  

$$a_\mathbb{Z}^* = \text{Hom}(T, \mathbb{C}^*) \quad \text{and} \quad W_0 = N(T)/T,$$

(1.1)

where $N(T)$ is the normalizer of $T$ in $G$. The simple reflections in $W_0$ correspond to a choice of Borel subgroup $B$ of $G$ which contains $T$. Let $R^+$ denote the positive roots. Let $\alpha_1, \ldots, \alpha_n$ be the simple roots and let $\alpha_1^\vee, \ldots, \alpha_n^\vee$ be the simple coroots. The dot action of $W_0$ on $a_\mathbb{Z}^*$ is given by  

$$w \circ \lambda = w(\lambda + \rho) - \rho, \quad \text{where} \quad \rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$$

(1.2)

is the half sum of the positive roots for $G$ (with respect to $B$).

Fix $\ell \in \mathbb{Z}_{>0}$. The Fock space $F_\ell$ is the $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$-module generated by $\{|\lambda\rangle \mid \lambda \in a_\mathbb{Z}^*\}$ with relations  

$$|s_i \circ \lambda\rangle = \begin{cases}  
-|\lambda\rangle, & \text{if } \langle \lambda + \rho, \alpha_i^\vee \rangle \in \ell \mathbb{Z}_{\geq 0}, \\
-t^{\frac{1}{2}}|\lambda\rangle, & \text{if } 0 < \langle \lambda + \rho, \alpha_i^\vee \rangle < \ell, \\
-t^{\frac{1}{2}}|s_i \circ \lambda|^{(1)}\rangle - |\lambda|^{(1)}\rangle - t^{\frac{1}{2}}|\lambda\rangle, & \text{if } \langle \lambda + \rho, \alpha_i^\vee \rangle > \ell \text{ and } \langle \lambda + \rho, \alpha_i^\vee \rangle \notin \ell \mathbb{Z},
\end{cases}$$

(1.3)

where $\lambda^{(1)} = \lambda - j \alpha_i$ if $\langle \lambda + \rho, \alpha_i^\vee \rangle = k\ell + j$ with $k \in \mathbb{Z}_{>0}$ and $j \in \{1, \ldots, \ell - 1\}$. $t$

The following picture illustrates the terms in (1.3). This is the case $G = SL_2$ with $\ell = 5$, $\langle \omega_1, \alpha_1^\vee \rangle = 1$ and $\alpha_1 = 2\omega_1$ and, in the picture, $\lambda$ corresponds to the third case of (1.3), $\mu$ to the
first case and \( \nu \) to the second case.

Define a \( \mathbb{Z} \)-linear involution \( \overline{} : \mathcal{F}_\ell \rightarrow \mathcal{F}_\ell \) by

\[
\overline{t^j} = t^{-\frac{j}{2}} \quad \text{and} \quad \overline{\langle \lambda \rangle} = (-1)^{\ell(w_0)}(t^{-\frac{1}{2}})^{\ell(w_0)-N_\lambda} |w_0 \circ \lambda|.
\]

where \( w_0 \) is the longest element of \( W_0 \), \( \ell(w_0) = \text{Card}(R^+) \) is the length of \( w_0 \), and \( N_\lambda = \text{Card}\{\alpha \in R^+ \mid \langle \lambda + \rho, \alpha^\vee \rangle \in \ell\mathbb{Z}\} \).

### 1.2 \( \mathcal{F}_\ell \) is a KL-module

The *dominant integral weights* with the *dominance partial order* \( \preceq \) are the elements of

\[
(a^+_2)^+ = \{ \lambda \in a^+_2 \mid \langle \lambda + \rho, \alpha_i^\vee \rangle > 0 \text{ for } i = 1, 2, \ldots, n \}
\]

with \( \mu \leq \lambda \) if \( \mu \in \lambda - \sum_{\alpha \in R^+} \mathbb{Z}_{\geq 0} \alpha \).

In combination, Theorem 1.1 and Proposition 2.1 below give that \( \mathcal{F}_\ell \) has bases

\[
\{ \langle \lambda \rangle \mid \lambda \in (a^+_2)^+ \} \quad \text{and} \quad \{ C_\lambda \mid \lambda \in (a^+_2)^+ \}
\]

where \( C_\lambda \) are determined by

\[
C_\lambda = C_\lambda \quad \text{and} \quad C_\lambda = \langle \lambda \rangle + \sum_{\mu \neq \lambda} p_{\mu \lambda} |\mu\rangle, \quad \text{with } p_{\mu \lambda} \in t^{\frac{j}{2}} \mathbb{Z}[t^{\frac{j}{2}}].
\]

**Theorem 1.1.** Let \( \mathcal{F}_\ell \) be defined as \( (2.3) \) and let \( \mathcal{L} = \{ \langle \lambda \rangle \mid \lambda \in (a^+_2)^+ \} \). Then, with the definition of KL-module as in Section 2, \( \mathcal{L} \) is a basis of \( \mathcal{F}_\ell \) and

\[
((a^+_2)^+, \mathcal{F}_\ell, \mathcal{L}, \overline{} : \mathcal{F}_\ell \rightarrow \mathcal{F}_\ell) \text{ is a KL-module.}
\]

**Proof.** (Sketch) If \( \lambda \in (a^+_2)^+ \) then there are only finitely many \( \mu \leq \lambda \) with the property that \( \mu \) is also dominant (see \([51, \text{Cor. 1.4}]\)).

Let \( i \in \{1, \ldots, n\} \) and let \( \lambda \in a^+_2 \) be such that \( 0 < \langle \lambda + \rho, \alpha_i^\vee \rangle \). Write

\[
\langle \lambda + \rho, \alpha_i^\vee \rangle = k \ell + j, \quad \text{with } k \in \mathbb{Z} \text{ and } j \in \{0, 1, \ldots, \ell - 1\}.
\]

When \( j \neq 0 \) define

\[
\lambda^{(1)} = \lambda - j \alpha_i \quad \text{and} \quad \lambda^{(j+1)} = (\lambda^{(j)})^{(1)}.
\]

Then induction on \( k \) using the third case in \( (1.3) \) gives

\[
|s_i \circ \lambda| = (-t^{\frac{j}{2}})|\lambda| + (-t^{\frac{j}{2}})t^{-\frac{1}{2}}|\lambda^{(1)}| + (-t^{\frac{j}{2}})|s_i \circ \lambda^{(1)}|
\]

\[
= (-t^{\frac{j}{2}})|\lambda| + (-t^{\frac{j}{2}})t^{-\frac{1}{2}}|\lambda^{(1)}|
\]

\[
+ (-t^{\frac{j}{2}})(-t^{\frac{j}{2}}) \left( |\lambda^{(1)}| - (t^{\frac{j}{2}} - t^{-\frac{1}{2}})|\lambda^{(2)}| - (t^{\frac{j}{2}} - t^{-\frac{1}{2}})(-t^{\frac{j}{2}})|\lambda^{(3)}| \right)
\]

\[
- \cdots - (t^{\frac{j}{2}} - t^{-\frac{1}{2}})(-t^{\frac{j}{2}})^{k-2}|\lambda^{(k)}| \right)
\]

\[
= (-t^{\frac{j}{2}}) \left( |\lambda| - (t^{\frac{j}{2}} - t^{-\frac{1}{2}})|\lambda^{(1)}| - (t^{\frac{j}{2}} - t^{-\frac{1}{2}})(-t^{\frac{j}{2}})|\lambda^{(2)}| \right)
\]

\[
- \cdots - (t^{\frac{j}{2}} - t^{-\frac{1}{2}})(-t^{\frac{j}{2}})^{k-1}|\lambda^{(k)}| \right). \quad (1.8)
\]
More generally, for \( \lambda \in \mathfrak{a}_z^\times \) such that \( \langle \lambda + \rho, \alpha_i^\vee \rangle \neq 0 \) for \( i \in \{1, \ldots, n\} \) let \( \lambda^+ \) be the dominant representative of \( W_0 \circ \lambda \) and let
\[
R(\lambda) = \{ \alpha \in R^+ \mid \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{\leq 0} \}, \\
R_\ell(\lambda) = \{ \alpha \in R^+ \mid \langle \lambda + \rho, \alpha^\vee \rangle \in \ell\mathbb{Z}_{\leq 0} \}.
\] (1.9)

Then iterating (1.8) produces \( c_\mu \in (t^{-\frac{i}{2}} - t^{\frac{j}{2}})\mathbb{Z}[t^{\frac{k}{2}}] \) so that
\[
|\lambda| = (-1)^{\text{Card}(R(\lambda))} (t^{\frac{j}{2}})^{\text{Card}(R(\lambda)) - \text{Card}(R_\ell(\lambda))} \left( |\lambda^+| + \sum_{\mu^+ \in (a_z^\pm)^+} c_\mu |\mu^+| \right). 
\] (1.10)

With (1.10) in hand all steps in a direct proof of Theorem 1.1 are straightforward except proving that \( \{ |\lambda^+| \mid \lambda^+ \in (a_z^\pm)^+ \} \) is a basis of \( \mathcal{F}_\ell \) (the linear independence is the issue). To prove this directly the unpleasant step is to show that if \( \lambda^+ \in (a_z^\pm)^+ \) and \( w \in W_0 \) then \( |w \circ \lambda^+| \) defined by \( |w \circ \lambda^+| = |s_{i_1} \circ (s_{i_2} \circ \cdots (s_{i_k} \circ \lambda^+))| \) for a reduced decomposition \( w = s_{i_1} s_{i_2} \cdots s_{i_k} \) will produce a well defined element of \( \mathcal{F}_\ell \) (independent of the choice of reduced decomposition). Alternatively, it is possible to use a Gröbner basis argument using the ordering \( \preceq \) on \( \mathfrak{a}_z^\times \) given by

\[
\mu \prec \lambda \quad \text{if } \mu^+ < \lambda^+ \text{ in dominance order and} \\
\mu \circ \lambda^+ \prec v \circ \lambda^+ \quad \text{if } u < v \text{ in Bruhat order},
\]

where \( \mu^+ \) denotes the dominant representative of \( W_0 \circ \mu \). However, we will not complete this sketch here as Theorem 1.1 is a consequence of the realization of \( \mathcal{F}_\ell \) provided by Corollary 4.7.

1.3 \( \mathcal{F}_\ell \) as a semi-infinite wedge space for the case \( G = \text{GL}_\infty \)

Fix \( \ell \in \mathbb{Z}_{>0} \). The semi-infinite wedge space considered by Kashiwara-Miwa-Stern [KMS] (43)-(45) is
\[
\mathcal{F}_\ell = \Lambda^\infty V = \mathbb{C}\text{-span} \left\{ v_{a_1} \wedge v_{a_2} \wedge \cdots \mid a_j \in \mathbb{Z} \text{ and, for all but a finite number of } j, \ a_j = -j + 1 \right\},
\] (1.11)

where \( v_a, a \in \mathbb{Z} \) are symbols, and if \( a < b \) then
\[
v_b \wedge v_a = \begin{cases} 
-(v_a \wedge v_b), & \text{if } a - b \in \ell\mathbb{Z}_{\geq 0}, \\
-t^{\frac{i}{2}}(v_a \wedge v_b), & \text{if } 0 < a - b < \ell, \\
-t^{\frac{j}{2}}(v_{b+j} \wedge v_{a-j}) - (v_{a-j} \wedge v_{b+j}) - t^{\frac{k}{2}}(v_a \wedge v_b), & \text{if } a - b = k\ell + j \text{ with } k \in \mathbb{Z} \text{ and } j \in \{0, 1, \ldots, \ell - 1\}.
\end{cases}
\]

From the point of view of (1.1) and (1.3), this is the case \( G = \text{GL}_\infty(\mathbb{C}) \) with \( \mathfrak{a}_z^\times = \mathbb{Z}\text{-span} \{ \varepsilon_1, \varepsilon_2, \ldots \} \) and \( W_0 \) the infinite symmetric group generated by \( s_1, s_2, s_3, \ldots \), where \( s_i \) is the simple transposition that switches \( \varepsilon_i \) and \( \varepsilon_{i+1} \). This framework illustrates that the straightening laws of (1.3) are generalizations of those that appear in [KMS] (43-45) and [LT] Prop. 5.11.

In the semi-infinite wedge space setting of (1.11) the bar involution appears in [LE] §3.6, and [LT] Prop. 5.9 and (85)]. Kashiwara-Miwa-Stern [KMS] already have the affine Hecke algebra playing a significant role in their story; in retrospect, this is not unrelated to the role that the
affine Hecke algebra takes for us in Corollary 4.7. Leclerc-Thibon [LT] also have the affine Hecke algebra playing an important role, essentially the same as in this paper.

The correspondence between partitions, semi-infinite wedges and Maya diagrams appears in [MJD] §4.3 and Fig. 9.3 (see also [Le] §2.2.1 and [Tin] Fig. 1). Following [Le] §2.2.1, the partition

\[ \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0) = (\lambda_1, \lambda_2, \ldots, \lambda_s, 0, 0, \ldots) \]

corresponds to

the semi-infinite wedge \( |\lambda\rangle = v_{\lambda_1-1+1} \wedge v_{\lambda_2-2+1} \wedge \cdots \).

The \( \rho \)-shift which appears in (1.3) also appears here since \( \rho \) can be taken to be

\[ \rho = (0, 1, 2, 3, \ldots) \]

for the case of \( G = GL_\infty(\mathbb{C}) \).

In the picture below, when following the bold boundary of the partition \( \lambda = (4, 4, 3, 3, 2, 1, 1, 1) \) the positive slope edges correspond to black dots in the Maya diagram and the black dots in the Maya diagram correspond to the indices in the corresponding wedge \( |\lambda\rangle = v_{i_1} \wedge v_{i_2} \wedge \cdots \).

\[ \lambda = (4, 4, 3, 3, 2, 1, 1, 1) \quad \text{with} \quad |\lambda\rangle = v_4 \wedge v_3 \wedge v_1 \wedge v_0 \wedge v_{-2} \wedge v_{-3} \wedge v_{-5} \wedge v_{-6} \wedge v_{-7} \wedge v_{-9} \wedge v_{-10} \wedge \cdots \]

2 KL-modules and bases

The \textit{bar involution} on the ring \( \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}] \) of Laurent polynomials in \( t^{\frac{1}{2}} \) is the ring isomorphism

\[ \overline{\cdot} : \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}] \rightarrow \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}] \quad \text{given by} \quad \overline{t^{\frac{1}{2}}} = t^{-\frac{1}{2}}. \quad (2.1) \]

A \textit{KL-module} over \( \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}] \) is a tuple \((\Lambda, M, \{T_w\}_{w \in \Lambda}, : M \rightarrow M)\) where

(a) \( \Lambda \) is a partially ordered set such that if \( w \in \Lambda \) then \( \{v \in \Lambda \mid v \leq w\} \) is finite,
(b) $M$ is a free $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$-module with basis $\{T_w \mid w \in \Lambda\}$,

(c) $\overline{\cdot}$: $M \to M$ is a $\mathbb{Z}$-module homomorphism such that if $m \in M$, $a \in \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ and $w \in \Lambda$ then

$$a \cdot m = \overline{a} \cdot m = \overline{a} \cdot m$$

and $T_w = T_w + \sum a_{vw} T_v$,

$$\overline{T_w} = T_w + \sum_{v < w} a_{vw} T_v$$  \hspace{1cm} (2.2)

where $\overline{a}$ is given by (2.1) and the coefficients $a_{v,w}$ in the expansion of $\overline{T_w}$ are elements of $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$.

