Yangians

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Abstract

Abstract.

1 Deformations

A *Lie bialgebra* is a pair $(\mathfrak{g}, \delta)$, where $\mathfrak{g}$ is a Lie algebra and $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is a linear map, such that

(a) $[x \otimes 1 + 1 \otimes x, \delta(y)] - [y \otimes 1 + 1 \otimes y, \delta(x)] = 0$, for all $x, y \in \mathfrak{g}$,

(b) $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket.

Alternatively, a *Lie bialgebra* is a pair $(\mathfrak{g}, \phi)$ where $\mathfrak{g}$ is a Lie algebra, $\phi\colon \mathfrak{g} \to \wedge^2\mathfrak{g}$ is a 1-cocycle so that $(\mathfrak{g}, \phi)$ is a Lie coalgebra.

A *Manin triple* is a Lie algebra $\mathfrak{p}$ with an invariant scalar product $\langle , \rangle \colon \mathfrak{p} \otimes \mathfrak{p} \to \mathfrak{p}$ and a decomposition $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ such that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are isotropic. If $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ is a Manin triple define

$$\phi\colon \mathfrak{p}_1 \to \wedge^2\mathfrak{p}_1 \qquad \text{by} \qquad \phi(x) = ???.$$

Then the map

$$\begin{array}{ccc} \{\text{Manin triples}\} & \longleftrightarrow & \{\text{Lie bialgebras}\} \\ (\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*) & \leftarrow & (\mathfrak{g}, \phi) \\ (\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2) & \longmapsto & (\mathfrak{p}_1, \phi) \end{array}$$

is a bijection.

Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra. A *deformation* of the universal enveloping algebra $U(\mathfrak{g})$ is a Hopf algebra $\mathcal{U}$ over $\mathbb{C}[[h]]$, such that $\mathcal{U} = U(\mathfrak{g})[[h]]$ as a $\mathbb{C}[[h]]-$ module with the following properties:

MSC 2000: 20C08 (05E10)
Keywords:
Research supported in part by National Security Agency grant MDA904-01-1-0032 and NSF Grant ????.

(a) $\mathcal{U}/h\mathcal{U} \simeq U(\mathfrak{g})$ as Hopf algebras, and

(b) $\dfrac{\Delta(a) - \Delta^{op}(a)}{h} \bmod h = \delta(a \bmod h)$, for $a \in \mathcal{U}$.

1.1 Existence of deformations

1.2 Equivalence of deformations and the trivial deformation

1.3 Obstructions

2 The Lie algebra $\mathfrak{g}[u]$

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra. Let $(,)$ be an ad-invariant bilinear form on $\mathfrak{g}$, and let $\{x_i\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to this form. The *Casimir element* of $\mathfrak{g}$ is the element of $\mathfrak{g} \otimes \mathfrak{g}$ given by

$$t = \sum_i x_i \otimes x_i, \qquad \text{in } \mathfrak{g} \otimes \mathfrak{g}.$$

The graded Lie algebra of polynomials in u with coefficients in $\mathfrak{g}$ is

$$\mathfrak{g}[u] = \mathfrak{g} \otimes \mathbb{C}[u] = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} \mathfrak{g} u^i, \qquad \text{with} \qquad [xu^i, yu^j] = [x, y]u^{i+j}, \quad \text{and} \quad \deg(xu^k) = k,$$

defining the bracket and grading on $\mathfrak{g}[u]$. Then $\mathfrak{g}[u]$ is a graded Lie bialgebra with cobracket defined by the map

$$\delta : \mathfrak{g}[u] \to \mathfrak{g}[u] \otimes \mathfrak{g}[u] = (\mathfrak{g} \otimes \mathfrak{g})[u, v] \quad \text{given by} \quad \delta(p(u)) = \left[p(u) \otimes 1 + 1 \otimes p(v), \frac{t}{u - v} \right],$$

for any $p(u) \in \mathfrak{g}[u]$. The map δ is well defined since $[x \otimes 1 + 1 \otimes x, t] = 0$ for $x \in \mathfrak{g}$. If $x \in \mathfrak{g}$ then

$$\begin{aligned}
\delta(xu^i) &= \left[xu^i \otimes 1 + 1 \otimes x\, v^i, \frac{t}{u - v} \right] = \frac{1}{u - v}([x \otimes 1, t]u^i + [1 \otimes x, t]v^i) \\
&= \frac{1}{u - v}([x \otimes 1, t]u^i - [x \otimes 1, t]v^i) = [x \otimes 1, t]\frac{u^i - v^i}{u - v} = [x \otimes 1, t](u^{i-1} + \ldots + v^{i-1}).
\end{aligned}$$

The last expression is a polynomial in u and v. In particular,

$$\delta(xu) = [x \otimes 1, t] \qquad \text{and} \qquad \deg(\delta) = -1.$$

The *classical r-matrix* is

$$r(u, v) = \frac{t}{u - v} \qquad \text{so that} \qquad \delta(p(u)) = \left[p(u) \otimes 1 + 1 \otimes p(v), \frac{t}{u - v} \right].$$

Since r satisfies the "triangle" or classical Yang-Baxter relation (CYBE),

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0,$$

but since r is not really an element of $\mathfrak{g}[u] \otimes \mathfrak{g}[u] = \mathfrak{g}[u, v]$, the pair $(\mathfrak{g}[u], r)$ is "pseudotriangular". If

$$\begin{array}{cccc} \gamma_\lambda : & \mathfrak{g}[u] & \longrightarrow & \mathfrak{g}[u] \\ & p(u) & \longmapsto & p(u + \lambda) \end{array} \qquad \text{then} \qquad (\gamma_\lambda \otimes \mathrm{id})(r) = \frac{t}{u + \lambda - v} = \sum_{k \in \mathbb{Z}_{\geq 0}} t(v - u)^k \lambda^{-k-1}$$

is a power series in λ^{-1} with coefficients in $\mathfrak{g}[u] \otimes \mathfrak{g}[u]$.

3 Definition of the Yangian

We will give five definitions of the Yangian:

(a) As a deformation of $U\mathfrak{g}$ where $\mathfrak{g} = \mathfrak{a}[u]$,

(b) By a presentation with generators $\mathfrak{a}$ and $\mathfrak{a}u$,

(c) By a presentation in loop form,

(d) RTT presentation,

(e) By degeneration from $U_q\hat{\mathfrak{g}}$.

The *Yangian* $Y_h(\mathfrak{g})$ is a graded Hopf algebra deformation of the graded Lie bialgebra $\mathfrak{g}[u]$ with

$$\deg(h) = 1 \qquad \text{and generators} \quad x \text{ and } J(x), \quad \text{for } x \in \mathfrak{g},$$

which have classical limits x and xu, respectively. The formulas

$$\Delta(x) = x \otimes 1 + 1 \otimes x \qquad \text{and} \qquad \Delta(J(x)) = J(x) \otimes 1 + 1 \otimes J(x) + \frac{h}{2}[x \otimes 1, t]$$

are forced by the degree condition and

$$\frac{\Delta(J(x)) - \Delta^{\text{op}}(J(x))}{h} = \frac{\frac{h}{2}[x \otimes 1 - 1 \otimes x, t]}{h} = \frac{h[x \otimes 1, t]}{h} = [x \otimes 1, t] = \delta(xu).$$

Then

$$[x, y]_h = [x, y]$$

is forced by the degree condition. It seems that we could have

$$[x, J(y)]_h = J([x, y]) + hz, \qquad \text{for some } z \in \mathfrak{g}.$$

Is there a reason why $z = 0$??

Note that the maps

$$\begin{array}{rccc} \gamma_\lambda\colon & Y_h(\mathfrak{g}) & \longrightarrow & Y_h(\mathfrak{g}) \\ & x & \longmapsto & x \\ & J(x) & \longmapsto & \lambda J(x) \\ & h & \longmapsto & \lambda h \end{array} \qquad \text{for } \lambda \in \mathbb{C}^*,$$

are Hopf algebra isomorphisms (essentially because $Y_h(\mathfrak{g})$ is a graded Hopf algebra). Hence

$$Y_a(\mathfrak{g}) \cong Y_b(\mathfrak{g}), \qquad \text{for any } a, b, \in \mathbb{C}^*.$$

We have $Y_0(\mathfrak{g}) = U(\mathfrak{g}[u]) \ncong Y_1(\mathfrak{g})$.

4 The automorphisms τ_λ

The automorphisms τ_λ, $\lambda \in \mathbb{C}$, of $\mathfrak{g}[u]$ given by

$$\begin{array}{rccc} \tau_\lambda\colon & \mathfrak{g}[u] & \longrightarrow & \mathfrak{g}[u] \\ & xu^k & \longmapsto & x(u+\lambda)^k, \end{array} \qquad \text{for } x \in \mathfrak{g},$$

have analogues for $Y(\mathfrak{g})$. For each $\lambda \in \mathbb{C}$ the map

$$\begin{array}{rccc} \tau_\lambda\colon & Y(\mathfrak{g}) & \longrightarrow & Y(\mathfrak{g}) \\ & x & \longmapsto & x \\ & J(x) & \longmapsto & J(x)+\lambda x, \end{array} \qquad \text{for } x \in \mathfrak{g},$$

is a Hopf algebra automorphism,

$$\Delta(\tau_\lambda(a)) = (\tau_\lambda \otimes \tau_\lambda)(\Delta(a)), \qquad \text{for } a \in Y(\mathfrak{g}).$$

Then

$$\tau_a(H_{i,r}) = \sum_{s=0}^{r} \binom{r}{s} a^{r-s} H_{i,s} \qquad \text{and} \qquad \tau(X^{\pm}_{i,r}) = \sum_{s=0}^{r} \binom{r}{s} a^{r-s} X^{\pm}_{i,s}$$

in $Y(\mathfrak{g})$.

5 The evaluation homomorphisms $\mathrm{ev}_\lambda\colon Y(\mathfrak{sl}_n) \to U\mathfrak{sl}_n$

By the Jacobi identity, the map

$$\begin{array}{ccc} \mathfrak{g}\otimes\mathfrak{g} & \longrightarrow & \mathfrak{g} \\ x\otimes y & \longmapsto & [x,y] \end{array}, \qquad \text{is a } \mathfrak{g}\text{-module homomorphism.}$$

If $\mathfrak{g} = \mathfrak{sl}_n$ then there is a *another* copy of $\mathfrak{g}$ in $\mathfrak{g}\otimes\mathfrak{g}$. Let $\pi\colon \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ be the projection onto this other copy and define

$$\begin{array}{rccc} \mathrm{ev}_\lambda\colon & Y(\mathfrak{sl}_n) & \longrightarrow & U\mathfrak{sl}_n \\ & x & \longmapsto & x \\ & J(x) & \longmapsto & ax + \frac{1}{4}\sum_i \pi(x\otimes x_i)x_i, \end{array} \qquad \text{for } \lambda \in \mathbb{C}.$$

These are algebra homomorphisms but *not* Hopf algebra homomorphisms.

