Affine and degenerate affine BMW algebras: The center

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Abstract

The degenerate affine and affine BMW algebras arise naturally in the context of Schur-Weyl duality for orthogonal and symplectic Lie algebras and quantum groups, respectively. Cyclotomic BMW algebras, affine Hecke algebras, cyclotomic Hecke algebras, and their degenerate versions are quotients. In this paper the theory is unified by treating the orthogonal and symplectic cases simultaneously; we make an exact parallel between the degenerate affine and affine cases via a new algebra which takes the role of the affine braid group for the degenerate setting. A main result of this paper is an identification of the centers of the affine and degenerate affine BMW algebras in terms of rings of symmetric functions which satisfy a "cancellation property" or "wheel condition" (in the degenerate case, a reformulation of a result of Nazarov). Miraculously, these same rings also arise in Schubert calculus, as the cohomology and K-theory of isotropic Grassmanians and symplectic loop Grassmanians. We also establish new intertwiner-like identities which, when projected to the center, produce the recursions for central elements given previously by Nazarov for degenerate affine BMW algebras, and by Beliakova-Blanchet for affine BMW algebras.

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Contents

1 Introduction

The degenerate affine BMW algebras $\mathcal{W}_k$ and the affine BMW algebras W_k arise naturally in the context of Schur-Weyl duality and the application of Schur functors to modules in category $\mathcal{O}$ for orthogonal and symplectic Lie algebras and quantum groups (using the Schur functors of [Ze], [AS], and [OR]). The degenerate algebras $\mathcal{W}_k$ were introduced in [Naz] and the affine versions W_k appeared in [OR], following foundational work of [Hä1]-[Hä3]. The representation theory of $\mathcal{W}_k$ and W_k contains the representation theory of any quotient: in particular, the degenerate cyclotomic BMW algebras $\mathcal{W}_{r,k}$, the cyclotomic BMW algebras $W_{r,k}$, the degenerate affine Hecke algebras $\mathcal{H}_k$, the affine Hecke algebras H_k, the degenerate cyclotomic Hecke algebras $\mathcal{H}_{r,k}$, and the cyclotomic Hecke algebras $H_{r,k}$ as quotients. The representation theory of the affine BMW algebras is an image of the representation theory of category $\mathcal{O}$ for orthogonal and symplectic Lie algebras and their quantum groups in the same way that the affine Hecke algebras arise in Schur-Weyl duality with the enveloping algebra of $\mathfrak{gl}_n$ and its Drinfeld-Jimbo quantum group.

In the literature, the algebras $\mathcal{W}_k$ and W_k have often been treated separately. One of the goals of this paper is to unify the theory. To do this we have begun by adjusting the definitions of the algebras carefully to make the presentations match, relation by relation. In the same way that the affine BMW algebra is a quotient of the group algebra of the affine braid group, we have defined a new algebra, the degenerate affine braid algebra which has the degenerate affine BMW algebra and the degenerate affine Hecke algebras as quotients. We have done this carefully, to ensure that the Schur-Weyl duality framework is completely analogous for both the degenerate affine and the affine cases. We have also added a parameter ϵ (which takes values ± 1) so that both the orthogonal and symplectic cases can be treated simultaneously. Our new presentations of the algebras $\mathcal{W}_k$ and W_k are given in section 2.

In section 3 we consider some remarkable recursions for generating central elements in the algebras $\mathcal{W}_k$ and W_k. These recursions were given by Nazarov [Naz] in the degenerate case, and then extended to the affine BMW algebra by Beliakova-Blanchet [BB]. Another proof in the affine cyclotomic case appears in [RX2, Lemma 4.21] and, in the degenerate case, in [AMR, Lemma 4.15]. In all of these proofs, the recursion is obtained by a rather mysterious and tedious computation. We show that there is an "intertwiner" like identity in the full algebra which, when "projected to the center" produces the Nazarov recursions. Our approach dramatically simplifies the proof and provides insight into where these recursions are coming from. Moreover, the proof is exactly analogous in both the degenerate and the affine cases, and includes the parameter ϵ, so that both the orthogonal and symplectic cases are treated simultaneously.

In section 4 we identify the center of the degenerate and affine BMW algebras. In the degenerate case this has been done in [Naz]. Nazarov stated that the center of the degenerate affine BMW algebra is the subring of the ring of symmetric functions generated by the odd power sums. We identify the ring in a different way, as the subring of symmetric functions with the Q-cancellation property, in the language of Pragacz [Pr]. This is a fascinating ring. Pragacz identifies it as the cohomology ring of orthogonal and symplectic Grassmannians; the same ring appears again as the cohomology of the loop Grassmannian for the symplectic group in [LSS, La]; and references for the relationship of this ring to the projective representation theory of the symmetric group, the BKP hierarchy of differential equations, representations of Lie superalgebras, and twisted Gelfand pairs are found in [Mac, Ch. II §8]. For the affine BMW algebra, the Q-cancellation property can be generalized well to provide a suitable description of the center. From our perspective, one would expect that the ring which appears as the center of the affine BMW algebra should also appear as the K-theory of the orthogonal and symplectic Grassmannians and as the K-theory of the loop Grassmannian for the symplectic group, but we are not aware that these identifications have yet been made in the literature.

This paper is part of a more comprehensive work on affine and degenerate affine BMW algebras. In future work [DRV] we may:

(a) set up the commuting actions between the algebras $\mathcal{W}_k$ and W_k and the enveloping algebras of orthogonal and symplectic Lie algebras and their quantum groups,

(b) show how the central elements which arise in the Nazarov recursions coincide with central elements studied in Baumann [Bau],

(c) provide a new approach to admissibility conditions by providing "universal admissible parameters" in an appropriate ground ring (arising naturally, from Schur-Weyl duality, as the center of the enveloping algebra, or quantum group),

(d) classify and construct the irreducible representations of $\mathcal{W}_k$ and W_k by multisegments, and

(e) define Khovanov-Lauda-Rouquier analogues of the affine BMW algebras.

Many parts of this program are already available in the works of Goodman, Rui, Wilcox-Yu, and others (see, for example, [RS1]-[RS2], [RX1]-[RX2], [Go1]-[Go3], [GH1]-[GH3], [WY1]-[WY2], [Yu]). Some parts of our work are also available at [Ra].

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2 Affine and degenerate affine BMW algebras

In this section, we define the affine Birman-Murakami-Wenzl (BMW) algebra W_k and its degenerate version $\mathcal{W}_k$. We have adjusted the definitions to unify the theory. In particular, in section 2.1, we define a new algebra, the degenerate affine braid algebra $\mathcal{B}_k$, which has the degenerate affine BMW algebras $\mathcal{W}_k$ and the degenerate affine Hecke algebras $\mathcal{H}_k$ as quotients. The motivation for the definition of $\mathcal{B}_k$ is that the affine BMW algebras W_k and the affine Hecke algebras H_k are quotients of the group algebra of affine braid group CB_k.

The definition of the degenerate affine braid algebra $\mathcal{B}_k$ also makes the Schur-Weyl duality framework completely analogous in both the affine and degenerate affine cases. Both $\mathcal{B}_k$ and CB_k are designed to act on tensor space of the form $M \otimes V^{\otimes k}$. In the degenerate affine case this is an action commuting with a complex semisimple Lie algebra $\mathfrak{g}$, and in the affine case this is an action commuting with the Drinfeld-Jimbo quantum group $U_q\mathfrak{g}$. The degenerate affine and affine BMW algebras arise when $\mathfrak{g}$ is $\mathfrak{so}_n$ or $\mathfrak{sp}_n$ and V is the first fundamental representation and the degenerate affine and affine Hecke algebras arise when $\mathfrak{g}$ is $\mathfrak{gl}_n$ or f_n and V is the first fundamental representation. In the case when M is the trivial representation and $\mathfrak{g}$ is $\mathfrak{so}_n$, the "Jucys-Murphy" elements $y_1, \ldots, y_k$ in $\mathcal{B}_k$ become the "Jucys-Murphy" elements for the Brauer algebras used in [Naz] and, in the case that $\mathfrak{g} = \mathfrak{gl}_n$, these become the classical Jucys-Murphy elements in the group algebra of the symmetric group. The Schur-Weyl duality actions are explained in [DRV] and [Ra].

2.1 The degenerate affine braid algebra $\mathcal{B}_k$

Let C be a commutative ring, and let S_k denote the symmetric group on $\{1, \ldots, k\}$. For $i \in \{1, \ldots, k\}$, write s_i for the transposition in S_k that switches i and $i+1$. The *degenerate affine braid algebra* is the algebra $\mathcal{B}_k$ over C generated by

$$t_u \quad (u \in S_k), \qquad \kappa_0, \kappa_1, \qquad \text{and} \qquad y_1, \ldots, y_k, \tag{2.1}$$

with relations

$$t_u t_v = t_{uv}, \qquad y_i y_j = y_j y_i, \quad \kappa_0 \kappa_1 = \kappa_1 \kappa_0, \quad \kappa_0 y_i = y_i \kappa_0, \quad \kappa_1 y_i = y_i \kappa_1, \tag{2.2}$$

$$\kappa_0 t_{s_i} = t_{s_i} \kappa_0, \qquad \kappa_1 t_{s_1} \kappa_1 t_{s_1} = t_{s_1} \kappa_1 t_{s_1} \kappa_1, \qquad \text{and} \qquad \kappa_1 t_{s_j} = t_{s_j} \kappa_1, \text{ for } j \neq 1, \tag{2.3}$$

$$t_{s_i}(y_i + y_{i+1}) = (y_i + y_{i+1}) t_{s_i}, \quad \text{and} \qquad y_j t_{s_i} = t_{s_i} y_j, \quad \text{for } j \neq i, i+1, \tag{2.4}$$

and

$$t_{s_i} t_{s_{i+1}} \gamma_{i,i+1} t_{s_{i+1}} t_{s_i} = \gamma_{i+1,i+2}, \qquad \text{where} \quad \gamma_{i,i+1} = y_{i+1} - t_{s_i} y_i t_{s_i} \text{ for } i = 1, \ldots, k-2. \tag{2.5}$$

In the degenerate affine braid algebra $\mathcal{B}_k$ let $c_0 = \kappa_0$ and

$$c_j = \kappa_0 + 2(y_1 + \ldots + y_j), \qquad \text{so that} \quad y_j = \tfrac{1}{2}(c_j - c_{j-1}), \quad \text{for } j = 1, \ldots, k. \tag{2.6}$$

Then $c_0, \ldots, c_k$ commute with each other, commute with κ_1, and the relations (2.4) are equivalent to

$$t_{s_i} c_j = c_j t_{s_i}, \qquad \text{for } j \neq i. \tag{2.7}$$

Theorem 2.1. *The degenerate affine braid algebra $\mathcal{B}_k$ has another presentation by generators*

$$t_u, \text{ for } u \in S_k, \qquad \kappa_0, \ldots, \kappa_k \qquad \text{and} \qquad \gamma_{i,j}, \quad \text{for } 0 \leq i, j \leq k \text{ with } i \neq j, \tag{2.8}$$

and relations

$$t_u t_v = t_{uv}, \qquad t_w \kappa_i t_{w^{-1}} = \kappa_{w(i)}, \qquad t_w \gamma_{i,j} t_{w^{-1}} = \gamma_{w(i),w(j)}, \tag{2.9}$$

$$\kappa_i \kappa_j = \kappa_j \kappa_i, \qquad \kappa_i \gamma_{\ell,m} = \gamma_{\ell,m} \kappa_i, \tag{2.10}$$

$$\gamma_{i,j} = \gamma_{j,i}, \qquad \gamma_{p,r} \gamma_{\ell,m} = \gamma_{\ell,m} \gamma_{p,r}, \qquad \text{and} \qquad \gamma_{i,j}(\gamma_{i,r} + \gamma_{j,r}) = (\gamma_{i,r} + \gamma_{j,r}) \gamma_{i,j}, \tag{2.11}$$

for $p \neq \ell$ and $p \neq m$ and $r \neq \ell$ and $r \neq m$ and $i \neq j$, $i \neq r$ and $j \neq r$.

The commutation relations between the κ_i and the $\gamma_{i,j}$ can be rewritten in the form

$$[\kappa_r, \gamma_{\ell,m}] = 0, \qquad [\gamma_{i,j}, \gamma_{\ell,m}] = 0, \qquad \text{and} \qquad [\gamma_{i,j}, \gamma_{i,m}] = [\gamma_{i,m}, \gamma_{j,m}], \tag{2.12}$$

for all r and all $i \neq \ell$ and $i \neq m$ and $j \neq \ell$ and $j \neq m$.

Proof. The generators in (2.8) are written in terms of the generators in (2.1) by the formulas

$$\kappa_0 = \kappa_0, \qquad \kappa_1 = \kappa_1, \qquad t_w = t_w, \tag{2.13}$$

$$\gamma_{0,1} = y_1 - \tfrac{1}{2}\kappa_1, \quad \text{and} \quad \gamma_{j,j+1} = y_{j+1} - t_{s_j} y_j t_{s_j}, \quad \text{for } j = 1, \ldots, k-1, \tag{2.14}$$

and

$$\kappa_m = t_u \kappa_1 t_{u^{-1}}, \qquad \gamma_{0,m} = t_u \gamma_{0,1} t_{u^{-1}} \qquad \text{and} \qquad \gamma_{i,j} = t_v \gamma_{1,2} t_{v^{-1}}, \tag{2.15}$$

for $u, v \in S_k$ such that $u(1) = m$, $v(1) = i$ and $v(2) = j$.

The generators in (2.1) are written in terms of the generators in (2.8) by the formulas

$$\kappa_0 = \kappa_0, \qquad \kappa_1 = \kappa_1, \qquad t_w = t_w, \qquad \text{and} \qquad y_j = \tfrac{1}{2}\kappa_j + \sum_{0 \le \ell < j} \gamma_{\ell,j}. \tag{2.16}$$

Let us show that relations in (2.2-5) follow from the relations in (2.9-2.11).

