Representation theory
Lecture Notes: Chapter 1

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1. Algebras and representations.

Algebras.

An *algebra* is a vector space (over $\mathbb{C}$) with a multiplication such that A is a ring with identity, i.e. there is a map $A \times A \to A$, $(a, b) \mapsto ab$, which is bilinear and satisfies the associative and distributive laws. The following are examples of algebras:

(1) The group algebra of a group G is the vector space $\mathbb{C}G$ with basis G and with multiplication forced by the multiplication in G (and the bilinearity).
(2) If M is a vector space (over $\mathbb{C}$) then the space $\mathrm{End}(M)$ of $\mathbb{C}$-linear transformations of M is an algebra under the multiplication given by composition of endomorphisms.
(3) Given a basis $B = \{b_1, \ldots, b_d\}$ of the vector space M the algebra $\mathrm{End}(M)$ can be idenitified with the algebra $M_d(\mathbb{C})$ of $d \times d$ matrices $T = (T_{ij})_{1\le i,j,\le d}$ with entries in $\mathbb{C}$ via

$$Tb_i = \sum_{i=1}^{d} b_j T_{ji}, \qquad \text{for } t \in \mathrm{End}(M).$$

Let A be an algebra. An *ideal* in A is a subspace $I \subset A$ such that $ar \in I$ and $ra \in I$, for all $a \in A$ and $r \in I$. A *minimal ideal* of A is a nonzero ideal I which cannot be written as a direct sum $I = I_1 \oplus I_2$ of nonzero ideals I_1 and I_2 of A. An *idempotent* is a nonzero element $p \in A$ such that $p^2 = p$. Two idempotents $p_1, p_2 \in A$ are *orthogonal* if $p_1p_2 = p_2p_1 = 0$. A *minimal idempotent* is an idempotent p that cannot be written as a sum $p = p_1 + p_2$ of orthogonal idempotents $p_1, p_2 \in A$. The *center* of A is

$$Z(A) = \{z \in A \mid az = za \text{ for all } a \in A\}.$$

A *central idempotent* is an idempotent in $Z(A)$ and a *minimal central idempotent* is a central idempotent z that cannot be written as a sum $z = z_1 + z_2$ of orthogonal central idempotents z_1 and z_2.

A *trace* on A is a linear map $\vec{t}\colon A \to \mathbb{C}$ such that

$$\vec{t}(a_1a_2) = \vec{t}(a_2a_1), \qquad \text{for all } a_1, a_2 \in A.$$

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A *character* of A is a trace on A. A trace $\vec{t}$ on A is *nondegenerate* if for each $b \in A$ there is an $a \in A$ such that $\vec{t}(ba) \neq 0$. The *radical* of a trace $\vec{t}$ is

$$\text{rad } t = \{b \in A \mid \vec{t}(ba) = 0 \text{ for all } a \in A.\} \tag{1.1}$$

Every trace $\vec{t}$ on A determines a symmetric bilinear form $\langle , \rangle : A \times A \to \mathbb{C}$ given by

$$\langle a_1, a_2 \rangle = \vec{t}(a_1 a_2), \qquad \text{for all } a_1, a_2 \in A. \tag{1.2}$$

The form $\langle , \rangle$ is nondegenerate if and only if the trace $\vec{t}$ is nondegenerate and the radical

$$\text{rad } \langle , \rangle = \{a \in A \mid \langle a, b \rangle = 0 \text{ for all } b \in A\}$$

of the form $\langle , \rangle$ is the same as rad $\vec{t}$.

Lemma 1.3. *Let $\vec{t}$ be a trace on A and let $\langle , \rangle$ be the bilinear form on A defined by the trace $\vec{t}$, as in ??. Let B be a basis of A. Let $G = \big(\langle b, b' \rangle\big)_{b,b' \in B}$ be the matrix of the form $\langle , \rangle$ with respect to B. The following are equivalent:*

(1) The trace $\vec{t}$ is nondegenerate.

(2) $\det G \neq 0$.

(3) The dual basis B^* to the basis B with respect to the form $\langle , \rangle$ exists.

Proof. (2) $\Leftrightarrow$ (1): The trace $\vec{t}$ is degenerate if there is an element $a \in A$, $a \neq 0$, such that $\vec{t}(ac) = 0$ for all $c \in B$. If $a_b \in \mathbb{C}$ are such that

$$a = \sum_{b \in B} a_b b, \qquad \text{then} \qquad 0 = \langle a, c \rangle = \sum_{b \in B} a_b \langle b, c \rangle$$

for all $c \in B$. So a exists if and only if the columns of G are linearly dependent, i.e. if an only if G is not invertible.

(3) $\Leftrightarrow$ (2): Let $B^* = \{b^*\}$ be the dual basis to $\{b\}$ with respect to $\langle , \rangle$ and let P be the change of basis matrix from B to B^*. Then

$$d^* = \sum_{b \in B} P_{db} b, \qquad \text{and} \qquad \delta_{bc} = \langle b, d^* \rangle = \sum_{d \in B} P_{dc} \langle b, c \rangle = (GP^t)_{b,c}.$$

So P^t, the transpose of P, is the inverse of the matrix G. So the dual basis to B exists if and only if G is invertible, i.e. if and only if $\det G \neq 0$. ∎

Proposition 1.4. *Let A be an algebra and let $\vec{t}$ be a nondegenerate trace on A. Define a symmetric bilinear form $\langle , \rangle : A \times A \to \mathbb{C}$ on A by $\langle a_1, a_2 \rangle = \vec{t}(a_1 a_2)$, for all $a_1, a_2 \in A$. Let B be a basis of A and let B^* be the dual basis to B with respect to $\langle \, , \, \rangle$. Let $a \in A$ and define*

$$[a] = \sum_{b \in B} b a b^*.$$

Then $[a]$ is an element of the center $Z(A)$ of A and $[a]$ does not depend on the choice of the basis B.

Proof. Let $c \in A$. Then

$$c[a] = \sum_{b\in B} cbab^* = \sum_{b\in B}\sum_{d\in B} \langle cb, d^*\rangle dab^* = \sum_{d\in B} da \sum_{b\in B}\langle d^*c, b\rangle b^* = \sum_{d\in B} dad^*c = [a]c,$$

since $\langle cb, d^*\rangle = \vec{t}(cbd^*) = \vec{t}(d^*cb) = \langle d^*c, b\rangle$. So $[a] \in Z(A)$.

Let D be another basis of A and let D^* be the dual basis to D with respect to $\langle , \rangle$. Let $P = \big(P_{db}\big)$ be the transition matrix from D to B and let P^{-1} be the inverse of P. Then

$$d = \sum_{b\in B} P_{db} b \qquad \text{and} \qquad d^* = \sum_{\tilde b\in B} (P^{-1})_{\tilde b d}\tilde b^*,$$

since

$$\langle d, \tilde d^*\rangle = \left\langle \sum_{b\in B} P_{db} b, \sum_{\tilde b\in B}(P^{-1})_{\tilde b\tilde d}\tilde b^*\right\rangle = \sum_{b,\tilde b\in B} P_{db}(P^{-1})_{\tilde b\tilde d}\delta_{b\tilde b} = \delta_{d\tilde d}.$$

So

$$\sum_{d\in D} dad^* = \sum_{d\in D}\sum_{b\in B} P_{db} ba \sum_{\tilde b\in B}(P^{-1})_{\tilde b d}\tilde b^* = \sum_{b,\tilde b\in B} ba\tilde b^*\delta_{b\tilde b} = \sum_{b\in B} bab^*.$$

So $[a]$ does not depend on the choice of the basis B. ∎

Representations.