The *evaluation map* is given by

$$\begin{array}{rccc} \mathrm{ev}_a\colon & U_h(L(\mathfrak{sl}_2)) & \longrightarrow & U_h(\mathfrak{sl}_2) \\ & \xi_0^+ & \longmapsto & \xi^+ \\ & \xi_0^- & \longmapsto & \xi^- \\ & \xi_1^+ & \longmapsto & q^{-1}a^{-1}e^{h\kappa}\xi^+ \\ & \xi_1^- & \longmapsto & q^{-1}a^{-1}\xi^- e^{h\kappa} \\ & \xi_{-1}^+ & \longmapsto & qae^{-h\kappa}\, xi^+ \\ & \xi_{-1}^- & \longmapsto & qa\xi^- e^{-h\kappa} \end{array}$$

On the representation $(\mathrm{ev}_a) * (L(m\omega_1))$

$$\xi^+ v_i = [m-i+1]v_{i-1} \qquad \text{and} \qquad \xi^- v_i = [i+1]v_{i+1},$$

and so

$$\begin{aligned} \xi_0^+ v_i &= [m-i+1]v_{i-1} & \xi_0^- v_i &= [i+1]v_{i+1}, \\ \xi_1^+ v_i &= q^{m-2i+2}[m-i+1]v_{i-1}(qa)^{-1}, & \xi_1^- v_i &= (qa)^{-1}q^{m-i}v_{i+1} \\ xi_{-1}^+ v_i &= (qa)q^{-(m-2(i-1))}[m-i+1]v_{i-1}, & \xi_{-1}^- v_i &= (qa)q^{-(m-i)}v_{i+1}, \end{aligned}$$

On the representation $(\mathrm{ev}_a)^*(L(m\omega_1))$ the algebra $Y(\mathfrak{sl}_2)$ acts as

$$x_k^+ v_i = \left(a + \frac{m-2i-1}{2}\right)^k (m-i-1)v_{i-1}.$$

6 The $\mathcal{R}$ matrix

Theorem 6.1. *There is a unique formal power series*

$$\mathcal{R}(\lambda) = 1 + \sum_{k\in\mathbb{Z}_{>0}} \mathcal{R}_k \lambda^{-k}, \qquad \textit{with} \quad \mathcal{R}_k \in Y(\mathfrak{g}) \otimes Y(\mathfrak{g}),$$

such that

$$(\Delta \otimes \mathrm{id})(\mathcal{R}(\lambda)) = \mathcal{R}^{13}(\lambda)\mathcal{R}^{23}(\lambda) \quad \textit{and} \quad (\tau_\lambda \otimes \mathrm{id})\Delta^{\mathrm{op}}(a) = \mathcal{R}(\lambda)((\tau_\lambda \otimes \mathrm{id})\Delta(a))\mathcal{R}(\lambda)^{-1},$$

for $a \in Y(\mathfrak{g})$.

Conceptually,

$$\mathcal{R}(\lambda) = (\tau_\lambda \otimes \mathrm{id})(\mathcal{R}), \quad \text{for some } \mathcal{R} \text{ such that} \quad \Delta^{\mathrm{op}}(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1},$$

$$(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}^{13}\mathcal{R}^{23}, \qquad (\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}^{13}\mathcal{R}^{12}, \quad \text{and} \quad \mathcal{R}^{12}\mathcal{R}^{21} = 1.$$

Facts about $\mathcal{R}(\lambda)$:

(a) $\mathcal{R}^{12}(\lambda_1-\lambda_2)\mathcal{R}^{13}(\lambda_1-\lambda_3)\mathcal{R}^{23}(\lambda_2-\lambda_3) = \mathcal{R}^{23}(\lambda_2-\lambda_3)\mathcal{R}^{13}(\lambda_1-\lambda_3)\mathcal{R}^{12}(\lambda_1-\lambda_2)$,

(b) $\mathcal{R}^{12}(\lambda)\mathcal{R}^{21}(-\lambda) = 1$,

(c) $(\tau_\mu \otimes \tau_\nu)(\mathcal{R}(\lambda)) = \mathcal{R}(\lambda+\mu-\nu)$,

(d) $\mathcal{R}_1 = t$.

(e) $\ln \mathcal{R}(\lambda) = \lambda^{-1}t + \lambda^{-2}\left(\sum_i J(x_i)\otimes x_i - x_i \otimes J(x_i)\right) + \cdots$

Proof. (c) Let

$$\Delta_{\lambda,\mu} = (\tau_\lambda \otimes \tau_\mu)\Delta \qquad \text{and} \qquad \Delta^{\mathrm{op}}_{\lambda,\mu} = (\tau_\lambda \otimes \tau_\mu)\Delta^{\mathrm{op}}.$$

Then the second defining relation for $\mathcal{R}(\lambda)$ can be written?

$$\Delta_{\lambda,\mu}(a)\mathcal{R}(\lambda-\mu) = \mathcal{R}(\lambda-\mu)\Delta^{\mathrm{op}}_{\lambda,\mu}(a).$$

Then $\mathcal{R}(\lambda)$ is characterized by the conditions

$$\mathcal{R}(\lambda)\Delta_{\lambda,0}(a) = \Delta^{\mathrm{op}}_{\lambda,0}\mathcal{R}(\lambda) \qquad \text{and} \qquad (\Delta\otimes\mathrm{id})\mathcal{R}(\lambda) = \mathcal{R}^{13}(\lambda)\mathcal{R}^{23}(\lambda).$$

Then, since τ_μ is an automorphism of $Y(\mathfrak{g})$,

$$(\tau_\mu \otimes \tau_\mu)(\mathcal{R}(\lambda))\Delta_{\lambda+\mu,\mu}(a) = \Delta^{\mathrm{op}}_{\lambda+\mu,\mu}(a)(\tau_\mu \otimes \tau_\mu)(\mathcal{R}(\lambda)).$$

So

$$(\tau_\mu \otimes \tau_\mu)(\mathcal{R}(\lambda))\Delta_{\lambda,0}(\tau_\mu(a)) = \Delta^{\mathrm{op}}_{\lambda,0}(\tau_\mu(a))(\tau_\mu \otimes \tau_\mu)(\mathcal{R}(\lambda)).$$

So

$$(\tau_\mu \otimes \tau_\mu)(\mathcal{R}(\lambda))\Delta_{\lambda,0}(b) = \Delta^{\mathrm{op}}_{\lambda,0}(b)(\tau_\mu \otimes \tau_\mu)(\mathcal{R}(\lambda)),$$

for all $b \in Y(\mathfrak{g})$. Furthermore,

$$\begin{aligned}(\Delta \otimes \mathrm{id})((\tau_\mu \otimes \tau_\mu)(\mathcal{R}(\lambda)) &= (\tau_\mu \otimes \tau_\mu \otimes \tau_\mu)(\Delta \otimes \mathrm{id})(\mathcal{R}(\lambda)) \\ &= (\tau_\mu \otimes \tau_\mu \otimes \tau_\mu)(\mathcal{R}^{13}(\lambda)\mathcal{R}^{23}(\lambda)) \\ &= (\tau_\mu \otimes \tau_\mu)(\mathcal{R}(\lambda))^{13}(\tau_\mu \otimes \tau_\mu)(\mathcal{R}(\lambda))^{23}.\end{aligned}$$

So

$$(\tau_\mu \otimes \tau_\mu)(\mathcal{R}(\lambda)) = \mathcal{R}(\lambda).$$

Then

$$\begin{aligned}(\tau_\mu \otimes \tau_\nu)(\mathcal{R}(\lambda))\Delta_{\lambda+\mu,\nu}(a) &= (\tau_\mu \otimes \tau_\nu)(\mathcal{R}(\lambda)\Delta_{\lambda,0}(a) \\ &= (\tau_\mu \otimes \tau_\nu)(\Delta^{\mathrm{op}}_{\lambda,0}(a)\mathcal{R}(\lambda)) \\ &= \Delta^{\mathrm{op}}_{\lambda+\mu,\nu}(a)(\tau_\mu \otimes \tau_\nu)(\mathcal{R}(\lambda))\end{aligned}$$

and

$$\begin{aligned}(\Delta \otimes \mathrm{id})(\tau_\mu \otimes \tau_\nu)\mathcal{R}(\lambda) &= (\tau_\mu \otimes \tau_\mu \otimes \tau_\nu)(\Delta \otimes \mathrm{id})\mathcal{R}(\lambda) \\ &= (\tau_\mu \otimes \tau_\mu \otimes \tau_\nu)(\mathcal{R}^{13}(\lambda)\mathcal{R}^{23}(\lambda)) \\ &= (\tau_\mu \otimes \tau_\nu)(\mathcal{R}(\lambda))^{13}(\tau_\mu \otimes \tau_\nu)(\mathcal{R}(\lambda))^{23}\end{aligned}$$

and so

$$(\tau_\mu \otimes \tau_\nu)\mathcal{R}(\lambda) = \mathcal{R}(\lambda + \mu - \nu).$$

(a)

$$\begin{aligned}&\mathcal{R}^{12}(\lambda_1 - \lambda_2)\mathcal{R}^{13}(\lambda_1 - \lambda_3)\mathcal{R}^{23}(\lambda_2 - \lambda_3) \\ &\quad = \mathcal{R}^{12}(\lambda_1 - \lambda_2)(\tau_{\lambda_1} \otimes \tau_{\lambda_2} \otimes \mathrm{id})(\mathcal{R}^{13}(-\lambda_3)\mathcal{R}^{23}(-\lambda_3)) \\ &\quad = \mathcal{R}^{12}(\lambda_1 - \lambda_2)(\tau_{\lambda_1} \otimes \tau_{\lambda_2} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(\mathcal{R}(-\lambda_3)) \\ &\quad = \mathcal{R}^{12}(\lambda_1 - \lambda_2)(\Delta_{\lambda_1,\lambda_2} \otimes \mathrm{id})(\mathcal{R}(-\lambda_3)) \\ &\quad = (\Delta^{\mathrm{op}}_{\lambda_1,\lambda_2} \otimes \mathrm{id})(\mathcal{R}(-\lambda_3))\mathcal{R}^{12}(\lambda_1 - \lambda_2) \\ &\quad = (\tau_{\lambda_1} \otimes \tau_{\lambda_2} \otimes \mathrm{id})(\Delta^{\mathrm{op}} \otimes \mathrm{id})(\mathcal{R}(-\lambda_3))\mathcal{R}^{12}(\lambda_1 - \lambda_2) \\ &\quad = (\tau_{\lambda_1} \otimes \tau_{\lambda_2} \otimes \mathrm{id})(\mathcal{R}^{23}(-\lambda_3)\mathcal{R}^{13}(-\lambda_3))\mathcal{R}^{12}(\lambda_1 - \lambda_2) \\ &\quad = \mathcal{R}^{23}(\lambda_2 - \lambda_3)\mathcal{R}^{13}(\lambda_1 - \lambda_3))\mathcal{R}^{12}(\lambda_1 - \lambda_2)\end{aligned}$$

(b)

$$\begin{aligned}\Delta_{\lambda,0}(a)\mathcal{R}^{12}(\lambda)\mathcal{R}^{21}(-\lambda) &= \mathcal{R}^{12}(\lambda)\Delta^{\mathrm{op}}_{\lambda,0}(a)\mathcal{R}^{21}(-\lambda) \\ &= \mathcal{R}^{12}(\lambda)(\tau_\lambda \otimes \tau_\lambda)\Delta^{\mathrm{op}}_{0,-\lambda}(a)\mathcal{R}^{21}(-\lambda) \\ &= \mathcal{R}^{12}(\lambda)(\tau_\lambda \otimes \tau_\lambda)(\Delta_{-\lambda,0,}(a)\mathcal{R}(-\lambda))^{\mathrm{op}} \\ &= \mathcal{R}^{12}(\lambda)(\tau_\lambda \otimes \tau_\lambda)(\mathcal{R}(-\lambda)\Delta^{\mathrm{op}}_{-\lambda,0,}(a))^{\mathrm{op}} \\ &= \mathcal{R}^{12}(\lambda)\mathcal{R}^{21}(-\lambda)\Delta_{\lambda,0}(a)).\end{aligned}$$

So???

$$\mathcal{R}^{12}(\lambda)\mathcal{R}^{21}(-\lambda) = 1.$$

$\square$

Theorem 6.2. *Fix a finite dimensional irreducible representation*

$$\rho\colon Y(\mathfrak{g}) \longrightarrow M_n(\mathbb{C}) \qquad \textit{and let} \qquad \mathcal{R}_\rho(\lambda) = (\rho\otimes\rho)\mathcal{R}(\lambda).$$