**Proposition 2.1.** Let $(\Lambda, M, \{T_w\}_v)$ be a KL-module over $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$. There is a unique basis $\{C_w \mid w \in \Lambda\}$ of $M$ characterized by

$$\overline{C_w} = C_w \quad \text{and} \quad C_w = T_w + \sum_{v < w} p_{vw} T_v, \quad \text{with} \quad p_{vw} \in t^{\frac{1}{2}} \mathbb{Z}[t^{\frac{1}{2}}] \text{ for } v < w.$$  \hspace{1cm} (2.3)

Let $d_{vw}$ be the coefficients in the expansion

$$T_w = C_w + \sum_{v < w} d_{vw} C_v, \quad \text{with } d_{vw} \in t^{\frac{1}{2}} \mathbb{Z}[t^{\frac{1}{2}}] \text{ for } v < w.$$  \hspace{1cm} (2.4)

The polynomials $p_{uw}$ and $d_{uw}$, $0 = 0$ are specified, inductively, by the equations $p_{uw} = d_{uw} = 0$ unless $u \leq w$, $p_{uw} = d_{uw} = 1$,

$$p_{uw} - \overline{p_{uw}} = \sum_{u < z \leq w} a_{uz} \overline{p_{zw}}, \quad \text{and} \quad d_{uw} - \overline{d_{uw}} = -\sum_{u < z < w} d_{uz} a_{zw}.$$  \hspace{1cm} (2.5)

**Proof.** The matrices $A = (a_{uw})$, $P = (p_{vw})$ and $D = (d_{vw})$ defined by (2.2) and (2.4) are all upper triangular with 1’s on the diagonal. Then

$$A \overline{A} = 1, \quad P = AP, \quad \overline{D} = DA \quad \text{and} \quad DP = 1 = PD,$$  \hspace{1cm} (2.6)

since

$$T_w = \overline{T_w} = \sum_{v} a_{vw} T_v = \sum_{u,v} a_{uv} \overline{a_{uv}} T_u,$$

$$\sum_{u} p_{uw} T_v = C_w = \overline{C_w} = \sum_{v} \overline{p_{vw}} T_v = \sum_{u,v} \overline{p_{uv}} a_{uv} T_u,$$

$$C_w + \sum_{v < w} d_{vw} C_v = \overline{T_w} = \sum_{u \leq w} a_{uw} T_u = \sum_{v \leq u \leq w} a_{uw} d_{vu} C_v.$$  

Letting $f = p_{uw} - \overline{p_{uw}} = \sum_{k \in \mathbb{Z}} f_k (t^{\frac{1}{2}})^k$,

$$f = p_{uw} - \overline{p_{uw}} = (P - \overline{P})_{uw} = ((A - 1)P)_{uw} = (AP - \overline{P})_{uw} = \sum_{u < z \leq w} a_{uz} \overline{p_{zw}},$$  \hspace{1cm} (2.7)

and the identity

$$\overline{f} = \overline{(p_{uw} - \overline{p_{uw}})} = p_{uw} - \overline{p_{uw}} = -f \quad \text{implies} \quad f_k = -f_{-k}, \quad \text{for } k \in \mathbb{Z}.$$

Thus $p_{uw} = \sum_{k \in \mathbb{Z} < 0} f_k (t^{\frac{1}{2}})^k$. The derivation of the formula for the entries of $D$ is similar using $D - \overline{D} = D - DA$ and $a_{uw} = 1$.  

\hspace{1cm} $\square$
2.1 KL modules associated to Hecke algebras of Coxeter groups

Let $W$ be a Coxeter group generated by $s_0, s_1, \ldots, s_n$ so that

$$s_i^2 = 1, \quad \text{and} \quad (s_is_j)^{m_{ij}} = 1, \quad \text{for} \ i \neq j \quad (2.8)$$

($m_{ij}$ is allowed to be $\infty$, in which case, the expression $(s_is_j)^{m_{ij}} = 1$ should be interpreted as “$s_is_j$ has infinite order”). Let $w \in W$. A reduced word for $w$ is a sequence $s_{i_1} \cdots s_{i_r}$ of generators with $w = s_{i_1} \cdots s_{i_r}$ and $r$ minimal. The length of $w$ is $\ell(w) = r$ if $s_{i_1} \cdots s_{i_r}$ is a reduced word for $w$. The Bruhat order $\leq$ on $W$ is given by $v \leq w$ if there is a reduced word $s_{j_1} \cdots s_{j_m}$ for $v$ which is a subword of a reduced word $s_{i_1} \cdots s_{i_r}$ for $w$.

The Hecke algebra of $W$ is the $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$-algebra $H$ with generators $T_0, T_1, \ldots, T_n$ and relations

$$T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_i + 1 \quad \text{and} \quad T_iT_j = T_jT_i \quad \text{if} \ \{i,j\} \text{is a} \ m_{ij} \text{-basis} \quad (2.9)$$

For $w \in W$ define $T_w = T_{s_{i_1}} \cdots T_{s_{i_r}}$ for a reduced word $w = s_{i_1} \cdots s_{i_r}$. By [Bou, Ch. 4, §2, Ex. 23]), $T_w$ does not depend on the choice of reduced word for $w$ and

$$\{T_w \mid w \in W\} \quad \text{is a} \ \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]-\text{basis of} \ H. \quad (2.10)$$

Define a $\mathbb{Z}$-algebra automorphism $\overline{\cdot}: H \to H$ by

$$\overline{t^{\frac{1}{2}}} = t^{-\frac{1}{2}} \quad \text{and} \quad T_{\overline{w}} = T_{w^{-1}} \quad \text{for} \ w \in W. \quad (2.11)$$

By the first relation in (2.9), $T_i^{-1} = T_i - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})$, so that if $w = s_{i_1} \cdots s_{i_r}$ is a reduced word for $w \in W$ then

$$T_w = T_{i_1} \cdots T_{i_r} = T_{i_1}^{-1} \cdots T_{i_r}^{-1} = \left(T_{i_1} - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\right) \cdots \left(T_{i_r} - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\right)$$

$$= T_w + \sum_{v < w} a_{vw} T_v, \quad \text{with} \ a_{vw} \in (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}].$$

With standard basis as in (2.10) indexed by the poset $W$ and with bar involution as in (2.11),

$$H$$

is a KL-module over $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ and, from Proposition 2.1 there is a unique basis $\{C_w \mid w \in W\}$ determined by

$$\overline{C_x} = C_x \quad \text{and} \quad C_x = \sum_{\substack{y \leq x \\text{in} \ W \ \text{with} \ y < x}} (-1)^{\ell(x) - \ell(y)} P_{y,x}(t^{\frac{1}{2}})T_y, \quad (2.12)$$

with $P_{y,x}(t^{\frac{1}{2}}) \in t^{\frac{1}{2}}\mathbb{Z}[t^{\frac{1}{2}}]$ for $y < x$. The polynomials $P_{y,x}$ are the Kazhdan-Lusztig polynomials for $H$.

2.2 Singular and parabolic KL polynomials

2.2.1 The projectors

Let $J, \gamma \subseteq \{0, 1, \ldots, n\}$ and let $W_j$ and $W_\gamma$ be the subgroups of $W$ generated by the corresponding simple reflections,

$$W_j = \langle s_j \mid j \in J \rangle \quad \text{and} \quad W_\gamma = \langle s_k \mid k \not\in \gamma \rangle, \quad \text{respectively.} \quad (2.13)$$
Assume that \( W_\nu \) and \( W_\gamma \) are both finite. Let \( w_\nu \) be the longest element of \( W_\nu \) and let \( w_\gamma \) be the longest element of \( W_\gamma \) and let

\[
W_\nu(t) = \sum_{z \in W_\nu} t^{\ell(z)} \quad \text{and} \quad W_\gamma(t) = \sum_{z \in W_\gamma} t^{\ell(z)}. \tag{2.14}
\]

Then

\[
1_\nu = \sum_{z \in W_\nu} \left( t^{-\frac{1}{2}} \right)^{\ell(w_\nu) - \ell(z)} T_z = \left( t^{-\frac{1}{2}} \right)^{\ell(w_\nu)} \sum_{z \in W_\nu} \left( t^{\frac{1}{2}} \right)^{\ell(z)} T_z, \quad \text{and}
\]

\[
\varepsilon_\gamma = \sum_{z \in W_\gamma} \left( -t^{\frac{1}{2}} \right)^{\ell(w_\gamma) - \ell(z)} T_z = \left( -t^{\frac{1}{2}} \right)^{\ell(w_\gamma)} \sum_{z \in W_\gamma} \left( -t^{-\frac{1}{2}} \right)^{\ell(z)} T_z, \tag{2.15}
\]

satisfy

\[
\overline{1}_\nu = 1_\nu, \quad T_j 1_\nu = t^{\frac{j}{2}} 1_\nu \quad \text{for} \quad j \in J, \quad \text{and} \quad 1_\nu^2 = \left( t^{-\frac{1}{2}} \right)^{\ell(w_\nu)} W_\nu(t) 1_\nu,
\]

\[
\varepsilon_\gamma \varepsilon_\gamma = \varepsilon_\gamma, \quad \varepsilon_\gamma T_k = -t^{-\frac{1}{2}} \varepsilon_\gamma \quad \text{for} \quad k \notin \gamma, \quad \text{and} \quad \varepsilon_\gamma = \left( -t^{-\frac{1}{2}} \right)^{\ell(w_\gamma)} W_\gamma(t) \varepsilon_\gamma,
\]

and

\[
1_\nu = T_{w_\nu} + \sum_{x < w_\nu} h_{x,w_\nu} T_x \quad \text{and} \quad \varepsilon_\gamma = T_{w_\gamma} + \sum_{x < w_\gamma} h_{x,w_\gamma} T_x,
\]

with coefficients \( h_{x,w_\nu} \in t^{-\frac{1}{2}} \mathbb{Z}[t^{-\frac{1}{2}}] \) and \( h_{x,w_\gamma} \in t^{\frac{1}{2}} \mathbb{Z}[t^{\frac{1}{2}}]. \)

### 2.2.2 Singular block KL polynomials

As in \( \text{(2.13)} \), let \( W_\nu = \langle s_j \mid j \in J \rangle \) and let \( W^\nu \) be the set of minimal length coset representatives of the cosets in \( W/W_\nu \). The \( \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}] \)-module

\[
H 1_\nu \quad \text{has basis} \quad \{ T_u 1_\nu \mid u \in W^\nu \} \quad \text{and} \quad \overline{\phantom{1}} : H 1_\nu \to H 1_\nu, \tag{2.16}
\]

since \( \overline{1}_\nu = 1_\nu \). The Bruhat order \( W^\nu \) is the restriction of the Bruhat order on \( W \) to \( W^\nu \) and, with these structures, \( H 1_\nu \) is a KL-module.

If \( \varphi : H \to H 1_\nu \) is the surjective KL-module homorphism defined by right multiplication by \( 1_\nu \) then

\[
H 1_\nu \quad \text{has KL-basis} \quad \{ C_u 1_\nu \mid u \in W^\nu \}, \tag{2.17}
\]

where \( \{ C_w \mid w \in W \} \) is the KL-basis of \( H \). With notation as in \( \text{(2.12)} \),

\[
C_x 1_\nu = \sum_{y \leq x} (-1)^{\ell(x) - \ell(y)} P_{y,x}(t^{\frac{1}{2}}) T_y 1_\nu, \quad \text{for} \quad x \in W^\nu, \tag{2.18}
\]

where the sum can contain several \( y \leq x \) which have the same coset \( yW_\nu \) (and this is how cancellation can occur in the sum \( \text{(2.18)} \)). Since

\[
T_z 1_\nu = (t^{\frac{1}{2}})^{\ell(z)} T_{xz} 1_\nu, \quad \text{for} \quad z \in W_\nu,
\]

the coefficients \( P^\nu_{y,x} \) in

\[
C_x 1_\nu = \sum_{y \in W^\nu} (-1)^{\ell(x) - \ell(y)} P^\nu_{y,x} T_y 1_\nu \quad \text{are} \quad P^\nu_{y,x} = \sum_{z \in W_\nu} (-1)^{\ell(y) - \ell(yz)} (t^{\frac{1}{2}})^{\ell(z)} P_{yz,x}. \tag{2.19}
\]
Since $C_w T_{s_i} = -t^{-\frac{1}{2}} C_w$ unless $ws_i > w$ (see \cite{Hu} Prop. 7.14(a)), it follows that $C_w (T_{s_i} + t^{-\frac{1}{2}}) = 0$ unless $ws_i > w$ so that

$$C_w 1_\nu = 0, \quad \text{unless } w \in W^\nu. \quad (2.20)$$

In summary, right multiplication by $1_\nu$ is a surjective homomorphism of $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$-modules

$$

e_{\gamma} H \quad \rightarrow \quad H 1_\nu,
T_w \quad \mapsto \quad (t^{\frac{1}{2}})^{\ell(v)} T_u \cdot 1_\nu, \quad \text{if } w = uv \text{ with } u \in W^\nu \text{ and } v \in W_\nu, \text{ and}
C_w \quad \mapsto \quad \begin{cases} C_w 1_\nu, & \text{if } w \in W^\nu, \\ 0, & \text{if } w \notin W^\nu. \end{cases} \quad (2.21)
$$

### 2.2.3 Parabolic KL polynomials

As in \cite{2.13}, let $W_\gamma = \langle s_k \mid k \notin \gamma \rangle$ and let $\gamma W$ be the set of minimal length coset representatives of the cosets in $W_\gamma \setminus W$. The $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$-module

$$e_{\gamma} H \text{ has basis } \{e_{\gamma} T_u \mid u \in \gamma W\} \quad \text{and} \quad \varepsilon : e_{\gamma} H \rightarrow e_{\gamma} H$$

since $\varepsilon \gamma = e_{\gamma}$. The Bruhat order $\gamma W$ is the restriction of the Bruhat order on $W$ to $\gamma W$ and, with these structures, $e_{\gamma} H$ is a KL-module.

Let $w_\gamma$ be the longest element of $W_\gamma$ and let $u \in \gamma W$. Since $T_{s_i} C_{w, u} = -t^{-\frac{1}{2}} C_{w, u}$ for simple reflections $s_i \in W_\gamma$ (see \cite{Hu} Prop. 7.14(a)), it follows that $C_{w, u} \in e_{\gamma} H$. Thus

$$e_{\gamma} H \text{ has KL-basis } \{C_{w, u} \mid u \in \gamma W\}, \quad (2.22)$$

where $\{C_w \mid w \in W\}$ is the KL-basis of $H$. In summary, there is an injective homomorphism of KL-modules

$$
e_{\gamma} H \quad \rightarrow \quad H,
\varepsilon_{\gamma} T_u \quad \mapsto \quad e_{\gamma} T_u
C_{w_\gamma u} \quad \mapsto \quad C_{w_\gamma u} \quad (2.23)
$$

where $u \in \gamma W$.