(a) The relation $t_u t_v = t_{uv}$ in (2.2) is the first relation in (2.9).

(b) The relation $y_i y_j = y_j y_i$ in (2.2): Assume that $i < j$. Using the relations in (2.10) and (2.11),

$$\begin{aligned} [y_i, y_j] &= \big[\tfrac{1}{2}\kappa_i + \sum_{\ell<i} \gamma_{\ell,i}, \tfrac{1}{2}\kappa_j + \sum_{m<j} \gamma_{m,j}\big] = \big[\sum_{\ell<i} \gamma_{\ell,i}, \sum_{m<j} \gamma_{m,j}\big] \\ &= \sum_{\ell<i} \big[\gamma_{\ell,i}, \sum_{m<j} \gamma_{m,j}\big] = \sum_{\ell<i} \big[\gamma_{\ell,i}, (\gamma_{\ell,j} + \gamma_{i,j}) + \sum_{\substack{m<j \\ m\neq\ell, m\neq i}} \gamma_{m,j}\big] = 0. \end{aligned}$$

(c) The relation $\kappa_0\kappa_1 = \kappa_1\kappa_0$ in (2.2) is part of the first relation in (2.10), and the relations $\kappa_0 y_i = y_i \kappa_0$ and $\kappa_1 y_i = y_i \kappa_1$ in (2.2) follow from the relations $\kappa_i\kappa_j = \kappa_j\kappa_i$ and $\kappa_i\gamma_{\ell,m} = \gamma_{\ell,m}\kappa_i$ in (2.10).

(d) The relations $\kappa_0 t_{s_i} = t_{s_i}\kappa_0$ and $\kappa_1 t_{s_j} = t_{s_j}\kappa_1$ for $j \neq 1$ from (2.3) follow from the relation $t_w \kappa_i t_w^{-1} = \kappa_{w(i)}$ in (2.9), and the relation $\kappa_1 t_{s_1} \kappa_1 t_{s_1} = t_{s_1}\kappa_1 t_{s_1}\kappa_2$ from (2.3) follows from $\kappa_1\kappa_2 = \kappa_2\kappa_1$, which is part of the first relation in (2.10).

(e) The relations in (2.4) and (2.5) all follow from the relations $t_w \kappa_i t_{w^{-1}} = \kappa_{w(i)}$ and $t_w \gamma_{i,j} t_{w^{-1}} = \gamma_{w(i),w(j)}$ in (2.9).

To complete the proof let us show that the relations of (2.9-11) follow from the relations in (2.2-5).

(a) The relation $t_u t_v = t_{uv}$ in (2.9) is the first relation in (2.2).

(b) The relations $t_w \kappa_i t_{w^{-1}} = \kappa_{w(i)}$ in (2.9) follow from the first and last relations in (2.3) (and force the definition of κ_m in (2.15)).

(c) Since $\gamma_{0,1} = y_1 - \frac{1}{2}\kappa_1$, the relations $t_w \gamma_{0,j} t_{w^{-1}} = \gamma_{0,w(j)}$ in (2.10) follow from the last relation in each of (2.3) and (2.4) (and force the definition of $\gamma_{0,m}$ in (2.15)).

(d) Since $\gamma_{1,2} = y_2 - t_{s_1} y_1 t_{s_1}$, the first relation in (2.4) gives

$$t_{s_1}\gamma_{1,2}t_{s_1} - \gamma_{1,2} = (t_{s_1} y_2 t_{s_1} - y_1) - y_2 + t_{s_1} y_1 t_{s_1} = t_{s_1}(y_1 + y_2)t_{s_1} - (y_1 + y_2) = 0. \tag{2.17}$$

The relations $t_w \gamma_{1,2} t_{w^{-1}} = \gamma_{w(1),w(2)}$ in (2.9) then follow from (2.17) and the last relation in (2.4) (and force the definitions $\gamma_{i,j} = t_v \gamma_{1,2} t_{v^{-1}}$ in (2.15)).

(e) The third relation in (2.2) is $\kappa_0\kappa_1 = \kappa_1\kappa_0$ and the second relation in (2.3) gives $\kappa_1\kappa_2 = \kappa_2\kappa_1$. The relations $\kappa_i\kappa_j = \kappa_j\kappa_i$ in (2.10) then follow from the second set of relations in (2.9).

(f) The second relation in (2.3) gives $[\kappa_1, \kappa_2] = 0$. Using this and the relations in (2.2),

$$[\kappa_1, \gamma_{0,2} + \gamma_{1,2}] = [\kappa_1, (y_2 - \tfrac{1}{2}\kappa_2 - \gamma_{1,2}) + \gamma_{1,2}] = [\kappa_1, \tfrac{1}{2}\kappa_2] = 0, \tag{2.18}$$

and

$$[\gamma_{0,1}, \gamma_{0,2} + \gamma_{1,2}] = [y_1 - \tfrac{1}{2}\kappa_1, y_2 - \tfrac{1}{2}\kappa_2] = \tfrac{1}{4}[\kappa_1, \kappa_2] = 0, \tag{2.19}$$

so that

$$[\gamma_{0,1}, \kappa_2] = [\gamma_{0,1}, 2y_2 - 2(\gamma_{0,2} + \gamma_{1,2})] = [\gamma_{0,1}, 2y_2] = [y_1 - \tfrac{1}{2}\kappa_1, 2y_2] = -[\kappa_1, y_2] = 0.$$

Conjugating the last relation by t_{s_1} gives

$$[\kappa_1, \gamma_{0,2}] = 0, \qquad \text{and thus} \qquad [\kappa_1, \gamma_{1,2}] = 0,$$

by (2.18). By the third and fourth relations in (2.2),

$$[\kappa_0, \gamma_{0,1}] = [\kappa_0, y_1 - \tfrac{1}{2}\kappa_1] = 0, \qquad \text{and} \qquad [\kappa_1, \gamma_{0,1}] = [\kappa_1, y_1 - \tfrac{1}{2}\kappa_1] = 0.$$

By the relations in (2.3) and (2.2),

$$[\kappa_0, \gamma_{1,2}] = [\kappa_0, y_2 - t_{s_1}y_1t_{s_1}] = 0 \qquad \text{and} \qquad [\kappa_1, \gamma_{2,3}] = [\kappa_1, y_3 - t_{s_2}y_2t_{s_2}] = 0.$$

Putting these together with the (already established) relations in (2.9) provides the second set of relations in (2.10).

(g) From the commutativity of the y_i and the second relation in (2.4)

$$\gamma_{1,2}\gamma_{3,4} = (y_2 - t_{s_1}y_1t_{s_1})(y_4 - t_{s_3}y_3t_{s_3}) = (y_4 - t_{s_3}y_3t_{s_3})(y_2 - t_{s_1}y_1t_{s_1}) = \gamma_{3,4}\gamma_{1,2}.$$

By the last relation in (2.2) and the last relation in (2.3),

$$[\gamma_{0,1}, \gamma_{2,3}] = [y_1 - \tfrac{1}{2}\kappa_1, y_3 - t_{s_2}y_2t_{s_2}] = 0.$$

Together with the (already established) relations in (2.9), we obtain the first set of relations in (2.11).

(h) Conjugating (2.19) by $t_{s_2}t_{s_1}t_{s_2}$ gives $[\gamma_{0,2}, \gamma_{0,3} + \gamma_{2,3}] = 0$, and this and the (already established) relations in (2.10) and the first set of relations in (2.11) provide

$$\begin{aligned} 0 = [y_2, y_3] &= [\tfrac{1}{2}\kappa_2 + \gamma_{0,2} + \gamma_{1,2}, \tfrac{1}{2}\kappa_3 + \gamma_{0,3} + \gamma_{1,3} + \gamma_{2,3}] \\ &= [\gamma_{0,2} + \gamma_{1,2}, \gamma_{0,3} + \gamma_{1,3} + \gamma_{2,3}] = [\gamma_{1,2}, \gamma_{0,3} + \gamma_{1,3} + \gamma_{2,3}] = [\gamma_{1,2}, \gamma_{1,3} + \gamma_{2,3}]. \end{aligned}$$

Note also that

$$\begin{aligned} [\gamma_{1,2}, \gamma_{1,0} + \gamma_{2,0}] &= [\gamma_{1,2}, \gamma_{0,1} + \gamma_{0,2}] = -[\gamma_{0,1}, \gamma_{1,2}] + [\gamma_{1,2}, \gamma_{0,2}] \\ &= [\gamma_{0,1}, \gamma_{0,2}] + [\gamma_{1,2}, \gamma_{0,2}] = t_{s_1}[\gamma_{0,2} + \gamma_{1,2}, \gamma_{0,1}]t_{s_1} = 0, \end{aligned}$$

by (two applications of) (2.19). The last set of relations in (2.11) now follow from the last set of relations in (2.9).

□

By the first formula in (2.6) and the last formula in (2.16),

$$c_j = \sum_{i=0}^{j} \kappa_i + 2 \sum_{0 \le \ell < m \le j} \gamma_{\ell,m}. \tag{2.20}$$

2.2 The degenerate affine BMW algebra $\mathcal{W}_k$

Let C be a commutative ring and let $\mathcal{B}_k$ be the degenerate affine braid algebra over C as defined in Section 2.1. Define e_i in the degenerate affine braid algebra by

$$t_{s_i}y_i = y_{i+1}t_{s_i} - (1-e_i), \qquad \text{for } i = 1, 2, \ldots, k-1, \tag{2.21}$$

so that, with $\gamma_{i,i+1}$ as in (2.5),

$$\gamma_{i,i+1}t_{s_i} = 1 - e_i. \tag{2.22}$$

Fix constants

$$\epsilon = \pm 1 \qquad \text{and} \qquad z_0^{(\ell)} \in C, \quad \text{for } \ell \in \mathbb{Z}_{\geq 0}.$$

The *degenerate affine Birman-Wenzl-Murakami (BMW) algebra* $\mathcal{W}_k$ (with parameters ϵ and $z_0^{(\ell)}$) is the quotient of the degenerate affine braid algebra $\mathcal{B}_k$ by the relations

$$e_i t_{s_i} = t_{s_i} e_i = \epsilon e_i, \qquad e_i t_{s_{i-1}} e_i = e_i t_{s_{i+1}} e_i = \epsilon e_i, \tag{2.23}$$

$$e_1 y_1^\ell e_1 = z_0^{(\ell)} e_1, \qquad e_i(y_i + y_{i+1}) = 0 = (y_i + y_{i+1})e_i. \tag{2.24}$$

Conjugating (2.21) by t_{s_i} and using the first relation in (2.23) gives

$$y_i t_{s_i} = t_{s_i} y_{i+1} - (1-e_i). \tag{2.25}$$

Then, by (2.22) and (2.5),

$$\gamma_{i,i+1} = t_{s_i} - \epsilon e_i, \qquad \text{and} \qquad e_{i+1} = t_{s_i} t_{s_{i+1}} e_i t_{s_{i+1}} t_{s_i}. \tag{2.26}$$

Multiply the second relation in (2.26) on the left and the right by e_i, and then use the relations in (2.23) to get

$$e_i e_{i+1} e_i = e_i t_{s_i} t_{s_{i+1}} e_i t_{s_{i+1}} t_{s_i} e_i = e_i t_{s_{i+1}} e_i t_{s_{i+1}} e_i = \epsilon\, e_i t_{s_{i+1}} e_i = e_i,$$

so that

$$e_i e_{i\pm 1} e_i = e_i. \qquad \text{Note that} \qquad e_i^2 = z_0^{(0)} e_i \tag{2.27}$$

is a special case of the first identity in (2.24). The relations

$$e_{i+1} e_i = e_{i+1} t_{s_i} t_{s_{i+1}}, \quad e_i e_{i+1} = t_{s_{i+1}} t_{s_i} e_{i+1}, \tag{2.28}$$

$$t_{s_i} e_{i+1} e_i = t_{s_{i+1}} e_i, \quad \text{and} \quad e_{i+1} e_i t_{s_{i+1}} = e_{i+1} t_{s_i} \tag{2.29}$$

result from

$$\begin{aligned}
e_{i+1} t_{s_i} t_{s_{i+1}} &= \epsilon\, e_{i+1} t_{s_i} e_{i+1} t_{s_i} t_{s_{i+1}} = e_{i+1} t_{s_{i+1}} t_{s_i} e_{i+1} t_{s_i} t_{s_{i+1}} = e_{i+1} e_i, \\
t_{s_{i+1}} t_{s_i} e_{i+1} &= \epsilon t_{s_{i+1}} t_{s_i} e_{i+1} t_{s_i} e_{i+1} = t_{s_{i+1}} t_{s_i} e_{i+1} t_{s_i} t_{s_{i+1}} e_{i+1} = e_i e_{i+1}, \\
t_{s_i} e_{i+1} e_i &= \epsilon\, t_{s_i} e_{i+1} t_{s_i} e_i = \epsilon\, t_{s_{i+1}} e_i t_{s_{i+1}} e_i = t_{s_{i+1}} e_i, \quad \text{and} \\
e_{i+1} e_i t_{s_{i+1}} &= \epsilon\, e_{i+1} t_{s_{i+1}} e_i t_{s_{i+1}} = \epsilon\, e_{i+1} t_{s_i} e_{i+1} t_{s_i} = e_{i+1} t_{s_i}.
\end{aligned}$$

Remark 2.2. A consequence (see (3.7)) of the defining relations of $\mathcal{W}_k$ is the equation

$$\left(z_0(-u) - \left(\tfrac{1}{2} + \epsilon\, u\right)\right)\left(z_0(u) - \left(\tfrac{1}{2} - \epsilon\, u\right)\right) e_1 = \left(\tfrac{1}{2} - \epsilon\, u\right)\left(\tfrac{1}{2} + \epsilon\, u\right) e_1,$$

where $z_0(u)$ is the generating function

$$z_0(u) = \sum_{\ell \in \mathbb{Z}_{\geq 0}} z_0^{(\ell)} u^{-\ell}.$$

This means that, unless the parameters $z_0^{(\ell)}$ are chosen carefully, it is likely that $e_1 = 0$ in $\mathcal{W}_k$.