An A-module is a vector space M (over $\mathbb{C}$) with an A-action, i.e. a map $A \times M \to M$, $(a, m) \mapsto am$, which is bilinear and such that

$$1_A m = m \qquad \text{and} \qquad a_1(a_2 m) = (a_1 a_2)m,$$

for all $a_1, a_2 \in A$ and $m \in M$ (1_A denotes the identity in the algebra A). A *representation* of A is an A-module. A *representation* of a group G is a representation of the group algebra $\mathbb{C}G$. The *character* of an A-module M is the map $\chi^M : A \to \mathbb{C}$ given by

$$\chi^M(a) = \mathrm{Tr}(M(a)), \qquad \text{for } a \in A,$$

where $M(a)$ is the linear transformation of M determined by the action of A and $\mathrm{Tr}(M(a))$ is the trace of $M(a)$. An *irreducible character* of A is the character of an irreducible representation of A.

An A-module M gives rise to a map

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{End}(M) \\ a & \longmapsto & M(a) \end{array} \tag{1.5}$$

where $M(a)$ is the linear transformation of M determined by the action of a on M. This map is linear and satisfies

$$M(1_A) = \mathrm{Id}_M,$$
$$M(a_1 a_2) = M(a_1)M(a_2),$$

for all $a_1, a_2 \in A$, i.e. $A \to \mathrm{End}(M)$ is a homomorphism of algebras. (Given a basis $B = \{b_1, \ldots, b_d\}$ of M the map $A \to \mathrm{End}(M)$ can be identified with a map $M : A \to M_d(\mathbb{C})$.) Conversely, an algebra homomorphism as in ??? and ??? determines an A-action on M by

$$am = M(a)m, \qquad \text{for all } a \in A \text{ and } m \in M.$$

Thus, the map $M: A \to \mathrm{End}(M)$ and the A-module M are equivalent data. Historically, the map $M: A \to \mathrm{End}(M)$ was the representation and M was the A-module, but now the terms representation and A-module are used interchangeably. This is the reason for the use of the letter M, both for the A-module and the corresponding algebra homomorphism $M: A \to \mathrm{End}(M)$.

A *submodule* of an A-module M is a subspace $N \subseteq M$ such that $an \in N$, for all $a \in A$ and $n \in N$. An A-module M is *simple* or *irreducible* if it has no submodules except 0 and itself. The *direct sum* of two A-modules M_1 and M_2 is the vector space $M = M_1 \oplus M_2$ with A-action given by

$$a(m_1, m_2) = (am_1, am_2), \qquad \text{for all } a \in A,\ m_1 \in M_1 \text{ and } m_2 \in M_2.$$

An A-module M is *semisimple* or *completely decomposable* if M can be written as a direct sum of simple submodules. An A-module M is *indecomposable* if M cannot be written as a direct sum $M = M_1 \oplus M_2$ of nonzero submodules $M_1 \subseteq M$ and $M_2 \subseteq M$.

Here we need a reference to the reader to look at the examples in Chapter 2 etc.

Homomorphisms

Let M and N be A-modules. Then define

$$\mathrm{Hom}_A(M, N) = \{\phi \in \mathrm{Hom}(M, N) \mid a\phi(m) = \phi(am), \text{ for all } a \in A \text{ and } m \in M\},$$

where $\mathrm{Hom}(M, N)$ is the set of $\mathbb{C}$-linear transformations from M to N. The proof of the following Proposition is identical to the proof of Proposition ??? except with a replaced by ϕ.

Proposition 1.6. *Let A be an algebra and let $\vec{t}$ be a nondegenerate trace on A. Define a symmetric bilinear form $\langle , \rangle: A \times A \to \mathbb{C}$ on A by $\langle a_1, a_2 \rangle = \vec{t}(a_1 a_2)$, for all $a_1, a_2 \in A$. Let B be a basis of A and let B^* be the dual basis to B with respect to $\langle\ ,\ \rangle$. Let M and N be A-modules and let $\phi \in \mathrm{Hom}(M, N)$. Define*

$$[\phi] = \sum_{b \in B} b\phi b^*.$$

Then $[\phi] \in \mathrm{Hom}_A(M, N)$ and $[\phi]$ does not depend on the choice of the basis B.

Direct sums of algebras

Proposition 1.7. *Let A and B be algebras and let A^λ, $\lambda \in \hat{A}$, and B^μ, $\mu \in \hat{B}$, be the irreducible representations of A and B, respectively. The irreducible representations of $A \oplus B$ are A^λ, $\lambda \in \hat{A}$, with $A \oplus B$ action given by*

$$(a, b)m = am, \qquad \text{for } a \in A,\ b \in B,\ m \in A^\lambda,$$

and B^μ, $\mu \in \hat{B}$, with $A \oplus B$ action given by

$$(a, b)n = bn, \qquad \text{for } a \in A,\ b \in B, \text{ and } n \in B^\mu.$$

Proof. The elements $(1, 0)$ and $(0, 1)$ in $A \oplus B$ are central idempotents of $A \oplus B$ such that $(1, 0)(0, 1) = (0, 0)$. If P is an $A \oplus B$-module then

$$P = (1, 0)P \oplus (0, 1)P,$$

and this is a decomposition as $A \oplus B$-modules. Since

$$(a,b)(1,0)p = (a,0)(1,0)p, \qquad \text{and} \qquad (a,b)(0,1)p = (0,b)(0,1)p,$$

for all $a \in A$, $b \in B$, and $p \in P$, the structure of $(1,0)P$ is determined completely by the A-action and the structure of $(0,1)P$ is determined by the action of B. If P is a simple module then $P = (1,0)P$ or $P = (0,1)P$. In the first case $P \cong A^\lambda$ for some $\lambda \in \hat{A}$ and in the second $P \cong B^\mu$ for some $\mu \in \hat{B}$. ∎

Similar arguments with the elements $(1,0)$ and $(0,1)$ in $A \oplus B$ yield the following.

(1) If A and B are algebras then the ideals of $A \oplus B$ are all of the form $I \oplus J$ where I is an ideal of A and J is an ideal of B.

(2) If A and B are algebras then $Z(A \oplus B) = Z(A) \oplus Z(B)$.

(3) If A and B are algebras and $\vec{t}$ is a trace on $A \oplus B$ then $\vec{t}$ is given by

$$\vec{t}(a,b) = \vec{t}_A(a) + \vec{t}_B(b),$$

where $\vec{t}_A$ is the trace on A given by $\vec{t}_A(a) = \vec{t}(a,0)$ and $\vec{t}_B$ is the trace on B given by $\vec{t}_B(b) = \vec{t}(0,b)$.