Then, up to a scalar factor, $\mathcal{R}_\rho(\lambda)$ is the unique solution to the system

$$\mathcal{R}_\rho(\lambda-\mu)P^+_{\rho,x}(\lambda,\mu) = P^-_{\rho,x}(\lambda,\mu)\mathcal{R}_\rho(\lambda-\mu), \qquad \textit{for } x\in\mathfrak{g},$$

where

$$P^+_{\rho,x}(\lambda,\mu) = (\rho\otimes\rho)\Delta_{\lambda,\mu}(J(x)) \qquad \textit{and} \qquad P^-_{\rho,x}(\lambda,\mu) = (\rho\otimes\rho)\Delta^{\mathrm{op}}_{\lambda,\mu}(J(x))$$

Note that

$$\begin{aligned}\Delta^{\mathrm{op}}_{\lambda,\mu}(J(x)) &= (\tau_\lambda\otimes\tau_\mu)(J(x)\otimes 1 + 1\otimes J(x) + \frac{1}{2}[x\otimes 1, t])^{\mathrm{op}}\\ &= (J(x)+\lambda x)\otimes 1 + 1\otimes(J(x)+\mu x) + \frac{1}{2}[x\otimes 1, t]^{\mathrm{op}}\\ &= (J(x)+\lambda x)\otimes 1 + 1\otimes(J(x)+\mu x) - \frac{1}{2}[x\otimes 1, t]\end{aligned}$$

Theorem 6.3. *?? Let $\rho\colon\mathfrak{g}\to\mathrm{End}(V)$ be a representation of $\mathfrak{g}$ and let*

$$\mathcal{R}(u) = 1 + u^{-1}(\rho\otimes\rho)(t) + \sum_{k\in\mathbb{Z}_{>1}} u^{-k}\mathcal{R}_k, \qquad \mathcal{R}_k\in\mathrm{End}(V\otimes V),$$

a solution to the QYBE. Then there is a representation

$$\pi\colon Y(\mathfrak{g})\to\mathrm{End}(V) \qquad \textit{such that} \qquad \mathcal{R}(u) = f(u)(\pi\otimes\pi)(\mathcal{R}(u)),$$

with $f(u)$ a constant. So every solution to the QYBE of the above power series form comes from a representation of $Y(\mathfrak{g})$.

Theorem 6.4. *Let $\hat\rho:\mathfrak{g}\to M_n(\mathbb{C})$ be an irreducible representation of $\mathfrak{g}$. Let $\hat R_{\hat\rho}(\lambda)\in M_n(\mathbb{C})\otimes M_n(\mathbb{C})$ be a solution of QYBE with the property*

$$\hat R_{\hat\rho}(\lambda) = 1\otimes 1 + \frac{1}{\lambda}\sum x_i\otimes x_i + \textit{higher terms in} \quad \lambda.$$

Then there is a unique extension of $\hat\rho$ to a representation $\rho: Y(\mathfrak{g})\to M_n(\mathbb{C})$ such that

$$\hat R_{\hat\rho}(\lambda) = f(\lambda)R_\rho(\lambda),$$

where $R_\rho(\lambda) = (\rho\otimes\rho)\mathcal{R}(\lambda)$, $f(\lambda)\in 1+\lambda^{-1}\mathbb{C}[[\lambda]]$.

ρ is unique up to the shift by τ_μ.

Let $\rho: Y(\mathfrak{g})\to M_n(\mathbb{C})$ be an irreducible nontrivial representation and let $R_\rho(\lambda) = (\rho\otimes\rho)\mathcal{R}(\lambda)$. Define an algebra A_ρ with generators $\{t^{(k)}_{ij} | 1\le i,j\le n,\ k\in\mathbb{Z}_{\ge 0}\}$ and the relations

$$R_\rho(\lambda-\mu)(T(\lambda)\otimes id)(id\otimes T(\mu)) = (id\otimes T(\mu))(T(\lambda)\otimes id)R_\rho(\lambda-\mu).$$

Here $T(\lambda)$ is the matrix with entries

$$t_{ij}(\lambda) = \delta_{ij} + \sum_k t_{ij}^{(k)} \lambda^{-k}$$

This is a Hopf algebra with the comultiplication

$$\Delta t_{ij}(\lambda) = \sum_l t_{il}(\lambda) \otimes t_{lj}(\lambda)$$

Theorem 6.5. *a) There is a surjective Hopf algebras homomorphism $\phi : A_\rho \to Y(\mathfrak{g})$, given by*

$$T(\lambda) \to (\rho \otimes id)\mathcal{R}(\lambda).$$

b) The kernel of ϕ is spanned by the elements $\{c_1, c_2, ...\}$ of the center of A_ρ c) The element

$$c(\lambda) = 1 + \sum c_k \lambda^{-k}$$

is group like: $\Delta(c(\lambda) = c(\lambda) \otimes c(\lambda)$

Example. Let $\mathfrak{g} = \mathfrak{sl}_n$ with inner product

$$\langle x, y \rangle = \mathrm{tr}(xy), \qquad \text{for } x, y \in \mathfrak{sl}_n.$$

Let

$$\begin{array}{rcccl} \rho\colon & Y(\mathfrak{sl}_n) & \longrightarrow & M_n(\mathbb{C}) & \\ & x & \longmapsto & x, & \qquad \text{for } x \in \mathfrak{sl}_n, \\ & J(x) & \longmapsto & 0, & \qquad \text{for } x \in \mathfrak{sl}_n. \end{array}$$

Then

$$\mathcal{R}_\rho(\lambda) = f(\lambda)(1 + \lambda^{-1}\sigma) \qquad \text{where} \qquad \begin{array}{rccc} \sigma\colon & \mathbb{C}^n \otimes \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \otimes \mathbb{C}^n \\ & m \otimes n & \longmapsto & n \otimes m \end{array}$$

and $f(\lambda) \in 1 + \lambda^{-1}\mathbb{C}[[\lambda^{-1}]]$ is determined by

$$f(\lambda - 1) f(\lambda - 2) \cdots f(\lambda - n) = 1 - \lambda^{-1}.$$

In this case $A_\rho = Y(\mathfrak{gl}_n(\mathbb{C}))$

7 The algebra $Y(\mathfrak{gl}_n)$

The matrix

$$R(u) = u\,\mathrm{id} + \sigma, \qquad \text{where} \qquad \begin{array}{rccc} \sigma\colon & V \otimes V & \longrightarrow & V \otimes V \\ & m \otimes n & \longmapsto & n \otimes m \end{array}$$

satisfies the QYBE

$$R_{12}(u) R_{13}(u + v) R_{23}(v) = R_{23}(v) R_{13}(u + v) R_{12}(u).$$

Let

$$T(u) = (t_{ij}(u)) \qquad \text{where} \qquad t_{ij}(u) = \delta_{ij} + \sum_{k \in \mathbb{Z}_{\geq 1}} t_{ij}^{(k)} u^{-k}.$$

We want

$$R(u - v)(T(u) \otimes \mathrm{id})(\mathrm{id} \otimes T(v)) = (\mathrm{id} \otimes T(v))(T(u) \otimes \mathrm{id}) R(u - v).$$

Define

$$\tau_v \colon Y(\mathfrak{gl}_n) \to Y(\mathfrak{gl}_n) \quad \text{by} \quad \tau_v(t_{ij}(u)) = t_{ij}(u+v), \quad \text{and let} \quad \tau_{u,v} = \tau_u \otimes \tau_v.$$

Define a coproduct on $Y(\mathfrak{gl}_n)$ by

$$\Delta(t_{ij}(u)) = \sum_{k=1}^{n} t_{ik}(u) \otimes t_{kj}(u).$$

Then

$$R(v-u)(\tau_{u,v}\Delta^{\mathrm{op}}(a)) = (\tau_{u,v}\Delta(a))R(v-u), \qquad \text{for all } a \in Y(\mathfrak{gl}_n).$$

Consider the irreducible $\mathfrak{sl}_N$-module $\mathbb{C}^N$. Then

$$R(u) = 1 \otimes 1 + \sum_{i,j=1}^{N} (E_{ij} \otimes E_{ji})u^{-1}$$

is a solution of the QYBE in $\mathrm{End}(\mathbb{C}^N \otimes \mathbb{C}^N)$ such that the u^{-1} term coincides with the action of the "Casimir" $t = \sum x_i \otimes x_i$, where $\{x_i\}$ is an orthnormal basis of $\mathfrak{sl}_N$. Hence we

$$\text{define } Y(\mathfrak{gl}_N) \text{ by} \quad \text{generators} \quad T_{ij}^{(r)}, \quad 1 \le i,j \le N, \quad r \in \mathbb{Z}_{\ge 0},$$

with relations

$$T_{ij}^{(0)} = \delta_{ij} \quad \text{and} \quad R(u-v)T(u) \otimes \mathrm{id})(\mathrm{id} \otimes T(v)) = (\mathrm{id} \otimes T(v))(T(u) \otimes \mathrm{id})R(u-v)$$

where

$$T(u) = (T_{ij}(u)) \quad \text{and} \quad T_{ij}(u) = \sum_{k \in \mathbb{Z}_{\ge 0}} T_{ij}^{(k)} u^{-k}.$$

Then there is a surjective map

$$\phi \colon Y(\mathfrak{gl}_N) \longrightarrow Y({}_N) \qquad \text{with} \qquad \ker\phi = Z(Y(\mathfrak{gl}_{\mathfrak{N}})) = \langle \mathfrak{c}_1, \mathfrak{c}_2, \ldots \rangle,$$

where

$$A_N(u) = 1 + c_1 u^{-1} + c_2 u^{-2} + \cdots = \sum_{\pi \in S_N} (-1)\ell(\pi) T_{1\pi(1)}(u) T_{2\pi(2)}(u-1) \cdots T_{N\pi(N)}(u-N+1).$$

The evaluation map

$$\begin{array}{rccc} \mathrm{ev}\colon & Y({}_N) & \longrightarrow & U\mathfrak{sl}_N \\ & x & \longmapsto & x \\ & J(x) & \longmapsto & 0, \end{array} \qquad \text{extends to} \qquad \begin{array}{ccc} \mathrm{ev}\colon \quad Y({}_N) & \longrightarrow & U\mathfrak{sl}_N \\ T_{ij}(u) & \longmapsto & \delta_{ij} + E_{ji}u^{-1} \end{array}$$

and the automorphisms

$$\begin{array}{rccc} \tau_\lambda\colon & Y(\mathfrak{sl}_N) & \longrightarrow & Y(\mathfrak{sl}_N) \\ & x & \longmapsto & x \\ & J(x) & \longmapsto & J(x) + ax \end{array} \qquad \text{extend to} \qquad \begin{array}{rccc} \tau_\lambda\colon & Y(\mathfrak{gl}_N) & \longrightarrow & Y(\mathfrak{gl}_N) \\ & T_{ij(u)} & \longmapsto & T_{ij}(u+\lambda) \end{array}$$

The evaluation map sends

$$\begin{array}{ccccccc} Z(Y(\mathfrak{gl}_1)) & & Z(Y(\mathfrak{gl}_2)) & & & & Z(Y(\mathfrak{gl}_N) \\ \cap| & & \cap| & & & & \cap| \\ Y(\mathfrak{gl}_1) & \subseteq & Y(\mathfrak{gl}_2) & \subseteq & \cdots & \subseteq & Y(\mathfrak{gl}_N) \end{array}$$

to

$$\begin{array}{ccccccc}
Z(U(\mathfrak{gl}_1)) & & Z(U(\mathfrak{gl}_2)) & & & & Z(U(\mathfrak{gl}_N)) \\
\cap| & & \cap| & & & & \cap| \\
U(\mathfrak{gl}_1) & \subseteq & U(\mathfrak{gl}_2) & \subseteq & \cdots & \subseteq & U(\mathfrak{gl}_N)
\end{array}$$

Let λ be a partition and let v_λ^+ be a highest weight vector for $U\mathfrak{gl}_N$ of weight λ. Then

$$\begin{aligned}
A_k(u)v_\lambda^+ &= \mathrm{ev}\left(\sum_{\pi\in S_k}(-1)^{\ell(\pi)}T_{1\pi(1)}(u)\cdots T_{k\pi(k)}(u-k+1)\right)v_\lambda^+ \\
&= \mathrm{ev}(T_{11}(u)\cdots T_{kk}(u-k+1))v_\lambda^+ \\
&= (1+E_{11}u^{-1})\cdots(1+E_{kk}(u-k+1)^{-1})v_\lambda^+ \\
&= \frac{1}{u(u-1)\cdots(u-k+1)}(u+E_{11})(u-1+E_{22})\cdots u-k+1+E_{kk})v_\lambda^+ \\
&= \frac{(\lambda_1+u)(\lambda_2+u-1)\cdots(\lambda_k+u-k+1)}{u(u-1)\cdots(u-k+1)}v_\lambda^+ \\
&= \prod_{i=1}^{k}\frac{u+c(r_i(\lambda))+1}{u+c(\ell_i(\lambda))}v_\lambda^+,
\end{aligned}$$

where $r_i(\lambda)$ is the rightmost box in row i of λ and $\ell_i(\lambda)$ is the leftmost box in row i of λ.