If $x \in \gamma W$ then, from the second formula in \cite{2.12},

$$C_{w_\gamma x} = \sum_{y \leq w_\gamma x \atop y \in \gamma W} (-1)^{\ell(w_\gamma x) - \ell(y)} P_{y, w_\gamma x} (t^{\frac{1}{2}}) T_y = \sum_{w_\gamma y \leq w_\gamma x \atop y \in \gamma W} (-1)^{\ell(w_\gamma x) - \ell(w_\gamma y)} P_{w_\gamma y, w_\gamma x} (t^{\frac{1}{2}}) e_{\gamma} T_y. \quad (2.24)$$

where, by the second formula in \cite{2.15}, if $w \in W$ and $w = vu$ with $u \in \gamma W$ and $v \in W_\gamma$ then

$$e_{\gamma} T_w = e_{\gamma} T_v T_u = (-t^{-\frac{1}{2}})^{\ell(v)} e_{\gamma} T_u = (-t^{\frac{1}{2}})^{\ell(v)} \sum_{z \in \gamma W} (-t^{\frac{1}{2}})^{\ell(w_\gamma) - \ell(z)} T_z. \quad (2.25)$$

### 2.2.4 Singular block parabolic KL polynomials

As in \cite{2.13},

$$W_\gamma = \langle s_k \mid k \notin \gamma \rangle \quad \text{and let} \quad W_\nu = \langle s_j \mid j \in J \rangle.$$
Let \( w_\gamma \) be the longest element of \( W_\gamma \) and let \( \varepsilon_\gamma \) and \( 1_\nu \) be as defined in (2.15). The composite of (2.21) and (2.23)

\[
\begin{align*}
\varepsilon_\gamma H & \rightarrow H \rightarrow H1_\nu \\
\varepsilon_\gamma T_u & \mapsto \varepsilon_\gamma T_u \mapsto \varepsilon_\gamma T_u 1_\nu \\
C_{w_\gamma,x} & \mapsto C_{w_\gamma,x} \mapsto C_{w_\gamma,x} 1_\nu
\end{align*}
\]

has image \( \varepsilon_\gamma H1_\nu \).

(2.26)

Let \( \gamma W \) be the set of minimal length coset representatives of the cosets in \( W_\gamma \setminus W \), and let \( W' \) be the set of minimal length coset representatives of the cosets in \( W/W_\nu \). From (2.20), \( C_{w_\gamma} 1_\nu = 0 \) unless \( w \in W' \), and so in (2.26),

\[
\text{if } u \in \gamma W \text{ then } C_{w_\gamma,u} 1_\nu = 0 \text{ unless } w_\gamma u \in W'.
\]

By [Bou, Ch. IV §1 Ex. 3]), the elements of \( \gamma W \cap W' \) are the minimal length elements of the double cosets in \( W_\gamma \setminus W/W_\nu \) and are a set of representatives of the double cosets in \( W_\gamma \setminus W/W_\nu \).

If \( W_\gamma aW_\nu \) is a double coset in \( W_\gamma \setminus W/W_\nu \) then there is a unique element \( u \in W_\gamma aW_\nu \) of minimal length and

\[
\text{if } w \in W_\gamma aW_\nu \text{ then } w = vu_z, \text{ with } v \in W_\gamma, z \in W_\nu \\
\text{and } \ell(w) = \ell(v) + \ell(u) + \ell(z).
\]

(2.27)

Note that (2.27) does not imply that \( \text{Card}(W_\gamma aW_\nu) = \text{Card}(W_\gamma)\text{Card}(W_\nu) \).

**Proposition 2.2.** Let \( u \in \gamma W \cap W' \) so that \( u \) is a minimal length element of a double coset in \( W_\gamma \setminus W/W_\nu \).

(a) If \( w_\gamma u \notin W' \) then \( \varepsilon_\gamma T_u 1_\nu = 0 \).

(b) If \( w_\gamma u \in W' \) then

\[
\varepsilon_\gamma T_u 1_\nu = \sum_{\nu \in W_\gamma, z \in W_\nu} (-t^{1})^{\ell(w)}(t^{-\frac{1}{2}})^{\ell(v)}(t^{\frac{1}{2}})^{\ell(z)}T_{vuz}.
\]

(2.28)

Proof. The group \( W_\gamma \) acts on the coset space \( W/W_\nu \). The coset space \( W/W_\nu \) can always be identified with the orbit \( W' \) for some element \( \nu \in a^* \), where \( a^* = a \otimes \mathbb{R} \). Thus a \( W_\gamma \) orbit is \( W_\gamma \lambda \) for some \( \lambda \in a^* \). We may take \( \lambda = w_\nu \) where \( u \) is minimal length in the orbit \( W_\gamma uW_\nu \).

Let \( \lambda = \text{Stab}_{W_\gamma}(\lambda) = uW_\nu u^{-1} \). Since the stabilizer of the \( W_\gamma \) action on \( \lambda \) is \( W_\gamma \cap W_\lambda \), the elements of the orbit \( W_\gamma \lambda \) are indexed by the set \( W_\gamma \) of minimal length representatives of the cosets in \( W_\gamma/(W_\gamma \cap W_\lambda) \). It follows that

\[
W_\gamma aW_\nu = \{xuy \mid x \in W_\gamma, y \in W_\nu\} \quad \text{with} \quad \text{Card}(W_\gamma aW_\nu) = \text{Card}(W_\gamma)^{\lambda}\text{Card}(W_\nu).
\]

(a) Assume \( w_\gamma u \notin W' \). Then there exists \( s_i \in W_\gamma \cap W_\lambda \). So \( s_i u = us_j \) with \( s_j \in W_\nu \) and it follows that

\[
\varepsilon_\gamma T_u 1_\nu = (-t^{\frac{1}{2}})\varepsilon_\gamma T_s, T_u 1_\nu = (-t^{\frac{1}{2}})\varepsilon_\gamma T_s 1_\nu = (-t^{\frac{1}{2}})^2 \varepsilon_\gamma T_\nu 1_\nu = -t^2 \varepsilon_\gamma T_\nu 1_\nu,
\]

giving that \( \varepsilon_\gamma T_u 1_\nu = 0 \).

(b) Continuing from the proof of (a), \( \varepsilon_\gamma T_u 1_\nu \neq 0 \) only when \( W_\gamma \cap W_\lambda = \{1\} \) so that

\[
W_\gamma^\lambda = W_\gamma, \quad \text{in which case} \quad \text{Card}(W_\gamma aW_\nu) = \text{Card}(W_\gamma)\text{Card}(W_\nu) \quad \text{and}
\]

\[
W_\gamma uW_\nu = \{xuy \mid x \in W_\gamma, y \in W_\nu\} \quad \text{and} \quad w_\gamma u \in W' \nu.
\]

11
Then
\[
\varepsilon_\gamma T_u 1_\nu = \left(-t^{\frac{1}{2}} T_u^\ell(\nu) \sum_{v \in W_\gamma} (-1)^{\ell(v)-\ell(v')} T_v \left((t^{-\frac{1}{2}} T_v^\ell(\nu) \sum_{z \in W_\nu} t_z \right)T_u \left(t^{\frac{1}{2}} T_v^\ell(\nu) \sum_{z \in W_\nu} t_z \right)T_u \right)
\]
\[
= \left(-t^{\frac{1}{2}} T_u^\ell(\nu) \left(t^{-\frac{1}{2}} T_v^\ell(\nu) \sum_{v \in W_\gamma, z \in W_\nu} (-1)^{\ell(v)} t_z \right)T_u \left(t^{\frac{1}{2}} T_v^\ell(\nu) \sum_{z \in W_\nu} t_z \right)T_u \right)
\]
\[
= \left(-t^{\frac{1}{2}} T_u^\ell(\nu) \left(t^{-\frac{1}{2}} T_v^\ell(\nu) \sum_{v \in W_\gamma, z \in W_\nu} (-1)^{\ell(v)} t_z \right)\right)
\]
where the first equality follows from (2.15) and the third equality follows from (2.27).

Since $\overline{\nu} = \epsilon_\gamma$, the restriction of $\overline{\cdot} : H \to H$ provides
\[
\overline{\cdot} : \varepsilon_\gamma H 1_\nu \to \varepsilon_\gamma H 1_\nu, \quad \text{and} \quad \varepsilon_\gamma H 1_\nu \text{ has basis } \left\{ \varepsilon_\gamma T_u 1_\nu \mid u \in \gamma W \text{ and } w_\gamma u \in W^\nu \right\}
\]
and the restriction of the Bruhat order on $W$ provides a partial order on the set $\left\{ u \in \gamma W \mid w_\gamma u \in W^\nu \right\}$. With these structures, $\varepsilon_\gamma H 1_\nu$ is a KL-module and, from (2.17) and (2.22),
\[
\varepsilon_\gamma H 1_\nu \text{ has KL-basis } \left\{ C_{w_\gamma u} 1_\nu \mid u \in \gamma W \text{ and } w_\gamma u \in W^\nu \right\}
\]
and, using (2.24) and Proposition 2.2,
\[
C_{w_\gamma u} 1_\nu = \sum_{w_\gamma u \leq w_\gamma, y \in \gamma W} (-1)^{\ell(w_\gamma, x) - \ell(w_\gamma, y)} P_{w_\gamma y, w_\gamma x} \left(t^{\frac{1}{2}} T_u \right)\varepsilon_\gamma T_y 1_\nu
\]
\[
= \sum_{w_\gamma u \leq w_\gamma, y \in \gamma W, w_\gamma u \in W^\nu} (-1)^{\ell(w_\gamma, x) - \ell(w_\gamma, y)} P_{w_\gamma y, w_\gamma x} \left(t^{\frac{1}{2}} T_u \right)\varepsilon_\gamma T_y 1_\nu,
\]
where, as in (2.19),
\[
P_{w_\gamma y, w_\gamma x} = \sum_{z \in W_\nu} (-1)^{\ell(w_\gamma, y) - \ell(w_\gamma, z)} P_{w_\gamma y z, w_\gamma x}.
\]

3 Decomposition numbers via Hecke algebras

3.1 Affine Kac-Moody and $\nu$ negative level rational

With $W_0$ and $a_0^\times$ as in (1.1), let $\tilde{g}$ be a finite dimensional complex reductive Lie algebra with Cartan subalgebra $a$ and Borel subalgebra $b$ containing $a$ such that the Weyl group is $W_0$, the weight lattice is $a_0^\times$ and the simple coroots are $\alpha_1^\vee, \ldots, \alpha_n^\vee$. Let $g$ be the corresponding affine Kac-Moody Lie algebra (see [Kac (7.2.2)]),
\[
g = (\tilde{g} \otimes \mathbb{C}[\epsilon, \epsilon^{-1}]) \oplus \mathbb{C}K \oplus \mathbb{C}d, \quad \text{with Cartan subalgebra } \mathfrak{h} = a \oplus \mathbb{C}K \oplus \mathbb{C}d
\]
and positive real roots $R^+_e$ and integral weight lattice $\mathfrak{h}_e^\mathbb{Z}$. Let $\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_n^\vee$ be the simple coroots of $\tilde{g}$ with respect to the Borel subalgebra $b = b \oplus \mathbb{C}K \oplus \mathbb{C}d \oplus \left(\tilde{g} \otimes \mathbb{C} e \mathbb{C}[\epsilon]\right)$ (see [Kac Theorem 7.4]) and let
\[
\hat{\rho} \in \mathfrak{h}^* \quad \text{such that } \langle \hat{\rho}, \alpha_i^\vee \rangle = 1, \text{ for } i \in \{0, 1, \ldots, n\}
\]
For \( \nu \in \mathfrak{h}^* \) define
\[
\Delta^+(\nu) = \{ \alpha \in R_{re}^+ \mid \langle \nu + \hat{\rho}, \alpha^\vee \rangle \in \mathbb{Z} \} \quad \text{and} \quad W(\nu) = \{ s_\alpha \mid \alpha \in \Delta^+(\nu) \}
\]
and define the *dot action of \( W \) on \( \mathfrak{h}^* \) by
\[
w \circ \lambda = w(\lambda + \hat{\rho}) - \hat{\rho}, \quad \text{for } w \in W \text{ and } \lambda \in \mathfrak{h}^*.
\]
If \( \nu \in \mathfrak{h}^*_R \) then \( W(\nu) = \{ s_\alpha \mid \alpha \in R_{re}^+ \} = W \) as defined in (3.2) is the full affine Weyl group.

A weight \( \nu \in \mathfrak{h}^* \) is *negative level rational* if \( \nu \) satisfies:

(a) (negativity/antidominance) If \( i \in \{ 0, 1, \ldots, n \} \) then \( \langle \nu + \hat{\rho}, \alpha_i^\vee \rangle \in \mathbb{Q}_{\leq 0} \).

(b) (negative level) \( \langle \nu + \hat{\rho}, K \rangle \in \mathbb{Q}_{< 0} \).

Given condition (a) the only additional content of (b) is that \( \langle \nu + \hat{\rho}, K \rangle \neq 0 \), (see the statement of [KT96, Theorem 3.3.6]).

**Theorem 3.1.** [KT96, Theorem 0.1] Let \( \mathfrak{g} \) be an affine Kac-Moody Lie algebra and let \( \nu \in \mathfrak{h}^* \) be negative level rational. Let \( w \in W \) be of minimal length in \( wW(\nu) \). Letting \( < \) denote the Bruhat order on \( W \), let \( x \in W(\nu) \) be such that
\[
\text{if } w' \in W \text{ and } w' < wx \quad \text{then} \quad w' \circ \nu \neq wx \circ \nu.
\]

Let \( \text{ch}(M) \) denote the character (weight space generating function) of a \( \mathfrak{g} \)-module \( M \). Then
\[
\text{ch}(L(wx \circ \nu)) = \sum_{y \leq_{\nu} x} (-1)^{\ell_{\nu}(x) - \ell_{\nu}(y)} P_{y,x}^\nu(1) \text{ch}(M(wy \circ \nu)),
\]
where \( \ell_{\nu} \) is the length function, \( \leq_{\nu} \) is the Bruhat order and \( P_{y,x}^\nu \) are the Kazhdan-Lusztig polynomials (see (2.12)) for the Coxeter group \( W(\nu) \), and the sum is over \( y \in W(\nu) \) such that \( y \leq_{\nu} x \).

This statement generalizes a conjecture of Lusztig [Lu90, Conj. 2.5c], proved by Kashiwara-Tanisaki in [KT95]. It is a negative level affine version of the original “Kazhdan-Lusztig conjecture” of [KL79, Conjecture 1.5]. A refinement of [KL79, Conjecture 1.5] is the Jantzen conjecture, which was proved by Beilinson-Bernstein [BB, Cor. 5.3.5]. The “Jantzen conjecture” result generalizes to the negative level affine setting, as proved by Shan [Sh, Proposition 5.5 and Theorem 6.4].

### 3.2 The Kashiwara-Tanisaki theorem in Hecke algebra notation

The purpose of this subsection is to repackage the result of Theorem 3.1 (in the strong “Jantzen conjecture” form) into the Hecke algebra notations of Section 2.2.

Keep the notations of Theorem 3.1 so that \( \mathfrak{g} \) is the affine Lie algebra, \( \mathfrak{h} \) is the Cartan subalgebra as in (3.1) and \( \nu \in \mathfrak{h}^* \) is negative level rational.

Let \( H \) be the Hecke algebra of the group \( W(\nu) \),
where $W(\nu)$ is as defined in (3.2) and $H$ is as defined in (2.9). Let

\[ K(O[\nu]) \] be the free $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$-module generated by symbols $[M(x \circ \nu)]$

for $x \in W^\nu$. Define elements $[L(y \circ \nu)]$, $y \in W^\nu$, by the equation

\[ [M(x \circ \nu)] = \sum_{y \leq x} \left( \sum_{i \in \mathbb{Z} \geq 0} \left[ \frac{M^{(i)}(x \circ \nu)}{M^{(i-1)}(x \circ \nu)} : L(y \circ \nu) \right] (t^{\frac{1}{2}})^i \right) [L(y \circ \nu)], \]

where $[M : L(\mu)]$ denotes the multiplicity of the simple $g$-module $L(\mu)$ of highest weight $\mu$ in a composition series of $M$ and

\[ M(\lambda) = M(\lambda)^{(0)} \supseteq M(\lambda)^{(1)} \supseteq \cdots \]

is the Jantzen filtration of $M(\lambda)$, see, for example, [OR (2.5)].