Tensor products

Let M and N be vector spaces and let

$$B_m = \{m_i\} \quad \text{and} \quad B_n = \{n_j\}$$

be bases of M and N, respectively. The tensor product $M \otimes N$ is the vector space with basis

$$B_{M\otimes N} = \{m_i \otimes n_j \mid m_i \in B_M, n_j \in B_n\}.$$

If $m = \sum_i c_i m_i$, and $n = \sum_j d_j n_j$, then write

$$m \otimes n = \left(\sum_i c_i m_i\right) \otimes \left(\sum_j d_j n_j\right) = \sum_{i,j} c_i d_j (m_i \otimes n_j).$$

If A and Z are algebras the *tensor product* is the vector space $A \otimes Z$ with multiplication determined by

$$(a_1 \otimes z_1)(a_2 \otimes z_2) = a_1 a_2 \otimes z_1 z_2, \qquad \text{for all } a_1, a_2 \in A,\ z_1, z_2 \in Z.$$

If M and N are vector spaces then

$$\mathrm{End}(M \otimes N) = \mathrm{End}(M) \otimes \mathrm{End}(N) \qquad \text{as algebras.}$$

This equality can be expressed in terms of matrices by choosing bases $\{m_1, \ldots, m_r\}$ and $\{n_1, \ldots, n_s\}$ of M and N, respectively. The $\mathrm{End}(M)$ is identified with $M_r(\mathbb{C})$ and $\mathrm{End}(N)$ is identified with $M_s(\mathbb{C})$ by

$$E_{ij} m_j = m_i \qquad \text{and} E_{k\ell} n_\ell = n_k, \qquad \text{for } 1 \le i,j \le r \text{ and } 1 \le k,\ell \le s.$$

Then

$$(E_{ij} \otimes E_{k\ell})(m_j \otimes n_\ell) = E_{ij}m_j \otimes E_{k\ell}n_\ell = m_i \otimes n_k.$$

Use the (ordered) basis

$$\{m_1 \otimes n_1, \ldots m_1 \otimes n_s, m_2 \otimes n_1, \ldots, m_2 \otimes n_s, \ldots, m_r \otimes n_1, \ldots, m_r \otimes n_s\}$$

of $M \otimes N$ to identify $\mathrm{End}(M \otimes N)$ with $M_{rs}(\mathbb{C})$. Then, if $a = (a_{ij}) \in M_r(\mathbb{C})$ and $b = (b_{k\ell}) \in M_s(\mathbb{C})$ then $a \otimes b$ is the $rs \times rs$ matrix

$$a \otimes b = \begin{pmatrix} a_{11}b & a_{12}b & \cdots & a_{1r}b \\ a_{21}b & a_{22}b & \cdots & a_{2r}b \\ \vdots & & \ddots & \vdots \\ a_{r1}b & a_{r2}b & \cdots & a_{rr}b \end{pmatrix}$$

Theorem 1.8. *Let A and B be algebras. Let A^λ, $\lambda \in \hat{A}$, be the simple A-modules and let B^μ, $\mu \in \hat{B}$, be the simple B-modules. The simple $A \otimes B$-modules are*

$$A^\lambda \otimes B^\mu, \quad \lambda \in \hat{A}, \mu \in \hat{B}, \qquad \textit{where} \qquad (a \otimes b)(m \otimes n) = am \otimes bn,$$

for $a \in A$, $b \in B$, $m \in A^\lambda$, $n \in B^\mu$.

Proof. There are two things to show:

(1) $A^\lambda \otimes B^\mu$ is a simple $A \otimes B$-module,

(2) If P is a simple $A \otimes B$-module then $P \cong A^\lambda \otimes B^\mu$ for some $\lambda \in \hat{A}$ and $\mu \in \hat{B}$.

(1) By Burnside's theorem $\mathrm{End}(A^\lambda) = A^\lambda(A)$ and $\mathrm{End}(B^\mu) = B^\mu(B)$ and therefore

$$\mathrm{End}(A^\lambda \otimes B^\mu) = \mathrm{End}(A^\lambda) \otimes \mathrm{End}(B^\mu) = A^\lambda(A) \otimes B^\mu(B) = (A^\lambda \otimes B^\mu)(A \otimes B).$$

So $A^\lambda \otimes B^\mu$ has no submodules. So $A^\lambda \otimes B^\mu$ is simple.

(2) Let P be a simple $(A \otimes B)$-module. Let A^λ be a simple A-submodule of P and let B^μ be a simple B-submodule of $\mathrm{Hom}_A(A^\lambda, P)$. We claim that $A^\lambda \otimes B^\mu \cong P$.

Consider the $(A \otimes B)$-module homomorphism

$$\begin{array}{rccccc} \Phi\colon & A^\lambda \otimes B^\mu & \hookrightarrow & A^\lambda \otimes \mathrm{Hom}_A(A^\lambda, P) & \longrightarrow & P \\ & & & m \otimes \phi & \longmapsto & \phi(m). \end{array}$$

This map is nonzero since the injection $\phi\colon A^\lambda \hookrightarrow P$ is a nonzero element of $\mathrm{Hom}_A(A^\lambda, P)$. Since $A^\lambda \otimes B^\mu)$ is simple $\ker\Phi = 0$ and since P is simple $\mathrm{im}\Phi = P$. So $A^\lambda \otimes B^\mu \cong P$. ∎

2. The algebra $M_d(\mathbb{C})$.

Let $A = M_d(\mathbb{C})$ be the algebra of $d \times d$ matrices with entries from $\mathbb{C}$. Set

$$E_{ij} = \text{ the matrix with 1 in the } (i,j) \text{ entry and all other entries 0.}$$

Then $\{E_{ij} \mid 1 \le i, j \le d\}$ is a basis of A and

$$E_{ij}E_{kl} = \delta_{jk}E_{il}, \qquad 1 \le i, j, k, l \le d,$$

describes the multiplication in A.

Theorem 2.1. *Let $M_d(\mathbb{C})$ be the algebra of $d \times d$ matrices with entries from $\mathbb{C}$.*

(a) Up to isomorphism, there is only one irreducible representation M of $M_d(\mathbb{C})$.

(b) $\dim(M) = d$.

(c) The character $\chi^M \colon A \to \mathbb{C}$ of M is given by

$$\chi^M(a) = \mathrm{Tr}(a), \qquad \text{for all } a \in A,$$

where $\mathrm{Tr}(a)$ is the trace of the matrix a.

(d) The irreducible representation M is the vector space

$$M = \{(c_1, \ldots, c_d)^t \mid c_i \in \mathbb{C}\}$$

of column vectors of length d with A-action given by left multiplication, or, equivalently, M is given by the map

$$\begin{array}{rccc} M\colon & A & \longrightarrow & M_d(\mathbb{C}) \\ & a & \longmapsto & a, \end{array}$$

Proof. There are two things to show:

(1) M, as defined in (d), is a simple A-module, and

(2) If C is a simple A-module then $C \cong M$.

(1) Let ϵ_i be the column vector which has 1 in the ith entry and 0 in all other entries. The set $\{\epsilon_1, \ldots, \epsilon_d\}$ is a basis of M. Let $N \subseteq M$ be a nonzero submodule of M and let $n = \sum_{i=1}^d n_i \epsilon_i$ be a nonzero vector in N. Then $n_j \neq 0$ for some j and so

$$\epsilon_k = \frac{1}{n_j} E_{kj} n \in N, \qquad \text{for all } 1 \leq k \leq d.$$

Thus $N = M$, since N contains a basis of M.

(2) Let C be a simple A-module and let c be a nonzero vector in C. Since $c = \mathrm{Id} \cdot c = \sum_{i=1}^d E_{ii} c \neq 0$, $E_{jj} c \neq 0$ for some j. Define an A-module homomorphism by

$$\begin{array}{rccc} \phi\colon & M & \longrightarrow & C \\ & \epsilon_k & \longmapsto & E_{kj} c. \end{array}$$

Since $\phi(\epsilon_j) \neq 0$, $\ker \phi \neq 0$. Since M is simple, $\ker \phi = M$ and so ϕ is injective. Since $\mathrm{im}\phi \neq 0$ and C is simple, $\mathrm{im}\phi = C$ and so ϕ is surjective. So ϕ is an isomorphism and $C \cong M$. ∎

Proposition 2.2. *Let $M_d(\mathbb{C})$ be the algebra of $d \times d$ matrices with entries from $\mathbb{C}$.*

(1) The only ideals of $M_d(\mathbb{C})$ are 0 and $M_d(\mathbb{C})$.