Remark 2.3. From the point of view of the Schur-Weyl duality for the degenerate affine BMW algebra (see [AS] and [Ra]) the natural choice of base ring is the center of the enveloping algebra of the orthogonal or symplectic Lie algebra which, by the Harish-Chandra isomorphism, is isomorphic to the subring of symmetric functions given by

$$C = \{z \in \mathbb{C}[h_1, \ldots, h_r]^{S_r} \mid z(h_1, \ldots, h_r) = z(-h_1, h_2, \ldots, h_r)\},$$

where the symmetric group S_r acts by permuting the variables $h_1, \ldots, h_r$. Here the constants $z_0^{(\ell)} \in C$ are given, explicitly, by setting the generating function

$$z_0(u) \text{ equal, up to a normalization, to } \quad \prod_{i=1}^{r} \frac{(u + \frac{1}{2} + h_i)(u + \frac{1}{2} - h_i)}{(u - \frac{1}{2} - h_i)(u - \frac{1}{2} + h_i)}.$$

This choice of C and the $z_0^{(\ell)}$ are the *universal admissible parameters* for $\mathcal{W}_k$. This point of view will be explained in [DRV].

Remark 2.4. Careful manipulation of the defining relations of $\mathcal{W}_k$ provides an inductive presentation of $\mathcal{W}_k$ as

$$\mathcal{W}_k = \mathcal{W}_{k-1}e_{k-1}\mathcal{W}_{k-1} + \mathcal{W}_{k-1}t_{s_{k-1}}\mathcal{W}_{k-1} + \sum_{\ell \in \mathbb{Z}_{\geq 0}} \mathcal{W}_{k-1}y_k^\ell \mathcal{W}_{k-1},$$

and provides that

$$e_k\mathcal{W}_k e_k = \mathcal{W}_{k-1}e_k, \qquad \text{and} \qquad \begin{matrix} \mathcal{W}_k & \longrightarrow & \mathcal{W}_{k-1}e_k \\ b & \longmapsto & e_k b e_k \end{matrix}$$

is a $(\mathcal{W}_{k-1}, \mathcal{W}_{k-1})$-bimodule homomorphism. These structural facts are important to the understanding of $\mathcal{W}_k$ by "Jones basic constructions". Under the conditions of Theorem 4.1(a) it is true, but not immediate from the defining relations, that the natural homomorphism $\mathcal{W}_{k-1} \to \mathcal{W}_k$ is injective so that $\mathcal{W}_{k-1}$ is a subalgebra of $\mathcal{W}_k$. These useful structural results for the algebras $\mathcal{W}_k$ are justified in [AMR].

2.2.1 Quotients of $\mathcal{W}_k$

The *degenerate affine Hecke algebra* $\mathcal{H}_k$ is the quotient of $\mathcal{W}_k$ by the relations

$$e_i = 0, \qquad \text{for } i = 1, \ldots, k-1. \tag{2.30}$$

Fix $b_1, \ldots, b_r \in C$. The *degenerate cyclotomic BMW algebra* $\mathcal{W}_{r,k}(b_1, \ldots, b_r)$ is the degenerate affine BMW algebra with the additional relation

$$(y_1 - b_1) \cdots (y_1 - b_r) = 0. \tag{2.31}$$

The *degenerate cyclotomic Hecke algebra* $\mathcal{H}_{r,k}(b_1, \ldots, b_r)$ is the degenerate affine Hecke algebra $\mathcal{H}_k$ with the additional relation (2.31).

Remark 2.5. Since the composite map $C[y_1, \ldots, y_k] \to \mathcal{B}_k \to \mathcal{W}_k \to \mathcal{H}_k$ is injective (see [Kl, Theorem 3.2.2]) and the last two maps are surjections, it follows that the polynomial ring $C[y_1, \ldots, y_k]$ is a subalgebra of $\mathcal{B}_k$ and $\mathcal{W}_k$.

Remark 2.6. A consequence of the relation (2.31) in $\mathcal{W}_{r,k}(b_1, \ldots, b_r)$ is

$$\left(z_0(u) + u - \tfrac{1}{2}\right) e_1 = \left(u - \tfrac{1}{2}(-1)^r\right) \left(\prod_{i=1}^{r} \frac{u + b_i}{u - b_i}\right) e_1. \tag{2.32}$$

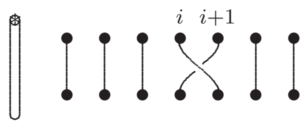
This equation makes the data of the values b_i almost equivalent to the data of the $z_0^{(\ell)}$.

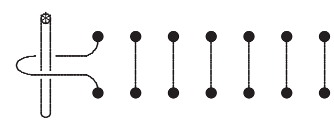
2.3 The affine braid group B_k

The *affine braid group* B_k is the group given by generators $T_1, T_2, \ldots, T_{k-1}$ and X^{ε_1}, with relations

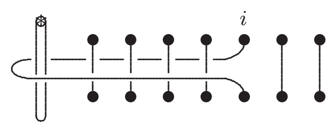
$$
\begin{aligned}
T_iT_j &= T_jT_i, && \text{if } j \neq i \pm 1, && (2.33)\\
T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}, && \text{for } i = 1, 2, \ldots, k-2, && (2.34)\\
X^{\varepsilon_1}T_1X^{\varepsilon_1}T_1 &= T_1X^{\varepsilon_1}T_1X^{\varepsilon_1}, && && (2.35)\\
X^{\varepsilon_1}T_i &= T_iX^{\varepsilon_1}, && \text{for } i = 2, 3, \ldots, k-1. && (2.36)
\end{aligned}
$$

The affine braid group is isomorphic to the group of braids in the thickened annulus, where the generators T_i and X^{ε_1} are identified with the diagrams

$T_i =$ 

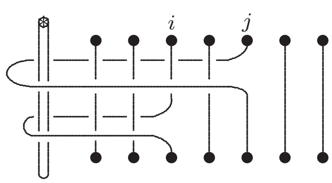
and $X^{\varepsilon_1} =$  . (2.37)

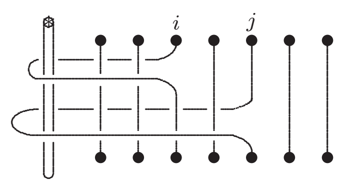
For $i = 1, \ldots, k$ define

$X^{\varepsilon_i} = T_{i-1}T_{i-2}\cdots T_2T_1X^{\varepsilon_1}T_1T_2\cdots T_{i-2}T_{i-1} =$ 

. (2.38)

The pictorial computation

$X^{\varepsilon_j}X^{\varepsilon_i} =$ 

$=$ 

$= X^{\varepsilon_i}X^{\varepsilon_j}$

shows that the X^{ε_i} all commute with each other.

2.4 The affine BMW algebra W_k

Let C be a commutative ring and let CB_k be the group algebra of the affine braid group. Fix constants

$$q, z \in C \qquad \text{and} \qquad Z_0^{(\ell)} \in C, \quad \text{for } \ell \in \mathbb{Z},$$

with q and z invertible. Let $Y_i = zX^{\varepsilon_i}$ so that

$$Y_1 = zX^{\varepsilon_1}, \qquad Y_i = T_{i-1}Y_{i-1}T_{i-1}, \qquad \text{and} \qquad Y_iY_j = Y_jY_i, \quad \text{for } 1 \leq i, j \leq k. \tag{2.39}$$

In the affine braid group

$$T_iY_iY_{i+1} = Y_iY_{i+1}T_i. \tag{2.40}$$

Assume that $q - q^{-1}$ is invertible in C and define E_i in the group algebra of the affine braid group by

$$T_iY_i = Y_{i+1}T_i - (q - q^{-1})Y_{i+1}(1 - E_i). \tag{2.41}$$

The *affine BMW algebra* W_k is the quotient of the group algebra CB_k of the affine braid group B_k by the relations

$$E_iT_i^{\pm1} = T_i^{\pm1}E_i = z^{\mp1}E_i, \qquad E_iT_{i-1}^{\pm1}E_i = E_iT_{i+1}^{\pm1}E_i = z^{\pm1}E_i, \tag{2.42}$$

$$E_1Y_1^{\ell}E_1 = Z_0^{(\ell)}E_1, \qquad E_iY_iY_{i+1} = E_i = Y_iY_{i+1}E_i. \tag{2.43}$$

Since $Y_{i+1}^{-1}(T_iY_i)Y_{i+1} = Y_{i+1}^{-1}Y_iY_{i+1}T_i = Y_iT_i$, conjugating (2.41) by Y_{i+1}^{-1} gives

$$Y_iT_i = T_iY_{i+1} - (q-q^{-1})(1-E_i)Y_{i+1}. \tag{2.44}$$

Left multiplying (2.41) by Y_{i+1}^{-1} and using the second identity in (2.39) shows that (2.41) is equivalent to $T_i - T_i^{-1} = (q-q^{-1})(1-E_i)$, so that

$$E_i = 1 - \frac{T_i - T_i^{-1}}{q-q^{-1}} \qquad \text{and} \qquad T_iT_{i+1}E_iT_{i+1}^{-1}T_i^{-1} = E_{i+1}. \tag{2.45}$$

Multiply the second relation in (2.45) on the left and the right by E_i, and then use the relations in (2.42) to get

$$E_iE_{i+1}E_i = E_iT_iT_{i+1}E_iT_{i+1}^{-1}T_i^{-1}E_i = E_iT_{i+1}E_iT_{i+1}^{-1}E_i = zE_iT_{i+1}^{-1}E_i = E_i,$$

so that

$$E_iE_{i\pm1}E_i = E_i, \qquad \text{and} \qquad E_i^2 = \left(1 + \frac{z-z^{-1}}{q-q^{-1}}\right)E_i \tag{2.46}$$

is obtained by multiplying the first equation in (2.45) by E_i and using (2.42). Thus, from the first relation in (2.43),

$$Z_0^{(0)} = 1 + \frac{z-z^{-1}}{q-q^{-1}} \qquad \text{and} \qquad (T_i - z^{-1})(T_i + q^{-1})(T_i - q) = 0, \tag{2.47}$$

since $(T_i - z^{-1})(T_i + q^{-1})(T_i - q)T_i^{-1} = (T_i - z^{-1})(T_i^2 - (q-q^{-1})T_i - 1)T_i^{-1} = (T_i - z^{-1})(T_i - T_i^{-1} - (q-q^{-1})) = (T_i - z^{-1})(q-q^{-1})(-E_i) = -(z^{-1} - z^{-1})(q-q^{-1}) = 0$. The relations

$$E_{i+1}E_i = E_{i+1}T_iT_{i+1}, \qquad E_iE_{i+1} = T_{i+1}^{-1}T_i^{-1}E_{i+1}, \tag{2.48}$$

$$T_iE_{i+1}E_i = T_{i+1}^{-1}E_i, \quad \text{and} \quad E_{i+1}E_iT_{i+1} = E_{i+1}T_i^{-1}, \tag{2.49}$$

follow from the computations

$$\begin{aligned}
&E_{i+1}T_iT_{i+1} = z(E_{i+1}T_i^{-1}E_{i+1})T_iT_{i+1} = z(z^{-1}E_{i+1}T_{i+1}^{-1})T_i^{-1}E_{i+1}T_iT_{i+1} = E_{i+1}E_i,\\
&T_{i+1}^{-1}T_i^{-1}E_{i+1} = T_{i+1}^{-1}T_i^{-1}(z^{-1}E_{i+1}T_iE_{i+1}) = T_{i+1}^{-1}T_i^{-1}z^{-1}E_{i+1}T_izT_{i+1}E_{i+1} = E_iE_{i+1},\\
&T_iE_{i+1}E_i = T_iE_{i+1}(T_i^{-1}E_iz^{-1}) = z^{-1}T_{i+1}^{-1}E_iT_{i+1}E_iz^{-1} = T_{i+1}^{-1}zE_iz^{-1} = T_{i+1}^{-1}E_i, \quad \text{and}\\
&E_{i+1}E_iT_{i+1} = E_{i+1}T_{i+1}^{-1}zE_iT_{i+1} = zE_{i+1}T_iE_{i+1}T_i^{-1} = zz^{-1}E_{i+1}T_i^{-1} = E_{i+1}T_i^{-1}.
\end{aligned}$$

Remark 2.7. A consequence (see (3.26)) of the defining relations of W_k is the equation

$$\left(Z_0^- - \frac{z}{q-q^{-1}} - \frac{u^2}{u^2-1}\right)\left(Z_0^+ + \frac{z^{-1}}{q-q^{-1}} - \frac{u^2}{u^2-1}\right)E_1 = \frac{-(u^2-q^2)(u^2-q^{-2})}{(u^2-1)(q-q^{-1})^2}E_1,$$

where Z_0^+ and Z_0^- are the generating functions

$$Z_0^+ = \sum_{\ell\in\mathbb{Z}_{\geq0}} Z_0^{(\ell)}u^{-\ell} \qquad \text{and} \qquad Z_0^- = \sum_{\ell\in\mathbb{Z}_{\leq0}} Z_0^{(\ell)}u^{-\ell}.$$

This means that, unless the parameters $Z_0^{(\ell)}$ are chosen carefully, it is likely that $E_1 = 0$ in W_k.