The map $\phi\colon Y(\mathfrak{gl}_N)\to Y(_N)$ satisfies

$$\phi\left(\frac{A_{i+1}(u)A_{i-1}(u-1)}{A_i(u)A_i(u-1)}\right)=H_i(u), \qquad \text{where} \qquad H_i(u)=1+\sum_{k\geq 0}H_{i,k}u^{-k-1},$$

and $A_0(u)=1$. So

$$\begin{aligned}
H_k(u) &= \frac{\prod_{i=1}^{k+1}\frac{u+c(r_i(\lambda))+1}{u+c(\ell_i(\lambda))}\cdot\prod_{i=1}^{k-1}\frac{u+c(r_i(\lambda))}{u+c(\ell_i(\lambda))-1}}{\prod_{i=1}^{k}\frac{u+c(r_i(\lambda))+1}{u+c(\ell_i(\lambda))}\cdot\prod_{i=1}^{k}\frac{u+c(r_i(\lambda))}{u+c(\ell_i(\lambda))-1}} \\
&= \frac{\frac{u+c(r_{k+1}(\lambda))+1}{u+c(\ell_{k+1}(\lambda))}}{\frac{u+c(r_k(\lambda))}{u+c(\ell_k(\lambda)-1}} = \frac{(u+c(r_{k+1}(\lambda))+1)(u+c(\ell_k(\lambda))-1)}{(u+c(\ell_{k+1}(\lambda)))(u+c(r_k(\lambda)))}
\end{aligned}$$

and this determines the Drinfeld polynomials of $L(\lambda)$.

The diagram

$$\begin{array}{ccc}
Y(\mathfrak{gl}_M)\otimes Y(\mathfrak{gl}_N) & \hookrightarrow & Y(\mathfrak{gl}_{M+N}) \\
\Big\downarrow{\scriptstyle \phi\otimes\phi} & & \Big\downarrow{\scriptstyle \phi} \\
U\mathfrak{gl}_M\otimes U\mathfrak{gl}_N & \hookrightarrow & U\mathfrak{gl}_{M+N}
\end{array}$$

commutes and

$$\text{if}\quad L(\lambda/\mu)=\left(\mathrm{Res}^{\mathfrak{gl}_{M+N}}_{\mathfrak{gl}_M}L(\lambda)\right)^+_\mu \qquad \text{then} \qquad L(\lambda)\cong\bigoplus_\mu L(\mu)\otimes L(\lambda/\mu)$$

as $\mathfrak{gl}_M\otimes\mathfrak{gl}_N$ modules. In this way $L(\lambda/\mu)$ is an $Y(\mathfrak{gl}_N)$-module.

8 Presentation of the Yangian

The *Yangian* $Y_h(\mathfrak{g})$ is a the graded Hopf algebra over $\mathbb{C}[[h]]$ given by generators

$$x \quad \text{and} \quad J(x), \qquad x \in \mathfrak{g},$$

with

$$\deg(x) = 0, \qquad \deg(J(x)) = 1, \qquad \deg(h) = 1,$$

and the relations

$$J(ax + by) = a(x) + bJ(y), \qquad \text{for } a, b \in \mathbb{C},$$

$$[x, y] = [x, y], \qquad [x, J(y)] = J([x, y]),$$

$$\begin{aligned}
&[J(x), J([y, z])] + [J(z), J([x, y])] + [J(y), J([z, x])] \\
&\qquad = \frac{h^2}{4} \sum_{\alpha,\beta,\gamma} \langle [x, I_\alpha], [[y, I_\beta], [z, I_\gamma]] \rangle \{I_\alpha, I_\beta, I_\gamma\},
\end{aligned}$$

$$\begin{aligned}
&[[J(x), J((y)], [z, J(w)]] + [[J(z), J(w)], [x, J(y)]] \\
&\qquad = \frac{h^2}{4} \sum_{\alpha,\beta,\gamma} \langle [x, I_\alpha], [[y, I_\beta], [[z, w], I_\gamma]] \rangle \{I_\alpha, I_\beta, J(I_\gamma)\},
\end{aligned}$$

where

$$\{z_1, z_2, z_3\} = \frac{1}{6} \sum_{\pi \in S_3} z_{\pi(1)} z_{\pi(2)} z_{\pi(3)}.$$

The Hopf algebra structure is given by

$$\Delta(x) = x \otimes 1 + 1 \otimes x,$$

$$\Delta(J(x)) = J(x) \otimes 1 + 1 \otimes J(x) + \frac{h}{2}[x \otimes 1, t],$$

$$\varepsilon(x) = \varepsilon(J(x)) = 0.$$

$$S(x) = -x, \qquad S(J(x)) = -J(x) + \frac{c}{4}x,$$

where c is the eigenvalue of the Casimir element t in the adjoint representation of $\mathfrak{g}$. This is a deformation of $\mathfrak{g}[u]$ where the classical limit of x is x and the classical limit of $J(x)$ is xu. Note that

$$\Delta^{\mathrm{op}}(J(x)) - J(x) \otimes 1 + 1 \otimes J(x) - \frac{1}{2}h[x \otimes 1, t].$$

The relations can also be written in the form

$$\sum_i [a_i, b_i] = 0 \quad \text{implies} \quad \sum_i [J(a_i), J(b_i)] = ???$$

and

$$\sum_i [[a_i, b_i], c_i] = 0 \quad \text{implies} \quad \sum_i [[J(a_i), J(b_i)], J(c_i)] = ???$$

If $\mathfrak{g} = \mathfrak{sl}_2$ the the first of these is unneeded and if $\mathfrak{g} \neq \mathfrak{sl}_2$ then the second of these is unneeded.

9 Loop presentation of the Yangian

There is another presentation of $Y(\mathfrak{g})$ by

$$\text{generators} \qquad X_{i,r}^{\pm} \quad \text{and} \quad H_{,i,r}, \quad \text{for } 1 \le i \le n,\ r \in \mathbb{Z}_{\ge 0},$$

with

$$\deg(X_{i,r}^{\pm}) = \deg(H_{i,r}) = r,$$

and such that the classical limit of $X_{i,r}^{\pm}$ is $X_i^{\pm}u^r$, and the classical limit of $H_{i,r}$ is H_iu^r. The relations are

$$[H_{i,r}, H_{i,s}] = 0, \qquad [H_{i,0}, X_{j,s}^{\pm}] = \pm 2\langle \alpha_i, \alpha_j^{\vee}\rangle X_{j,s}^{\pm}, \qquad [X_{i,r}^{+}, X_{j,s}^{-}] = \delta_{ij} H_{i,r+s},$$

$$[H_{i,r+1}, X_{j,s}^{\pm}] - [H_{i,r}, X_{j,s+1}^{\pm}] = \pm h\langle \alpha_i^{\vee}, \alpha_j^{\vee}\rangle (H_{i,r}X_{j,s}^{\pm} + X_{j,s}^{\pm}H_{i,r}),$$

$$[X_{i,r+1}^{\pm}, X_{j,s}^{\pm}] - [X_{i,r}^{\pm}, X_{j,s+1}^{\pm}] = \pm h\langle \alpha_i^{\vee}, \alpha_j^{\vee}\rangle (X_{i,r}^{\pm}X_{j,s}^{\pm} + X_{j,s}^{\pm}X_{i,r}^{\pm}),$$

and, if $i \neq j$ and $m = 1 - \langle \alpha_i, \alpha_j^{\vee}\rangle$ and $r_1, \ldots, r_n \in \mathbb{Z}_{\ge 0}$ then

$$\sum_{\pi \in S_m} [X_{i,r_{\pi(1)}}^{\pm}, [X_{i,r_{\pi(2)}}^{\pm}, [\ldots, [X_{i,r_{\pi(m)}}^{\pm}, X_{j,s}^{\pm}] \cdots] = 0.$$

The relations imply that

$$X_{i,r+1}^{\pm} = \pm\frac{1}{2}[H_{i,1}, X_{i,r}^{\pm}] - \frac{1}{2}\frac{2}{\langle \alpha_i, \alpha_i\rangle}(H_{i,0}X_{i,r}^{\pm} + X_{i,r}^{\pm}H_{i,0}) \qquad \text{and} \qquad H_{i,r+1} = [X_{i,r+1}^{+}, X_{i,0}^{-}].$$

Remark. Perhaps formulas for $\Delta(X_{i,r}^{\pm})$ and $\Delta(H_{i,r})$ are not known.

The relation between the two presentations is given by

$$H_i = \tfrac{\langle \alpha_i, \alpha_i\rangle}{2} H_{i,0}, \qquad X_i^{\pm} = X_{i,0}^{\pm},$$

$$J(H_i) = \frac{\langle \alpha_i, \alpha_j\rangle}{2}\left(H_{i,1} - \left(\frac{2H_i}{\langle \alpha_i, \alpha_i\rangle}\right)^2 + \frac{1}{4}\sum_{\beta \in R^+} \langle \alpha_i, \beta^{\vee}\rangle \big(X_\beta^{+}X_\beta^{-} + X_\beta^{-}X_\beta^{+}\big)\right),$$

$$J(X_i^{\pm}) = X_{i,1}^{\pm} - \frac{1}{4}\frac{2}{\langle \alpha_i, \alpha_i\rangle}(X_i^{\pm}H_i + H_iX_i^{\pm}) \pm \frac{1}{4}\sum_{\beta \in R^+} \frac{2}{\langle \beta, \beta\rangle}\big([X_i^{\pm}X_\beta^{\pm}]X_\beta^{\mp} + X_\beta^{\mp}[X_i^{\pm}, X_\beta^{\pm}]\big).$$

10 Definition of the affine quantum group

11 Presentation of the affine quantum group

12 Loop presentation of the affine quantum group

13 Spectral algebras

Let A be a Hopf algebra with an invertible element

$$\mathcal{R} \in A \otimes A \qquad \text{such that} \qquad \mathcal{R}\Delta(a)\mathcal{R}^{-1} = \Delta^{\mathrm{op}}(a),$$

for all $a \in A$. Let $R = \sum a_i \otimes a^i$. If M and N are A-modules, define the operator

$$\begin{array}{rrcl} \check{R}_{MN}\colon & M \otimes N & \longrightarrow & N \otimes M \\ & m \otimes n & \longmapsto & \sum a^i n \otimes a_i m \end{array} \qquad \text{where} \quad \mathcal{R} = \sum a_i \otimes a^i,$$

is an A-module isomorphism since

$$\begin{aligned} \check{R}_{MN}(a(m \otimes n)) &= \check{R}_{MN}(\Delta(a)(m \otimes n)) = \sigma R \Delta(a)(m \otimes n) \\ &= \sigma \Delta^{op}(a) \sigma \sigma^{-1} R(m \otimes n) = \Delta(a) \check{R}_{MN}(m \otimes n) \end{aligned} \tag{13.1}$$

The pair $(A, \mathcal{R})$ is a *quasitriangular Hopf algebra* if

$$(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}^{13}\mathcal{R}^{23} \qquad \text{and} \qquad (\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}^{13}\mathcal{R}^{12}.$$

These relations say that if M, N and P are A-modules then

$$\check{R}_{M \otimes N, P} = (\check{R}_{MP} \otimes \mathrm{id})(\mathrm{id} \otimes \check{R}_{NP}) \qquad \text{and} \qquad \check{R}_{M, N \otimes P} = (\mathrm{id} \otimes \check{R}_{MP})(\check{R}_{MN} \otimes \mathrm{id}),$$

as operators on $M \otimes N \otimes P$.