(2) $Z(M_d(\mathbb{C})) = \mathbb{C} \cdot \mathrm{Id}$ and Id is the only central idempotent in $M_d(\mathbb{C})$.

(3) Up to constant multiples, the trace $\mathrm{Tr} \colon M_d(\mathbb{C}) \to \mathbb{C}$ given by

$$\mathrm{Tr}(a) = \sum_{i=1}^d a_{ii}, \qquad \text{for all } a = (a_{ij}) \in M_d(\mathbb{C}),$$

is the unique trace on $M_d(\mathbb{C})$.

Proof. Let E_{ij} denote the matrix in $M_d(\mathbb{C})$ which has a 1 in the (i,j) entry and 0 everywhere else.
(1) Let I be a nonzero ideal of $M_d(\mathbb{C})$ and let $r = (r_{ij}) \in I$, $r \neq 0$. Let r_{ij} be a nonzero entry of r. Then

$$\frac{1}{r_{ij}} E_{ki} r E_{jl} = E_{kl} \in I, \qquad \text{for all } 1 \le k, l \le d.$$

So I contains a basis of $M_d(\mathbb{C})$. So $I = M_d(\mathbb{C})$.
(2) Clearly $\mathbb{C}\mathrm{Id} \subseteq Z(M_d(\mathbb{C})$. Let $z = (z_{ij}) \in Z(M_d(\mathbb{C}))$. If $i \neq j$ then

$$z_{ij} E_{ij} = E_{ii} z E_{jj} = z E_{ii} E_{jj} = 0.$$

So $z_{ij} = 0$ if $i \neq j$. Further

$$z_{ii} E_{ii} = E_{ii} z E_{ii} = E_{i1} z E_{1i} E_{ii} = z_{11} E_{ii},$$

so $z_{ii} = z_{11}$ for all $1 \le i \le d$. So $z = z_{11}\mathrm{Id}$. So $Z(M_d(\mathbb{C})) \subseteq \mathbb{C}\mathrm{Id}$. So $Z(M_d(\mathbb{C})) = \mathbb{C}\mathrm{Id}$.
(3) Let $\chi \colon M_d(\mathbb{C}) \to \mathbb{C}$ be a trace on $M_d(\mathbb{C})$. If $a = (a_{ij}) \in M_d(\mathbb{C})$ then

$$\chi(E_{ii} a E_{jj}) = a_{ij}\chi(E_{ij}) = a_{ij}\chi(E_{i1}E_{1j}) = a_{ij}\chi(E_{1j}E_{i1}) = a_{ij}\delta_{ij}\chi(E_{11}).$$

Thus

$$\chi(a) = \chi\left(\left(\sum_{i=1}^{d} E_{ii}\right) a \left(\sum_{j=1}^{d} E_{jj}\right)\right) = \sum_{i,j=1}^{d} a_{ij}\delta_{ij}\chi(E_{11}) = \chi(E_{11})\mathrm{Tr}(a).$$

So χ is a multiple of the trace Tr. ∎

3. The algebra $\bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C})$.

Let $\hat{A}$ be a finite set and let d_λ be positive integers indexed by the elements of $\hat{A}$. Let

$$A = \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C}),$$

be the algebra of block diagonal matrices with blocks $M_{d_\lambda}(\mathbb{C})$. Let E_{ij}^{λ} be the matrix which has a 1 in the (i,j) entry of the λth block and 0 everywhere else. Then $\{E_{ij}^{\lambda} \mid \lambda \in \hat{A}, 1 \le i, j, \le d_\lambda\}$ is a basis of A and the relations

$$E_{ij}^{\lambda} E_{kl}^{\mu} = \delta_{\lambda\mu}\delta_{ij} E_{il}^{\lambda}$$

determine the multiplication in A.

The following theorems are consequences of Theorems ?? and Proposition ???.

Theorem 3.1. *Let $\hat{A}$ be a finite set and let d_λ be positive integers indexed by the elements of $\hat{A}$. Let*

$$A = \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C}),$$

be the algebra of block diagonal matrices with blocks $M_{d_\lambda}(\mathbb{C})$.

(1) *The irreducible representations* A^λ *of* A *are indexed by the elements of* $\hat{A}$.
(2) $\dim(A^\lambda) = d_\lambda$.
(3) *The character* $\chi^\lambda\colon A \to \mathbb{C}$ *of* A^λ *is given by*

$$\chi^\lambda(a) = \mathrm{Tr}(A^\lambda(a)), \qquad a \in A,$$

where $A^\lambda(a)$ *is the* λ*th block of the matrix* a.
(4) *The irreducible representation* A^λ *is given by the map*

$$\begin{array}{rccc} A^\lambda\colon & A & \longrightarrow & M_{d_\lambda}(\mathbb{C}) \\ & a & \longmapsto & A^\lambda(a), \end{array}$$

where $A^\lambda(a)$ *is the* λ*th block of the matrix* A*, or, equivalently, by the vector space* A^λ *of column vectors of length* d_λ *and* A*-action given by*

$$am = A^\lambda(a)m, \qquad \text{for } a \in A \text{ and } m \in A^\lambda.$$

Theorem 3.2. *Let* $\hat{A}$ *be a finite set and let* d_λ *be positive integers indexed by the elements of* $\hat{A}$*. Let*

$$A = \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C}),$$

be the algebra of block diagonal matrices with blocks $M_{d_\lambda}(\mathbb{C})$*. If* $a \in A$ *let* $A^\lambda(a)$ *denote the* λ*th block of the matrix* a*. Let* E^λ_{ij} *be the matrix which has a* 1 *in the* (i,j) *entry of the* λ*th block and* 0 *everywhere else.*

(1) *The minimal ideals of* A *are given by*

$$I^\lambda = \{a \in A \mid A^\mu(a) = 0 \text{ for all } \mu \neq \lambda\}, \qquad \lambda \in \hat{A},$$

and every ideal of A *is of the form* $I = \bigoplus_{\lambda \in S} I^\lambda$*, for some subset* $S \subseteq \hat{A}$.
(2) *The minimal central idempotents of* A *are*

$$z_\lambda = \sum_{i=1}^{d_\lambda} E^\lambda_{ii}, \qquad \lambda \in \hat{A},$$

and $\{z_\lambda \mid \lambda \in \hat{A}\}$ *is a basis of the center* $Z(A)$ *of* A.
(3) *The irreducible characters* χ^λ*,* $\lambda \in \hat{A}$*, of* A *are given by*

$$\chi^\lambda(a) = \mathrm{Tr}(A^\lambda(a)), \qquad a \in A,$$

and every trace $\vec{t}\colon A \to \mathbb{C}$ *on* A *can be written uniquely in the form*

$$\vec{t} = \sum_{\lambda \in \hat{A}} t_\lambda \chi^\lambda, \qquad t_\lambda \in \mathbb{C}.$$

Let A be an algebra which is isomorphic to a direct sum of matrix algebras and fix an isomorphism

$$\phi\colon A \cong \bigoplus_{\lambda\in\hat{A}} M_{d_\lambda}(\mathbb{C}). \tag{3.3}$$