**Case R: regular $\nu$.** Let $\nu \in \mathfrak{h}^*$ such that $\langle \nu + \hat{\rho}, \alpha_i^\vee \rangle \in \mathbb{Q}_{< 0}$. Then $\text{Stab}(\nu) = \{1\}$ under the dot action of (3.3). In this case the strong “Jantzen conjecture” version of Theorem 3.1 (see [Sh Theorem 6.4 and Proposition 5.5]) is equivalent to a $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$-module isomorphism

\[
\frac{K(O[\nu])}{[M(y \circ \nu)]} \to H \quad \text{where } T_y \text{ and } C_x \text{ are as in (2.12).} \quad (3.7)
\]

**Case S: singular $\nu$.** Let $\nu \in \mathfrak{h}^*$ such that $\langle \nu + \hat{\rho}, \alpha_i^\vee \rangle \in \mathbb{Q}_{\leq 0}$ and let

\[ J = \{ j \in \{0, 1, \ldots, n\} \mid \langle \nu + \hat{\rho}, \alpha_j^\vee \rangle = 0 \} \quad \text{so that} \quad W_\nu = \{ s_j \mid j \in J \} \]

is the stabilizer of the dot action of $W$ on $\nu$. Let $1_\nu$ be the element of $H$ defined in (2.15). Then the strong “Jantzen conjecture” version of Theorem 3.1 (see [Sh Theorem 6.4 and Proposition 5.5]) is equivalent to a $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$-module isomorphism

\[
\frac{K(O[\nu])}{[M(y \circ \nu)]} \to H 1_\nu \quad \text{where } T_y 1_\nu \text{ and } C_x 1_\nu \text{ are as in (2.18).} \quad (3.8)
\]

### 3.3 Decomposition numbers for parabolic $O$

Keep the notations for the affine Lie algebra as in (3.1), and let $e_0, \ldots, e_n, f_0, \ldots, f_n, a$ and $d$ be Kac-Moody generators for $g$. Let $\gamma \subseteq \{0, 1, \ldots, n\}$ with $\gamma \neq \emptyset$ and define, following [Soe98 §7], a $\mathbb{Z}$-grading on $g$ by $\text{deg}(d) = 0$, $\text{deg}(h) = 0$ for $h \in a$,

\[
\text{deg}(e_i) = \begin{cases} 0, & \text{if } i \in \gamma, \\ 1, & \text{if } i \notin \gamma, \end{cases} \quad \text{and} \quad \text{deg}(f_i) = \begin{cases} 0, & \text{if } i \in \gamma, \\ -1, & \text{if } i \notin \gamma. \end{cases}
\]

Let

\[ g_\gamma = \{ x \in g \mid \text{deg}(x) = 0 \} \quad \text{and} \quad b_\gamma = \{ x \in g \mid \text{deg}(x) \geq 0 \}. \quad (3.9)\]

Following the first two paragraphs of [Soe98 §3], the parabolic category $O$ (with respect to $\text{deg}$) is the category $O_{b_\gamma}$, of $g$-modules $M$ such that
(a) \( M \) is \( \mathfrak{g}_\gamma \)-semisimple,

(b) \( M \) is \( \mathfrak{b}_\gamma \)-locally finite, i.e. If \( m \in M \) then \( \dim(U\mathfrak{b}_\gamma \cdot m) < \infty \).

Let \((\mathfrak{a}^*)_\gamma^+\) be an index set for the finite dimensional simple \( \mathfrak{g}_\gamma \)-modules \( \{L_{\mathfrak{g}_\gamma}(\lambda) \mid \lambda \in (\mathfrak{a}^*)_\gamma^+\} \).

The standard modules in \( \mathcal{O}_{\mathfrak{g}_\gamma}^\mathfrak{g} \) are

\[
\Delta_{\mathfrak{g}_\gamma}^\mathfrak{g}(\lambda) = U\mathfrak{g} \otimes_{U\mathfrak{b}_\gamma} L_{\mathfrak{g}_\gamma}(\lambda), \quad \text{for } \lambda \in (\mathfrak{a}^*)_\gamma^+, \tag{3.10}
\]

where \( L_{\mathfrak{g}_\gamma}(\lambda) \) becomes a \( \mathfrak{b}_\gamma \)-module by setting \( x \mathfrak{b}_\gamma = 0 \) if \( n \in L_{\mathfrak{g}_\gamma}(\lambda) \) and \( x \in \mathfrak{g} \) is homogeneous with \( \deg(x) > 0 \). The simple modules in \( \mathcal{O}_{\mathfrak{g}_\gamma}^\mathfrak{g} \) are the quotients

\[
L(\lambda) = \frac{\Delta_{\mathfrak{g}_\gamma}^\mathfrak{g}(\lambda)}{(\max. \text{ proper submodule)}}, \quad \text{for } \lambda \in (\mathfrak{a}^*)_\gamma^+.
\]

Let \( W_\gamma \) be the Weyl group corresponding to \( \gamma \) as in (2.13). Since \( \gamma \neq \emptyset \) and \( \mathfrak{g} \) is an affine Kac-Moody Lie algebra, the Lie algebra \( \mathfrak{g}_\gamma \) is finite dimensional and the integrable simple module \( L_{\mathfrak{g}_\gamma}(\lambda) \) for the Lie algebra \( \mathfrak{g}_\gamma \) has a BGG-resolution (see [Dx, Ex. 7.8.14]),

\[
0 \rightarrow \Delta_{\mathfrak{b}_\gamma}^\mathfrak{g}(w_\gamma \circ \lambda) \rightarrow \cdots \rightarrow \bigoplus_{z \in W_\gamma} \Delta_{\mathfrak{b}_\gamma}^\mathfrak{g}(z \circ \lambda) \rightarrow \cdots \rightarrow \Delta_{\mathfrak{b}_\gamma}^\mathfrak{g}(\lambda) \rightarrow L_{\mathfrak{g}_\gamma}(\lambda) \rightarrow 0. \tag{3.11}
\]

where \( \Delta_{\mathfrak{b}_\gamma}^\mathfrak{g}(\mu) \) denotes the Verma module of highest weight \( \mu \) for \( \mathfrak{g}_\gamma \) and \( w_\gamma \) is the longest element of \( W_\gamma \) (since \( \gamma \neq \emptyset \) then \( W_\gamma \) is a finite Coxeter group and \( w_\gamma \) exists). (The dot action in (3.11) coincides with the dot action defined in (3.3) since \( W_\gamma \) is generated by \{\( s_i \mid i \not\in \gamma \)\} and both actions satisfy \( s_i \circ \lambda = \lambda - (\langle \lambda, \alpha_i^\vee \rangle + 1)\alpha_i \) for \( i \not\in \gamma \).)

As in [Soe98, paragraph before Prop. 7.5], parabolic induction of the resolution (3.11) to \( \mathfrak{g} \) gives

\[
0 \rightarrow \Delta_{\mathfrak{b}_\gamma}^\mathfrak{g}(w_\gamma \circ \lambda) \rightarrow \cdots \rightarrow \bigoplus_{z \in W_\gamma} \Delta_{\mathfrak{b}_\gamma}^\mathfrak{g}(z \circ \lambda) \rightarrow \cdots \rightarrow \Delta_{\mathfrak{b}_\gamma}^\mathfrak{g}(\lambda) \rightarrow \Delta_{\mathfrak{b}_\gamma}^\mathfrak{g}(\lambda) \rightarrow 0, \tag{3.12}
\]

where \( \Delta_{\mathfrak{b}_\gamma}^\mathfrak{g}(\mu) = M(\mu) \) is the Verma module for \( \mathfrak{g} \) as in (3.4). Thus the multiplicity of a simple \( \mathfrak{g} \)-module \( L(\mu) \) in the standard module \( \Delta_{\mathfrak{b}_\gamma}^\mathfrak{g}(\lambda) \) is

\[
[\Delta_{\mathfrak{b}_\gamma}^\mathfrak{g}(\lambda) : L(\mu)] = \sum_{z \in W_\gamma} (-1)^{\ell(z)}[\Delta_{\mathfrak{b}_\gamma}^\mathfrak{g}(z \circ \lambda) : L(\mu)]. \tag{3.13}
\]

In the correspondence to the Hecke algebra as in (3.8),

\[
[M(z\gamma \circ \nu)] = [\Delta_{\mathfrak{b}_\gamma}^\mathfrak{g}(z\gamma \circ \nu)] \quad \rightarrow \quad T_{z\gamma}1_\nu
\]

\[
[\Delta_{\mathfrak{b}_\gamma}^\mathfrak{g}(w_\gamma y \circ \nu)] \quad \rightarrow \quad \varepsilon_y T_{y}1_\nu
\]

\[
[L(w \circ \nu)] \quad \rightarrow \quad C_w1_\nu
\]

so that the identity in (3.13) (which comes from the BGG resolution) corresponds to the Hecke algebra identity (see (2.28))

\[
\varepsilon_y T_{x}1_\nu = \sum_{z \in W_\gamma} (-1)^{\ell(z)}T_{z\gamma}1_\nu, \quad \text{where } w_\gamma \text{ is the longest element of } W_\gamma.
\]
Let $\gamma W$ be the set of minimal length representatives of cosets in $W_\gamma \backslash W$. Let

$$K(O^0_{\mathfrak{g}, \omega}(w_{\gamma} \circ \nu))$$

be the free $\mathbb{Z}[t^{1/2}, t^{-1/2}]$-module generated by symbols $[\Delta^0_{\mathfrak{g}, \omega}(w_{\gamma} x \circ \nu)]$, for $x \in \gamma W$ such that $w_{\gamma} x \in W^\nu$. Define elements $[L(w_{\gamma} y \circ \nu)]$, for $y \in \gamma W$ such that $w_{\gamma} y \in W^\nu$, by the equation

$$[\Delta^0_{\mathfrak{g}, \omega}(w_{\gamma} x \circ \nu)] = \sum_{y \leq x} \left( \sum_{i \in \mathbb{Z}_{\geq 0}} \left[ \Delta^0_{\mathfrak{g}, \omega}(w_{\gamma} x \circ \nu)^{(i)} \right] : L(w_{\gamma} y \circ \nu) \right) \left[ (t^{1/2})^i \right] [L(w_{\gamma} y \circ \nu)],$$

where $[M : L(\mu)]$ denotes the multiplicity of the simple $\mathfrak{g}$-module $L(\mu)$ of highest weight $\mu$ in a composition series of $M$ and

$$\Delta^0_{\mathfrak{g}, \omega}(\lambda) = \Delta^0_{\mathfrak{g}, \omega}(\lambda^{(0)}) \supseteq \Delta^0_{\mathfrak{g}, \omega}(\lambda^{(1)}) \supseteq \cdots$$

is the Jantzen filtration of $\Delta^0_{\mathfrak{g}, \omega}(\lambda)$ (see, for example, [Sh] §1.4, §2.3 and §2.10 for the Jantzen filtration in this context).

Case PR: Parabolic $O$, regular $\nu$. Let $\nu \in \mathfrak{h}^*$ such that $\langle \nu + \rho, \alpha_i \rangle \in \mathbb{Q}_{<0}$. Let $\gamma \subseteq \{0, 1, \ldots, n\}$ and let $\mathfrak{g}_0$ be the corresponding “standard” Levi subalgebra of $\mathfrak{g}$ as defined in (3.9) with Weyl group $W_{\gamma} = \langle s_k | k \in \gamma \rangle$ as defined in (2.13). Then Theorem 3.1 (or (3.8)) combined with (3.13) and (2.23) is equivalent, in the strong “Jantzen conjecture” form (see [Sh] Theorem 6.4 and Proposition 5.5)) to a $\mathbb{Z}[t^{1/2}, t^{-1/2}]$-module isomorphism

$$\begin{array}{cccc}
K(O^0_{\mathfrak{g}, \omega}[w_{\gamma} \circ \nu]) & \sim \rightarrow & \varepsilon_{\gamma} H \\
[\Delta^0_{\mathfrak{g}, \omega}(w_{\gamma} y \circ \nu)] & \longrightarrow & \varepsilon_{\gamma} T_y \\
[L(w_{\gamma} x \circ \nu)] & \longrightarrow & C_{w_{\gamma} x}
\end{array}$$

(3.14)

Case PS: parabolic $O$, singular $\nu$. Let $\nu \in \mathfrak{h}^*$ such that $\langle \nu + \rho, \alpha_i \rangle \in \mathbb{Q}_{\leq 0}$. The maps in (3.8) and (3.14) can be packaged into a single statement as follows: If $\nu \in \mathfrak{h}^*$ is such that $\langle \nu + \rho, \alpha_i \rangle \in \mathbb{Q}_{\leq 0}$ and $W_{\nu} = \text{Stab}(\nu)$ is the stabilizer of $\nu$ in $W$ under the dot action then

$$\begin{array}{cccc}
K(O^0_{\mathfrak{g}, \omega}[w_{\gamma} \circ \nu]) & \sim \rightarrow & \varepsilon_{\gamma} H 1_{\nu} \\
[\Delta^0_{\mathfrak{g}, \omega}(w_{\gamma} y \circ \nu)] & \longrightarrow & \varepsilon_{\gamma} T_y 1_{\nu} \\
[L(w_{\gamma} x \circ \nu)] & \longrightarrow & C_{w_{\gamma} x} 1_{\nu}
\end{array}$$

(3.15)