Remark 2.8. From the point of view of Schur-Weyl duality for the affine BMW algebra (see [OR] and [Ra]) the natural choice of base ring is the center of the quantum group corresponding to the orthogonal or symplectic Lie algebra which, by the (quantum version) of the Harish-Chandra isomorphism, is isomorphic to the subring of symmetric Laurent polynomials given by

$$C = \{z \in \mathbb{C}[L_1^{\pm 1}, \ldots, L_r^{\pm 1}]^{S_r} \mid z(L_1, L_2, \ldots, L_r) = z(L_1^{-1}, L_2, \ldots, L_r)\},$$

where the symmetric group S_r acts by permuting the variables $L_1, \ldots, L_r$. Here the constants $Z_0^{(\ell)} \in C$ are given, explicitly, by setting the generating functions Z_0^+ and Z_0^- equal, up to a normalization, to

$$\prod_{i=1}^{r} \frac{(u - qL_i)}{(u - q^{-1}L_i)} \cdot \frac{(u - qL_i^{-1})}{(u - q^{-1}L_i^{-1})} \qquad \text{and} \qquad \prod_{i=1}^{r} \frac{(u - q^{-1}L_i)}{(u - qL_i)} \cdot \frac{(u - q^{-1}L_i^{-1})}{(u - qL_i^{-1})},$$

respectively. This choice of C and the $Z_0^{(\ell)}$ are the *universal admissible parameters* for W_k. This point of view will be explained in [DRV].

Remark 2.9. Careful manipulation of the defining relations of W_k provides an inductive presentation of W_k as

$$W_k = W_{k-1}E_{k-1}W_{k-1} + W_{k-1}T_{k-1}W_{k-1} + W_{k-1}T_{k-1}^{-1}W_{k-1} + \sum_{\ell \in \mathbb{Z}} W_{k-1}Y_k^{\ell}W_{k-1}$$

(see [GH1, Prop. 3.16] or [Hä3]), and provides that

$$E_kW_kE_k = W_{k-1}E_k \qquad \text{and} \qquad \begin{matrix} W_k & \longrightarrow & W_{k-1}E_k \\ b & \longmapsto & E_kbE_k \end{matrix}$$

is a (W_{k-1}, W_{k-1})-bimodule homomorphism (see [GH1, Prop. 3.17]). These structural facts are important to the understanding of W_k by "Jones basic constructions". Under the conditions of Theorem 4.4(a) it is true, but not immediate from the defining relations, that the natural homomorphism $W_{k-1} \to W_k$ is injective so that W_{k-1} is a subalgebra of W_k (see [GH1, Cor. 6.15]).

2.4.1 Quotients of W_k

The *affine Hecke algebra* H_k is the affine BMW algebra W_k with the additional relations

$$E_i = 0, \qquad \text{for } i = 1, \ldots, k-1. \tag{2.50}$$

Fix $b_1, \ldots, b_r \in C$. The *cyclotomic BMW algebra* $W_{r,k}(b_1, \ldots, b_r)$ is the affine BMW algebra W_k with the additional relation

$$(Y_1 - b_1) \cdots (Y_1 - b_r) = 0. \tag{2.51}$$

The *cyclotomic Hecke algebra* $H_{r,k}(b_1, \ldots, b_r)$ is the affine Hecke algebra H_k with the additional relation (2.51).

Remark 2.10. Since the composite map $C[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}] \to CB_k \to W_k \to H_k$ is injective and the last two maps are surjections, it follows that the Laurent polynomial ring $C[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]$ is a subalgebra of CB_k and W_k.

Remark 2.11. A consequence of the relation (2.51) in $W_{r,k}(b_1,\dots,b_r)$ is

$$\left(Z_0^+ + \frac{z^{-1}}{q-q^{-1}} - \frac{u^2}{u^2-1}\right)E_1 = \left(\frac{z}{q-q^{-1}} + \frac{uz}{u^2-1}\right)\left(\prod_{j=1}^r \frac{u-b_j^{-1}}{u-b_j}\right)E_1. \tag{2.52}$$

This equation makes the data of the values b_i almost equivalent to the data of the $Z_0^{(\ell)}$.

3 Identities in affine and degenerate affine BMW algebras

In [Naz], Nazarov defined some naturally occurring central elements in the degenerate affine BMW algebra $\mathcal{W}_k$ and proved a remarkable recursion for them. This recursion was generalized to analogous central elements in the affine BMW algebra W_k by Beliakova-Blanchet [BB]. In both cases, the recursion was accomplished with an involved computation. In this section, we provide a new proof of the Nazarov and Beliakov-Blanchet recursions by lifting them out of the center, to intertwiner-like identities in $\mathcal{W}_k$ and W_k (Propositions 3.1and 3.5). These intertwiner-like identities for the degenerate affine and affine BMW algebras are reminiscent of the intertwiner identities for the degenerate affine and affine Hecke algebras found, for example, in [KR, Prop. 2.5(c)] and [Ra1, Prop. 2.14(c)], respectively. The central element recursions of [Naz] and [BB] are then obtained by multiplying the intertwiner-like identities by the projectors e_k and E_k, respectively. We have carefully arranged the proofs so that the degenerate affine and the affine cases are exactly in parallel.

3.1 The degenerate affine case

Let u be a variable,

$$u_i^+ = \frac{1}{u-y_i}, \qquad \text{and note that} \qquad u_i^+u_{i+1}^+ = \frac{1}{2u-(y_i+y_{i+1})}(u_i^+ + u_{i+1}^+). \tag{3.1}$$

By (2.25) and the definition of e_i in (2.21),

$$(u-y_{i+1})t_{s_i} = t_{s_i}(u-y_i) - (1-e_i) \quad \text{and} \quad (u-y_i)t_{s_i} = t_{s_i}(u-y_{i+1}) + (1-e_i),$$

which give

$$t_{s_i}u_i^+ = u_{i+1}^+t_{s_i} + u_{i+1}^+e_iu_i^+ - u_{i+1}^+u_i^+, \quad \text{and} \quad t_{s_i}u_{i+1}^+ = u_i^+t_{s_i} - u_i^+e_iu_{i+1}^+ + u_{i+1}^+u_i^+, \tag{3.2}$$

respectively.

Proposition 3.1. *In the degenerate affine BMW algebra $\mathcal{W}_{i+1}$,*

$$\begin{aligned}\left(e_i\frac{1}{1-y_{i+1}} - t_{s_i} - \frac{1}{2u-(y_i+y_{i+1})}\right)\left(e_i\frac{1}{1-y_i} + t_{s_i} - \frac{1}{2u-(y_i+y_{i+1})}\right)\\ = \frac{-(2u-(y_i+y_{i+1})+1)(2u-(y_i+y_{i+1})-1)}{(2u-(y_i+y_{i+1}))^2},\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}&\left(u_{i+1}^+ + t_{s_i} - e_i\frac{1}{2u-(y_i+y_{i+1})}\right) - u_i^+\left(u_{i+1}^+ + t_{s_i} - e_i\frac{1}{2u-(y_i+y_{i+1})}\right)u_i^+ \\ &= \left(t_{s_i}u_i^+t_{s_i} + t_{s_i} - e_i\frac{1}{2u-(y_i+y_{i+1})}\right) - u_{i+1}^+\left(e_iu_i^+e_i + \epsilon e_i - e_i\frac{1}{2u-(y_i+y_{i+1})}\right)u_{i+1}^+.\end{aligned} \tag{3.4}$$

Proof. Putting (3.1) into the first identity in (3.2) says that if

$$A = t_{s_i} + \frac{1}{2u-(y_i+y_{i+1})} \qquad \text{and} \qquad B = e_i u_i^+ + t_{s_i} - \frac{1}{2u-(y_i+y_{i+1})}$$

then

$$Au_i^+ = u_{i+1}^+ B, \qquad \text{and} \qquad Ae_i = e_i A$$

follows from (2.23) and (2.24). So

$$\begin{aligned}
&\left(e_i u_{i+1}^+ - t_{s_i} - \frac{1}{2u-(y_i+y_{i+1})}\right)\left(e_i u_i^+ + t_{s_i} - \frac{1}{2u-(y_i+y_{i+1})}\right) \\
&\quad = e_i u_{i+1}^+ B - AB = e_i A u_i^+ - AB = Ae_i u_i^+ - AB = A(e_i u_i^+ - B) \\
&\quad = -\left(t_{s_i} + \frac{1}{2u-(y_i+y_{i+1})}\right)\left(t_{s_i} - \frac{1}{2u-(y_i+y_{i+1})}\right),
\end{aligned}$$

and multiplying out the right hand side gives (3.3).

Multiplying the second relation in (3.2) by t_{s_i} gives

$$u_{i+1}^+ - t_{s_i} u_{i+1}^+ u_i^+ = t_{s_i} u_i^+ t_{s_i} - t_{s_i} u_i^+ e_i u_i^-$$

and again using the relations in (3.2) gives

$$u_{i+1}^+ - u_i^+(t_{s_i} - e_i u_{i+1}^+ + u_{i+1}^+)u_i^+ = t_{s_i} u_i^+ t_{s_i} - u_{i+1}^+(t_{s_i} + e_i u_i^+ - u_i^+)e_i u_{i+1}^+.$$

Using (3.1) and

$$\text{adding} \quad t_{s_i} - e_i\left(\frac{1}{2u-(y_i+y_{i+1})}\right) - \frac{1}{2u-(y_i+y_{i+1})} u_i^+ e_i u_{i+1}^+ \quad \text{to each side}$$

gives

$$\begin{aligned}
&\left(u_{i+1}^+ + t_{s_i} - e_i\frac{1}{2u-(y_i+y_{i+1})}\right) - u_i^+\left(u_{i+1}^+ + t_{s_i} - e_i\frac{1}{2u-(y_i+y_{i+1})}\right)u_i^+ \\
&\quad = t_{s_i} u_i^+ t_{s_i} + t_{s_i} - e_i\frac{1}{2u-(y_i+y_{i+1})} - u_{i+1}^+\left(e_i u_i^+ + t_{s_i} - \frac{1}{2u-(y_i+y_{i+1})}\right)e_i u_{i+1}^+ \\
&\quad = \left(t_{s_i} u_i^+ t_{s_i} + t_{s_i} - e_i\frac{1}{2u-(y_i+y_{i+1})}\right) - u_{i+1}^+\left(e_i u_i^+ e_i + \epsilon e_i - e_i\frac{1}{2u-(y_i+y_{i+1})}\right)u_{i+1}^+,
\end{aligned}$$

completing the proof of (3.4). □

Introduce notation $z_{i-1}^{(\ell)} e_i$ and the generating function $z_{i-1}(u)e_i$ by

$$z_{i-1}(u)e_i = \sum_{\ell\in\mathbb{Z}_{\geq 0}} z_{i-1}^{(\ell)} e_i u^{-\ell} = e_i\left(\sum_{\ell\in\mathbb{Z}_{\geq 0}} y_i^\ell u^{-\ell}\right)e_i = e_i\frac{1}{1-y_i u^{-1}}e_i, \tag{3.5}$$

By [AMR, Lemma 4.15], or the identity (3.9) below, $z_{i-1}^{(\ell)} \in \mathcal{W}_{i-1}$ for $\ell \in \mathbb{Z}_{\geq 0}$. If

$$u_i^- = \frac{1}{u+y_i} \qquad \text{then} \qquad e_i u_{i+1}^+ = e_i u_i^-, \quad u_{i+1}^+ e_i = u_i^- e_i, \quad e_i u_i^\pm e_i = \frac{z_{i-1}(\pm u)}{u} e_i, \tag{3.6}$$

where, for $i = 1$, the last identity is a restatement of the first identity in (2.24). The identities (3.7), (3.8), and (3.9) of the following theorem are [Naz, Lemma 2.5], [Naz, Prop. 4.2] and [Naz, Lemma 3.8], respectively.

Theorem 3.2. *Let $z_{i-1}^{(\ell)}$ and $z_{i-1}(u)$ be as defined in* (3.5). *Then $z_{i-1}^{(\ell)} \in Z(\mathcal{W}_{i-1})$,*

$$\big(z_{i-1}(-u) - (\tfrac{1}{2} + \epsilon\, u)\big)\,\big(z_{i-1}(u) - (\tfrac{1}{2} - \epsilon\, u)\big)\, e_i = \big(\tfrac{1}{2} - \epsilon\, u\big)\big(\tfrac{1}{2} + \epsilon\, u\big) e_i, \tag{3.7}$$

$$\big(z_i(u) + \epsilon\, u - \tfrac{1}{2}\big)e_{i+1} = \big(z_{i-1}(u) + \epsilon\, u - \tfrac{1}{2}\big)\left(\frac{\big((u+y_i)^2 - 1\big)(u-y_i)^2}{\big((u-y_i)^2 - 1\big)(u+y_i)^2}\right) e_{i+1}, \quad \textit{and} \tag{3.8}$$

$$(z_{k-1}(u) + \epsilon\, u - \tfrac{1}{2})e_{i+1} = \big(z_0(u) + \epsilon\, u - \tfrac{1}{2}\big) \prod_{i=1}^{k-1} \frac{(u+y_i-1)(u+y_i+1)(u-y_i)^2}{(u+y_i)^2(u-y_i+1)(u-y_i-1)} e_{i+1}. \tag{3.9}$$

Proof. Since the generators $t_{s_1}, \dots, t_{s_{i-2}}$, $e_1, \dots, e_{i-2}$ and $y_1, \dots, y_{i-1}$ of $\mathcal{W}_{i-1}$ all commute with e_i and y_i it follows that $z_{i-1}^{(\ell)} \in Z(\mathcal{W}_{i-1})$.

Multiply (3.3) on the right by e_i to get (3.7), since $(\frac{1}{2} - u)(\frac{1}{2} + u) = (\frac{1}{2} - \epsilon u)(\frac{1}{2} + \epsilon u)$.