The elements

$$e_{ij}^{\lambda} = \phi^{-1}(E_{ij})^{\lambda}, \qquad \lambda\in\hat{A},\quad 1\le i,j\le d_\lambda,$$

are *matrix units* in A, i.e. $\{e_{ij}^{\lambda} \mid \lambda\in\hat{A}, 1\le i,j\le d_\lambda\}$ is a basis of A and

$$e_{ij}^{\lambda}e_{kl}^{\mu} = \delta_{\lambda\mu}\delta_{ij}e_{il}^{\lambda},$$

for all $\lambda,\mu\in\hat{A}$, $1\le i,j\le d_\lambda$, $1\le k,l\le d_\mu$. If $a\in A$, let $A^\lambda(a)_{ij}\in\mathbb{C}$ be defined by the expansion

$$a = \sum_{\lambda\in\hat{A}}\sum_{i,j=1}^{d_\lambda} A^\lambda(a)_{ij}e_{ij}^{\lambda}.$$

It follows from Theorem ??? that the maps

$$\begin{array}{rccc} A^\lambda\colon & A & \longrightarrow & M_{d_\lambda}(\mathbb{C}) \\ & a & \longmapsto & A^\lambda(a) = \big(A^\lambda(a)_{ij}\big) \end{array} \qquad \text{and} \qquad \begin{array}{rccc} \chi^\lambda\colon & A & \longrightarrow & \mathbb{C} \\ & a & \longmapsto & \mathrm{Tr}(A^\lambda(a)), \end{array} \qquad \lambda\in\hat{A},$$

are the irreducible representations and the irreducible characters of A, respectively. The homomorphisms A^λ depend on the choice of ϕ but the irreducible characters χ^λ do not. The *weights* of a trace $\vec{t}$ on A are the constants t_λ, $\lambda\in\hat{A}$, defined by the expansion in ???. The trace $\vec{t}$ is nondegenerate if and only if the t_λ are all nonzero.

Theorem 3.4. *Let A be an algebra which is isomorphic to a direct sum of matrix algebras, indexed by $\lambda\in\hat{A}$. Let $\vec{t}$ be a nondegenerate trace on A and let $\langle,\rangle$ be the corresponding bilinear form. Let $B=\{b\}$ be a basis of A and let $B^*=\{b^*\}$ be the dual basis to B with respect to $\langle,\rangle$. Let χ^λ, $\lambda\in\hat{A}$, be the irreducible characters of A, t_λ be the weights of $\vec{t}$, d_λ the dimensions of the irreducible representations, $\{e_{ij}^{\lambda}\}$ a set of matrix units of A, and A^λ the corresponding irreducible representations of A.*

(a) (Fourier inversion formula)

$$e_{ij}^{\lambda} = \sum_{b\in B} t_\lambda A_{ji}^{\lambda}(b^*)b.$$

(b) The minimal central idempotent z_λ in A indexed by $\lambda\in\hat{A}$ is given by

$$z_\lambda = \sum_{b\in B} t_\lambda\chi^\lambda(b^*)b.$$

(c) (Orthogonality of characters) For all $\lambda,\mu\in\hat{A}$,

$$\sum_{b\in B}\chi^\lambda(b^*)\chi^\mu(b) = \delta_{\lambda\mu}\frac{d_\lambda}{t_\lambda}.$$

Proof. (a) Since $\vec{t}$ is nondegenerate, the equation $\vec{t}(e_{ij}^\lambda) = \sum_{\mu\in\hat{A}} t_\mu \chi^\mu(e_{ij}^\lambda) = t_\lambda \delta_{ij}$ implies that

$$\left\{\frac{e_{ji}^\lambda}{t_\lambda}\right\} \quad \text{is the dual basis to} \quad \{e_{ij}^\lambda\} \quad \text{with respect to } \langle,\rangle.$$

Thus, by (???), $A_{ij}^\lambda(a) = \frac{1}{t_\lambda}\langle a, e_{ji}^\lambda\rangle$, and so $e_{ij}^\lambda = \sum_{b\in B}\langle e_{ij}^\lambda, b^*\rangle b = \sum_{b\in B} t_\lambda A_{ji}^\lambda(b^*)b.$

(b) By part (a), $z_\lambda = \sum_{i=1}^{d_\lambda} e_{ii}^\lambda = \sum_{b\in B} t_\lambda \mathrm{Tr}(A^\lambda(b^*))b.$

(c) By part (b), $d_\lambda \delta_{\lambda\mu} = \chi^\mu(z_\lambda) = \sum_{b\in B} t_\lambda \chi^\lambda(b^*)\chi^\mu(b).$ ∎

Example 1. Let $A = \bigoplus_{\lambda\in\hat{A}} M_{d_\lambda}(\mathbb{C})$.

(1) As a left A-module under the action of A by left multiplication

$$A \cong \bigoplus_{\lambda\in\hat{A}} (A^\lambda)^{\oplus d_\lambda},$$

where A^λ is the irreducible A-module of column vectors of length d_λ.

(2) As an (A, A) bimodule under the action of A by left and right multiplication

$$A \cong \bigoplus_{\lambda\in\hat{A}} A^\lambda \otimes \overleftarrow{A}^\lambda,$$

where A^λ is the left A-module of column vectors of length d_λ and $\overleftarrow{A}^\lambda$ is th right A-module of row vectors of length d_λ.

(3) Let $a, b \in A$. If a acts on A by left multiplication and b acts on A by right multiplication then

$$\mathrm{Tr}(a\otimes b) = \sum_{\lambda\in\hat{A}} \chi^\lambda(a)\chi^\lambda(b),$$

where χ^λ, $\lambda \in \hat{A}$, are the irreducible characters of A.

Example 2. Let G be a finite group and let $\mathbb{C}G$ be the group algebra of G. The trace of the regular representation of $\mathbb{C}G$ is given by

$$\mathrm{tr}(g) = \sum_{h\in G} gh\big|_h = \begin{cases} |G|, & \text{if } g = 1, \\ 0, & \text{otherwise.}\end{cases}$$

So, (provided $|G| \neq 0$ in $\mathbb{C}$) the basis

$$\left\{\frac{g^{-1}}{|G|}\right\}_{g\in G} \qquad \text{is the dual basis to} \quad G$$

with respect to the form $\langle,\rangle$ defined by tr. Since tr is nondegenerate

$$\mathbb{C}G \cong \bigoplus_{\lambda\in\hat G} M_{d_\lambda}(\mathbb{C}),$$

for some set $\hat G$ and positive integers d_λ. Then

$$\mathrm{tr} = \sum_{\lambda\in\hat G} d_\lambda \chi^\lambda,$$

where χ^λ, $\lambda\in\hat G$, are the irreducible characters of G and, by (???),

$$z_\lambda = \frac{1}{|G|}\sum_{g\in G} d_\lambda \chi^\lambda(g^{-1})g, \qquad \lambda\in\hat G,$$

are the minimal central idempotents in $\mathbb{C}G$. The *orthogonality relation for characters* of G (???) is

$$\frac{1}{|G|}\sum_{g\in G}\chi^\lambda(g^{-1})\chi^\mu(g) = \delta_{\lambda\mu}, \qquad \text{for } \lambda,\mu\in\hat G.$$

If $G^\lambda\colon \mathbb{C}G\to M_{d_\lambda}(\mathbb{C})$ are the irreducible representations of G then

$$e^\lambda_{ij} = \frac{1}{|G|}\sum_{g\in G} d_\lambda G^\lambda(g^{-1})_{ji}g, \qquad \lambda\in\hat G, 1\le i,j\le d_\lambda,$$

are a set of matrix units in $\mathbb{C}G$, i.e.

$$e^\lambda_{ij}e^\mu_{k\ell} = \delta_{\lambda\mu}\delta_{kj}e^\lambda_{\subset\ell}$$

and $\{e^\lambda_{ij} \mid \lambda\in\hat G, 1\le i,j\le d_\lambda\}$ is a basis of $\mathbb{C}G$.