Then

$$C_0 = \{\mu \in A^* \mid \mu(xy) = \mu(yx)\} \quad \text{is a commutative algebra,}$$

since, if $\ell_1, \ell_2 \in C_0$ and $a \in A$ then

$$\begin{aligned} (\ell_2\ell_1)(a) &= (\ell_1 \otimes \ell_2)\Delta^{\mathrm{op}}(a) = (\ell_1 \otimes \ell_2)\mathcal{R}\Delta(a)\mathcal{R}^{-1} \\ &= (\ell_1 \otimes \ell_2)\Delta(a)\mathcal{R}^{-1}\mathcal{R} = (\ell_1 \otimes \ell_2)\Delta(a) = (\ell_1\ell_2)(a), \end{aligned}$$

where the third equality uses the definition of C_0.

If $(A, \mathcal{R})$ is a quasitriangular Hopf algebra then $\mathcal{R}$ satisfies the *quantum Yang-Baxter equation* (QYBE),

$$\mathcal{R}^{12}\mathcal{R}^{13}\mathcal{R}^{23} = \mathcal{R}^{12}(\Delta \otimes \mathrm{id})(\mathcal{R}) = (\Delta^{\mathrm{op}} \otimes \mathrm{id})(\mathcal{R})\mathcal{R}^{12} = \mathcal{R}^{23}\mathcal{R}^{13}\mathcal{R}^{12}. \tag{13.2}$$

Since

$$\begin{aligned} \mathcal{R} &= (\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(\mathcal{R}) = (\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})\mathcal{R}^{13}\mathcal{R}^{23} = (\varepsilon \otimes \mathrm{id})(\mathcal{R}) \cdot \mathcal{R}, \quad \text{and} \\ \mathcal{R} &= (\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)(\mathrm{id} \otimes \Delta)(\mathcal{R}) = (\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)\mathcal{R}^{13}\mathcal{R}^{23} = (\mathrm{id} \otimes \varepsilon)(\mathcal{R}) \cdot \mathcal{R}, \end{aligned}$$

and so

$$(\varepsilon \otimes \mathrm{id})(\mathcal{R}) = 1 \qquad \text{and} \qquad (\mathrm{id} \otimes \varepsilon)(\mathcal{R}) = 1. \tag{13.3}$$

Then, since

$$\mathcal{R}(S \otimes \mathrm{id})(\mathcal{R}) = (m \otimes \mathrm{id})(\mathrm{id} \otimes S \otimes \mathrm{id})(\mathcal{R}^{13}\mathcal{R}^{23}) = (m \otimes \mathrm{id})(\mathrm{id} \otimes S \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(\mathcal{R}) = (\varepsilon \otimes \mathrm{id})(\mathcal{R}) = 1,$$

it follows that

$$(S \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}^{-1}. \tag{13.4}$$

Applying this to the pair $(A^{\mathrm{op}}, \mathcal{R}^{21})$ gives $(S^{-1} \otimes \mathrm{id})(\mathcal{R}^{21}) = (\mathcal{R}^{21})^{\mathrm{op}}$, and so

$$(\mathrm{id} \otimes S^{-1})(\mathcal{R}) = \mathcal{R}^{-1}. \tag{13.5}$$

Then

$$(S \otimes S)(\mathcal{R}) = (\mathrm{id} \otimes S)(S \otimes \mathrm{id})(\mathcal{R}) = (\mathrm{id} \otimes S)(\mathcal{R}^{-1}) = (\mathrm{id} \otimes S)(\mathrm{id} \otimes S^{-1}(\mathcal{R}) = \mathcal{R}. \tag{13.6}$$

The map $\phi\colon C \to Z(A)$ in the following proposition is an analogue of the Harish-Chandra homomorphism.

Proposition 13.1. *Let $(A, \mathcal{R})$ be a quasitriangular Hopf algebra. Then*

$$C = \{\lambda \in A^* \mid \lambda(xy) = \lambda(yS^2(x))\} \quad \textit{is a commutative algebra}$$

and the map

$$\begin{array}{rccc} \phi\colon & C & \longrightarrow & Z(A) \\ & \ell & \longmapsto & (\mathrm{id} \otimes \ell)(\mathcal{R}_{21}\mathcal{R}) \end{array}$$

is a well defined algebra homomorphism.

Proof. If $\ell_1, \ell_2 \in A^*$ and $a \in A$ then

$$\begin{aligned} (\ell_2\ell_1)(a) &= (\ell_1 \otimes \ell_2)\Delta^{\mathrm{op}}(a) = (\ell_1 \otimes \ell_2)(\mathcal{R}\Delta(a)\mathcal{R}^{-1}) \\ &= (\ell_1 \otimes \ell_2)(\Delta(a)\mathcal{R}^{-1}(S^2 \otimes S^2)(\mathcal{R})), \quad \text{by definition of } C, \\ &= (\ell_1 \otimes \ell_2)(\Delta(a)\mathcal{R}^{-1}\mathcal{R}), \qquad \text{by (???)}, \\ &= (\ell_1 \otimes \ell_2)(\Delta(a)) \\ &= (\ell_1\ell_2)(a), \end{aligned}$$

and hence C is a commutative algebra.

Let $a \in A$. First note that

$$\begin{aligned} a \otimes 1 &= (\mathrm{id} \otimes \varepsilon)\Delta(a) = (\mathrm{id} \otimes m)(\mathrm{id} \otimes S^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta^{\mathrm{op}})\Delta(a) \\ &= \sum_a a_{(1)} \otimes S^{-1}(a_{(3)})a_{(2)} = \sum_a (1 \otimes S^{-1}(a_{(2)}))(a_{(11)} \otimes a_{(12)}) \\ &= \sum_a (1 \otimes S^{-1}(a_{(2)}))\Delta(a), \end{aligned}$$

since S^{-1} is the antipode of A^{op}, and

$$\begin{aligned} a \otimes 1 &= (\mathrm{id} \otimes \varepsilon)\Delta(a) = (\mathrm{id} \otimes m)(\mathrm{id} \otimes \mathrm{id} \otimes S)(\mathrm{id} \otimes \Delta)\Delta(a) \\ &= \sum_a a_{(1)} \otimes a_{(2)}S(a_{(3)}) = \sum_a (a_{(11)} \otimes a_{(12)})(1 \otimes S(a_{(2)})) \\ &= \sum_a \Delta(a_{(1)})(1 \otimes S(a_{(2)})). \end{aligned}$$

Then, since

$$\mathcal{R}^{21}\mathcal{R}\Delta(a) = \mathcal{R}^{21}\Delta^{\mathrm{op}}(a)\mathcal{R} = \Delta(a)\mathcal{R}^{21}\mathcal{R},$$

$$\begin{aligned} a\phi(\ell) &= a(\mathrm{id} \otimes \ell)(\mathcal{R}^{21}\mathcal{R}) = (\mathrm{id} \otimes \ell)((a \otimes 1)\mathcal{R}^{21}\mathcal{R}^{12}) \\ &= (\mathrm{id} \otimes \ell)\left(\sum_a (1 \otimes S^{-1}(a_{(2)}))\Delta(a_{(1)})\mathcal{R}^{21}\mathcal{R}\right) \\ &= (\mathrm{id} \otimes \ell)\left(\sum_a \Delta(a_{(1)})\mathcal{R}^{21}\mathcal{R}(1 \otimes S(a_{(2)})\right), \quad \text{by definition of } C, \\ &= (\mathrm{id} \otimes \ell)\left(\mathcal{R}^{21}\mathcal{R}\sum_a \Delta(a_{(1)})(1 \otimes S(a_{(2)}))\right) \\ &= (\mathrm{id} \otimes \ell)(\mathcal{R}^{21}\mathcal{R}(a \otimes 1) = (\mathrm{id} \otimes \ell)(\mathcal{R}^{21}\mathcal{R})a = \phi(\ell)a, \end{aligned}$$

and so $\phi(\ell) \in Z(A)$. Since

$$\begin{aligned}\phi(\ell_1\ell_2) &= (\mathrm{id} \otimes \ell_1\ell_2)(\mathcal{R}^{21}\mathcal{R}) = (\mathrm{id} \otimes \ell_1 \otimes \ell_2)((\mathrm{id} \otimes \Delta)(\mathcal{R}^{21}\mathcal{R})) \\ &= (\mathrm{id} \otimes \ell_1 \otimes \ell_2)(\mathcal{R}^{21}\mathcal{R}^{31}\mathcal{R}^{13}\mathcal{R}^{12}) = (\mathrm{id} \otimes \ell_1)(\mathcal{R}^{21}(\phi(\ell_2) \otimes 1)\mathcal{R}^{12}) \\ &= (\mathrm{id} \otimes \ell_1)(\mathcal{R}^{21}\mathcal{R}(\phi(\ell_2) \otimes 1)), \quad \text{since } \phi(\ell_2) \in Z(A), \\ &= \phi(\ell_1)\phi(\ell_2),\end{aligned}$$

and so ϕ is a homomorphism. □

14 RTT realizations

Let A be a Hopf algebra with an invertible element

$$\mathcal{R} = \sum_r a_r \otimes b_r \quad \in A \otimes A \qquad \text{such that} \qquad \mathcal{R}\Delta(a)\mathcal{R}^{-1} = \Delta^{\mathrm{op}}(a),$$

for $a \in A$. The dual A^* of A is a Hopf algebra. Fix a positive integer n and an index set $\hat{T}$. Let

$$\{\rho^\lambda \colon A \to M_n(\mathbb{C}) \mid \lambda \in \hat{T}\}$$

be a set of representations of A. Their matrix entries

$$\rho^\lambda_{ij} \colon A \to \mathbb{C} \quad \text{are elements of } A^*.$$

On the ρ^λ_{ij}, the coproduct $\Delta \colon A^* \to A^* \otimes A^*$ has values

$$\Delta(\rho^\lambda_{ij}) = \sum_{k=1}^n \rho^\lambda_{ik} \otimes \rho^\lambda_{kj}, \qquad \text{since} \qquad \rho^\lambda_{ij}(u_1u_2) = \sum_{k=1}^n \rho^\lambda_{ik}(u_1)\rho^\lambda_{kj}(u_2),$$

for $u_1, u_2 \in A$. Let

$$\mathcal{R}(\lambda,\mu) = (\rho^\lambda \otimes \rho^\mu)(\mathcal{R}) \qquad \text{and} \qquad T(\lambda) = (\rho^\lambda_{ij}),$$

so that $T(\lambda)$ is a matrix in $M_n(A^*)$. Then

$$T(\lambda) \otimes \mathrm{id} = \textstyle\sum_{i,j,k} t^\lambda_{ij}(E_{ij} \otimes E_{kk}), \qquad \mathrm{id} \otimes T(\mu) = \sum_{i,j,k} t^\mu_{k\ell}(E_{ii} \otimes E_{k\ell}), \qquad \text{and}$$

$$\mathcal{R}(\lambda,\mu) = \sum_{i,j,k,\ell} \rho^\lambda_{ij}(a_r)\rho^\mu_{k\ell}(b_r)(E_{ij} \otimes E_{k\ell}).$$

