The ring $\mathbb{Z}[P]^W$

Arun Ram
Department of Mathematics
University of Wisconsin
Madison, WI 53706
ram@math.wisc.edu

July 18, 2005

1 The ring $\mathbb{Z}[P]^W$

Proposition 1.1. *$\mathbb{Z}[P]^W$ is a polynomial ring, i.e. there are algebraically independent elements $e_1, \ldots, e_n \in \mathbb{Z}[P]^W$ such that*

$$\mathbb{Z}[P]^W = \mathbb{Z}[e_1, \ldots, e_n].$$

Proof. Let $e_1, \ldots, e_n \in \mathbb{K}[P]^W$ be such that

$$e_i = x^{\omega_i} + (\text{lower terms in dominance order}).$$

If $\lambda = \ell_1\omega_1 + \cdots + \ell_n\omega_n \in P^+$ then

$$e_1^{\ell_1} e_2^{\ell_2} \cdots e_n^{\ell_n} = x^\lambda + (\text{lower terms in dominance order}).$$

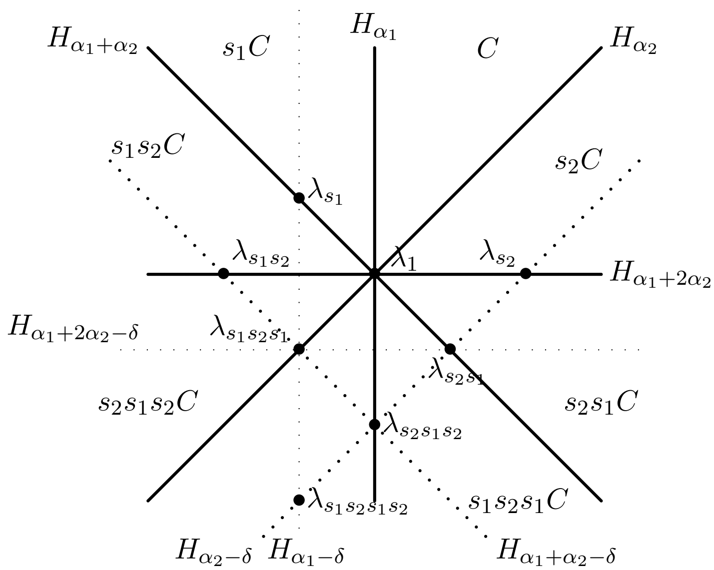
Thus $\{e_1^{\ell_1} \cdots e_n^{\ell_n} \mid \ell_1, \ldots, \ell_n \in \mathbb{Z}_{\geq 0}\}$ is a basis of $\mathbb{K}[P]^W$ and the result follows. $\square$

The *Shi arrangement* $\mathcal{A}^-$ is the arrangement of (affine) hyperplanes given by

$$\mathcal{A}^- = \{H_\alpha, H_{\alpha-\delta} \mid \alpha \in R^+\} \quad \text{where} \quad \begin{array}{ll} H_\alpha & = \{x \in \mathbb{R}^n \mid \langle x, \alpha^\vee \rangle = 0\}, \\ H_{\alpha-\delta} & = \{x \in \mathbb{R}^n \mid \langle x, \alpha^\vee \rangle = -1\}, \end{array} \qquad (1.1)$$

Consider the partition of $\mathfrak{h}_{\mathbb{R}}^*$ formed determined by the Shi arrangement. Each chamber $w^{-1}C$, $w \in W$, contains a unique region of $\mathcal{A}^-$ which is a cone, and the vertex of this cone is the point

$$\lambda_w = w^{-1}\left(\sum_{s_i w < w} \omega_i\right). \qquad (1.2)$$



The arrangement $\mathcal{A}^-$

Theorem 1.2. *$\mathbb{Z}[P]$ is a free $\mathbb{Z}[P]^W$ module of rank $|W|$ with basis $\{x^{\lambda_w} \mid w \in W\}$.*

Proof. The proof is accomplished by establishing three facts:

(1) Let f_y, $y \in W$, be a family of elements of $\mathbb{Z}[P]$. Then $\det(zf_y)$ is divisible by $\prod_{\alpha\in R^+}(1-x^{-\alpha})^{|W|/2}$.