Let $g,h\in G$ and let g act on $\mathbb{C}G$ by left multiplication and let h act on $\mathbb{C}G$ by right multiplication. Then

$$\mathrm{Tr}(g\otimes h) = \sum_{k\in G} gkh\big|_k = \sum_{k\in G} khk^{-1}\big|_{g^{-1}} = \begin{cases}\mathrm{Card}(\mathcal{C}_h), & \text{if } h \text{ is conjugate to } g^{-1},\\ 0, & \text{otherwise,}\end{cases}$$

where $\mathcal{C}_h$ is the conjugacy class of h. Thus, by (???),

$$\sum_{\lambda\in\hat G}\chi^\lambda(g)\chi^\lambda(h) = \begin{cases}\mathrm{Card}(\mathcal{C}_h), & \text{if } h \text{ is conjugate to } g^{-1},\\ 0, & \text{otherwise,}\end{cases}$$

which is the *second orthogonality relation for characters* of G.

The elements

$$c_g = \sum_{x\in\mathcal{C}_g} x$$

are a basis of the center of $\mathbb{C}G$. Since $\{z_\lambda \mid \lambda\in\hat G\}$ is also a basis of $Z(\mathbb{C}G)$ we have that

$$\mathrm{Card}(\hat G) = \#\text{ of conjugacy classes of } G,$$

though there is no (known) natural bijection between the irreducible representations of G and the conjugacy classes of G.

It follows from ??? that

$$|G| = \sum_{\lambda\in\hat{G}} d_\lambda^2.$$

Every trace $\vec{t}$ on $\mathbb{C}G$ has a unique decomposition

$$\vec{t} = \sum_{\lambda\in\hat{G}} t_\lambda \chi^\lambda, \qquad t_\lambda \in \mathbb{C}.$$

So, since every G-module is semisimple, its decomposition is determined by its character. So

Two G-modules are isomorphic if and only if they have the same character.

and

$$\begin{aligned}\dim(Z(\mathbb{C}G)) &= (\#\text{ of irreducible representations of } G)\\ &= (\#\text{ of conjugacy classes of } G).\end{aligned}$$

4. Centralizers.

Let A be an algebra and let M be an A-module. The *centralizer* or *commutant* of M is the algebra

$$\mathrm{End}_A(M) = \{T \in \mathrm{End}(M) \mid Ta = aT \text{ for all } a \in A\}.$$

If M and N are A-modules then $\mathrm{Hom}_A(M,N)$ is a left $\mathrm{End}_A(M)$-module and a right $\mathrm{End}_A(N)$-module.

Theorem 4.1. *(Schur's Lemma) Let A be an algebra.*

(1) Let A^λ be a simple A-module. Then $\mathrm{End}_A(A^\lambda) = \mathbb{C}\cdot \mathrm{Id}_{A^\lambda}$.

(2) If A^λ and A^μ are nonisomorphic simple A-modules then $\mathrm{Hom}_A(A^\lambda, A^\mu) = 0$.

Proof. Let $T\colon A^\lambda \to A^\mu$ be a nonzero A-module homomorphism. Since A^λ is simple, $\ker T = 0$ and so T is injective. Since A^μ is simple, $\mathrm{im} T = A^\mu$ and so T is surjective. So T is an isomorphism. Thus we may assume that $T\colon A^\lambda \to A^\lambda$.

When A^λ is finite dimensional: Since $\mathbb{C}$ is algebraically closed T has an eigenvector and a corresponding eigenvalue $\alpha \in \mathbb{C}$. Then $T - \alpha\cdot Id \in \mathrm{Hom}_A(A^\lambda, A^\lambda)$ and so $T - \alpha\cdot Id$ is either 0 an isomorphism. However, since $\det(T-\alpha\cdot Id) = 0$, $T-\alpha\cdot Id$ is not invertible. So $T - \alpha\cdot Id = 0$. So $T = \alpha\cdot Id$. So $\mathrm{End}_A(A^\lambda) = \mathbb{C}\cdot Id$.

When A^λ is countable dimensional: We shall show that there exists a $\lambda\in\mathbb{C}$ such that $T - \lambda\cdot\mathrm{Id}$ is not invertible. Suppose $T-\lambda\cdot\mathrm{Id}$ is invertible for all $\lambda\in\mathbb{C}$. Then $p(T)$ is invertible for all polynomials $p(t)\in\mathbb{C}[t]$. So $p(T)/q(T)$ is well defined for all $p(t), q(t) \in \mathbb{C}[t]$.

Let $v \in A^\lambda$ be nonzero. Then the map

$$\begin{array}{ccccc} \mathbb{C}(t) & \longrightarrow & \mathrm{End}(V) & \longrightarrow & V \\ \frac{p(t)}{q(t)} & \longmapsto & \dfrac{p(T)}{q(T)} & \longmapsto & \dfrac{p(T)}{q(T)}v \end{array}$$

is injective. Since $\dim \mathbb{C}(t)$ is uncountable and $\dim V$ is countable thsi is a contradiction. So $T - \lambda \cdot \mathrm{Id}$ is invertibel for some $\lambda \in \mathbb{C}$. Then the same proof as in the finite dimensional case shows that $T = \lambda \cdot \mathrm{Id}$.

If A^λ is unitary: Let

$$A = \frac{T + T^*}{2} \qquad \text{and} \qquad B = \frac{T - T^*}{2i}$$

where T^* is defined by $\langle Tv_1, v_2 \rangle = \langle v_1, T^* v_2 \rangle$ for all $v_1, v_2 \in A^\lambda$. Then

$$A = A^*, \qquad B = B^*, \qquad T = A + iB, \qquad \text{and} \qquad A, B, T \in \mathrm{Hom}_A(A^\lambda, A^\lambda).$$

Then the spectral theorem for self adjoint operators says that A and B can be diagonalized [Rudin, Thm. 12.2],

$$A = \sum_i \lambda_i P_i \qquad \text{and} B = \sum_j \mu_j Q_j, \qquad \text{with } P_i^2 = P_i,\ Q_j^2 = Q_j,\ P_i, Q_j \in \mathrm{Hom}_A(A^\lambda, A^\lambda),\ \lambda_i, \mu_j \in \mathbb{C}.$$

Then $P_i A^\lambda$ is a submodule of A^λ. So $P_i A^\lambda = A^\lambda$. So $A = \lambda \cdot \mathrm{Id}$. ∎

Lemma 4.2. *Suppose that V is a unitary representation. Then*

$$\mathrm{Hom}_A(V, V) = \mathbb{C} \cdot \mathrm{Id}_V \qquad \textit{implies that} \qquad V \textit{ is irreducible.}$$

Proof. Suppose that V is not irreducible. Then let $W \subseteq V$ be a sumodule of V. Let

$$W^\perp = \{ v \in V \mid \langle v, w \rangle = 0, \text{ for all } w \in W \}.$$

Then $W^\perp$ is a submodule since, if $v \in W^\perp$ and $w \in W$, then $\langle av, w \rangle = \langle v, a^* w \rangle = 0$ because $a^* w \in W$. Now, for Hilbert spaces, we have $V = W \oplus W^\perp$ and we can define a

$$\begin{array}{ccccc} V & \xrightarrow{p} & V & & \\ w & \longmapsto & w, & & \text{if } w \in W, \\ w^\perp & \longmapsto & 0, & & \text{if } w \in W^\perp, \end{array}$$

This map is a nonidentity A-module homomorphism. So $\mathrm{Hom}_A(V, V) \neq \mathbb{C} \cdot \mathrm{Id}$. ∎

Theorem 4.3. *Let A be an algebra. Let M be a semisimple A-module and set $Z = \mathrm{End}_A(M)$. Suppose that*

$$M \cong \bigoplus_{\lambda \in \hat{M}} (A^\lambda)^{\oplus m_\lambda},$$

where $\hat{M}$ is an index set for the irreducible A-modules A^λ which appear in M and the m_λ are positive integers.