Since

$$\mathcal{R}(\lambda,\mu)(T(\lambda) \otimes \mathrm{id})(\mathrm{id} \otimes T(\mu)) = \sum_{\substack{i,j,k,\ell \\ x,y}} \rho^\lambda_{ix}(a_r)t^\lambda_{xj}\rho^\mu_{xy}(b_r)t^\mu_{y\ell}\ (E_{ij} \otimes E_{k\ell}), \qquad \text{and}$$

$$(\mathrm{id} \otimes T(\mu))(T(\lambda) \otimes \mathrm{id})\mathcal{R}(\lambda,\mu) = \sum_{\substack{i,j,k\ell \\ \alpha,\beta}} t^\mu_{k\beta}t^\lambda_{i\alpha}\rho^\lambda_{\alpha j}(a_s)\rho^\mu_{\beta\ell}(b_s),$$

the equation

$$\mathcal{R}(\lambda,\mu)(T(\lambda) \otimes \mathrm{id})(\mathrm{id} \otimes T(\mu)) = (\mathrm{id} \otimes T(\mu))(T(\lambda) \otimes \mathrm{id})\mathcal{R}(\lambda,\mu)$$

is a concise way of encoding the relations

$$
\begin{aligned}
\left(\sum_{x,y} \rho^\lambda_{ix}(a_r)\rho^\mu_{ky}(b_r)\rho^\lambda_{xj}\rho^\mu_{y\ell}\right)(a) &= \sum_{x,y,a} \rho^\lambda_{ix}(a_r)\rho^\mu_{ky}(b_r)\rho^\lambda_{xj}(a_{(1)}\rho^\mu_{y\ell}(a_{(2)}) \\
&= \sum_a \rho^\lambda_{ij}(a_r a_{(1)})\rho^\mu_{k\ell}(b_r a_{(2)}) \\
= (\rho^\lambda_{ij}\otimes\rho^\mu_{k\ell})(\mathcal{R}\Delta(a)) &= (\rho^\lambda_{ij}\otimes\rho^\mu_{k\ell})(\Delta^{\mathrm{op}}(a)\mathcal{R}) \\
&= \sum_a \rho^\lambda_{ij}(a_{(2)}a_s)\rho^\mu_{k\ell}(a_{(1)}b_s) \\
&= \sum_{\alpha,\beta} \rho^\lambda_{i\alpha}(a_{(2)})\rho^\lambda_{\alpha j}(a_s)\rho^\mu_{k\beta}(a_{(1)})\rho^\mu_{\beta\ell}(b_s) \\
&= \left(\sum_{\alpha,\beta} \rho^\mu_{k\beta}\rho^\lambda_{i\alpha}\rho^\lambda_{\alpha_j}(a_s)\rho^\mu_{\beta\ell}(b_s)\right)(a)
\end{aligned}
$$

which are satisfied by the ρ^λ_{ij} in A^*.

Let B be the Hopf algebra given by

$$\text{generators} \qquad t^\lambda_{ij}, \qquad 1\le i,j\le n, \quad \lambda\in\hat{T},$$

and relations

$$\mathcal{R}(\lambda,\mu)(T(\lambda)\otimes\mathrm{id})(\mathrm{id}\otimes T(\mu)) = (\mathrm{id}\otimes T(\mu))(T(\lambda)\otimes\mathrm{id})\mathcal{R}(\lambda,\mu)$$

with comultiplication given by

$$\Delta(t^\lambda_{ij}) = \sum_{k=1}^n t^\lambda_{ik}\otimes t^\lambda_{kj}.$$

The the map

$$\begin{array}{ccc} B & \longrightarrow & A^* \\ t^\lambda_{ij} & \longmapsto & \rho^\lambda_{ij} \end{array}$$

is a Hopf algebra homomorphism.

We really want a map $B\to A$, not $B\to A^*$. But it is "easy" to make maps $A^*\to A$. For example, one can construct a map $A^*\to A$ by

$$l\to(id\otimes l)(R) \quad\text{or}\quad l\to(id\otimes l)(R_{21}^{-1}) \quad\text{or}\quad l\to(id\otimes l)(R_{21}R).$$

In the case of Yangian or $U_q(\mathfrak{g})$, the composition $\Phi: B\to A^*\to A$ is surjective and ker Φ is generated by the elments of the center of B.

15 Finite dimensional representations

Let M be a $Y(\mathfrak{g})$-module. Let

$$\mu_{i,r}\in\mathbb{C}, \qquad 1\le i\le n, \quad r\in\mathbb{Z}_{\ge 0}.$$

The *μ-weight space* of M is

$$M_\mu = \{m\in M \mid H_{i,r}m = \mu_{i,r}m, \text{ for } 1\le i\le n,\ r\in\mathbb{Z}_{\ge 0}\}.$$

A *highest weight vector* is a weight vector $v^+ \in M$ such that

$$X^+_{i,r} v = 0, \qquad 1 \le i \le n, \quad r \in \mathbb{Z}_{\ge 0}.$$

The *Verma module* $M(\mu)$ is the $Y(\mathfrak{g})$-module generated by v^+_μ with relations

$$H_{i,r} v^+_\mu = \mu_{i,r} v^+_\mu \qquad \text{and} \qquad X^+_{i,r} v^+_\mu = 0,$$

for $1 \le i \le n$, $r \in \mathbb{Z}_{\ge 0}$. Define

$$L(\mu) \quad \text{to be the unique simple quotient of} \quad M(\mu).$$

I DON'T LIKE THIS SETUP. THIS SHOULD BEGIN WITH A TRIANGULAR DECOMPOSITION OF $Y(\mathfrak{g})$.

Theorem 15.1. *The simple module $L(\mu)$ is finite dimensional if and only if there are monic polynomials $P_1, \ldots, P_n \in \mathbb{C}[u]$ such that*

$$\frac{P_i(u + d_i)}{P_i(u)} = 1 + \sum_{r \in \mathbb{Z}_{\ge 0}} \mu_{i,r} u^{-(r+1)}, \qquad \textit{for } 1 \le i \le n.$$

Proof. The module $\mathrm{Res}^{Y(\mathfrak{g})}_{U\mathfrak{g}} L(\mu)$ has a $U\mathfrak{g}$ submodule generated by v^+ and this is isomorphic to $L_{\mathfrak{g}}(\mu)$, and

$$(X^-_{i,0})^{\lambda_{i,0}+1} v^+ = 0, \quad \text{for } 1 \le i \le n.$$

So we want

$$P_i(u) = \sum_{r \in \mathbb{Z}_{\ge 0}} p_{i,r} u^r, \qquad \text{with } p_{i,r} \in Y(\mathfrak{g}),$$

such that

$$P_i(u + d_i) = P_i(u) \left(1 + \sum_{r \in \mathbb{Z}_{\ge 0}} H_{i,r} u^{-r-1} \right).$$

Solving for $P_i(u)$ is an $Y(\mathfrak{sl}_2)$ computation. □

Theorem 15.2. *For the affine quantum group with*

$$\mathcal{X}^+_{i,r} v^+ = 0 \qquad \textit{and} \qquad \Phi^\pm_{i,r} v^+ = \phi^\pm_{i,r} v^+.$$

the simple module $L(\phi)$ is finite dimensional if and only if there are monic polynomials $P_1, \ldots, P_n \in \mathbb{C}[z]$ with nonzero constant term such that

$$q_i^{\deg(P_i)} \frac{P_i(q_i^{-2} z)}{P_i(z)} = \sum_{r \in \mathbb{Z}_{\ge 0}} \phi^+_{i,r} z^r = \sum_{r \in \mathbb{Z}_{\ge 0}} \phi^-_{i,r} z^r, \qquad \textit{for } 1 \le i \le n.$$

16 The case $\mathfrak{sl}_2$

Let $\{x^\pm, h\}$ be a basis of $\mathfrak{sl}_2$ with

$$[h, x^\pm] = \pm 2x^\pm, \qquad [x^+, x^-] = h.$$

Then $Y(\mathfrak{sl}_2)$ has

$$\text{generators} \qquad x_k^\pm, \quad h_k, \qquad \text{for } k \in \mathbb{Z}_{\geq 0},$$

with relations

$$\begin{gathered}
[h_k, h_\ell] = 0, \qquad [h_0, x_k^\pm] = \pm 2x_k^\pm, \qquad [x_k^+, x_\ell^-] = h_{k+\ell}, \\
[h_{k+1}, x_\ell^-] - [h_k, x_{\ell+1}^\pm] = \pm(h_k x_\ell^\pm + x_\ell^\pm h_k) \\
[x_{k+1}^\pm, x_\ell^\pm] - [x_k^\pm, x_{\ell+1}^\pm] = \pm(x_k^\pm x_\ell^\pm + x_\ell^\pm x_k^\pm)
\end{gathered}$$

Let

$$x^\pm(u) = 0 + \sum_{k \in \mathbb{Z}_{\geq 0}} x_k^\pm u^{-k-1}, \qquad \text{and} \qquad h(u) = 1 + \sum_{k \in \mathbb{Z}_{\geq 0}} h_k u^{-k-1}.$$

Then the relations become

$$\begin{aligned}
&[h(u), h(v) = 0, \\
&[x^+(u), x^-(v)] = \frac{h(u) - h(v)}{u - v}, \\
&[x^-(u), x^-(v)] = \frac{-(x^-(u) - x^-(v))^2}{u - v}, \\
&[x^+(u), x^+(v)] = \frac{(x^+(u) - x^+(v))^2}{u - v}, \\
&[h(u), x^-(v)] = \frac{[h(u), x^-(u) - x^-(v)]_+}{u - v} \\
&[h(u), x^+(v)] = \frac{[h(u), x^+(u) - x^+(v)]_+}{u - v}
\end{aligned}$$

where $[a, b]_+ = ab + ba$.

The connection between the presentations is that

$$\begin{aligned}
&h = h_0, && x^\pm = x_0^\pm, \\
&J(h) = h_1 + \tfrac{1}{2}(x_0^+ x_0^- + x_0^- x_0^+ - h_0^2), && J(x^\pm) = x_1^\pm - \tfrac{1}{4}(x_0^\pm h_0 + h_0 x_0^\pm)
\end{aligned}$$

For $\lambda \in \mathbb{C}$ the maps

$$\begin{array}{rccc}
\tau_\lambda\colon & Y(\mathfrak{sl}_2) & \longrightarrow & Y(\mathfrak{sl}_2) \\
& x & \longmapsto & x \\
& J(x) & \longmapsto & J(x) + \lambda x
\end{array}$$

are Hopf algebra automorphisms of $Y(\mathfrak{sl}_2)$ and the map

$$\begin{array}{rccc}
\mathrm{ev}\colon & Y(\mathfrak{sl}_2) & \longrightarrow & Y(\mathfrak{sl}_2) \\
& x & \longmapsto & x \\
& J(x) & \longmapsto & 0
\end{array}$$

is an algebra homomorphism but not a Hopf algebra homomorphism.