Multiplying (3.4) on the left and right by e_{i+1} and using the relations in (2.27), (2.28) and (2.29),

$$e_{i+1} t_{s_i} u_i^+ t_{s_i} e_{i+1} = e_{i+1} t_{s_i} t_{s_{i+1}} u_i^+ t_{s_{i+1}} t_{s_i} e_{i+1} = e_{i+1} e_i u_i^+ e_i e_{i+1}, \text{ and}$$
$$e_{i+1} u_{i+1}^+ e_i u_{i+1}^+ e_{i+1} = e_{i+1} u_i^- e_i u_i^- e_{i+1} = u_i^- e_{i+1} e_i e_{i+1} u_i^- = (u_i^-)^2 e_{i+1},$$

gives

$$\left(\frac{z_i(u)}{u} + \epsilon - \frac{1}{2u}\right)\big(1 - (u_i^+)^2\big)\, e_{i+1} = \left(\frac{z_{i-1}(u)}{u} + \epsilon - \frac{1}{2u}\right)\big(1 - (u_i^-)^2\big)\, e_{i+1}.$$

So (3.8) follows from

$$\frac{1-(u_i^-)^2}{1-(u_i^+)^2} = \frac{1 - \big(\frac{1}{u+y_i}\big)^2}{1 - \big(\frac{1}{u-y_i}\big)^2} = \frac{(u^2 + 2y_i u + y_i^2 - 1)(u-y_i)^2}{(u^2 - 2y_i u + y_i^2 - 1)(u+y_i)^2} = \frac{(u+y_i-1)(u+y_i+1)(u-y_i)^2}{(u-y_i-1)(u-y_i+1)(u+y_i)^2}.$$

Finally, relation (3.9) follows, by induction, from (3.8). ∎

Remark 3.3. Using the expansion

$$\frac{1}{u-a} = \frac{u^{-1}}{1 - au^{-1}} = \sum_{\ell \in \mathbb{Z}_{\geq 1}} a^{\ell-1} u^{-\ell},$$

and taking the coefficient of $u^{-(\ell+1)}$ on each side of the relations in (3.2) gives

$$t_{s_i} y_i^\ell = y_{i+1}^\ell t_{s_i} - (y_{i+1}^{\ell-1}(1-e_i) + y_{i+1}^{\ell-2}(1-e_i) y_i + \cdots + (1-e_i) y_i^{\ell-1}), \quad \text{and} \tag{3.10}$$

$$t_{s_i} y_{i+1}^\ell = y_i^\ell t_{s_i} + y_i^{\ell-1}(1-e_i) + y_i^{\ell-2}(1-e_i) y_{i+1} + \cdots + (1-e_i) y_{i+1}^{\ell-1}, \tag{3.11}$$

respectively.

Remark 3.4. Taking the coefficient of u^{-s} on each side of (3.7) gives a trivial identity for even s but, for odd $s = 2\ell + 1$, gives

$$\Big(2z_{i-1}^{(2\ell+1)} + z_{i-1}^{(2\ell)} - (z_{i-1}^{(2\ell)} z_{i-1}^{(0)} - z_{i-1}^{(2\ell-1)} z_{i-1}^{(1)} + \cdots + z_{i-1}^{(0)} z_{i-1}^{(2\ell)})\Big)\, e_i = 0 \tag{3.12}$$

which is the admissibility relation in [AMR, Remark 2.11] (see also [Naz, (4.6)].)

3.2 The affine case

Let u be a variable,

$$U_i^+ = \frac{Y_i}{u - Y_i}, \qquad \text{and note that} \qquad U_i^+ U_{i+1}^+ = \frac{Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}} (U_i^+ + U_{i+1}^+ + 1). \tag{3.13}$$

By the definition of E_i in (2.41),

$$(u - Y_{i+1}) T_i = T_i (u - Y_i) - (q - q^{-1}) Y_{i+1} (1 - E_i),$$

and, by (2.44),

$$(u - Y_i) T_i = T_i (u - Y_{i+1}) + (q - q^{-1})(1 - E_i) Y_{i+1},$$

so that

$$T_i \frac{1}{u - Y_i} = \frac{1}{u - Y_{i+1}} T_i - (q - q^{-1}) \frac{Y_{i+1}}{u - Y_{i+1}} (1 - E_i) \frac{1}{u - Y_i}, \quad \text{and} \tag{3.14}$$

$$T_i \frac{1}{u - Y_{i+1}} = \frac{1}{u - Y_i} T_i + (q - q^{-1}) \frac{1}{u - Y_i} (1 - E_i) \frac{Y_{i+1}}{u - Y_{i+1}}. \tag{3.15}$$

The relations

$$\begin{aligned} T_i U_i^+ &= U_{i+1}^+ T_i^{-1} - (q - q^{-1}) U_{i+1}^+ (1 - E_i) U_i^+ \\ &= U_{i+1}^+ \left(T_i^{-1} - (q - q^{-1})(1 - E_i) U_i^+ \right), \quad \text{and} \end{aligned} \tag{3.16}$$

$$\begin{aligned} T_i^{-1} U_{i+1}^+ &= U_i^+ T_i - (q - q^{-1}) U_i^+ E_i U_{i+1}^+ + (q - q^{-1}) U_{i+1}^+ U_i^+ \\ &= U_i^+ \left(T_i + (q - q^{-1})(1 - E_i) U_{i+1}^+ \right) \end{aligned} \tag{3.17}$$

are obtained by multiplying (3.14) and (3.15) on the right (resp. left) by Y_i and using the relation $T_i Y_i = Y_{i+1} T_i^{-1}$.

Proposition 3.5. *Let $Q = q - q^{-1}$. Then, in the affine BMW algebra W_{i+1},*

$$\begin{aligned} &\left(E_i \frac{Y_{i+1}}{u - Y_{i+1}} - \frac{T_i}{Q} - \frac{Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}} \right) \left(E_i \frac{Y_i}{u - Y_i} + \frac{T_i^{-1}}{Q} - \frac{Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}} \right) \\ &\qquad = \frac{-(u^2 - q^2 Y_i Y_{i+1})(u^2 - q^{-2} Y_i Y_{i+1})}{Q^2 (u^2 - Y_i Y_{i+1})^2}, \qquad \textit{and} \end{aligned} \tag{3.18}$$

$$\begin{aligned} &\left(U_{i+1}^+ + \frac{T_i}{Q} - E_i \frac{Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}} \right) - Q^2 (U_i^+ + 1) \left(U_{i+1}^+ + \frac{T_i}{Q} - E_i \frac{Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}} \right) U_i^+ \\ &= \left(T_i U_i^+ T_i^{-1} + \frac{T_i}{Q} - E_i \frac{Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}} \right) - Q^2 U_{i+1}^+ \left(E_i U_i^+ E_i + z \frac{E_i}{Q} - E_i \frac{Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}} \right) (U_{i+1}^+ + 1). \end{aligned} \tag{3.19}$$

Proof. Putting (3.13) into (3.16) says that if

$$A = \frac{T_i}{Q} + \frac{Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}} \quad \text{and} \quad B = E_i U_i^+ + \frac{T_i^{-1}}{Q} - \frac{Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}}$$

then

$$A U_i^+ = U_{i+1}^+ B - \frac{Y_i Y_{i+1}}{u^2 - Y_i Y_{i+1}}. \qquad \text{Next,} \quad A E_i = E_i A$$

follows from (2.42) and (2.43). So

$$
\begin{aligned}
&\left(E_i\frac{Y_{i+1}}{u-Y_{i+1}}-\frac{T_i}{Q}-\frac{Y_iY_{i+1}}{u^2-Y_iY_{i+1}}\right)\left(E_i\frac{Y_i}{u-Y_i}+\frac{T_i^{-1}}{Q}-\frac{Y_iY_{i+1}}{u^2-Y_iY_{i+1}}\right)\\
&=E_i(U_{i+1}^+B)-AB=E_i\left(AU_i^+ +\frac{Y_iY_{i+1}}{u^2-Y_iY_{i+1}}\right)-AB=A(E_iU_i^+-B)+E_i\frac{Y_iY_{i+1}}{u^2-Y_iY_{i+1}}\\
&=-\left(\frac{T_i}{Q}+\frac{Y_iY_{i+1}}{u^2-Y_iY_{i+1}}\right)\left(\frac{T_i^{-1}}{Q}-\frac{Y_iY_{i+1}}{u^2-Y_iY_{i+1}}\right)+E_i\frac{Y_iY_{i+1}}{u^2-Y_iY_{i+1}},
\end{aligned}
$$

and, by (2.45), multiplying out the right hand side gives (3.18).

Rewrite $T_i^{-1}U_{i+1}^+ = U_i^+T_i^{-1}+QU_i^+(1-E_i)(U_{i+1}^+ +1)$ as

$$T_i^{-1}U_{i+1}^+ - Q(U_{i+1}^+ +1)U_i^+ = U_i^+T_i^{-1} - QU_i^+E_i(U_{i+1}^+ +1),$$

and multiply on the left by T_i to get

$$U_{i+1}^+ - QT_i(U_{i+1}^+ +1)U_i^+ = T_iU_i^+T_i^{-1} - QT_iU_i^+E_i(U_{i+1}^+ +1). \tag{3.20}$$

Then, since $T_i = T_i^{-1}+Q(1-E_i)$, equations (3.17) and (3.16) imply

$$T_i(U_{i+1}^+ +1) = Q(U_i^+ +1)\left(\frac{T_i}{Q}+(1-E_i)U_{i+1}^+\right) \quad \text{and} \quad T_iU_i^+ = QU_{i+1}^+\left(\frac{T_i^{-1}}{Q}-(1-E_i)U_i^+\right),$$

and so (3.20) is

$$
\begin{aligned}
&U_{i+1}^+ - Q^2(U_i^+ +1)\left(\frac{T_i}{Q}+(1-E_i)U_{i+1}^+\right)U_i^+ \\
&\qquad = T_iU_i^+T_i^{-1} - Q^2U_{i+1}^+\left(\frac{T_i^{-1}}{Q}-(1-E_i)U_i^+\right)E_i(U_{i+1}^+ +1).
\end{aligned} \tag{3.21}
$$

Using (3.13) and

$$\text{adding} \quad \frac{T_i}{Q} - E_i\frac{Y_iY_{i+1}}{u^2-Y_iY_{i+1}} - Q^2\frac{Y_iY_{i+1}}{u^2-Y_iY_{i+1}}(U_i^+ +1)E_i(U_{i+1}^+ +1) \quad \text{to each side}$$

of (3.21) gives

$$
\begin{aligned}
&U_{i+1}^+ + \frac{T_i}{Q} - E_i\frac{Y_iY_{i+1}}{u^2-Y_iY_{i+1}} - Q^2(U_i^+ +1)\left(U_{i+1}^+ + \frac{T_i}{Q} - E_i\frac{Y_iY_{i+1}}{u^2-Y_iY_{i+1}}\right)U_i^+\\
&= T_iU_i^+T_i^{-1} + \frac{T_i}{Q} - E_i\frac{Y_iY_{i+1}}{u^2-Y_iY_{i+1}} - Q^2U_{i+1}^+\left(E_iU_i^+ + \frac{T_i^{-1}}{Q} - \frac{Y_iY_{i+1}}{u^2-Y_iY_{i+1}}\right)E_i(U_{i+1}^+ +1)\\
&= T_iU_i^+T_i^{-1} + \frac{T_i}{Q} - E_i\frac{Y_iY_{i+1}}{u^2-Y_iY_{i+1}} - Q^2U_{i+1}^+\left(E_iU_i^+E_i + z\frac{E_i}{Q} - E_i\frac{Y_iY_{i+1}}{u^2-Y_iY_{i+1}}\right)(U_{i+1}^+ +1),
\end{aligned}
$$

completing the proof of (3.19). □

Introduce notation $Z_{i-1}^{(\ell)}E_i$ and generating functions $Z_{i-1}^+E_i$ and $Z_{i-1}^-E_i$ by

$$Z_{i-1}^+E_i = \sum_{\ell\in\mathbb{Z}_{\geq 0}} Z_{i-1}^{(\ell)}E_iu^{-\ell} = E_i\left(\sum_{\ell\in\mathbb{Z}_{\geq 0}} Y_i^\ell u^{-\ell}\right)E_i, = E_i\frac{1}{1-Y_iu^{-1}}E_i, \tag{3.22}$$

$$Z_{i-1}^-E_i = \sum_{\ell\in\mathbb{Z}_{\geq 0}} Z_{i-1}^{(-\ell)}E_iu^{-\ell} = E_i\left(\sum_{\ell\in\mathbb{Z}_{\geq 0}} Y_i^{-\ell} u^{-\ell}\right)E_i = E_i\frac{1}{1-Y_i^{-1}u^{-1}}E_i. \tag{3.23}$$

By [GH1, Lemma 3.15(1)], or the identity (3.28) below, $Z_{i-1}^{(\ell)} \in W_{i-1}$ for $\ell \in \mathbb{Z}$. If

$$U_i^- = \frac{Y_i^{-1}}{u - Y_i^{-1}} \qquad \text{then} \qquad Z_{i-1}^{(0)} = 1 + \frac{z - z^{-1}}{q - q^{-1}}, \tag{3.24}$$

by the second relation in (2.46), and

$$E_i U_{i+1}^+ = E_i U_i^-, \qquad U_{i+1}^+ E_i = U_i^- E_i, \qquad E_i U_i^\pm E_i = (Z_{i-1}^\pm - Z_{i-1}^{(0)}) E_i, \tag{3.25}$$

where, for $i = 1$, the last identity is a restatement of the first identity in (2.43). In the following theorem, the identity (3.26) is equivalent to [GH1, Lemma 2.8, parts (2)and (3)] or [GH2, Lemma 2.6(4)] (see Remark 3.8) and the identity (3.27) is found in [BB, Lemma 7.4].