3.4 Decomposition numbers for quantum groups

In 1989 and 1990, Lusztig made conjectures that the decomposition numbers for representations of quantum groups can be picked up by Kazhdan-Lusztig polynomials for the affine Weyl group. Let $q \in \mathbb{C}^*$ and let $U_q(\mathfrak{g})$ be the Drinfel’d-Jimbo quantum group corresponding to $\mathfrak{g}$. Let

$$M_q(\lambda)$$

the Verma module of highest weight $\lambda$ for $U_q(\mathfrak{g})$,

$$\Delta_q(\lambda)$$

the Weyl module for $U_q(\mathfrak{g})$ of highest weight $\lambda$,

$$L_q(\lambda)$$

the simple module for $U_q(\mathfrak{g})$ of highest weight $\lambda$,

the conjectures [Lu90] Conj. 2.5 and [Lu89] Conj. 8.2] are

$$L_q(x \circ \nu) = \sum_{y \in W_0 \atop y \leq x} (-1)^{\ell(\nu) + \ell(w)} P_{y,x}(1) M_q(y \circ \nu),$$

if $q = 1$ or $q^2$ is not a root of unity,

$$L_q(x \circ \nu) = \sum_{y \in W \atop y \leq x} (-1)^{\ell(\nu) + \ell(w)} P_{y,x}(1) M_q(y \circ \nu),$$

if $q^2$ is a primitive $\ell$-th root of unity,

$$L_q(x \circ \nu) = \sum_{y \in W \atop y \leq x} (-1)^{\ell(\nu) + \ell(w)} P_{y,x}(1) \Delta_q(y \circ \nu),$$

if $q^2$ is a primitive $\ell$-th root of unity,
Thus, if \( \lambda \) where, with the notation for affine Lie algebras as in (3.1)-(3.3). Following [Kac, Lu90, Conjecture 2.3]. Theorem 3.2 provides a connection between the representations of affine Lie algebras and the representations of quantum groups. Let us first sketch this relation on the level of weights. Keep the notation for affine Lie algebras as in (3.1)-(3.3). Following [Kac, §6.2] and coordinatizing \( \mathfrak{h}^* = \mathbb{C}A_0 + \mathfrak{a}^* + \mathbb{C}\delta \) with \( \langle \mathfrak{a}^*, K \rangle = 0, \langle \mathfrak{a}^*, d \rangle = 0, \)

\[
\langle \Lambda_0, K \rangle = 1, \quad \langle \Lambda_0, a \rangle = 0, \quad \langle \Lambda_0, d \rangle = 0, \quad \langle \delta, K \rangle = 0, \quad \langle \delta, a \rangle = 0, \quad \langle \delta, d \rangle = 1,
\]

then \( \varphi^\vee \in \mathfrak{a} \) and the dual Coxeter number \( h \) are such that

\[
\alpha_i^\vee = -\varphi^\vee + K, \quad \text{and} \quad \hat{\rho} = \rho + h\Lambda_0,
\]

where \( \hat{\rho} \in \mathfrak{h}^* \) and \( \rho \in \mathfrak{a}^* \) are as in (3.3) and (1.2), respectively. Let \( \ell \in \mathbb{Z}_{\geq 0}. \) Weights of \( \mathfrak{g} \)-modules that are level \( -\ell - h \) are elements of \((-\ell - h)\Lambda_0 + \mathfrak{a}^* + \mathbb{C}\delta. \) Restricting modules in \( \mathcal{O}_{\mathfrak{h}_0}^0 \) to the subalgebra \( \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] \) loses the information of \( \mathbb{C}\delta, \) and in the diagram

\[
(-\ell - h)\Lambda_0 + \mathfrak{a}^* + \mathbb{C}\delta \quad \longrightarrow \quad (-\ell - h)\Lambda_0 + \mathfrak{a}^* \quad \longleftrightarrow \quad \mathfrak{a}^* \quad \longrightarrow \quad (-\ell - h)\Lambda_0 + \lambda + a\delta \quad \longrightarrow \quad (-\ell - h)\Lambda_0 + \lambda\]  

the second map is a bijection. Using the definition of negative level rational from just before Theorem 3.1

\[
\{ \nu \in \mathfrak{h}_0^* | \ nu + \hat{\rho} \text{ is level } -\ell \text{ and } \nu \text{ is negative level rational} \}
\]

\[
= \{ \nu \in \mathfrak{h}_0^* | \ \langle \nu + \hat{\rho}, K \rangle = -\ell \text{ and } \langle \nu + \hat{\rho}, \alpha_i^\vee \rangle \in \mathbb{Q}_{\leq 0} \text{ for } i \in \{0, \ldots, n\} \}
\]

\[
= (-\ell - h)\Lambda_0 + \{ \nu \in \mathfrak{a}_z | \ \langle \nu, \varphi^\vee \rangle \in \mathbb{Q}_{\leq 0} \text{ for } i \in \{0, \ldots, n\} \}
\]

\[
= (-\ell - h)\Lambda_0 + \{ \nu \in \mathfrak{a}_z | \ \langle \nu - \ell\Lambda_0, -\varphi^\vee + K \rangle \in \mathbb{Q}_{\geq 0} \text{ and } \langle \nu, \alpha_i^\vee \rangle \in \mathbb{Q}_{\leq 0} \text{ for } i \in \{1, \ldots, n\} \}
\]

\[
= (-\ell - h)\Lambda_0 + \{ \nu \in \mathfrak{a}_z | \ \langle \nu, \varphi^\vee \rangle \geq -\ell \text{ and } \langle \nu, \alpha_i^\vee \rangle \leq 0 \text{ for } i \in \{1, \ldots, n\} \}
\]

\[
= (-\ell - h)\Lambda_0 + A_{-\ell-h},
\]

and, in light of Theorem 3.2 below, the “source” of the alcove \( A_{-\ell-h} \) in (3.16) is the negative level rational condition for weights of the affine Lie algebra.

Next we compare the dot action from (3.3) to the dot action from (1.2). Following [Kac, (6.5.2)], the action of a translation \( t_\mu \) on \( \mathfrak{h}^* = \mathbb{C}\delta + \mathfrak{a}^* + \mathbb{C}\Lambda_0 \) is given by

\[
t_\mu(a\delta + \lambda + m\Lambda_0) = (a - \langle \lambda, \mu \rangle - \frac{1}{2}m\langle \mu, \mu \rangle)d + \lambda + m\mu + m\Lambda_0, \quad \text{and} \quad w(a\delta + \lambda + m\Lambda_0) = a\delta + w\lambda + m\Lambda_0, \quad \text{for } w \in W_0, \text{ the finite Weyl group}.
\]

Thus, if \( \lambda \in \mathfrak{a}^* \) then

\[
(t_\mu w) \circ (\lambda + (-\ell - h)\Lambda_0) = (t_\mu w)(\lambda + (-\ell - h)\Lambda_0 + \hat{\rho}) - \hat{\rho}
\]

\[
= (t_\mu w)(\lambda + (-\ell - h)\Lambda_0 + \rho + h\Lambda_0) - (\rho + h\Lambda_0)
\]

\[
= t_\mu(w(\lambda + \rho) - \ell\Lambda_0) - \rho - h\Lambda_0
\]

\[
= (w(\lambda + \rho) - \ell\Lambda_0 - \ell\mu) - \rho - h\Lambda_0 \text{ mod } \delta
\]

\[
= (w \circ \lambda) - \ell\mu + (-\ell - h)\Lambda_0 \text{ mod } \delta,
\]
where it is important to note that that the $\circ$ on the left side of this equation is the dot action of (3.3) and the $\circ$ on the right hand side is the dot action of (1.2). This computation is the basis for using (3.18) to obtain an action of the affine Weyl group $W$ on $a^*$ and define the level $(\ell - h)$ dot action of $W$ on $a^*$ by
\[
(t_{\mu}w) \circ \lambda = (w \circ \lambda) - \ell \mu = w(\lambda + \rho) - \rho - \ell \mu,
\]

(3.19)

Now let us state the Kazhdan-Lusztig theorem relating representations of affine Lie algebras to representations of quantum groups at root of unity. Let
\[
\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] = \hat{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[\epsilon, \epsilon^{-1}] + \mathbb{C}K.
\]

In the context of (2.13) and (3.9), let $\gamma = \{0\}$ so that
\[
\mathfrak{g}_\gamma = \mathfrak{g}_0 = \hat{\mathfrak{g}} \quad \text{and} \quad \varepsilon_\gamma = \varepsilon_0 = \sum_{w \in W_0} (-1)^{\frac{1}{2}(\ell(w_0) - \ell(w)T_z),}
\]

where $w_0$ is the longest element of $W_0$, the Weyl group of $\hat{\mathfrak{g}}$. By restriction, the modules in $\mathcal{O}^\theta_{\mathfrak{g}_0}$ are $\mathfrak{g}'$-modules.

**Theorem 3.2.** [KL94] Theorem 38.1 There is an equivalence of categories

\[
\begin{align*}
\{ \text{finite length $\mathfrak{g}'$-modules} & \} \quad \leftrightarrow \quad \{ \text{finite dimensional $U_q(\hat{\mathfrak{g}})$-modules} \} \\
\Delta^0_{\mathfrak{g}_0}((-\ell - h)\Lambda_0 + \lambda) & \quad \leftrightarrow \quad \Delta_q(\lambda) \\
\mathcal{L}((-\ell - h)\Lambda_0 + \lambda) & \quad \leftrightarrow \quad L_q(\lambda)
\end{align*}
\]

This statement of Theorem 3.2 is for the simply-laced (symmetric) case. With the proper modifications to this statement the result holds for non-simply laced cases as well, see [Lu94 §8.4] and [Lu95].

Let
\[
K(\text{fd}U_q(\hat{\mathfrak{g}})\text{-mod}) \quad \text{be the free } \mathbb{Z}[t^\frac{1}{2}, t^{-\frac{1}{2}}]\text{-module generated by symbols } [\Delta_q(\lambda)],
\]

for $\lambda \in \mathfrak{a}_\mathbb{Z}^\ast$. Define elements $[L_q(w_0y \circ \nu)]$, for $\nu \in \mathcal{L}_{-\ell - h}$ and $y \in 0W$ such that $w_0y \in W^\nu$, by the equation
\[
[\Delta_q(w_0x \circ \nu)] = \sum_{y \leq x} \left( \sum_{i \in \mathbb{Z}_{\geq 0}} \left[ \frac{\Delta_q(w_0x \circ \nu)}{\Delta_q(w_0x \circ \nu)} \right]^{(i)} \cdot \frac{L_q(w_0y \circ \nu)^{(i+1)}(t^\frac{1}{2})}{L_q(w_0y \circ \nu)} \right) [L_q(w_0y \circ \nu)],
\]

where $[M : L_q(\mu)]$ denotes the multiplicity of the simple $\mathfrak{g}$-module $L_q(\mu)$ of highest weight $\mu$ in a composition series of $M$ and
\[
[\Delta_q(\lambda)]^{(0)} \supseteq [\Delta_q(\lambda)]^{(1)} \supseteq \cdots \text{ is the Jantzen filtration of } [\Delta_q(\lambda)]
\]

(see, for example, [Sh] §1.4, §2.3 and §2.10 and Cor. 2.14] and [JM] §4 for the Jantzen filtration in this context).

**Case QG: quantum groups, integral weights.** The maps in (3.15) combined with the result of Theorem 3.2 can be packaged in terms of the affine Hecke algebra as follows: Let $\nu \in \mathcal{L}_{-\ell - h}$ and let $W_{\nu} = \text{Stab}(\nu)$ is the stabilizer of $\nu$ in $W$ under the level $-\ell - h$ dot action. Then
\[
K(\text{fd}U_q(\hat{\mathfrak{g}})\text{-mod}) \quad \xrightarrow{\sim} \quad \bigoplus_{\nu \in \mathcal{L}_{-\ell - h}} \varepsilon_0 H_{1_{\nu}} \quad \text{and}
\]

(3.20)
4 The Fock space Hecke KL-module in the general setting

Keep the notation for the finite Weyl group $W_0$, the simple reflections $s_1, \ldots, s_n$ and the weight lattice $\mathfrak{a}_Z^*$ as in (1.1). The affine Weyl group is

$$W = \{ t_\mu w \mid \mu \in \mathfrak{a}_Z^*, w \in W_0 \}, \quad \text{with} \quad t_\mu t_\nu = t_{\mu + \nu}, \quad \text{and} \quad wt_\mu = t_{w_0}\mu, \quad (4.1)$$

for $\mu, \nu \in \mathfrak{a}_Z^*$ and $w \in W_0$.

Let $\ell \in \mathbb{Z}_{>0}$. Following (3.19), the level $(-\ell - h)$ dot action of $W$ on $\mathfrak{a}_Z^*$ is given by

$$(t_\mu w) \circ \lambda = (w \circ \lambda) - \ell\mu = w(\lambda + \rho) - \rho - \ell\mu, \quad (4.2)$$

for $\mu \in \mathfrak{a}_Z^*$, $w \in W_0$ and $\lambda \in \mathfrak{a}_Z^*$.

4.1 The affine Hecke algebra $H$

Keep the notation for the finite Weyl group $W_0$, the simple reflections $s_1, \ldots, s_n$ and the weight lattice $\mathfrak{a}_Z^*$ as in (1.1). For $i, j \in \{1, \ldots, n\}$ with $i \neq j$, let

$$m_{ij} \quad \text{denote the order of} \quad s_is_j \quad \text{in} \quad W_0$$

so that $s_i^2 = 1$ and $(s_is_j)^{m_{ij}} = 1$ are the relations for the Coxeter presentation of $W_0$. The affine Hecke algebra is

$$H = \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]-\text{span}\{X^\mu T_w \mid \mu \in \mathfrak{a}_Z^*, w \in W_0\}, \quad (4.3)$$

with $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ basis $\{X^\mu T_w \mid \mu \in \mathfrak{a}_Z^*, w \in W_0\}$ and relations

$$(T_{s_i} - t^{\frac{1}{2}})(T_{s_i} + t^{-\frac{1}{2}}) = 0, \quad \underbrace{T_{s_i}T_{s_j}T_{s_i} \cdots}_{m_{ij} \text{ factors}} = \underbrace{T_{s_j}T_{s_i}T_{s_j} \cdots}_{m_{ij} \text{ factors}}, \quad (4.4)$$

$$X^{\lambda + \mu} = X^{\lambda}X^{\mu}, \quad \text{and} \quad T_{s_i}X^{\lambda} - X^{s_i\lambda}T_{s_i} = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{X^{\lambda + s_i\lambda} - X^{s_i\lambda}}{1 - X^{-\alpha_i}}, \quad (4.5)$$

for $i, j \in \{1, \ldots, n\}$ with $i \neq j$ and $\lambda, \mu \in \mathfrak{a}_Z^*$. The bar involution on $H$ is the $\mathbb{Z}$-linear automorphism $\overline{} : H \to H$ given by

$$t^{\frac{1}{2}} = t^{-\frac{1}{2}}, \quad T_{s_i} = T_{s_i}^{-1}, \quad \text{and} \quad \overline{X^{\lambda}} = T_{w_0}X^{w_0\lambda}T_{w_0}^{-1}, \quad (4.6)$$

for $i = 1, \ldots, n$ and $\lambda, \mu \in \mathfrak{a}_Z^*$. For $\mu \in \mathfrak{a}_Z^*$ and $w \in W_0$ define

$$X^{t_\mu w} = X^{\mu}(T_{w^{-1}})^{-1} \quad \text{and} \quad T_{t_\mu w} = T_xX^{\mu^+}(T_{x^{-1}}w)^{-1}, \quad (4.7)$$

where $\mu^+$ is the dominant representative of $W_0\mu$ and $x \in W_0$ is minimal length such that $\mu = x\mu^+$.