(2) $\det\left(zx^{\lambda_y}\right)_{z,y\in W} = \left(\prod_{\alpha>0} x^{\rho}(1-x^{-\alpha})\right)^{|W|/2}$.

(3) If $f \in \mathbb{Z}[P]$ then there is a unique solution to the equation

$$\sum_{w\in W} a_w x^{\lambda_w} = f, \qquad \text{with} \qquad a_w \in \mathbb{Z}[P]^W.$$

(1) For each $\alpha \in R^+$ subtract row zf_y from row $s_\alpha z f_y$. Then this row is divisible by $(1 - x^{-\alpha})$. Since there are $|W|/2$ pairs of rows $(zf_y, s_\alpha z f_y)$ the whole determinant is divisible by $(1-x^{-\alpha})^{|W|/2}$. For $\alpha, \beta \in R^+$ the factors $(1-x^{-\alpha})$ and $(1-X^{-\beta})$ are coprime, and so $\det(zf_y)$ is divisible by $\prod_{\alpha\in R^+}(1-x^{-\alpha})^{|W|/2}$. (2) Since $y\lambda_y$ is dominant, $y\lambda_y \geq zy\lambda_y$. So all the entries in the yth column are (weakly) less than the entry on the diagonal. If $y\lambda_y = zy\lambda_y$ then z is in the stabilizer of

$$\sum_{\alpha_i\in R(y)} \omega_i.$$

Thus $\ell(y^{-1}) = \ell(z^{-1}) + \ell(zy^{-1})$. If $\ell(y) \leq \ell(z)$ then $\ell(zy^{-1}) = 0$ and so $z = y$. Thus, if the rows are ordered so that the yth row is above the zth row when $\ell(y) \leq \ell(z)$ then all terms above the diagonal are strictly less than the diagonal entry.

Thus the top coefficient of $\det(zx^{\lambda_y})$ is equal to

$$\prod_{z\in W} zx^{\lambda_z} = \prod_{z\in W} \prod_{\substack{i \\ s_i z < z}} x^{\omega_i} = \prod_{i=1}^{n} x^{(|W|/2)\omega_i} = (x^\rho)^{|W|/2}.$$

Since $s_i \det(zx^{\lambda_y}) = -\det(zx^{\lambda_y})$ the lowest term of $\det(zx^{\lambda_y})$ is $w_0(x^\rho)^{|W|/2} = (x^\rho)^{-|W|/2}$. These are the same as the highest and lowest terms of $\left(x^\rho \prod_{\alpha\in R^+}(1-x^{-\alpha})\right)^{|W|/2}$ and so (2) follows from (1). (3) Assume that $a_y \in \mathbb{Z}[P]^W$ are solutions of the equation $\sum_{y\in W} x^{\lambda_y} a_y = f$. Act on this equation by the elements of W to obtain the system of $|W|$ equations

$$\sum_{y\in W} (zx^{\lambda_y}) a_y = zf, \qquad z \in W.$$

By (1) the matrix $(zx^{\lambda_y})_{z,y\in W}$ is invertible and so this system has a unique solution with $a_y \in \mathbb{Z}[P]^W$. Cramer's rule provides an expression for a_y as a quotient of two determinants. By (1) and (2) the denominator divides the numerator to give an element of $\mathbb{Z}[P]$. Since each determinant is an alternating function (an element of Fock space), the quotient is an element of $\mathbb{Z}[P]^W$. $\square$

Remark. In [Sb2] Steinberg proves this type of result in full generality without the assumptions that W acts irreducibly on $\mathfrak{h}^*_{\mathbb{R}}$ and $L = P$. Note also that the proof given above is sketchy, particularly in the aspect that the top coefficient of the determinant is what we have claimed it is. See [Sb2] for a proper treatment of this point.

References

[GL] S. Gaussent, P. Littelmann, *LS galleries, the path model, and MV cycles*, Duke Math. J. **127** (2005), no. 1, 35–88.