Theorem 3.6. *Let $Z_{i-1}^{(\ell)}$ and the generating functions Z_{i-1}^+ and Z_{i-1}^- be as defined in (3.22) and (3.23). Then $Z_{i-1}^{(\ell)} \in Z(W_{i-1})$,*

$$\begin{aligned} &\left(Z_{i-1}^- - \frac{z}{q-q^{-1}} - \frac{u^2}{u^2-1}\right)\left(Z_{i-1}^+ + \frac{z^{-1}}{q-q^{-1}} - \frac{u^2}{u^2-1}\right) E_i \\ &\qquad = \frac{-(u^2-q^2)(u^2-q^{-2})}{(u^2-1)^2(q-q^{-1})^2} E_i, \end{aligned} \tag{3.26}$$

$$\begin{aligned} &\left(Z_i^+ + \frac{z^{-1}}{q-q^{-1}} - \frac{u^2}{u^2-1}\right) E_{i+1} \\ &= \left(Z_{i-1}^+ + \frac{z^{-1}}{q-q^{-1}} - \frac{u^2}{u^2-1}\right) \frac{(u-Y_i)^2(u-q^{-2}Y_i^{-1})(u-q^2Y_i^{-1})}{(u-Y_i^{-1})^2(u-q^2Y_i)(u-q^{-2}Y_i)} E_{i+1}, \quad \text{and} \end{aligned} \tag{3.27}$$

$$\begin{aligned} &\left(Z_{k-1}^+ + \frac{z^{-1}}{q-q^{-1}} - \frac{u^2}{u^2-1}\right) E_{i+1} \\ &= \left(Z_0^+ + \frac{z^{-1}}{q-q^{-1}} - \frac{u^2}{u^2-1}\right) \left(\prod_{i=1}^{k-1} \frac{(u-Y_i)^2(u-q^{-2}Y_i^{-1})(u-q^2Y_i^{-1})}{(u-Y_i^{-1})^2(u-q^2Y_i)(u-q^{-2}Y_i)}\right) E_{i+1}. \end{aligned} \tag{3.28}$$

Proof. Since the generators $T_1, \ldots, T_{i-2}$, $E_1, \ldots, E_{i-2}$ and $Y_1, \ldots, Y_{i-1}$ of W_{i-1} all commute with E_i and Y_i, it follows that $Z_{i-1}^{(\ell)} \in Z(W_{i-1})$.

Multiply (3.18) on the right by E_i and use $Z_{i-1}^{(0)} = 1 + (z - z^{-1})/(q - q^{-1})$ to get (3.26).

Multiply (3.19) on the left and right by E_{i+1} and use the relations in (2.42), (2.43), (2.46), and

$$E_{i+1} T_i U_i^+ T_i^{-1} E_{i+1} = E_{i+1} T_i T_{i+1} U_i^+ T_{i+1}^{-1} T_i^{-1} E_{i+1} = E_{i+1} E_i U_i^+ E_i E_{i+1},$$

to obtain

$$\begin{aligned} &\left(Z_i^+ - Z_i^{(0)} + \frac{z}{q-q^{-1}} - \frac{1}{u^2-1}\right)\left(1 - (q-q^{-1})^2 U_i^+(U_i^+ + 1)\right) E_{i+1} \\ &\quad = \left(Z_{i-1}^+ - Z_{i-1}^{(0)} + \frac{z}{q-q^{-1}} - \frac{1}{u^2-1}\right)\left(1 - (q-q^{-1})^2 U_i^-(U_i^- + 1)\right) E_{i+1}. \end{aligned}$$

Then (3.27) follows from

$$\begin{aligned} \frac{1-(q-q^{-1})^2 U_i^-(U_i^-+1)}{1-(q-q^{-1})^2 U_i^+(U_i^+ +1)} &= \frac{1-(q-q^{-1})^2 \frac{Y_i^{-1}}{u-Y_i^{-1}}\left(\frac{Y_i^{-1}}{u-Y_i^{-1}}+1\right)}{1-(q-q^{-1})^2 \frac{Y_i}{u-Y_i}\left(\frac{Y_i}{u-Y_i}+1\right)} \\ = \frac{((u-Y_i^{-1})^2 - (q-q^{-1})^2 Y_i^{-1} u)\frac{1}{(u-Y_i^{-1})^2}}{((u-Y_i)^2 - (q-q^{-1})^2 Y_i u)\frac{1}{(u-Y_i)^2}} &= \frac{(u-q^{-2}Y_i^{-1})(u-q^2Y_i^{-1})(u-Y_i)^2}{(u-q^{-2}Y_i)(u-q^2Y_i)(u-Y_i^{-1})^2} \end{aligned}$$

and $Z_i^{(0)} = Z_{i-1}^{(0)} = 1 + (z - z^{-1})/(q - q^{-1})$. Finally, relation (3.28) follows, by induction, from (3.27). $\square$

Remark 3.7. Taking the coefficient of $u^{-(\ell+1)}$ on each side of (3.14) and (3.15) gives

$$T_iY_i^\ell = Y_{i+1}^\ell T_i - (q-q^{-1})(Y_{i+1}^\ell(1-E_i) + Y_{i+1}^{\ell-1}(1-E_i)Y_i + \cdots + Y_{i+1}(1-E_i)Y_i^{\ell-1}), \quad (3.29)$$

$$T_iY_{i+1}^\ell = Y_i^\ell T_i + (q-q^{-1})(Y_i^{\ell-1}(1-E_i)Y_{i+1} + Y_i^{\ell-2}(1-E_i)Y_{i+1}^2 + \cdots + (1-E_i)Y_{i+1}^\ell), \quad (3.30)$$

respectively, for $\ell \in \mathbb{Z}_{\geq 0}$. Therefore,

$$T_iY_i^{-\ell} = Y_{i+1}^{-\ell}T_i + (q-q^{-1})\left(Y_{i+1}^{-(\ell-1)}(1-E_i)Y_i^{-1} + \cdots + (1-E_i)Y_i^{-\ell}\right), \quad (3.31)$$

$$T_iY_{i+1}^{-\ell} = Y_i^{-\ell}T_i - (q-q^{-1})\left(Y_i^{-\ell}(1-E_i) + \cdots + Y_i^{-1}(1-E_i)Y_{i+1}^{-(\ell-1)}\right). \quad (3.32)$$

Remark 3.8. Combining (3.26) and (3.28) yields a formula for Z_{k-1}^- in terms of Z_0^+ and $Y_1, Y_2, \ldots, Y_{k-1}$. Using $Z_{i-1}^{(0)} = 1 + \frac{z-z^{-1}}{q-q^{-1}}$, rewrite (3.26) as

$$\begin{aligned}&\left(zZ_{i-1}^- - z^{-1}Z_{i-1}^+ - (z-z^{-1})Z_{i-1}^{(0)}\right)E_i \\ &\quad = (q-q^{-1})\left(\frac{1}{u^2-1}(Z_{i-1}^+ + Z_{i-1}^- - Z_{i-1}^{(0)}) - \left(Z_{i-1}^- - Z_{i-1}^{(0)}\right)\left(Z_{i-1}^+ - Z_{i-1}^{(0)}\right)\right)E_i,\end{aligned} \quad (3.33)$$

and take the coefficient of $u^{-\ell}$ in (3.26) to get

$$\begin{aligned}&\left(zZ_{i-1}^{(-\ell)} - z^{-1}Z_{i-1}^{(\ell)}\right)E_i \\ &\quad = (q-q^{-1})\begin{pmatrix} Z_{i-1}^{(\ell-2)} + Z_{i-1}^{(\ell-4)} + \cdots + Z_{i-1}^{(-(\ell-2))} \\ -\left(Z_{i-1}^{(\ell-1)}Z_{i-1}^{(-1)} + Z_{i-1}^{(\ell-2)}Z_{i-1}^{(-2)} + \cdots + Z_{i-1}^{(1)}Z_{i-1}^{(-(\ell-1))}\right)\end{pmatrix}E_i,\end{aligned} \quad (3.34)$$

from [GH2, Lemma 2.6(4)].

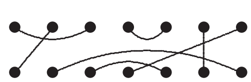
4 The center of the affine and degenerate affine BMW algebras

In this section, we identify the center of $\mathcal{W}_k$ and W_k. Both centers arise as algebras of symmetric functions with a "cancellation property" (in the language of [Pr]) or "wheel condition" (in the language of [FJ+]). In the degenerate case, $Z(\mathcal{W}_k)$ is the ring of symmetric functions in $y_1, \ldots, y_k$ with the Q-cancellation property of Pragacz. By [Pr, Theorem 2.11(Q)], this is the same ring as the ring generated by the odd power sums, which is the way that Nazarov [Naz] identified $Z(\mathcal{W}_k)$.

The cancellation property in the case of W_k is analogous, exhibiting the center of the affine BMW algebra $Z(W_k)$ as a subalgebra of the ring of symmetric Laurent polynomials. At the end of this section, in an attempt to make the theory for the affine BMW algebra completely analogous to that for the degenerate affine BMW algebra, we have formulated an alternate description of $Z(W_k)$ as a ring generated by "negative" power sums.

4.1 A basis of $\mathcal{W}_k$

A *(Brauer) diagram* on k dots is a graph with k dots in the top row, k dots in the bottom row and k edges pairing the dots. For example,

$d =$  is a Brauer diagram on 7 dots. (4.1)

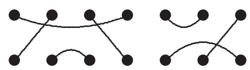
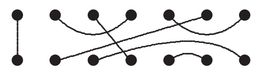
Number the vertices of the top row, left to right, with $1, 2, \ldots, k$ and the vertices in the bottom row, left to right, with $1', 2', \ldots, k'$ so that the diagram in (4.1) can be written

$$d = (13)(21')(45)(66')(74')(2'7')(3'5').$$

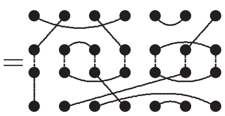
The *Brauer algebra* is the vector space

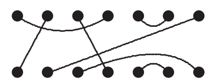
$$\mathcal{W}_{1,k} \quad \text{with basis} \qquad D_k = \{ \text{ diagrams on } k \text{ dots } \}, \tag{4.2}$$

and product given by stacking diagrams and changing each closed loop to x. For example,

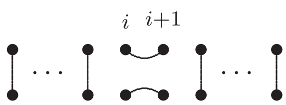
if $d_1 =$  and $d_2 =$  then

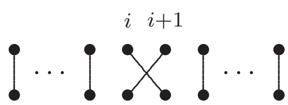
$$d_1 d_2 = \text{[diagram]} = x \, \text{[diagram]}. \tag{4.3}$$





The Brauer algebra is generated by

$e_i =$ 

, $\qquad s_i =$ 

, $\qquad 1 \le i \le k-1. \quad (4.4)$

Setting

$$x = z_0^{(0)} \qquad \text{and} \qquad s_i = \epsilon t_{s_i}$$

realizes the Brauer algebra as a subalgebra of the degenerate affine BMW algebra $\mathcal{W}_k$. The Brauer algebra is also the quotient of $\mathcal{W}_k$ by $y_1 = 0$ and, hence, can be viewed as the degenerate cyclotomic BMW algebra $\mathcal{W}_{1,k}(0)$.

Theorem 4.1. *Let $\mathcal{W}_k$ be the degenerate affine BMW algebra and let $\mathcal{W}_{r,k}(b_1, \ldots, b_r)$ be the degenerate cyclotomic BMW algebra as defined in* (2.23)-(2.24) *and* (2.31)*, respectively. For $n_1, \ldots, n_k \in \mathbb{Z}_{\ge 0}$ and a diagram d on k dots let*

$$d^{n_1,\ldots,n_k} = y_{i_1}^{n_1} \cdots y_{i_\ell}^{n_\ell} d y_{i_{\ell+1}}^{n_{\ell+1}} \cdots y_{i_k}^{n_k},$$

where, in the lexicographic ordering of the edges $(i_1, j_1), \ldots, (i_k, j_k)$ of d, $i_1, \ldots, i_\ell$ are in the top row of d and $i_{\ell+1}, \ldots, i_k$ are in the bottom row of d. Let D_k be the set of diagrams on k dots, as in (4.2)*.*

(a) If $\kappa_0, \kappa_1 \in C$ and

$$\left(z_0(-u) - \left(\tfrac{1}{2} + \epsilon\, u\right)\right)\left(z_0(u) - \left(\tfrac{1}{2} - \epsilon\, u\right)\right) = \left(\tfrac{1}{2} - \epsilon\, u\right)\left(\tfrac{1}{2} + \epsilon\, u\right) \tag{4.5}$$

then $\{d^{n_1,\ldots,n_k} \mid d \in D_k,\ n_1, \ldots, n_k \in \mathbb{Z}_{\ge 0}\}$ is a C-basis of $\mathcal{W}_k$.

(b) If $\kappa_0, \kappa_1 \in C$, (4.5) *holds, and*

$$\left(z_0(u) + u - \tfrac{1}{2}\right) = \left(u - \tfrac{1}{2}(-1)^r\right)\left(\prod_{i=1}^{r} \frac{u + b_i}{u - b_i}\right) \tag{4.6}$$

then $\{d^{n_1,\ldots,n_k} \mid d \in D_k,\ 0 \le n_1, \ldots, n_k \le r-1\}$ is a C-basis of $\mathcal{W}_{r,k}(b_1, \ldots, b_r)$.

Part (a) of Theorem 4.1 is [Naz, Theorem 4.6] (see also [AMR, Theorem 2.12]) and part (b) is [AMR, Prop. 2.15 and Theorem 5.5]. We refer to these references for the proof, remarking only that one key point in showing that $\{d^{n_1,\ldots,n_k} \mid d \in D_k, n_1, \ldots, n_k \in \mathbb{Z}_{\geq 0}\}$ spans $\mathcal{W}_k$ is that if (i,j) is a top-to-bottom edge in d, then

$$y_i d = d y_j + (\text{terms with fewer crossings}), \tag{4.7}$$

and if (i,j) is a top-to-top edge in d then

$$y_i d = -y_j d + (\text{terms with fewer crossings}). \tag{4.8}$$

This is illustrated in the affine case in (4.24).