Let $L(m)$ be the irreducible $\mathfrak{sl}_2$-module with basis $\{e_0, \ldots, e_m\}$ and $\mathfrak{sl}_2$ action

$$x^+ e_i = (i+1)e_{i+1}, \qquad x^- e_i = (m-i+1)e_{i-1}, \qquad he_i = (2i-m)e_i.$$

Then $(\mathrm{ev}\tau_a)^*(L(m))$ has $Y(\mathfrak{sl}_2)$ action given by

$$J(x^+)e_i = a(i+1)e_{i+1}, \qquad J(x^-)e_i = a(m-i+1)e_{i-1}, \qquad J(h)e_i = a(2i-m)e_i,$$

and

$$\begin{aligned} x_k^+ e_i &= (a - \frac{1}{2}m + i - \frac{1}{2})^k (i+1)e_{i+1}, \\ x_k^- e_i &= (a - \frac{1}{2}m + i - \frac{1}{2})^k (m-i+1)e_{i+1}, \\ h_k e_i &= \left((a - \frac{1}{2}m + i - \frac{1}{2})^k i(m-i+1) - (a - \frac{1}{2}m + i + \frac{1}{2})^k i(m-i) \right) e_i, \end{aligned}$$

Then $x_k^+ e_m = 0$ and $h_k e_m = m(a + (m-1)/2)^k e_m$, for $k \in \mathbb{Z}_{\geq 0}$. Then

$$\begin{aligned} \frac{P(u+1)}{P(u)} &= 1 + \sum_{k \in \mathbb{Z}_{\geq 0}} m \left(a + \frac{m-1}{2} \right)^k u^{-k-1} \\ &= 1 + m \sum_{k \in \mathbb{Z}_{\geq 0}} m \left(\frac{a + (m-1)/2}{u} \right)^k u^{-k} \\ &= 1 + mu^{-1} \left(\frac{1}{1 - (a + (m-1)/2)u^{-1}} \right) \\ &= 1 + \frac{m}{u - a - (m-1)/2} = \frac{u - a - \frac{m}{2} + \frac{1}{2} + m}{u - a - \frac{m}{2} + \frac{1}{2}} \\ &= \frac{u - a - \frac{m-1}{2} + m}{u - a - \frac{m-1}{2} + m - 1} \cdot \frac{u - a - \frac{m-1}{2} + m - 1}{u - a - \frac{m-1}{2} + m - 2} \cdots \frac{u - a - \frac{m-1}{2} + 1}{u - a - \frac{m-1}{2}} \end{aligned}$$

and

$$P(u) = \big(u - a - \frac{m-1}{2}\big)\big(u - a - \frac{m-3}{2}\big) \cdots \big(u - a + \frac{m-1}{2}\big).$$

For the quantum group

$$t_0 v_i = q^{(m-2s)/2} v_i, \quad t_1 v_i = q^{-(m-2s)/2} v_i, \quad x_1^+ v_i = [m-s+1] v_{i-1}, \quad x_1^- v_i = [i+1] v_{i+1},$$

and

$$P(z) = (z - a^{-1}q^{-(r-1)})(z - a^{-1}q^{-(r-3)}) \cdots (z - a^{-1}q^{(r-1)}).$$

17 Schur-Weyl duality

The graded Hecke algebra produces the Schur- Weyl duality for $Y(sl_N)$.

Definition. The graded Hecke algebra of type $GL_k(\mathbb{C})$ is the algebra H_k, generated by the group algebra $\mathbb{C}S_k$ of the symmetric group S_k and the elements $y_1, \ldots, y_k$ with the following relations:

$$\begin{aligned} \sigma y_i &= y_{\sigma(i)} \sigma, \qquad \sigma \in S_k, \\ [y_i, y_j] &= \frac{1}{4} \sum_{r \neq i,j} ((i,j,r) - (j,i,r)). \end{aligned}$$

The algebra H_k has the grading:

$$\deg(y_i) = 1, \quad \deg(\sigma) = 0.$$

Define

$$u_i = y_i - \frac{1}{2}\sum_{j=1}^{k} \operatorname{sgn}(i-j) s_{ij},$$

where s_{ij} is the transposition in S_k which switches i and j. Then for $s_i = (i, i+1)$

$$[u_i, u_j] = 0, \tag{17.1}$$
$$u_i s_i = s_i u_{i+1} + 1, \tag{17.2}$$
$$u_{i+1} s_i = s_i u_i - 1, \tag{17.3}$$
$$u_j s_i = s_i u_j, \quad \text{if} \quad j \neq i, i+1, \tag{17.4}$$

So $\mathbb{C}[u_1, \ldots, u_k] \subseteq H_k$, and there is an isomorphism of vector spaces

$$H_k = \mathbb{C}S_k \otimes \mathbb{C}[u_1, \ldots, u_k].$$

But this is not an isomorphism of algebras: u_i's and s_i' s do not commute.

Proof. Proof that the u_i generators and relations determine the others. □

Remark. Some history

Definition. (must be checked!) The affine Hecke algebra of type $GL_k(\mathbb{C})$ is the algebra $\tilde{H}_k$, generated by the Laurent polynomials $\mathbb{C}[x_1^{\pm 1}, \ldots, x_k^{\pm 1}]$ and elements $T_1, ..., T_{k-1}$ with the following relations:

$$\begin{aligned} T_i^2 &= (p-1)T_i + p \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \\ T_i T_j = T_j T_i, &\quad \text{if} \quad |i-j| > 1, \\ x_i T_i &= T_i x_{i+1} - (p-1) x_{i+1}, \\ x_{i+1} T_i &= T_i x_i + (p-1) x_{i+1} \\ x_j T_i = T_i x_j, &\quad \text{if} \quad j \neq i, i+1. \end{aligned} \tag{17.5}$$

Remark. Some history

Let $x_i = 1 - (p-1)U_i$, $s_i = T_i|_{p=1}$. Then the fourth equation in () becomes

$$(1-(p-1)U_i)T_i = T_i(1-(p-1)U_{i+1}) - (p-1)(1-(p-1)U_{i+1}).$$

Subtract T_i from each side and divide by $(p-1)$ to get

$$U_i T_i = T_i U_{i+1} + 1 - (p-1)U_{i+1}.$$

Observe that if we set $p = 1$, we get the relation (...) in the graded Hecke algebra H_k. This process is called degeneration. To make it precise is a pain (see Luzstig, he wrote it). If we complete the graded Hecke algebra, one gets the affine Hecke algebra (Lusztig). The representations of these algebras are the same.

For any $a \in \mathbb{C}$ one can define an automorphism

$$\begin{array}{rccccl} \tau_a\colon & H_k & \longrightarrow & H_k & & \\ & \sigma & \mapsto & \sigma, & \quad \text{for} & \sigma \in S_k, \\ & y_i & \mapsto & y_i + a & & \\ & u_i & \mapsto & u_i + a. & & \end{array} \tag{17.6}$$

Proof. It is easy to check that this map satisfies the relations for the s_i and u_i which are in (???). The image of y_i is then determined by the equation (???). $\square$

Remark. Recall that we have similar automorphisms in Yangians.

We also can construct an algebra homomorphism $H_l \otimes H_k \to H_{k+l}$, which extends the homomorphism of groups $S_k \times S_l \to S_{k+l}$. Recall that the last one allows us to induce the representations of $S_k \times S_l$ to the representations of S_{k+l} and corresponds to the multiplication of Schur functions. This is also the source of the classical Schur - Weyl dualitiy.

The algebra homomorphism

$$\begin{array}{ccc} H_l \otimes H_k & \longrightarrow & H_{k+l} \\ s_i \otimes 1 & \mapsto & s_i \\ 1 \otimes s_i & \mapsto & s_{i+l} \\ u_j \otimes 1 & \mapsto & u_j \\ 1 \otimes u_j & \mapsto & u_{j+l} \\ & & \\ y_j \otimes 1 & \mapsto & y_j + \frac{1}{2}\sum_{r=l+1}^{l+k}(j,r) \\ & & \\ 1 \otimes y_j & \mapsto & y_{j+l} - \frac{1}{2}\sum_{r=1}^{l}(j+l,r) \end{array} \tag{17.7}$$

Proof. It is easy to check that this map satisfies the relations for the s_i and u_i which are in (???). The images of $1 \otimes y_i$ and $y_i \otimes 1$ then follow from equation (???). $\square$

Using this map we can define

$$\mathrm{Ind}_{H_l \otimes H_k}^{H_{k+l}}(P \otimes Q)$$

There are 4 objects in the picture:

1. Graded Hecke algebra H_k,
2. Yangian $Y(sl_N)$
3. Affine Hecke algebra $?H_k$
4. p-adic group $GL_n(\mathbb{Q}_p)$

Roughly speaking, we have the equivalence of the following categories:

1. All finite-dimensional modules of H_k
2. Finite-dimensional $Y(sl_N)$-modules M, such that components of $\mathrm{Res}_{U(sl_N)}^{Y(sl_N)}(M)$ are in $(\mathbb{C}^N)^{\otimes k}$
3. Finite-dimensional modules of $\tilde{H}_k$
4. Weakly ramified admissible representations of $GL_k(\mathbb{Q}_p)$ (i.e. admissible representations with an Iwahori fixed vector).

We have the correspondence (?) of the following twistings:

1. Twisting by τ_a in H_k (defined above)
2. Twisting by τ_a in $Y(sl_N)$ (defined in sec....)
3. Twisting by τ_a in $\tilde{H}_k$ (here $x_k \to e^a x_k$)
4. If χ is a character of $\mathbb{Q}_p^* = GL_1(\mathbb{Q}_p)$ and π is some representation $\pi : GL_k(\mathbb{Q}_p) \to \mathrm{End}(M)$, then

$$(\chi \otimes \pi)(g) = \chi(det(g))\pi(g)$$

is the twisted representaion.

The modules can be multiplied ? induced? by the following means:

1. $\mathrm{Ind}_{H_l\otimes H_k}^{H_{k+l}}(P\otimes Q)$
2. Tensor product via $\delta : Y(sl_N)\to Y(sl_N)\otimes Y(sl_N)$
3. $\mathrm{Ind}_{\tilde{H}_l\otimes\tilde{H}_k}^{\tilde{H}_{k+l}}(P\otimes Q)$
4. Parabolic induction (= Harish-Chandra induction)

$$M\cdot N=\mathrm{Ind}_{P_{l,k}}^{GL_{k+l}}(\mathrm{Ind}_{GL_l\times GLk}^{P_{l,k}}(M\otimes N))$$

In each case we can ask, whether the product of modules $P\cdot Q$ is isomorphic to $Q\cdot P$. In all cases the answer is yes, but the isomorphism is not just permutation $p\cdot q\to q\cdot p$. These isomorphisms can be called $R-$matrices. From the $R-$matrix of any of these 4 objects one can get the information about any other.

The Schur-Weyl duality for Yangians is the correspondance

$$\begin{array}{cccc} F_k: & \{H_k-\text{modules}\} & \longrightarrow & \{Y(sl_N)\text{modules such that }\dots\} \\ & M & \longmapsto & M\otimes_{S_K}V^{\otimes k}\end{array}$$

where the $Y(_N)$ action on $M\otimes_{S_k}V^{\otimes k}$ is given by

$$I_\mu(m\otimes v_1\otimes ...\otimes v_k)=\sum_{i=1}^k m\otimes v_1\otimes ...\otimes I_\mu v_i\otimes ...\otimes v_k,$$

$$J_\mu(m\otimes v_1\otimes ...\otimes v_k)=\sum_{i=1}^k y_i m\otimes v_1\otimes ...\otimes I_\mu v_i\otimes ...\otimes v_k,$$

Remark. Note that in terms of y_i this is a nice expression but not in terms of the u_i.

How one can get these formulas? We want two things:

$$F_k(\tau_a(M))=\tau_a(F_k(M))\qquad\text{and}\qquad F_{l+k}(\mathrm{Ind}_{H_l\otimes H_k}^{H_{l+k}}(M\otimes N))=F_l(M)\otimes F_k(N).$$

18 Degeneration

Example: Degeneration from the double affine Hecke algebra to the double graded Hecke algebra

Example: Degeneration from the affine Hecke algebra to the affine Hecke algebra

To get this right we should really match up the finite dimensional representations of each algebra.