Remark 4.1. Formulas (4.6) and (4.7) are just a reformulation of the usual bar involution and the conversion between the Bernstein and Coxeter presentations of the affine Hecke algebra (see for example [NR Lemma 2.8 and (1.22)]).
4.2 Definition of $\mathcal{P}_{-\ell-h}^+$

Following $(3.16)$ and $(3.20)$, define

$$A_{-\ell-h} = \{ \nu \in \mathfrak{a}_2^* \ | \ \langle \nu, \varphi^\vee \rangle \geq -\ell \text{ and } \langle \nu, \alpha_i^\vee \rangle \leq 0 \text{ for } i \in \{1, \ldots, n\} \},$$

(4.8)

and

$$\mathcal{P}_{-\ell-h}^+ = \bigoplus_{\nu \in A_{-\ell-h}} \varepsilon_0 H \mathbf{1}_\nu,$$

(4.9)

where $\varepsilon_0$ and $\mathbf{1}_\nu$ are formal symbols satisfying $\bar{\varepsilon}_0 = \varepsilon_0$, $\bar{\mathbf{1}}_\nu = \mathbf{1}_\nu$,

$$\varepsilon_0 T_w = (-t^{-\frac{1}{2}})^{\ell(w)} \varepsilon_0 \text{ for } w \in W_0 \quad \text{and} \quad T_y \mathbf{1}_\nu = (t^{\frac{1}{2}})^{\ell(y)} \mathbf{1}_\nu \text{ for } y \in W_\nu,$$

where $W_\nu = \text{Stab}_W(\nu)$ under the level $(-\ell - h)$ dot action of $W$ on $\mathfrak{a}_2^*$. It is important to note that here the $\mathbf{1}_\nu$ are formal symbols (and not elements of the Hecke algebra as in the case of (2.15)) so that $\mathbf{1}_\nu \neq \mathbf{1}_\gamma$ if $\nu \neq \gamma$ (even though it may be that $W_\nu = W_\gamma$). Define a bar involution

$$\overline{-} : \mathcal{P}_{-\ell-h}^+ \rightarrow \mathcal{P}_{-\ell-h}^+ \quad \text{by} \quad \overline{\varepsilon_0 h \mathbf{1}_\nu} = \varepsilon_0 h \mathbf{1}_\nu, \quad \text{for } \nu \in A_{-\ell-h} \text{ and } h \in H.$$ (4.10)

For $\lambda \in \mathfrak{a}_2^*$ define

$$[T_\lambda] = [T_{w_0 y_0 \nu}] = \varepsilon_0 T_y \mathbf{1}_\nu \quad \text{and} \quad [X_\lambda] = [X_{w_0 y_0 \nu}] = \varepsilon_0 X^y \mathbf{1}_\nu,$$

(4.11)

where

$$\lambda = w_0 y \circ \nu = w_0 v \circ \nu, \quad \text{with } \nu \in A_{-\ell-h}, \quad \text{and} \quad (4.12)$$

(T) $y \in W$ is such that $T_{yu} = T_y T_u$ for any $u \in W_\nu$ and

(X) $v \in W$ is such that $X^{vu} = X^v T_u$ for any $u \in W_\nu$.

The condition (T) is equivalent to $y$ being a minimal length representative of the coset $yW_\nu$, i.e. $y \in W_\nu$.

4.3 The straightening laws for $[T_\lambda]$

The following Proposition is a special case of the situation in Proposition 2.2. As in Proposition 2.2, when $\lambda \in (\mathfrak{a}_2^*)^+$ (\lambda is a dominant integral weight) then the element $[T_\lambda]$ has an expansion in $H$ as a sum over the double coset $W_0 u W_\nu$, where $\lambda^+ = w_0 u \circ \nu$ with $\nu \in A_{-\ell-h}$ and $u$ is minimal length in $W_0 u W_\nu$. The properties in Proposition 4.2 determine $[T_\lambda]$ for $\lambda \in \mathfrak{a}_2^*$ (all integral weights).

**Proposition 4.2.** Let $\lambda \in \mathfrak{a}_2^*$. Let $\lambda^+$ be the maximal element of $W_0 \circ \lambda$ and let $\lambda^-$ be the minimal element of $W_0 \circ \lambda$ in dominance order. Let $u \in W$ and $x \in W_0$ be of minimal length such that

$$\lambda^- = u \circ \nu \quad \text{and} \quad \lambda = x \circ \lambda^+.$$  

Then $[T_\lambda] = (-t^{-\frac{1}{2}})^{\ell(x)} [T_{\lambda^+}]$ and

$$[T_{\lambda^+}] = \begin{cases} \varepsilon_0 T_u \mathbf{1}_\nu, & \text{if } \langle \lambda^+ + \rho, \alpha_i^\vee \rangle \neq 0 \text{ for } i \in \{1, \ldots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$
Proof. As in (1.11), let $y \in W^\nu$ be such that $\lambda = w_0 y \circ \nu$. Then
\[
\lambda^- = u \circ \nu \quad \text{and} \quad \lambda^+ = w_0 u \circ \nu \quad \text{and} \quad y = (w_0 x w_0) u,
\]
since $\lambda = x \circ \lambda^+ = x w_0 u \circ \nu = w_0 (w_0 x w_0) u \circ \nu$. Thus, using the definition in (1.11),
\[
[T_{\lambda^+}] = [T_{w_0 u \circ \nu}] = \varepsilon_0 T_u 1_\nu, \quad [T_{\lambda^-}] = [T_{u \circ \nu}] = [T_{w_0 (w_0 u) \circ \nu}] = \varepsilon_0 T_m 1_\nu,
\]
where $m$ is the minimal length representative of the coset $w_0 u W_\ell$ and
\[
[T_\lambda] = [T_{w_0 (w_0 x w_0) u \circ \nu}] = \varepsilon_0 T_{w_0 x w_0} 1_\nu = \varepsilon_0 T_{w_0 x w_0} T_u 1_\nu = (-t^{-\frac{1}{2}}) (w_0 x w_0) \varepsilon_0 T_u 1_\nu = (-t^{-\frac{1}{2}}) (x) \varepsilon_0 T_u 1_\nu.
\]
If $i \in \{1, \ldots, n\}$ and $\langle \lambda^+ + \rho, \alpha_i^\vee \rangle = 0$ then $s_j \in W_{\lambda^-}$ where $s_j = w_0 s_i w_0$. Since $W_{\lambda^-} = W_{u \circ \nu} = u W_\ell u^{-1}$, then $s u_\alpha_j = u^{-1} s_j u \in W_\ell$. Since $\nu \in A_{-\ell-k}$ then $u^{-1} s_j u = s u_\alpha_j = s_k$ with $k \in \{0, \ldots, n\}$. Thus $s_j u = u s_k$ and
\[
[T_{\lambda^+}] = \varepsilon_0 T_u 1_\nu = (-t^{\frac{1}{2}}) \varepsilon_0 T_s T_u 1_\nu = (-t^{\frac{1}{2}}) \varepsilon_0 T_{s_k} 1_\nu = (-t^{\frac{1}{2}}) \varepsilon_0 T_{s_k} T_u 1_\nu = (-t^{\frac{1}{2}}) \varepsilon_0 T_{s_k} T_u 1_\nu = -t \varepsilon_0 T_{s_k} 1_\nu = -t [T_{\lambda^+}],
\]
so that $[T_{\lambda^+}] = 0$. \qed

Remark 4.3. The following “straightening laws” for $[T_\lambda]$ follow from Proposition 4.2. Let $\lambda \in a^*_Z$ and let $i \in \{1, \ldots, n\}$. Then
\[
[T_{s_i \circ \lambda}] = \begin{cases} -t^{\frac{1}{2}} [T_\lambda], & \text{if } \langle \lambda + \rho, \alpha_i^\vee \rangle < 0, \\ \varepsilon_0 T_\lambda, & \text{if } \langle \lambda + \rho, \alpha_i^\vee \rangle = 0. \end{cases}
\]

4.4 The straightening laws for $[X_\lambda]$

In parallel with the case for $[T_\lambda]$, the properties in Proposition 4.4 determine $[X_\lambda]$ for $\lambda \in a^*_Z$ (all integral weights) in terms of $[X_{\lambda^+}]$ for $\lambda^+ \in (a^*_Z)^+$ (dominant integral weights). Proposition 4.4 is the same as [Gri, Prop. 6.3(ii)] (see also [Lan, Prop. 5.11]).

Proposition 4.4. Let $\lambda \in a^*_Z$ and let $\lambda^+$ and $\lambda^-$ be the dominant and the antidominant representatives of $W_0 \circ \lambda$, respectively.

(a) If $i \in \{1, \ldots, n\}$ and $\langle \lambda + \rho, \alpha_i^\vee \rangle = 0$ then $[X_\lambda] = 0$.

(b) If $\langle \lambda + \rho, \alpha_i^\vee \rangle \neq 0$ for $i \in \{1, \ldots, n\}$ then $[X_{\lambda^+}] = [T_{\lambda^+}]$.

(c) Let $i \in \{1, \ldots, n\}$. Then
\[
[X_{s_i \circ \lambda}] = \begin{cases} -[X_\lambda], & \text{if } \langle \lambda + \rho, \alpha_i^\vee \rangle \in \ell \mathbb{Z}_{\geq 0}, \\ -t^{\frac{1}{2}} [X_\lambda], & \text{if } 0 < \langle \lambda + \rho, \alpha_i^\vee \rangle < \ell, \\ -t^{\frac{1}{2}} [X_{s_i \circ \lambda}] - [X_{\lambda^+}] - t^{\frac{1}{2}} [X_\lambda], & \text{if } \langle \lambda + \rho, \alpha_i^\vee \rangle > \ell \text{ and } \langle \lambda + \rho, \alpha_i^\vee \rangle \notin \ell \mathbb{Z}, \end{cases}
\]
where
\[
\lambda^{(1)} = \lambda - j \alpha_i \quad \text{if } \langle \lambda + \rho, \alpha_i^\vee \rangle = k \ell + j, \quad \text{with } k \in \mathbb{Z}_{\geq 0} \text{ and } j \in \{1, \ldots, \ell - 1\}.
\]
The weight \( \epsilon \) \((c)\) The proof depends on the following identities in \( H \) = \( \{ \nu \in \mathfrak{a}_k^+ \} \) = \( 0 \) then \( t_\nu \lambda = \lambda \) and \( [X_\lambda] = [X_{s_i \circ \lambda}] = -[X_\lambda] \), so that \( 2[X_\lambda] = 0 \).

(b) Assume \( \langle \lambda + \rho, \alpha_i^\vee \rangle \neq 0 \) for all \( i \in \{1, \ldots, n\} \). Let \( u \in W \) be of minimal length such that \( \lambda^- = u \circ \nu \). Then \( \lambda^+ = w_0 u \circ \nu \) and, by the definition in \( (4.11) \), \( [X_{\lambda^+}] = [X_{w_0 \circ \nu}] = \epsilon_0 X^u_1 \nu \).

Write \( u = t^\rho \nu \) with \( \rho \in \mathfrak{a}_k^+ \) and \( x \in W_0 \). By \( (4.15) \), \( \mu^+ \) is dominant since \( \lambda^- \) is in the antidominant chamber, and \( (4.7) \) then gives that \( X^u = T_\nu \). Thus

\[
[X_{\lambda^+}] = [X_{w_0 \circ \nu}] = \epsilon_0 X^u_1 \nu = \epsilon_0 T_\nu 1_\nu = [T_{w_0 \circ \nu}] = [T_{\lambda^+}].
\]

(c) The proof depends on the following identities in \( H \), which we refer to as “lifted straightening laws”. The equality \( 0 = \epsilon_0(t^{1/2} + T_{s_k}^{-1}) \) is used to establish the “right half of the hexagon lifted straightening law”: If \( s_k w > w \) then

\[
0 = \epsilon_0(t^{1/2} + T_{s_k}^{-1})(X^{s_k \mu} + X^\mu)(T_{w^{-1}})^{-1} = \epsilon_0(X^{s_k \mu} + X^\mu)(t^{1/2} + T_{s_k}^{-1})T_{w^{-1}}^{-1}
\]

\[
= \epsilon_0(X^{s_k \mu}T_{w^{-1}})^{-1}T_{(s_k w)^{-1}} + t^{1/2} X^{s_k \mu}T_{w^{-1}}^{-1} + X^\mu T_{w^{-1}}^{-1} = \frac{1}{2}X^\mu T_{w^{-1}}^{-1}.
\]

The equality

\[
0 = T_{s_k} X^{s_k \mu} - X^{s_k \mu + \alpha_k}T_{s_k}^{-1} + T_{s_k} X^\mu - X^\mu T_{s_k}^{-1},
\]

is proved by the computation

\[
T_{s_k} X^{s_k \mu} - X^{s_k \mu + \alpha_k}T_{s_k}^{-1} + T_{s_k} X^\mu - X^\mu T_{s_k}^{-1}
\]

\[
= T_{s_k} X^{s_k \mu} - X^{s_k \mu + \alpha_k}(T_{s_k} - (t^{1/2} - t^{-1/2}) + T_{s_k} X^\mu - X^\mu (T_{s_k} - (t^{1/2} - t^{-1/2}))
\]

\[
= (t^{1/2} - t^{-1/2}) X^{s_k \mu} - X^\mu + X^{s_k \mu + \alpha_k}(t^{1/2} - t^{-1/2}) + (t^{1/2} - t^{-1/2}) X^\mu - X^{s_k \mu + \alpha_k} + X^\mu (t^{1/2} - t^{-1/2})
\]

\[
= (t^{1/2} - t^{-1/2}) (X^{s_k \mu} - X^\mu + (1 - X^{-\alpha_k}) X^\mu - X^{s_k \mu + \alpha_k} + (1 - X^{-\alpha_k}) X^{s_k \mu + \alpha_k})
\]

\[
= 0.
\]

The identity \( (4.17) \) is the source of the “left half of the hexagon lifted straightening law”: If \( s_k w > w \) then

\[
0 = \epsilon_0(T_{s_k} X^{s_k \mu} - X^{s_k \mu + \alpha_k}T_{s_k}^{-1} + T_{s_k} X^\mu - X^\mu T_{s_k}^{-1})T_{w^{-1}}^{-1}
\]

\[
= \epsilon_0(-t^{1/2} X^{s_k \mu}T_{w^{-1}}^{-1} - X^{s_k \mu + \alpha_k}T_{s_k w^{-1}}^{-1} - t^{-1/2} X^\mu T_{s_k w^{-1}}^{-1}) - X^\mu T_{s_k w^{-1}}^{-1}.
\]

Case 1R: \( 0 \leq \langle \ell(-w_0 \mu), \alpha^\vee_i \rangle - \ell \leq \langle \ell(-w_0 \mu), \alpha^\vee_i \rangle \leq \langle \lambda + \rho, \alpha^\vee_i \rangle < \langle \ell(-w_0 \mu), \alpha^\vee_i \rangle + \ell \).
First assume that \( \langle \ell(-w_0\mu), \alpha_i^\vee \rangle - \ell < \langle \ell(-w_0\mu), \alpha_i^\vee \rangle < \langle \lambda + \rho, \alpha_i^\vee \rangle < \langle \ell(-w_0\mu), \alpha_i^\vee \rangle + \ell \). Then (see the upper picture for Case 1R)

\[
[X_{s_i\circ\lambda}] = \varepsilon_0 X^{s_k\mu} T_{(s_k w)_1}^{-1} 1_{\nu}, \quad [X_{\lambda}] = \varepsilon_0 X^{\mu} T_{w_1}^{-1} 1_{\nu},
\]

\[
[X_{s_i\circ\lambda(1)}] = \varepsilon_0 X^{s_k\mu} T_{w_1}^{-1} 1_{\nu}, \quad [X_{\lambda(1)}] = \varepsilon_0 X^{\mu} T_{(s_k w)_1}^{-1} 1_{\nu}.
\]

Since

\[
\langle w \circ \nu + \rho, \alpha_k^\vee \rangle = \langle w_0 w \circ \nu + \rho, \alpha_i^\vee \rangle = \langle (\lambda - \ell(-w_0\mu)) + \rho, \alpha_i^\vee \rangle > 0
\]

then \( s_k w > w \) and so equation \([\text{R}]\) gives

\[
0 = \varepsilon_0 (X^{s_k\mu} T_{(s_k w)_1}^{-1} + t^\frac{1}{2} X^{s_k\mu} T_{w_1}^{-1} + X^{\mu} T_{(s_k w)_1}^{-1} + t^\frac{1}{2} X^{\mu} T_{w_1}^{-1}) 1_{\nu}
\]

\[
= [X_{s_i\circ\lambda}] + t^\frac{1}{2} [X_{s_i\circ\lambda(1)}] + [X_{\lambda(1)}] + t^\frac{1}{2} [X_{\lambda}]. \quad \text{(1Rreg)}
\]