(a) $Z \cong \bigoplus_{\lambda \in \hat{M}} M_{m_\lambda}(\mathbb{C})$.

(b) As an $(A \otimes Z)$-module

$$M \cong \bigoplus_{\lambda \in \hat{M}} A^\lambda \otimes Z^\lambda,$$

where the Z^λ, $\lambda \in \hat{M}$, are the simple Z-modules.

Proof. Index the components in the decomposition of M by dummy variables ϵ_i^λ so that we may write

$$M \cong \bigoplus_{\lambda\in\hat{M}} \bigoplus_{i=1}^{m_\lambda} A^\lambda \otimes \epsilon_i^\lambda.$$

For each $\lambda \in \hat{M}$, $1 \le i, j \le m_\lambda$ let $\phi_{ij}^\lambda \colon A^\lambda \otimes \epsilon_j \to A^\lambda \otimes \epsilon_i$ be the A-module isomorphism given by

$$\phi_{ij}^\lambda(m \otimes \epsilon_j^\lambda) = m \otimes \epsilon_i^\lambda, \qquad \text{for } m \in A^\lambda.$$

By Schur's Lemma,

$$\begin{aligned}
\mathrm{End}_A(M) = \mathrm{Hom}_A(M, M) &\cong \mathrm{Hom}_A\left(\bigoplus_\lambda \bigoplus_j A^\lambda \otimes \epsilon_j^\lambda, \bigoplus_\mu \bigoplus_i A^\mu \otimes \epsilon_i^\mu\right) \\
&\cong \bigoplus_{\lambda,\mu} \bigoplus_{i,j} \delta_{\lambda\mu} \mathrm{Hom}_A(A^\lambda \otimes \epsilon_j^\lambda, A^\mu \otimes \epsilon_i^\mu) \\
&\cong \bigoplus_\lambda \bigoplus_{i,j=1}^{m_\lambda} \mathbb{C}\phi_{ij}^\lambda.
\end{aligned}$$

Thus each element $z \in \mathrm{End}_A(M)$ can be written as

$$z = \sum_{\lambda\in\hat{M}} \sum_{i,j=1}^{m_\lambda} z_{ij}^\lambda \phi_{ij}^\lambda, \qquad \text{for some } z_{ij}^\lambda \in \mathbb{C},$$

and identified with an element of $\oplus_\lambda M_{m_\lambda}(\mathbb{C})$. Since $\phi_{ij}^\lambda \phi_{kl}^\mu = \delta_{\lambda\mu}\delta_{jk}\phi_{il}^\lambda$ it follows that

$$\mathrm{End}_A(M) \cong \bigoplus_{\lambda\in\hat{M}} M_{m_\lambda}(\mathbb{C}).$$

(b) As a vector space $Z^\mu = \mathrm{span}\{\epsilon_i^\mu \mid 1 \le i \le m_\mu\}$ is isomorphic to the simple $\oplus_\lambda M_{m_\lambda}(\mathbb{C})$ module of column vectors of length m_μ. The decomposition of M as $A \otimes Z$ modules follows since

$$(a \otimes \phi_{ij}^\lambda)(m \otimes \epsilon_k^\mu) = \delta_{\lambda\mu}\delta_{jk}(a \otimes \epsilon_i^\mu), \qquad \text{for all } m \in A^\mu,\ a \in A, \qquad \blacksquare$$

If A is an algebra then A^{op} is the algebra A except with the opposite multiplication, i.e.

$$A^{\mathrm{op}} = \{a^{\mathrm{op}} \mid a \in A\} \qquad \text{with} \qquad a_1^{\mathrm{op}} a_2^{\mathrm{op}} = (a_2 a_1)^{\mathrm{op}}, \quad \text{for all } a_1, a_2 \in A.$$

Let left *regular representation* of A is the vector space A with A action given by left multiplication. Here A is serving both as an algebra and as an A-module. It is often useful to distinguish the two roles of A and use the notation $\vec{A}$ for the A-module, i.e. $\vec{A}$ is the vector space

$$\vec{A} = \{\vec{b} \mid b \in A\} \qquad \text{with } A\text{-action} \qquad a\vec{b} = \overrightarrow{ab}, \qquad \text{for all } a \in A,\ \vec{b} \in \vec{A}.$$

Proposition 4.4. *Let A be an algebra and let $\vec{A}$ be the regular representation of A. Then* $\mathrm{End}_A(\vec{A}) \cong A^{\mathrm{op}}$. *More precisely,*

$$\mathrm{End}_A(\vec{A}) = \{\phi_b \mid b \in A\}, \qquad \text{where } \phi_b \text{ is given by} \qquad \phi_b(\vec{a}) = \vec{ab}, \quad \text{for all } \vec{a} \in \vec{A}.$$

Proof. Let $\phi \in \mathrm{End}_A(\vec{A})$ and let $b \in A$ be such that $\phi(\vec{1}) = \vec{b}$. For all $\vec{a} \in \vec{A}$,

$$\phi(\vec{a}) = \phi(a \cdot \vec{1}) = a\phi(\vec{1}) = a\vec{b} = \vec{ab},$$

and so $\phi = \phi_b$. Then $\mathrm{End}_A(\vec{A}) \cong A^{\mathrm{op}}$ since

$$(\phi_{b_1} \circ \phi_{b_2})(\vec{a}) = \overrightarrow{ab_2b_1} = \phi_{b_2b_1}(\vec{a}),$$

for all $b_1, b_2 \in A$ and $\vec{a} \in \vec{A}$. ∎

5. Characterizing algebras isomorphic to $\bigoplus_\lambda M_{d_\lambda}(\mathbb{C})$

Theorem 5.1. *Suppose that A is an algebra such that the regular representation $\vec{A}$ of A is completely decomposable. Then A is isomorphic to a direct sum of matrix algebras, i.e.*

$$A \cong \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C}),$$

for some set $\hat{A}$ and some positive integers d_λ, indexed by the elements of $\hat{A}$.