Let $U_h(L(\mathfrak{g}))$ be $U_h\mathfrak{g}$ without D and with $c=0$. Let A be the $\mathbb{C}[[h]]$-subalgebra of $U_h(L(\mathfrak{g}))\otimes_{\mathbb{C}[[h]]}\mathbb{C}((h))$ generated by

$$U_h(L(\mathfrak{g}))\qquad\text{and}\qquad\frac{1}{h}\ker f,$$

where

$$\begin{array}{cccccc} f: & U_h(L(\mathfrak{g})) & \longrightarrow & U(L(\mathfrak{g})) & \longrightarrow & U\mathfrak{g} \\ & h & \longmapsto & 0 & & \\ & & & t & \longmapsto & 1\end{array}$$

19 The double affine Hecke algebra

For type A_1, the *double graded Hecke algebra* $\mathbb{H}_{1,c}$ has generators t_s, x, y with

$$t_s^2 = 1, \qquad t_s x = -x t_s, \qquad t_s y = -y t_s, \qquad [y, x] = 1 - 2cs.$$

The *double affine Hecke algebra* $\tilde{K}_T$ has generators T, X and Y with

$$(T-\tau)(T+\tau^{-1}) = 0, \qquad TXT = x^{-1}, \qquad T^{-1}YT^{-1} = Y^{-1}, \qquad \text{and} \qquad Y^{-1}X^{-1}YXT^2 = q.$$

The conversion is

$$X = e^{hx}, \qquad Y = e^{hy}, \qquad T = se^{h^2cs}, \qquad q = e^{h^2}, \tau = e^{h^2c}.$$

Then

$$\frac{\tilde{K}_T}{h\tilde{K}_T} = \mathbb{H}_{1,c}.$$

20 The graded Hecke algebra

The *graded Hecke algebra* $\mathbb{H}$ is given by

$$\mathbb{H} = \mathbb{C}W \otimes S(\mathfrak{h}^*)$$

with multiplication determined by the multiplication in $S(\mathfrak{h}^*)$ and $\mathbb{C}W$ and the relations

$$pt_{s_i} = t_{s_i}(s_i p) + c_{\alpha_i}\Delta_i(p), \qquad \text{where} \quad \Delta_i(p) = \frac{p - s_i p}{\alpha_i},$$

for $p \in S(\mathfrak{h}^*)$. Equivalently, $t_{s_i} p = (s_i p)t_{s_i} + c_{\alpha_i}\Delta_i(p)$.

The *affine Hecke algebra* K_T is given by

$$K_T = H \otimes \mathbb{C}[P],$$

where

$$H \text{ is the finite Hecke algebra,} \qquad \text{and} \qquad \mathbb{C}[P] = \text{span}\{X^\lambda \mid \lambda \in P\}.$$

and

$$T_i X^\lambda = X^{s_i\lambda}T_i + (q_i - q_i^{-1})\frac{X^\lambda - X^{s_i\lambda}}{1 - X^{-\alpha_i}}.$$

In fact, one can convert from one to the other by the formulas

$$X^\lambda = e^{h\lambda}, \qquad q_i = e^{hc_i}, \qquad T_{s_i} = \frac{e^{hc_i} - e^{-hc_i}}{1 - e^{-h\alpha_i}} - \frac{c_i}{\alpha_i} + t_{s_i}.$$

For the graded Hecke algebra

$$\text{the } \tau\text{-operators are} \qquad \begin{array}{rccc} \tau_i\colon & M_\gamma^{\text{gen}} & \longrightarrow & M_{s_i\gamma}^{\text{gen}} \\ & m & \longmapsto & \left(t_{s_i} - \dfrac{c_{\alpha_i}}{\alpha_i}\right) m \end{array}$$

and the action on calibrated representations is given by

$$xv_w = (w\gamma)(x)v_w, \qquad t_{s_i}v_w = \frac{c_{\alpha_i}}{(w\gamma)(\alpha_i)}v_w + \left(1 + \frac{c_{\alpha_i}}{(w\gamma)(\alpha_i)}\right)v_{s_iw}.$$

For the affine Hecke algebra

$$\text{the } \tau\text{-operators are} \qquad \begin{array}{rccc} \tau_i\colon & M_t^{\mathrm{gen}} & \longrightarrow & M_{s_it}^{\mathrm{gen}} \\ & m & \longmapsto & \left(T_{s_i} - \dfrac{q-q^{-1}}{1-X^{-\alpha_i}}\right)m \end{array}$$

and the action on calibrated representations is given by

$$X^\lambda v_w = (wt)(X^\lambda)v_w, \qquad T_{s_i}v_w = \frac{q-q^{-1}}{1-(wt)(X^{-\alpha_i})}v_w + \left(q^{-1} + \frac{q-q^{-1}}{1-(wt)(X^{-\alpha_i})}\right)v_{s_iw}.$$

So

$$t(X^\lambda) = q^{\langle\gamma,\lambda\rangle} \qquad \text{and} \qquad X^\lambda = q^\lambda = e^{ch\lambda}.$$

Then

$$\left(T_{s_i} - \frac{q-q^{-1}}{1-X^{-\alpha_i}}\right) = t_{s_i} = \frac{c}{\alpha_i}.$$

So

$$T_{s_i} = \frac{q-q^{-1}}{1-e^{-h\alpha_i}} - \frac{c}{\alpha_i} + t_{s_i} = \frac{e^{hc}-e^{-hc}}{1-e^{-h\alpha_i}} - \frac{c}{\alpha_i} + t_{s_i} = \frac{e^{\frac{h}{2}(2c+\alpha_i)} - e^{-\frac{h}{2}(2c-\alpha_i)}}{e^{\frac{h}{2}\alpha_i} - e^{-\frac{h}{2}\alpha_i}} - \frac{c}{\alpha_i} + t_{s_i}.$$

So

$$\begin{aligned}
T_{s_i}X^\lambda &= \left(t_{s_i} - \frac{c}{\alpha_i} + \frac{e^{\frac{h}{2}(2c+\alpha_i)} - e^{-\frac{h}{2}(2c-\alpha_i)}}{e^{\frac{h}{2}\alpha_i} - e^{-\frac{h}{2}\alpha_i}}\right)e^{h\lambda} \\
&= e^{hs_i\lambda}t_{s_i} + c\frac{e^{h\lambda}-e^{hs_i\lambda}}{\alpha_i} - \frac{c}{\alpha_i} + \frac{e^{\frac{h}{2}(2c+\alpha_i)} - e^{-\frac{h}{2}(2c-\alpha_i)}}{e^{\frac{h}{2}\alpha_i} - e^{-\frac{h}{2}\alpha_i}}e^{h\lambda} \\
&= e^{hs_i\lambda}t_{s_i} - e^{hs_i\lambda}\frac{c}{\alpha_i} + e^{hs_i\lambda}\frac{e^{hc}-e^{-hc}}{1-e^{-h\alpha}} + \frac{c}{\alpha_i}(e^{h\lambda}-e^{hs_i\lambda}) \\
&\quad + e^{hs_i\lambda}\frac{c}{\alpha_i} - e^{hs_i\lambda}\frac{e^{hc}-e^{-hc}}{1-e^{-h\alpha_i}} - e^{h\lambda}\frac{c}{\alpha_i} + e^{h\lambda}\frac{e^{hc}-e^{-hc}}{1-e^{-h\alpha_i}} \\
&= e^{hs_i\lambda}T_{s_i} + \left(\frac{e^{h\lambda}-e^{hs_i\lambda}}{1-e^{-h\alpha_i}}\right)(e^{hc}-e^{-hc}).
\end{aligned}$$

21 The classical case: generators and relations

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra. The *Killing form* $\langle,\rangle\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ is given by

$$\langle x,y\rangle = \mathrm{Tr}(\mathrm{ad}_x\mathrm{ad}_y), \qquad \text{for } x,y\in\mathfrak{g}.$$

Up to constant multiples, this is the unique nondegenerate symmetric bilinear form on $\mathfrak{g}$. The standard presentation of $\mathfrak{g}$ by Chevalley generators and Serre relations is given by the generators

$$e_i, \qquad f_i, \qquad h_i, \qquad 1\le i\le n,$$

and relations

$$
\begin{gathered}
[h_i, h_j] = 0, \qquad 1 \le i, j \le n, \\
[h_i, e_j] = \alpha_j(h_i) e_j, \qquad [h_i, f_j] = -\alpha_j(h_i) f_j, \\
[e_i, f_j] = \delta_{ij} h_i, \\
\underbrace{[e_i, [e_i, [e_i, \cdots, [e_i}_{-a_{ij}+1\text{factors}}, e_j]] \cdots] = 0, \qquad i \ne j, \\
\underbrace{[f_i, [f_i, [f_i, \cdots, [f_i}_{-a_{ij}+1\text{factors}}, f_j]] \cdots] = 0, \qquad i \ne j.
\end{gathered}
$$

EXPLAIN what is a_{ij}, and what is $\alpha_i(h_j)$.

Define

$$
\begin{gathered}
\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d, \qquad \text{with bracket} \\
[x \otimes t^k + \lambda_1 c + \mu_1 d, y \otimes t^\ell + \lambda_2 2 + \mu_2 d] = [x, y] \otimes t^{k+\ell} + k\delta_{k,-\ell}\langle x, y\rangle c + ????,
\end{gathered}
$$

where $\langle , \rangle \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is the Killing form on $\mathfrak{g}$. The subalgebras

$$
L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \qquad \text{and} \qquad \hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,
$$

are, respectively, the *loop algebra* and the *affine Lie algebra* associated to $\mathfrak{g}$.

The Lie algebra $\tilde{\mathfrak{g}}$ can be given by generators

$$
e_{i,r} = e_i \otimes t^r, \quad f_{i,r} = f_i \otimes t^r, \quad h_{i,r} = h_i \otimes t^r, \qquad r \in \mathbb{Z}, i = 1, 2, \ldots, n,
$$

and the relations

????

It is often helpful to write these relations in a more compact form by using the generating functions

$$
e_i(u) = \sum_{r \in \mathbb{Z}} e_{i,r} u^r, \qquad f_i(v) = \sum_{r \in \mathbb{Z}} f_{i,r} v^r, \qquad h_i(z) = \sum_{r \in \mathbb{Z}} h_{i,r} z^r, \qquad 1 \le i \le n.
$$

With these notations the relations in ??? take the form

?????

The algebra $\hat{\mathfrak{g}}$ has an alternative presentation by generators

$$
c, \quad d, \quad e_i, \quad f_i, \quad h_i, \qquad 0 \le i \le n,
$$

and relations

$$
\begin{gathered}
[h_i, h_j] = 0, \qquad 1 \le i, j \le n, \\
[h_i, e_j] = \alpha_j(h_i) e_j, \qquad [h_i, f_j] = -\alpha_j(h_i) f_j, \\
[e_i, f_j] = \delta_{ij} h_i, \\
\underbrace{[e_i, [e_i, [e_i, \cdots, [e_i}_{-a_{ij}+1\text{factors}}, e_j]] \cdots] = 0, \qquad i \ne j, \\
\underbrace{[f_i, [f_i, [f_i, \cdots, [f_i}_{-a_{ij}+1\text{factors}}, f_j]] \cdots] = 0, \qquad i \ne j.
\end{gathered}
$$

where a_{ij} and $\alpha_i(h_j)$ are as in ??? and $a_{0i}, a_{i0}, \alpha_0(h_i), \alpha_i(h_0)$ are given by

????

.

In order to obtain the second presentation of $\tilde{\mathfrak{g}}$ from the first set

$$\begin{aligned} e_i &= e_i \otimes 1, \qquad f_i = f_i \otimes 1, \qquad h_i = h_i \otimes 1, \qquad \text{for } 1 \leq i \leq n, \text{ and} \\ e_0 &= f_\theta \otimes t, \qquad f_0 = e_\theta \otimes t^{-1}, \qquad h_0 = h_\theta \otimes 1 + \frac{2}{\langle \theta, \theta \rangle} c, \end{aligned}$$

where θ is the highest root of the root system of $\mathfrak{g}$ and e_θ, f_θ and h_θ are given by

?????

In order to obtain the first presentation of $\tilde{\mathfrak{g}}$ for the second presentation set

$$e_{i,r} = t^r_{\omega_i}(e_0), f_{i,r} = t^r_{\omega_i}(f_0), h_{i,r} = t^r_{\omega_i}(h_0), ???????? \tag{21.1}$$

where t_{ω_i} is the element of the extended affine Weyl group $\tilde{W}$ given by translating by the fundamental weight ω_i.

References

[Dr1] .G. Drinfel'd, *A new realization of Yangians and quantized affine algebras*, Soviet Math. Dokl. **36** No, 2 (1998), 212–216.