In the limiting case \( \langle \ell(-w_0\mu), \alpha_i^\vee \rangle - \ell < \langle \ell(-w_0\mu), \alpha_i^\vee \rangle = \langle \lambda + \rho, \alpha_i^\vee \rangle < \langle \ell(-w_0\mu), \alpha_i^\vee \rangle + \ell \), then (see the lower picture for Case 1R)

\[
[X_{s_i\circ\lambda}] = \varepsilon_0 X^{s_k\mu} T_{w_1}^{-1} 1_{\nu}, \quad \text{and} \quad [X_{\lambda}] = \varepsilon_0 X^{\mu} T_{w_1}^{-1} 1_{\nu} \quad \text{(cen)}
\]

Since

\[
\langle w \circ \nu + \rho, \alpha_k^\vee \rangle = \langle w_0 w \circ \nu + \rho, \alpha_i^\vee \rangle = \langle (\lambda - \ell(-w_0\mu)) + \rho, \alpha_i^\vee \rangle = 0
\]

then \( s_k \in W_{w_0\nu} \) and \( w^{-1} s_k w \in W_{\nu} \). Let

\[
s_j = w^{-1} s_k w \in W_{\nu} \quad \text{and} \quad x = s_k w = w s_j,
\]

so that \( s_k x > x \) and \( x s_j > x \) and

\[
X^{\mu} T_{(s_k w)_1}^{-1} T_{s_j}^{-1} = X^{\mu} T_{w_1}^{-1} \quad \text{and} \quad X^{s_k\mu} T_{(s_k w)_1}^{-1} T_{s_j}^{-1} = X^{s_k\mu} T_{w_1}^{-1}.
\]

Since \( s_k x > x \) then equation \([\text{R}]\) gives

\[
0 = \varepsilon_0 (X^{s_k\mu} T_{(s_k w)_1}^{-1} + t^\frac{1}{2} X^{s_k\mu} T_{x_1}^{-1} + X^{\mu} T_{(s_k w)_1}^{-1} + t^\frac{1}{2} X^{\mu} T_{x_1}^{-1}) 1_{\nu}
\]

\[
= \varepsilon_0 (X^{s_k\mu} T_{(s_k w)_1}^{-1} + t X^{s_k\mu} T_{x_1}^{-1} T_{s_j}^{-1} + X^{\mu} T_{(s_k w)_1}^{-1} + t X^{\mu} T_{x_1}^{-1} T_{s_j}^{-1}) 1_{\nu}
\]

\[
= \varepsilon_0 (X^{s_k\mu} T_{(s_k w)_1}^{-1} + t X^{s_k\mu} T_{(s_k w)_1}^{-1} + X^{\mu} T_{(s_k w)_1}^{-1} + t X^{\mu} T_{(s_k w)_1}^{-1}) 1_{\nu}
\]

\[
= \varepsilon_0 (X^{s_k\mu} T_{x_1}^{-1} + t X^{s_k\mu} T_{x_1}^{-1} + X^{\mu} T_{x_1}^{-1} + t X^{\mu} T_{x_1}^{-1}) 1_{\nu}
\]

\[
= (1 + t)([X_{s_i\circ\lambda}] + [X_{\lambda}]). \quad \text{(1Rsing)}
\]

\text{Case 1L:} 0 \leq \langle \ell(-w_0\mu), \alpha_i^\vee \rangle - \ell \leq \langle \lambda + \rho, \alpha_i^\vee \rangle < \langle \ell(-w_0\mu), \alpha_i^\vee \rangle < \langle \ell(-w_0\mu), \alpha_i^\vee \rangle + \ell.

First assume that \( \langle \ell(-w_0\mu), \alpha_i^\vee \rangle - \ell < \langle \lambda + \rho, \alpha_i^\vee \rangle < \langle \ell(-w_0\mu), \alpha_i^\vee \rangle < \langle \ell(-w_0\mu), \alpha_i^\vee \rangle + \ell \).

With \( x = s_k w \), (see the upper picture for Case 1L)

\[
[X_{s_i\circ\lambda(1)}] = \varepsilon_0 X^{s_k\mu + \alpha_k} T_{(s_k w)_1}^{-1} 1_{\nu}, \quad [X_{\lambda(1)}] = \varepsilon_0 X^{\mu - \alpha_k} T_{w_1}^{-1} 1_{\nu},
\]

\[
[X_{s_i\circ\lambda}] = \varepsilon_0 X^{s_k\mu} T_{x_1}^{-1} 1_{\nu}, \quad [X_{\lambda}] = \varepsilon_0 X^{\mu} T_{(s_k w)_1}^{-1} 1_{\nu}.
\]

Since

\[
\langle w \circ \nu + \rho, \alpha_k^\vee \rangle = \langle w_0 w \circ \nu + \rho, \alpha_i^\vee \rangle = \langle (\lambda - \ell(-w_0\mu)) + \rho, \alpha_i^\vee \rangle < 0
\]
then $s_k w < w$ and so $x < s_k x$. Then equation (L) gives
\[
0 = \varepsilon_0 (-t^{-\frac{1}{2}} s_{k\mu}^{w-1} T_{w-1}^{T-1} s_{k\mu} - t^{-\frac{1}{2}} X^{\mu - \alpha_k T_{w-1}^{T-1} - X^{T-1}_{(s_k\mu)^{-1}}} ) 1_{\nu} = -t^{-\frac{1}{2}} ([X_{s_i\alpha}] + t^{\frac{1}{2}} [X_{s_i\alpha}]+ [X_{\alpha}] + t^{\frac{1}{2}} [X_{\alpha}]).
\] (1Reg)

In the limiting case $\langle \ell(-w_0) , x^\nu \rangle - \ell = \langle \lambda + \rho, x^\nu \rangle < \langle \ell(-w_0), x^\nu \rangle + \ell$
\[
\gamma = \mu + \alpha_k,
\]
then (see the lower picture for Case 1L)
\[
[X_{s_i\alpha}] = \varepsilon_0 X^{s_{k\mu} - \alpha_k T_{w-1}^{T-1} 1_{\nu}} = \varepsilon_0 X^{s_{k\gamma} T_{w-1}^{T-1} 1_{\nu}}
\]
and
\[
[X_{\lambda}] = \varepsilon_0 X^{s_{\mu} T_{w-1}^{T-1} 1_{\nu}} = \varepsilon_0 X^{s_{\nu} T_{w-1}^{T-1} 1_{\nu}}.
\]
Since
\[
\langle w \circ x + \rho, x^\nu \rangle = \langle w_0 w \circ x + \rho, x^\nu \rangle = \langle (\lambda - \ell(-w_0)) + \rho, x^\nu \rangle > 0
\]
then $s_k w > w$. Since
\[
\langle w \circ x + \rho, x^\nu \rangle - \ell = \langle w_0 w \circ x + \rho, x^\nu \rangle - \ell = \langle (\lambda - \ell(-w_0)) + \rho, x^\nu \rangle - \ell = 0
\]
then $s_{\alpha_k} w \in W_w w_\nu$ and $s_0 = w s_{\alpha_k} w^{-1} = s_{w(-\alpha_k w)} w^{-1} \in W_\nu$. Then
\[
X^{s_k \gamma + \alpha_k T_{(s_k w)^{-1}}} = X^{s_k \gamma T_{(w-1)^{-1} T_{s_0}}} \quad \text{and} \quad X^{\gamma T_{(s_k w)^{-1}}} = X^{\gamma - \alpha_k T_{(w-1)^{-1} T_{s_0}}}.
\]
and equation (L) gives
\[
0 = \varepsilon_0 (-t^{-\frac{1}{2}} s_{k\gamma} T_{w-1}^{T-1} s_{k\gamma} - t^{-\frac{1}{2}} X^{\gamma - \alpha_k T_{w-1}^{T-1} - X^{T-1}_{(s_k w)^{-1}}} ) 1_{\nu} = -t^{-\frac{1}{2}} ([X_{s_i\alpha}] + [X_{\lambda}]).
\] (1Lsing)

Case 2R: $0 = \langle \ell(-w_0), x^\nu \rangle$ and $0 = \langle \ell(-w_0), x^\nu \rangle - \ell = \langle (\lambda + \rho, x^\nu \rangle + \ell$. This case is really a special case of Case 1R, with
\[
\gamma = \mu + \alpha_k,
\]
In the case that $0 < \langle (\lambda + \rho, x^\nu \rangle < \ell$ then (see the top picture in Case 2R)
\[
[X_{s_i\alpha}] = \varepsilon_0 X^{s_{\mu} T_{(s_k w)^{-1}}^{T-1} 1_{\nu}} \quad \text{and} \quad [X_{\lambda}] = \varepsilon_0 X^{s_{\mu} T_{w-1}} 1_{\nu}
\]
and (1Rreg) becomes
\[
0 = \varepsilon_0 (s_k T_{(s_k w)^{-1}}^{T-1} + t^{\frac{1}{2}} X^{s_k T_{w-1}^{T-1} X^{T-1}_{(s_k w)^{-1}}} + t^{\frac{1}{2}} X^{T-1} 1_{\nu}) = [X_{s_i\alpha}] + t^{\frac{1}{2}} [X_{s_i\alpha}] + t^{\frac{1}{2}} [X_{\lambda}] = 2(t^{\frac{1}{2}} [X_{\lambda}]) (2Rreg).
\]
For the limiting case where $0 = \langle (\lambda + \rho, x^\nu \rangle < \ell$ (this is analogous to (cent))
\[
[X_{\lambda}] = [X_{s_i\alpha}] = \varepsilon_0 X^{s_{\mu} T_{w-1}^{T-1} 1_{\nu}}.
\]
and \((1Rsing)\) becomes

\[
0 = \varepsilon_0(X^{t_k\mu} T_{w_{-1}}^{-1} + t X^{s_k\mu} T_{w_{-1}}^{-1} + X^{\mu} T_{w_{-1}}^{-1} + t X^{\mu} T_{w_{-1}}^{-1})1_\nu = (1 + \ell)2[X_\lambda]. \tag{2Rsing}
\]

Case 2L: \(\ell = \langle \ell(\text{w}_0\mu), \alpha_i^\vee \rangle\) and 0 = \(\langle \ell(\text{w}_0\mu), \alpha_i^\vee \rangle - \ell \leq \langle \lambda + \rho, \alpha_i \rangle < \langle \ell(\text{w}_0\mu), \alpha_i^\vee \rangle = \ell\). This case is really a special case of Case 1L, with

\[s_k\mu = \mu - \alpha_k, \quad \text{since} \quad 1 = \frac{1}{\ell} \langle \ell(\text{w}_0\mu), \alpha_i^\vee \rangle = \langle -\mu, \alpha_i^\vee \rangle.
\]

In the case that \(0 < \langle \lambda + \rho, \alpha_i^\vee \rangle < \ell\) then (see the bottom picture in Case 2L)

\[ [X_{s_i\alpha}] = \varepsilon_0 X^{\mu - \alpha_k} T_{w_{-1}}^{-1} 1_\nu \quad \text{and} \quad [X_\lambda] = \varepsilon_0 X^{\mu} T_{(s_k x)}^{-1} 1_\nu.
\]

and \((1Reg)\) becomes

\[
0 = \varepsilon_0(-t^{-\frac{1}{2}} X^{s_k\mu} T_{w_{-1}}^{-1} - X^{s_k\mu + \alpha_k} T_{(s_k x)}^{-1} - t^{-\frac{1}{2}} X^{\mu - \alpha_k} T_{w_{-1}}^{-1} - X^{\mu} T_{(s_k x)}^{-1})1_\nu
= -t^{-\frac{1}{2}} [X_{s_i\alpha}] - [X_\lambda] - t^{-\frac{1}{2}} [X_{s_i\alpha}] - [X_\lambda] = -2t^{-\frac{1}{2}} ([X_{s_i\alpha}] + t^\frac{1}{2} [X_\lambda]). \tag{2Reg}
\]

For the limiting case where \(0 = \langle \lambda + \rho, \alpha_i^\vee \rangle < \ell\) (this is analogous to \((bdy)\))

\[ [X_\lambda] = [X_{s_i\alpha}] = \varepsilon_0 X^{\mu} T_{(s_k x)}^{-1} 1_\nu = \varepsilon_0 X^{\gamma - \alpha_k} T_{w_{-1}}^{-1} 1_\nu = \varepsilon_0 X^{s_k\gamma} T_{w_{-1}}^{-1} 1_\nu
\]

and \((1Sing)\) becomes

\[
0 = \varepsilon_0(-t^{-\frac{1}{2}} X^{s_k\gamma} T_{w_{-1}}^{-1} - t^\frac{1}{2} X^{s_k\gamma} T_{w_{-1}}^{-1} - t^{-\frac{1}{2}} X^{\gamma - \alpha_k} T_{w_{-1}}^{-1} - t^\frac{1}{2} X^{\gamma - \alpha_k} T_{w_{-1}}^{-1})1_\nu
= -(t^{-\frac{1}{2}} + t^\frac{1}{2})2[X_\lambda]. \tag{2Sing}
\]

Together these computations complete the proof of part \((c)\): the third case follows from \((1Reg)\) and \((1Reg)\), the second case from \((2Reg)\) and \((2Reg)\), and the first case from \((1Sing)\) and \((1Sing)\), with \((2Sing)\) and \((2Sing)\) specifically treating the statement in \((a)\).

\[\square\]

Remark 4.5. If \(\lambda \in \ell \alpha_2^+ - \rho\) then there is a unique \(\mu \in \alpha_2^+\) such that

\[
\lambda = \ell \text{w}_0\mu - \rho = t_{w_0\mu} \circ (-\rho) = w_0 t_\mu \circ (-\rho),
\]

so that \(\lambda = w_0 \circ \nu\) with \(\nu = -\rho\) and \(v = t_\mu\). Since \(\nu = -\rho\) then \(1_\nu = 1_0\) with \(T_{s_i} 1_0 = t^\frac{1}{2} 1_0\) for \(i \in \{1, \ldots, n\}\). Thus,

\[\varepsilon_0 T_{\nu} \quad \text{and} \quad [X_\lambda] = \varepsilon_0 X^{\mu} 1_0,
\]

so that the \([X_\lambda]\), \(\lambda \in \ell \alpha_2^+ - \rho\), are the elements \(A_\mu\) studied in \([NR \ \S 2]\). In this case the first case of Proposition 4.4 \((c)\) is the straightening law and this coincides with the equality \(A_{s_i\mu} = -A_\mu\) proved in \([NR \ \text{Prop. 2.1}]\).