Proof. If $\vec{A}$ is completely decomposable then, by Theorem ???, $\mathrm{End}_A(\vec{A})$ is isomorphic to a direct sum of matrix algebras. By Proposition ??,

$$A^{\mathrm{op}} \cong \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C}),$$

for some set $\hat{A}$ and some positive integers d_λ, indexed by the elements of $\hat{A}$. The map

$$\begin{array}{ccc} \left(\bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C})\right)^{\mathrm{op}} & \longrightarrow & \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C}) \\ a & \longmapsto & a^t, \end{array}$$

where a^t is the transpose of the matrix a, is an algebra isomorphism. So A is isomorphic to a direct sum of matrix algebras. ∎

Proposition 5.2. *Let $A = \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C})$. Then the trace* tr *of the regular representation of A is nondegenerate.*

Proof. As A-modules, the regular representation

$$\vec{A} \cong \bigoplus_{\lambda \in \hat{A}} (A^\lambda)^{\oplus d_\lambda},$$

where A^λ is the irreducible A-module consisting of column vectors of length d_λ. So the trace tr of the regular representation is given by

$$tr = \sum_{\lambda\in\hat{A}} d_\lambda \chi^\lambda,$$

where χ^λ are the irreducible characters of A. Since the d_λ are all nonzero the trace tr is nondegenerate. ∎

Theorem 5.3. *(Maschke's theorem) Let A be an algebra such that the trace tr of the regular representation of A is nondegenerate. Then every representation of A is completely decomposable.*

Proof. Let B be a basis of A and let B^* be the dual basis of A with respect to the form $\langle,\rangle\colon A\times A \to \mathbb{C}$ defined by

$$\langle a_1, a_2\rangle = tr(a_1a_2), \qquad \text{for all } a_1, a_2 \in A.$$

The dual basis B^* exists because the trace tr is nondegenerate.

Let M be an A-module. If M is irreducible then the result is vacuously true, so we may assume that M has a proper submodule N. Let $p \in \mathrm{End}(M)$ be a projection onto N, i.e. $pM = N$ and $p^2 = p$. Let

$$[p] = \sum_{b\in B} bpb^*, \qquad \text{and} \qquad e = \sum_{b\in B} bb^*.$$

For all $a \in A$,

$$\mathrm{tr}(ea) = \sum_{b\in B} \mathrm{tr}(bb^*a) = \sum_{b\in B} \langle ab, b^*\rangle = \sum_{b\in B} ab\big|_b = \mathrm{tr}(a),$$

So $\mathrm{tr}((e-1)a) = 0$, for all $a \in A$. Thus, since tr is nondegenerate, $e = 1$.

Let $m \in M$. Then $pb^*m \in N$ for all $b \in B$, and so $[p]m \in N$. So $[p]M \subseteq N$. Let $n \in N$. Then $pb^*n = b^*n$ for all $b \in B$, and so $[p]n = en = 1\cdot n = n$. So $[p]M = N$ and $[p]^2 = [p]$, as elements of $\mathrm{End}(M)$.

Note that $[1-p] = [1] - [p] = e - [p] = 1 - [p]$. So

$$M = [p]M \oplus (1-[p])M = N \oplus [1-p]M,$$

and, by Proposition ??, $[1-p]M$ is an A-module. So $[1-p]M$ is an A-submodule of M which is complementary to M. By induction on the dimension of M, N and $[1-p]M$ are completely decomposable, and therefore M is completely decomposable. ∎

Together, Theorems ???, ??? and Proposition ??? yield the following theorem.

Theorem 5.4. *(Artin-Wedderburn) Let A be a finite dimensional algebra over $\mathbb{C}$. The following are equivalent:*

(1) Every representation of A is completely decomposable.
(2) The trace of the regular representation of A is nondegenerate.
(3) The regular representation of A is completely decomposable.

Example 1. Let A be the algebra with basis $\{1, e\}$ and mulitplication given by $e^2 = 0$. Then

$$\vec{t}\colon A \to \mathbb{C} \qquad \text{given by} \qquad \vec{t}(a+be) = a+b$$

is a nondegenerate trace on A. The regular representation of A is given by

$$\vec{A}(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \vec{A}(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and $\mathbb{C}e$ is the only submodule of $\vec{A}$. Thus, $\vec{A}$ is not completely decomposable. The trace tr of the regular representation of A is given by

$$\mathrm{tr}(a + be) = 2a, \qquad \text{for } a, b \in \mathbb{C}.$$

Theorem 5.5. *(Burnside's Theorem) Let A be an algebra and let $M : A \to \mathrm{End}(M)$ be an irreducible representation of A. Then $M(A) = \mathrm{End}(M)$.*

Proof. Clearly, $M(A) \subseteq \mathrm{End}(M)$ and M is both a simple $M(A)$-module and a simple $\mathrm{End}(M)$-module. As $\mathrm{End}(M)$-modules

$$\overrightarrow{\mathrm{End}(M)} \cong M^{\oplus d},$$

and so, by restriction, this is also true as an $M(A)$-module. Thus, by Schur's lemma,

$$\mathrm{End}_{M(A)}(\overrightarrow{\mathrm{End}(M)}) = M_d(\mathbb{C}).$$

Let us label the summands in the decomposition by dummy variables ϵ_i,

$$\overrightarrow{\mathrm{End}(M)} = \bigoplus_{i=1}^{d} M \otimes \epsilon_i, \qquad \text{so that} \qquad E_{ii}(\overrightarrow{\mathrm{End}(M)}) = M \otimes \epsilon_i.$$

Now $\overrightarrow{M(A)} \subseteq \overrightarrow{\mathrm{End}(M)}$ is an $M(A)$ submodule of $\overrightarrow{\mathrm{End}(M)}$. However,

$$E_{ii}(\overrightarrow{\mathrm{End}(M)}) \subseteq M \otimes \epsilon_i \qquad \text{and} \qquad \overrightarrow{M(A)} = E_{11}\overrightarrow{M(A)} \oplus \cdots \oplus E_{dd}\overrightarrow{M(A)} \subseteq M \otimes \epsilon_1 \oplus \cdots \oplus M \otimes \epsilon_d.$$

Since M is a simple $M(A)$ module, each $E_{ii}\overrightarrow{M(A)}$ is isomorphic to M or 0. So

$$\overrightarrow{M(A)} \cong M^{\oplus k}, \qquad \text{for some } 1 \le k \le d.$$

So the regular representation of $M(A)$ is semisimple and $M(A) \cong M_k(\mathbb{C})$. Since $\dim(M) = d$ and M is a simple module for $M(A)$ we have $M(A) \cong M_d(\mathbb{C})$. So $M(A) = \mathrm{End}(M)$. ∎

Remark 1. We used Schur's lemma in a crucial way so we are assuming that $\mathbb{C}$ is algebraically closed. In general we can say:

If M is a simple A-module then $M(A) = \mathrm{End}_Z(M)$ where $Z = \mathrm{End}_A(M)$.

The proof is similar to that given above and is called the *Jacobson density theorem.*

Example. Assume that A is a commutative algebra and let M be a simple A-module. Then $M(A)$ is commutative and $M(A) = \mathrm{End}(M) \cong M_d(\mathbb{C})$, where $d = \dim(M)$. However, $M_d(\mathbb{C})$ is commutative if and only if $d = 1$. This shows that every irreducible representation of a commutative algebra is one dimensional.

Example 2. Explain what the error is in the following proof of Burnside's theorem: If M is an irreducible A-module then $M(A) = \mathrm{End}(M)$.

Proof. Let $\{m_1, \ldots, m_d\}$ be a basis of M. Since M is irreducible, for any i and j there is an $a \in A$ such that $M(a)m_j = m_i$. So the matrix $E_{ji} \in M(A)$ for all $1 \le i, j \le n$. So $\mathrm{End}(M) \subseteq M(A)$. So $M(A) = \mathrm{End}(M)$. ∎