4.2 The center of $\mathcal{W}_k$

The degenerate affine BMW algebra is the algebra $\mathcal{W}_k$ over C defined in Section 2.2 and the polynomial ring $C[y_1, \ldots, y_k]$ is a subalgebra of $\mathcal{W}_k$ (see Remark 2.5). The symmetric group S_k acts on $C[y_1, \ldots, y_k]$ by permuting the variables and the ring of symmetric functions is

$$C[y_1, \ldots, y_k]^{S_k} = \{f \in C[y_1, \ldots, y_k] \mid wf = f, \text{ for } w \in S_k\}.$$

A classical fact (see, for example, [Kl, Theorem 3.3.1]) is that the center of the degenerate affine Hecke algebra $\mathcal{H}_k$ is

$$Z(\mathcal{H}_k) = C[y_1, \ldots, y_k]^{S_k}.$$

Theorem 4.2 gives an analogous characterization of the center of the degenerate affine BMW algebra.

Theorem 4.2. *The center of the degenerate affine BMW algebra $\mathcal{W}_k$ is*

$$\mathcal{R}_k = \{f \in C[y_1, \ldots, y_k]^{S_k} \mid f(y_1, -y_1, y_3, \ldots, y_k) = f(0, 0, y_3, \ldots, y_k)\}.$$

Proof. Step 1: $f \in \mathcal{W}_k$ commutes with all $y_i \Leftrightarrow f \in C[y_1, \ldots, y_k]$:
Assume $f \in \mathcal{W}_k$ and write

$$f = \sum c_d^{n_1,\ldots,n_k} d^{n_1,\ldots,n_k}$$

in terms of the basis in Theorem 4.1. Let $d \in D_k$ with the maximal number of crossings such that $c_d^{n_1,\ldots,n_k} \neq 0$ and, using the notation before (4.2), suppose there is an edge (i,j) of d such that $j \neq i'$. Then, by (4.7) and (4.8),

$$\text{the coefficient of} \quad y_i d^{n_1,\ldots,n_k} \quad \text{in} \quad y_i f \quad \text{is} \quad c_d^{n_1,\ldots,n_k}$$

and

$$\text{the coefficient of} \quad y_i d^{n_1,\ldots,n_k} \quad \text{in} \quad f y_i \quad \text{is} \quad 0.$$

If $y_i f = f y_i$, it follows that there is no such edge, and so $d = 1$. Thus $f \in C[y_1, \ldots, y_k]$. Conversely, if $f \in C[y_1, \ldots, y_k]$ then $y_i f = f y_i$.

Step 2: $f \in C[y_1, \ldots, y_k]$ commutes with all $t_{s_i} \Leftrightarrow f \in \mathcal{R}_k$:
Assume $f \in C[y_1, \ldots, y_k]$ and write

$$f = \sum_{a,b \in \mathbb{Z}_{\geq 0}} y_1^a y_2^b f_{a,b}, \qquad \text{where } f_{a,b} \in C[y_3, \ldots, y_k].$$

Then $f(0,0,y_3,\ldots,y_k) = \sum_{a,b\in\mathbb{Z}_{\ge 0}} f_{a,b}$ and

$$f(y_1,-y_1,y_3,\ldots,y_k) = \sum_{a,b\in\mathbb{Z}_{\ge 0}} (-1)^b y_1^{a+b} f_{a,b} = \sum_{\ell\in\mathbb{Z}_{\ge 0}} y_1^\ell \left(\sum_{b=0}^{\ell} (-1)^b f_{\ell-b,b} \right). \tag{4.9}$$

By direct computation using (3.10) and (3.11),

$$t_{s_1} y_1^a y_2^b = s_1(y_1^a y_1^b) t_{s_1} - \frac{y_1^a y_2^b - s_1(y_1^a y_2^b)}{y_1 - y_2} + (-1)^a \sum_{r=1}^{a+b} (-1)^r y_1^{a+b-r} e_1 y_1^{r-1},$$

and it follows that

$$t_{s_1} f = (s_1 f) t_{s_1} - \frac{f - s_1 f}{y_1 - y_2} + \sum_{\ell\in\mathbb{Z}_{>0}} \left(\left(\sum_{r=1}^{\ell} (-1)^r y_1^{\ell-r} e_1 y_1^{r-1} \right) \left(\sum_{b=0}^{\ell} (-1)^{\ell-b} f_{\ell-b,b} \right) \right). \tag{4.10}$$

Thus, if $f(y_1,-y_1,y_3,\ldots,y_k) = f(0,0,y_3,\ldots,y_k)$, then

$$\sum_{b=0}^{\ell} (-1)^b f_{\ell-b,b} = 0, \qquad \text{for } \ell \neq 0. \tag{4.11}$$

Hence, if $f \in C[y_1,\ldots,y_k]^{S_k}$ and $f(y_1,-y_1,y_3,\ldots,y_k) = f(0,0,y_3,\ldots,y_k)$ then $s_1 f = f$ and, by (4.9), (4.11) holds so that, by (4.10), $t_{s_1} f = f t_{s_1}$. Similarly, f commutes with all t_{s_i}.
Conversely, if $f \in C[y_1,\ldots,y_k]$ and $t_{s_i} f = f t_{s_i}$ then

$$s_i f = f \qquad \text{and} \qquad \sum_{b=0}^{\ell} (-1)^{\ell-b} f_{\ell-b,b} = 0, \qquad \text{for } \ell \neq 0,$$

so that $f \in C[y_1,\ldots,y_k]^{S_k}$ and $f(y_1,-y_1,y_3,\ldots,y_k) = f(0,0,y_3,\ldots,y_k)$.

It follows from (2.21) that $\mathcal{R}_k = Z(\mathcal{W}_k)$. □

The *power sum symmetric functions* p_i are given by

$$p_i = y_1^i + y_2^i + \cdots + y_k^i, \qquad \text{for } i \in \mathbb{Z}_{>0}.$$

The *Hall-Littlewood polynomials* (see [Mac, Ch. III (2.1)]) are given by

$$P_\lambda(y;t) = P_\lambda(y_1,\ldots,y_k;t) = \frac{1}{v_\lambda(t)} \sum_{w\in S_k} w \left(y_1^{\lambda_1} \cdots y_k^{\lambda_k} \prod_{1\le i<j\le k} \frac{x_i - t x_j}{x_i - x_j} \right),$$

where $v_\lambda(t)$ is a normalizing constant (a polynomial in t) so that the coefficient of $y_1^{\lambda_1}\cdots y_k^{\lambda_k}$ in $P_\lambda(y;t)$ is equal to 1. The *Schur Q-functions* (see [Mac, Ch. III (8.7)]) are

$$Q_\lambda = \begin{cases} 0, & \text{if } \lambda \text{ is not strict,} \\ 2^{\ell(\lambda)} P_\lambda(y;-1), & \text{if } \lambda \text{ is strict,} \end{cases}$$

where $\ell(\lambda)$ is the number of (nonzero) parts of λ and the partition λ is *strict* if all its (nonzero) parts are distinct. Let $\mathcal{R}_k$ be as in Theorem 4.2. Then (see [Naz, Cor. 4.10], [Pr, Theorem 2.11(Q)] and [Mac, Ch. III §8])

$$\mathcal{R}_k = C[p_1,p_3,p_5,\ldots] = C\text{span-}\{Q_\lambda \mid \lambda \text{ is strict}\}. \tag{4.12}$$

More generally, let $r \in \mathbb{Z}_{>0}$ and let ζ be a primitive rth root of unity. Define

$$\mathcal{R}_{r,k} = \{f \in \mathbb{Z}[\zeta][y_1, \ldots, y_k]^{S_k} \mid f(y_1, \zeta y_1, \ldots, \zeta^{r-1} y_1, y_{r+1}, \ldots, y_k) = f(0, 0, \ldots, 0, y_{r+1}, \ldots, y_k)\}.$$

Then

$$\mathcal{R}_{r,k} \otimes_{\mathbb{Z}[\zeta]} \mathbb{Q}(\zeta) = \mathbb{Q}(\zeta)[p_i \mid i \neq 0 \bmod r], \tag{4.13}$$

and

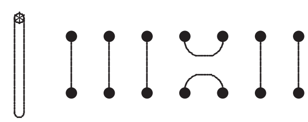
$$\mathcal{R}_{r,k} \text{ has } \mathbb{Z}[\zeta]\text{-basis} \qquad \{P_\lambda(y;\zeta) \mid m_i(\lambda) < r \text{ and } \lambda_1 \leq k\}, \tag{4.14}$$

where $m_i(\lambda)$ is the number parts of size i in λ. The ring $\mathcal{R}_{r,k}$ is studied in [Mo], [LLT], [Mac, Ch. III Ex. 5.7 and Ex. 7.7], [To], [FJ+], and others. The proofs of (4.13) and (4.14) follow from [Mac, Ch. III Ex. 7.7], [To, Lemma 2.2 and following remarks] and the arguments in the proofs of [FJ+, Lemma 3.2 and Proposition 3.5].

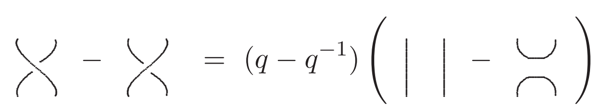
Remark 4.3. The left ideal of $\mathcal{W}_2$ generated by e_1 is $C[y_1]e_1$. This is an infinite dimensional (generically irreducible) $\mathcal{W}_2$-module on which $Z(\mathcal{W}_2)$ acts by constants. Thus, as noted by [AMR, par. before Ex. 2.17], it follows that $\mathcal{W}_2$ is not finitely generated as a $Z(\mathcal{W}_2)$-module.

4.3 A basis of W_k

An affine tangle has k strands and a flagpole just as in the case of an affine braid, but there is no restriction that a strand must connect an upper vertex to a lower vertex. Let X^{ε_1} and T_i be the affine braids given in (2.37) and let

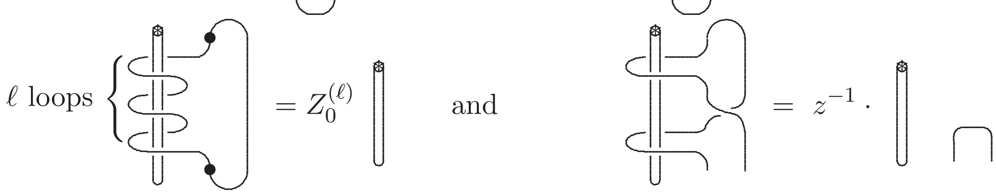
$E_i =$  . (4.15)

Goodman and Hauchschild [GH1, Cor. 6.14(b)] have shown that the affine BMW algebra W_k is the algebra of linear combinations of tangles generated by $X^{\varepsilon_1}, T_1, \ldots, T_{k-1}, E_1, \ldots, E_{k-1}$ and the relations (2.42), (2.43) and (2.45) expressed in the form



(4.16)

(4.17)



(4.18)

 $= \dfrac{z - z^{-1}}{q - q^{-1}} + 1 = Z_0^{(0)}.$ (4.19)

Theorem 4.4. *Let W_k be the affine BMW algebra and let $W_{r,k}(b_1, \ldots, b_r)$ be the cyclotomic BMW algebra as defined in Section 2.4. Let $d \in D_k$ be a Brauer diagram, where D_k is as in (4.2). Choose a minimal length expression of d as a product of $e_1, \ldots, e_{k-1}, s_1, \ldots, s_{k-1}$,*

$$d = a_1 \cdots a_\ell, \qquad a_i \in \{e_1, \ldots, e_{k-1}, s_1, \ldots, s_{k-1}\},$$

such that the number of s_i in this product is the number of crossings in d. For each a_i which is in $\{s_1, \ldots, s_{k-1}\}$ fix a choice of sign $\epsilon_j = \pm 1$ and set

$$T_d = A_1 \cdots A_\ell, \qquad \textit{where} \quad A_j = \begin{cases} E_i, & \textit{if } a_j = e_i, \\ T_i^{\epsilon_j}, & \textit{if } a_j = s_i. \end{cases}$$

For $n_1, \ldots, n_k \in \mathbb{Z}$ let

$$T_d^{n_1,\ldots,n_k} = Y_{i_1}^{n_1} \cdots Y_{i_\ell}^{n_\ell} T_d Y_{i_{\ell+1}}^{n_{\ell+1}} \cdots Y_{i_k}^{n_k},$$

where, in the lexicographic ordering of the edges $(i_1, j_1), \ldots, (i_k, j_k)$ of d, $i_1, \ldots, i_\ell$ are in the top row of d and $i_{\ell+1}, \ldots, i_k$ are in the bottom row of d.

(a) If

$$\left(Z_0^- - \frac{z}{q-q^{-1}} - \frac{u^2}{u^2-1}\right)\left(Z_0^+ + \frac{z^{-1}}{q-q^{-1}} - \frac{u^2}{u^2-1}\right) = \frac{-(u^2-q^2)(u^2-q^{-2})}{(u^2-1)^2(q-q^{-1})^2} \tag{4.20}$$

then $\{T_d^{n_1,\ldots,n_k} \mid d \in D_k,\ n_1, \ldots, n_k \in \mathbb{Z}\}$ is a C-basis of W_k.

(b) If (4.20) holds and

$$Z_0^+ + \frac{z^{-1}}{q-q^{-1}} - \frac{u^2}{u^2-1} = \left(\frac{z}{q-q^{-1}} + \frac{uz}{u^2-1}\right)\prod_{j=1}^{r} \frac{u-b_j^{-1}}{u-b_j} \tag{4.21}$$

then $\{T_d^{n_1,\ldots,n_k} \mid d \in D_k,\ 0 \le n_1, \ldots, n_k \le r-1\}$ is a C-basis of $W_{r,k}(b_1, \ldots, b_r)$.