Remark 4.6. Following the definition of \([X_\lambda]\) in \((4.11)\),

\[
\text{if} \quad \lambda = w_0 \circ \nu \quad \text{then} \quad w_0 \circ \lambda = w_0(\text{w}_0\nu) \circ \nu = w_0(\text{w}_0\nu) \circ \nu
\]

and we have \([X_{w_0\alpha}] = \varepsilon_0 X^\nu 1_\nu\) and \([X_{w_0\circ \alpha}] = \varepsilon_0 X^{w_0\nu} w_0 \circ \nu\). With \(X^\nu = X^t w\) then

\[X^w = X^{t w} = X^\mu (T_{w_{-1}}) = X^\mu T_w = T_{w_0} X^{w_0\mu} T_{w_{-1}} T_w = T_{w_0} X^{w_0\mu} T_{w_{-1}}^{-1} w_0
= T_{w_0} X^{w_0\mu} w_0 \circ \nu w_0 \circ \nu = T_{w_0} X^{w_0\nu} T_{w_{-1}}^{-1} w_0
\]

By the previous computation, \(X^{w_0\nu} = T_{w_0} X^{w_0\nu} T_{w_{-1}}^{-1}, \) so that

\[X_{w_0\circ \alpha} = \varepsilon_0 X^{w_0\nu} 1_\nu = \varepsilon_0 T_{w_0} X^{w_0\nu} T_{w_{-1}}^{-1} 1_\nu = (-t^{-\frac{1}{2}} - \ell(w_0)(t^\frac{1}{2}) - \ell(w_0) \varepsilon_0 X^{w_0\nu} 1_\nu
= (-1)^{\ell(w_0)} (t^{-\frac{1}{2}} - \ell(w_0) + \ell(w_0) \varepsilon_0 X^{w_0\nu} 1_\nu
= (-1)^{\ell(w_0)} (t^{-\frac{1}{2}} - \ell(w_0) + \ell(w_0) \varepsilon_0 X^{w_0\nu} 1_\nu
\]

Hence

\[\varepsilon_0 X^{w_0\nu} 1_\nu = (-1)^{\ell(w_0)} (t^{-\frac{1}{2}} - \ell(w_0) + \ell(w_0) \varepsilon_0 X^{w_0\nu} 1_\nu
\]

(4.18)
4.5 Relating the KL-modules $\mathcal{P}_{-\ell-h}^+$ and $\mathcal{F}_\ell$

In this subsection we tie together our components: the module with bar involution $\mathcal{P}_{-\ell-h}^+$ from [4.9] which was built from the affine Hecke algebra and the abstract Fock space $\mathcal{F}_\ell$ from (1.3). Because of the way that we arrived at $\mathcal{P}_{-\ell-h}^+$ from representation theory (see (QG) at the end of Section 3) the isomorphism between $\mathcal{P}_{-\ell-h}^+$ and $\mathcal{F}_\ell$ will allow us to prove that the abstract Fock space $\mathcal{F}_\ell$ captures decomposition numbers of Weyl modules for quantum groups at roots of unity.

**Theorem 4.7.** Let $\leq$ be the dominance order on the set $(\mathfrak{a}_\mathbb{Z}^*)^+$ of dominant integral weights.

Let $\mathcal{P}_{-\ell-h}^+$ with basis $B = \{ [X_\lambda] \mid \lambda \in (\mathfrak{a}_\mathbb{Z}^*)^+ \}$ and bar involution as in (4.10), and let $\mathcal{F}_\ell$ with basis $\mathcal{L} = \{ |\lambda\rangle \mid \lambda \in (\mathfrak{a}_\mathbb{Z}^*)^+ \}$ and bar involution as in (4.4).

Then $\mathcal{P}_{-\ell-h}^+$ is a KL-module and

$$\begin{align*}
\mathcal{P}_{-\ell-h}^+ & \hookrightarrow \mathcal{F}_\ell, \\
\left[ X_\lambda \right] & \mapsto |\lambda\rangle
\end{align*}$$

is a KL-module isomorphism.

**Proof.** By definition (see [4.9]), $\mathcal{P}_{-\ell-h}^+ = \bigoplus_{\nu \in A_{-\ell-h}} \varepsilon_0 H_{1\nu}$. By §2.2.4 each summand is a KL-module and so $\mathcal{P}_{-\ell-h}^+$ is a KL-module.

The $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$-module $\mathcal{F}_\ell$ is generated by $|\lambda\rangle$, $\lambda \in \mathfrak{a}_\mathbb{Z}^*$. By definition, these symbols satisfy the relations in (1.3). The $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$-module $\mathcal{P}_{-\ell-h}^+$ is generated by the symbols $[X_\lambda]$, $\lambda \in \mathfrak{a}_\mathbb{Z}^*$. By comparison of the relations in (1.3) with those in Proposition 4.4(c), there is a surjective $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$-module homomorphism

$$\Phi : \mathcal{F}_\ell \to \mathcal{P}_{-\ell-h}^+$$

given by $\Phi(|\lambda\rangle) = [X_\lambda]$, \hspace{1cm} (4.19)

for $\lambda \in \mathfrak{a}_\mathbb{Z}^*$. This homomorphism respects the bar involution since, by (4.4) and (4.18),

$$\Phi(\overline{\lambda}) = \Phi((-1)^{\ell(w_0)}(t^{-\frac{1}{2}})^{\ell(w_0)}-N_\lambda |w_0 \circ \lambda\rangle) = (-1)^{\ell(w_0)}(t^{-\frac{1}{2}})^{\ell(w_0)}-N_\lambda \Phi(|w_0 \circ \lambda\rangle) = (-1)^{\ell(w_0)}(t^{-\frac{1}{2}})^{\ell(w_0)}-N_\lambda [X_{w_0 \circ \lambda}] = (-1)^{\ell(w_0)}(t^{-\frac{1}{2}})^{\ell(w_0)}-N_\lambda (-1)^{\ell(w_0)}(t^{-\frac{1}{2}})^{\ell(w_0)}(t^{-\frac{1}{2}})^{-\ell(w_0)}+\ell(w_0) [X_\lambda] = (-1)^{\ell(w_0)}(t^{-\frac{1}{2}})^{\ell(w_0)}-N_\lambda (-1)^{\ell(w_0)}(t^{-\frac{1}{2}})^{-\ell(w_0)}+\ell(w_0) \Phi(|\lambda\rangle) = \overline{\Phi(|\lambda\rangle)}.
$$

If $\lambda \in (\mathfrak{a}_\mathbb{Z}^*)^+$ then $[X_\lambda] = [T_\lambda]$. Thus, by Proposition 4.2 (see also Proposition 2.2), the set $\{ [X_\lambda] \mid \lambda \in (\mathfrak{a}_\mathbb{Z}^*)^+ \}$ is a basis of $\mathcal{P}_{-\ell-h}^+$. Since the $\Phi_\ell$ image of $\{ |\lambda\rangle \mid \lambda \in (\mathfrak{a}_\mathbb{Z}^*)^+ \}$ is linearly independent in $\mathcal{P}_{-\ell-h}^+$ this set must be linearly independent in $\mathcal{F}_\ell$ and $\Phi_\ell$ is injective. Since $\mathcal{F}_\ell$ is spanned by $\{ |\lambda\rangle \mid \lambda \in (\mathfrak{a}_\mathbb{Z}^*)^+ \}$ then $\Phi_\ell$ is a KL-module isomorphism.

The KL-module $\mathcal{F}_\ell$ has

standard basis $\{ |\lambda\rangle \mid \lambda \in (\mathfrak{a}_\mathbb{Z}^*)^+ \}$ and KL-basis $\{ C_\lambda \mid \lambda \in (\mathfrak{a}_\mathbb{Z}^*)^+ \}$.

For $\mu, \lambda \in (\mathfrak{a}_\mathbb{Z}^*)^+$ define $p_{\mu \lambda}, d_{\lambda \mu} \in \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ by

$$C_\mu = |\mu\rangle + \sum_\mu p_{\mu \lambda} |\lambda\rangle, \quad \text{and} \quad |\lambda\rangle = C_\lambda + \sum_\mu d_{\lambda \mu} C_\mu.$$ \hspace{1cm} (4.21)
Theorem 4.8. Fix \( \ell \in \mathbb{Z}_{>0} \) and let \( q^\ell \) be a primitive \( \ell \)th root of unity in \( \mathbb{C} \). Let \( U_q(\mathfrak{g}) \) be the Drinfeld-Jimbo quantum group corresponding to the weight lattice \( \mathfrak{a}^*_{\mathbb{Z}} \), the Weyl group \( W_0 \) and the positive roots \( R^+ \). Let \( L_q(\mu) \) be the simple module of highest weight \( \mu \) for the quantum group \( U_q(\mathfrak{g}) \) and let

\[
\Delta_q(\lambda) = \Delta_q(\lambda)^{(0)} \supseteq \Delta_q(\lambda)^{(1)} \supseteq \cdots
\]

be the Jantzen filtration of the Weyl module \( \Delta_q(\lambda) \) of highest weight \( \lambda \) for \( U_q(\mathfrak{g}) \).

Let \( W \) be the affine Weyl group and let \( \lambda, \mu \in (\mathfrak{a}^*_{\mathbb{Z}})^+ \) and let \( p_{\mu\lambda} \) and \( d_{\mu\lambda} \) be as given in (4.21).

1. If \( \lambda \) and \( \mu \) are not in the same \( W \)-orbit for the level \((-\ell - h)\) dot-action of \( W \) on \( \mathfrak{a}^*_{\mathbb{Z}} \) then \( d_{\mu\lambda} = 0 \) and \( p_{\mu\lambda} = 0 \).

2. If \( \lambda \) and \( \mu \) are in the same \( W \)-orbit then let \( \nu \in A_{-\ell - h} \) and \( x, y \in W \) be such that

\[
\lambda = w_0x \circ \nu, \quad \mu = w_0y \circ \nu, \quad x, y \in ^0W \quad \text{and} \quad w_0x, w_0y \in W^\nu,
\]

where \( w_0 \) is the longest element of the Weyl group \( W_0 \), \( W_\nu \) is the stabilizer of \( \nu \) under the dot action of \( W \), \( W^\nu \) is the set of minimal length representatives of cosets in \( W/W_\nu \) and \(^0W \) is the set of minimal length representatives of cosets in \( W_0/W \).

Then

\[
p_{\mu\lambda}(-1)^{\ell(w_0x) - \ell(w_0y)}P^\nu_{w_0y, w_0x}(t^{\frac{1}{\ell}}) \quad \text{and} \quad d_{\mu\lambda} = \left( \sum_{j \in \mathbb{Z}_{\geq 0}} t^j \dim \left( \text{Hom} \left( \frac{\Delta_q(\lambda)(j)}{\Delta_q(\lambda)(j+1)}, L_q(\mu) \right) \right) \right),
\]

where \( P^\nu_{w_0y, w_0x}(t^{\frac{1}{\ell}}) \) is the (parabolic singular) Kazhdan-Lusztig polynomial (see (2.30)) for the affine Hecke algebra \( H \) corresponding to \( W \) (see (2.12) and (4.3)).

Proof. By definition (see (4.9)), \( \mathcal{P}^+_{-\ell - h} = \bigoplus_{\nu \in A_{-\ell - h}} \varepsilon_0H^1_\nu \). The analysis in (2.4.4) applies to each of the summands \( \varepsilon_0H^1_\nu \) to give that, for \( \lambda, \mu \in (\mathfrak{a}^*_{\mathbb{Z}})^+ \),

\[
\Phi([\lambda]) = [X_\lambda] = [T_\lambda] = \varepsilon_0T_{y_\nu}1_\nu \quad \text{and} \quad \Phi(C_\mu) = C_{w_0x}1_\nu,
\]

where \( \Phi : \mathcal{F}_\ell \to \mathcal{P}^+_{-\ell - h} \) is the KL-module isomorphism from (4.19) and \( x, y \in W \) and \( \nu \in A_{-\ell - h} \) are as defined in the statement of (b). In particular, by (2.30), the transition matrix between these bases is given by

\[
\Phi(C_\mu) = C_{w_0x}1_\nu = \sum_{\substack{y \in W_0 \cap W_\nu \cap W^\nu \ni x \leq y \leq w_0x \cap y \leq w_0x \cap y \in ^0W_\nu \cap W^\nu}} (-1)^{\ell(w_0x) - \ell(w_0y)}P^\nu_{w_0y, w_0x}(t^{\frac{1}{\ell}})\varepsilon_0T_{y_\nu}1_\nu
\]

and, since \( \Phi \) is an isomorphism, this establishes the formula for \( p_{\mu\lambda} \). The formula for \( d_{\mu\lambda} \) is then a consequence of the isomorphism of (QG) given at the end of Section 3. \( \square \)
Case 1R: $0 \leq \langle \ell(-w_0\mu), \alpha_i^\vee \rangle - \ell < \langle \ell(-w_0\mu), \alpha_i^\vee \rangle < \langle \lambda + \rho, \alpha_i^\vee \rangle < \langle \ell(-w_0\mu), \alpha_i^\vee \rangle + \ell$ and $0 < \langle \ell(-w_0\mu), \alpha_i^\vee \rangle - \ell < \langle \ell(-w_0\mu), \alpha_i^\vee \rangle = \langle \lambda + \rho, \alpha_i^\vee \rangle < \langle \ell(-w_0\mu), \alpha_i^\vee \rangle + \ell$. 
Case 1L: 

\[ [X_{\ell \cdot \mu}] = \varepsilon_0 X^{s_k \mu} T^{-1}_{w_0} \mathbf{1}_v \]

\[ [X_{s_i \cdot \lambda}] = \varepsilon_0 X^{s_k \mu} T^{-1}_{w_0} \mathbf{1}_v \]

\[ w_0 t_{s_k \mu} - a_k = w_0 t_{s_k \gamma} \]

\[ w_0 t_{s_k \mu} = w_0 t_{s_k \gamma} + a_k \]

\[ w_0 t_{s_k \mu} = w_0 t_{\gamma - a_k} \]

\[ w_0 t_{s_k \mu} = w_0 t_{\gamma + a_k} \]

\[ w_0 t_{s_k \gamma} = w_0 t_{s_k \gamma} + a_k \]

\[ w_0 t_{s_k \gamma} - a_k = w_0 t_{s_k \gamma} - a_k \]

\[ w_0 t_{s_k \gamma} w = w_0 t_{s_k \gamma} w s_0 \]

\[ w_0 t_{s_k \gamma}(s_k w) = w_0 t_{s_k \gamma} w s_0 \]

\[ w_0 t_{s_k \gamma} + a_k (s_k w) = w_0 t_{s_k \gamma} w s_0 \]

\[ w_0 t_{s_k \mu} w = w_0 t_{s_k \mu} w \]

\[ w_0 t_{s_k \mu} = w_0 t_{s_k \mu} \]

\[ w_0 t_{s_k \mu} = w_0 t_{s_k \mu} \]

\[ w_0 t_{s_k \mu} = w_0 t_{s_k \mu} \]

\[ 0 < \langle \ell(-w_0 \mu), \alpha_i^\gamma \rangle - \ell < \langle \lambda + \rho, \alpha_i^\gamma \rangle < \langle \ell(-w_0 \mu), \alpha_i^\gamma \rangle + \ell \] and

\[ 0 < \langle \ell(-w_0 \mu), \alpha_i^\gamma \rangle - \ell = \langle \lambda + \rho, \alpha_i^\gamma \rangle < \langle \ell(-w_0 \mu), \alpha_i^\gamma \rangle < \langle \ell(-w_0 \mu), \alpha_i^\gamma \rangle + \ell. \]
Case 2R: $0 = \langle \ell(-w_0\mu + \rho, \alpha^\gamma) \rangle < \langle \lambda + \rho, \alpha^\gamma \rangle < \ell = \langle \ell(-w_0\mu + \rho, \alpha^\gamma) \rangle + \ell$ and

Case 2L: $0 = \langle \ell(-w_0\mu + \rho, \alpha^\gamma) \rangle - \ell < \langle \lambda + \rho, \alpha^\gamma \rangle < \ell = \langle \ell(w_0\mu + \rho, \alpha^\gamma) \rangle$
References