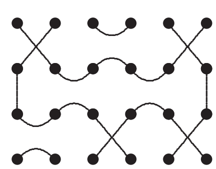
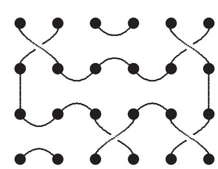
Part (a) of Theorem 4.4 is [GH2, Theorem 2.25] and part (b) is [GH2, Theorem 5.5] and [WY2, Theorem 8.1]. We refer to these references for proof, remarking only that one key point in showing that $\{T_d^{n_1,\ldots,n_k} \mid d \in D_k,\ n_1, \ldots, n_k \in \mathbb{Z}\}$ spans W_k is that if (i,j) is a top-to-bottom edge in d then

$$Y_i T_d = T_d Y_j + (\text{terms with fewer crossings}), \tag{4.22}$$

and, if (i,j) is a top-to-top edge in d then

$$Y_i T_d = Y_j^{-1} T_d + (\text{terms with fewer crossings}). \tag{4.23}$$

As an example, let $d = s_1 e_3 s_5 e_2 e_4 e_1 s_3 s_5$ and choose $\epsilon_1 = \epsilon_3 = -\epsilon_7 = \epsilon_8 = 1$. Then

$d =$  $=$  and $T_d =$ 

so that $T_d = T_1 E_3 T_5 E_2 E_4 E_1 T_3^{-1} T_5$ and $T_d^{5,3,-2,0,3,0} = Y_1^5 Y_2^3 Y_3^{-2} T_d Y_1^3$. Then, since $(1,6)$ is a horizontal edge in d, (4.23) is illustrated by the computation

$$\begin{aligned} Y_6 T_d &= T_1 E_3 Y_6 T_5 E_2 E_4 E_1 T_3^{-1} T_5 = T_1 E_3 (T_5 Y_5 + (q-q^{-1}) Y_6 (1-E_5)) E_2 E_4 E_1 T_3^{-1} T_5 \\ &= T_1 E_3 T_5 E_2 Y_5 E_4 E_1 T_3^{-1} T_5 + \cdots = T_1 E_3 T_5 E_2 Y_4^{-1} E_4 E_1 T_3^{-1} T_5 + \cdots \\ &= T_1 E_3 Y_4^{-1} T_5 E_2 E_4 E_1 T_3^{-1} T_5 + \cdots = T_1 E_3 Y_3 T_5 E_2 E_4 E_1 T_3^{-1} T_5 + \cdots \\ &= T_1 E_3 T_5 Y_3 E_2 E_4 E_1 T_3^{-1} T_5 + \cdots = T_1 E_3 T_5 Y_2^{-1} E_2 E_4 E_1 T_3^{-1} T_5 + \cdots \\ &= T_1 Y_2^{-1} E_3 T_5 E_2 E_4 E_1 T_3^{-1} T_5 + \cdots = Y_1^{-1} T_1 E_3 T_5 E_2 E_4 E_1 T_3^{-1} T_5 + \cdots, \end{aligned} \tag{4.24}$$

where $+\cdots$ is always a linear combination of terms with fewer crossings.

4.4 The center of W_k

The affine BMW algebra is the algebra W_k over C defined in Section 2.4 and the ring of Laurent polynomials $C[Y_1^{\pm 1},\ldots,Y_k^{\pm 1}]$ is a subalgebra of W_k (see Remark 2.10). The symmetric group S_k acts on $C[Y_1^{\pm 1},\ldots,Y_k^{\pm 1}]$ by permuting the variables and the ring of symmetric functions is

$$C[Y_1^{\pm 1},\ldots,Y_k^{\pm 1}]^{S_k} = \{f\in C[Y_1^{\pm 1},\ldots,Y_k^{\pm 1}] \mid wf=f, \text{ for } w\in S_k\}.$$

A classical fact (see, for example, [GV, Proposition 2.1]) is that the center of the affine Hecke algebra H_k is

$$Z(H_k) = C[Y_1^{\pm 1},\ldots,Y_k^{\pm 1}]^{S_k}.$$

Theorem 4.5 is a characterization of the center of the affine BMW algebra.

Theorem 4.5. *The center of the affine BMW algebra W_k is*

$$R_k = \{f\in C[Y_1^{\pm 1},\ldots,Y_k^{\pm 1}]^{S_k} \mid f(Y_1,Y_1^{-1},Y_3,\ldots,Y_k) = f(1,1,Y_3,\ldots,Y_k)\}.$$

Proof. Step 1: $f\in W_k$ commutes with all Y_i $\Leftrightarrow$ $f\in C[Y_1^{\pm 1},\ldots,Y_k^{\pm 1}]$:
Assume $f\in W_k$ and write

$$f = \sum c_d^{n_1,\ldots,n_k} T_d^{n_1,\ldots,n_k},$$

in terms of the basis in Theorem 4.4. Let $d\in D_k$ with the maximal number of crossings such that $c_d^{n_1,\ldots,n_k}\neq 0$ and, using the notation before (4.2), suppose there is an edge (i,j) of d such that $j\neq i'$. Then, by (4.22) and (4.23),

$$\text{the coefficient of}\quad Y_iT_d^{n_1,\ldots,n_k}\quad\text{in}\quad Y_if\quad\text{is}\quad c_d^{n_1,\ldots,n_k}$$

and

$$\text{the coefficient of}\quad Y_iT_d^{n_1,\ldots,n_k}\quad\text{in}\quad fY_i\quad\text{is}\quad 0.$$

If $Y_if=fY_i$ it follows that there is no such edge, and so $d=1$ (and therefore $T_d=1$). Thus $f\in C[Y_1^{\pm 1},\ldots,Y_k^{\pm 1}]$. Conversely, if $f\in C[Y_1^{\pm 1},\ldots,Y_k^{\pm 1}]$, then $Y_if=fY_i$.
Step 2: $f\in C[Y_1^{\pm 1},\ldots,Y_k^{\pm 1}]$ commutes with all T_i $\Leftrightarrow$ $f\in R_k$:
Assume $f\in C[Y_1^{\pm 1},\ldots,Y_k^{\pm 1}]$ and write

$$f=\sum_{a,b\in\mathbb{Z}} Y_1^aY_2^b f_{a,b},\qquad \text{where } f_{a,b}\in C[Y_3^{\pm 1},\ldots,Y_k^{\pm 1}].$$

Then $f(1,1,Y_3,\ldots,Y_k)=\sum_{a,b\in\mathbb{Z}} f_{a,b}$ and

$$f(Y_1,Y_1^{-1},Y_3,\ldots,Y_k) = \sum_{a,b\in\mathbb{Z}} Y_1^{a-b}f_{a,b} = \sum_{\ell\in\mathbb{Z}} Y_1^\ell\left(\sum_{b\in\mathbb{Z}} f_{\ell+b,b}\right). \tag{4.25}$$

By direct computation using (3.30) and (3.32),

$$T_1Y_1^aY_2^b = Y_1^aY_2^aT_1Y_2^{b-a} = s_1(Y_1^aY_2^b)T_1 + (q-q^{-1})\frac{Y_1^aY_2^b - s_1(Y_1^aY_2^b)}{1-Y_1Y_2^{-1}} + \mathcal{E}_{b-a},$$

where

$$\mathcal{E}_\ell = \begin{cases} -(q-q^{-1})\sum_{r=1}^{\ell} Y_1^{\ell-r}E_1Y_1^{-r}, & \text{if } \ell>0, \\ (q-q^{-1})\sum_{r=1}^{-\ell} Y_1^{\ell+r-1}E_1Y_1^{r-1}, & \text{if } \ell<0, \\ 0, & \text{if } \ell=0. \end{cases}$$

It follows that

$$T_1 f = (s_1 f)T_1 + (q-q^{-1})\frac{f-s_1 f}{1-Y_1Y_2^{-1}} + \sum_{\ell\in\mathbb{Z}_{\neq 0}} \mathcal{E}_\ell \left(\sum_{b\in\mathbb{Z}} f_{\ell+b,b}\right). \tag{4.26}$$

Thus, if $f(Y_1,Y_1^{-1},Y_3,\ldots,Y_k) = f(1,1,Y_3,\ldots,Y_k)$ then, by (4.25),

$$\sum_{b\in\mathbb{Z}} f_{\ell+b,b} = 0, \qquad \text{for } \ell\neq 0. \tag{4.27}$$

Hence, if $f\in C[Y_1^{\pm1},\ldots,Y_k^{\pm1}]^{S_k}$ and $f(Y_1,Y_1^{-1},Y_3,\ldots,Y_k) = f(1,1,Y_3,\ldots,Y_k)$ then $s_1 f = f$ and (4.27) holds so that, by (4.26), $T_1 f = fT_1$. Similarly, f commutes with all T_i.
Conversely, if $f\in C[Y_1^{\pm1},\ldots,Y_k^{\pm1}]$ and $T_i f = fT_i$ then

$$s_i f = f \qquad \text{and} \qquad \sum_{b\in\mathbb{Z}} f_{\ell+b,b} = 0, \qquad \text{for } \ell\neq 0,$$

so that $f\in C[Y_1^{\pm1},\ldots,Y_k^{\pm1}]^{S_k}$ and $f(Y_1,Y_1^{-1},Y_3,\ldots,Y_k) = f(1,1,Y_3,\ldots,Y_k)$.
It follows from (2.41) that $R_k = Z(W_k)$. □

The symmetric group S_k acts on $\mathbb{Z}^k$ by permuting the factors. The ring

$$C[Y_1^{\pm1},\ldots,Y_k^{\pm1}]^{S_k} \qquad \text{has basis} \qquad \{m_\lambda \mid \lambda\in\mathbb{Z}^k \text{ with } \lambda_1\ge\lambda_2\ge\cdots\ge\lambda_k\},$$

where

$$m_\lambda = \sum_{\mu\in S_k\lambda} Y_1^{\mu_1}\cdots Y_k^{\mu_k}.$$

The *elementary symmetric functions* are

$$e_r = m_{(1^r,0^{k-r})} \quad \text{and} \quad e_{-r} = m_{(0^{k-r},(-1)^r)}, \qquad \text{for } r=0,1,\ldots,k,$$

and the *power sum symmetric functions* are

$$p_r = m_{(r,0^{k-1})} \quad \text{and} \quad p_{-r} = m_{(0^{k-1},-r)}, \qquad \text{for } r\in\mathbb{Z}_{>0}.$$

The Newton identities (see [Mac, Ch. I (2.11′)]) say

$$\ell e_\ell = \sum_{r=1}^{\ell}(-1)^{r-1}p_r e_{\ell-r} \qquad \text{and} \qquad \ell e_{-\ell} = \sum_{r=1}^{\ell}(-1)^{r-1}p_{-r}e_{-(\ell-r)}, \tag{4.28}$$

where the second equation is obtained from the first by replacing Y_i with Y_i^{-1}. For $\ell\in\mathbb{Z}$ and $\lambda=(\lambda_1,\ldots,\lambda_k)\in\mathbb{Z}^k$,

$$e_k^\ell m_\lambda = m_{\lambda+(\ell^k)}, \qquad \text{where} \quad \lambda+(\ell^k) = (\lambda_1+\ell,\ldots,\lambda_k+\ell).$$

In particular,

$$e_{-r} = e_k^{-1} e_{k-r}, \qquad \text{for } r = 0, \ldots, k. \tag{4.29}$$

Define

$$p_i^+ = p_i + p_{-i} \qquad \text{and} \qquad p_i^- = p_i - p_{-i}, \quad \text{for } i \in \mathbb{Z}_{>0}. \tag{4.30}$$

The consequence of (4.29) and (4.28) is that

$$\begin{aligned} \mathbb{C}[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]^{S_k} &= \mathbb{C}[e_k^{\pm 1}, e_1, \ldots, e_{k-1}] \\ &= \mathbb{C}[e_k^{\pm 1}][e_1, e_2, \ldots, e_{\lfloor \frac{k}{2} \rfloor}, e_k e_{-\lfloor \frac{k-1}{2} \rfloor}, \ldots, e_k e_{-2}, e_k e_{-1}] \\ &= \mathbb{C}[e_k^{\pm 1}][e_1, e_2, \ldots, e_{\lfloor \frac{k}{2} \rfloor}, e_{-\lfloor \frac{k-1}{2} \rfloor}, \ldots, e_{-2}, e_{-1}] \\ &= \mathbb{C}[e_k^{\pm 1}][p_1, p_2, \ldots, p_{\lfloor \frac{k}{2} \rfloor}, p_{-\lfloor \frac{k-1}{2} \rfloor}, \ldots, p_{-2}, p_{-1}] \\ &= \mathbb{C}[e_k^{\pm 1}][p_1^+, p_2^+, \ldots, p^+_{\lfloor \frac{k}{2} \rfloor}, p^-_{\lfloor \frac{k-1}{2} \rfloor}, \ldots, p_2^-, p_1^-]. \end{aligned}$$

For $\nu \in \mathbb{Z}^k$ with $\nu_1 \geq \cdots \geq \nu_\ell > 0$ define

$$p_\nu^+ = p_{\nu_1}^+ \cdots p_{\nu_\ell}^+ \qquad \text{and} \qquad p_\nu^- = p_{\nu_1}^- \cdots p_{\nu_\ell}^-.$$

Then

$$\mathbb{C}[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]^{S_k} \qquad \text{has basis} \qquad \{e_k^\ell p_\lambda^+ p_\mu^- \mid \ell \in \mathbb{Z}, \ell(\lambda) \leq \lfloor \tfrac{k}{2} \rfloor, \ell(\mu) \leq \lfloor \tfrac{k-1}{2} \rfloor\}. \tag{4.31}$$

In analogy with (4.12) we expect that if R_k is as in Theorem 4.5 then

$$R_k = C[e_k^{\pm 1}][p_1^-, p_2^-, \ldots]. \tag{4.32}$$

Remark 4.6. The left ideal of W_2 generated by E_1 is $C[Y_1^{\pm 1}]E_1$. This is an infinite dimensional (generically irreducible) W_2-module on which $Z(W_2)$ acts by constants. It follows that W_2 is not a finitely generated $Z(W_2)$-module.

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