The Virasoro algebra

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1 The Virasoro algebra

Let A be an algebra over $\mathbb{C}$. A *derivation* of A is a linear map $d\colon A \to A$ such that

$$d(a_1a_2) = d(a_1)a_2 + a_1d(a_2), \qquad \text{for all } a_1, a_2 \in A.$$

The vector space of derivations on A is a Lie algebra with bracket

$$[d_1, d_2] = d_1d_2 - d_2d_1.$$

Let $\mathfrak{g}$ be a Lie algebra. A *central extension* of $\mathfrak{g}$ is a short exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{c} \longrightarrow \mathfrak{g}_1 \xrightarrow{\varphi_1} \mathfrak{g} \longrightarrow 0 \qquad \text{such that} \qquad \mathfrak{c} \subseteq Z(\mathfrak{g}_1),$$

the center of $\tilde{\mathfrak{g}}$. A *morphism* of central extensions is a Lie algebra homomorphism $\psi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\varphi_2\psi = \varphi_1$. A *universal central extension* is a central extension $\tilde{\mathfrak{g}}$ such that there is a unique morphism from $\tilde{\mathfrak{g}}$ to every other central extension of $\mathfrak{g}$. The *Schur multiplier* is the kernel of the universal central extension of $\mathfrak{g}$. It classifies the projective representations of $\mathfrak{g}$. (at least this is right for GROUPS, see Steinberg). Isomorphism classes of one-dimensional central extensions are in bijection with elements $c \in H^2(\mathfrak{g}; \mathbb{F})$ via the formula

$$[x, y]^{\sim} = [x, y] + c(x, y)\mathbf{c}, \qquad \text{for } x, y \in \mathfrak{g},$$

where $\mathbf{c}$ is a basis element of $Z(\tilde{\mathfrak{g}})$.

The *Witt algebra* is the Lie algebra of derivations of $\mathbb{C}[t, t^{-1}]$. If $d\colon \mathbb{C}[t, t^{-1}] \to \mathbb{C}[t, t^{-1}]$ is a derivation then

$$d(1) = 0, \quad d(t^k) = kt^{k-1}d(t), \quad \text{for all } k \in \mathbb{Z},$$

and hence d is determined by the value $d(t)$. Thus

$$W \qquad \text{has basis} \qquad \{d_j \mid j \in \mathbb{Z}\}, \qquad \text{where} \quad d_j = -t^{j+1}\frac{d}{dt},$$

and

$$[d_n, d_m] = (n - m)d_{n+m}.$$

Note that $\mathbb{C}[t, t^{-1}]$ is the complexification of the ring of smooth functions on the circle S^1.

The *Virasoro algebra* is the universal central extension of W. It has basis

$$\{c, d_i \mid i \in \mathbb{Z}\} \qquad \text{with} \qquad [c, d_i] = 0, \quad [d_n, d_m] = (n-m)d_{n+m} + \delta_{n,-m}\frac{n^3-n}{12}c.$$

To try to prove this note that if

$$[d_n, d_m] = (n-m)d_{n+m} + c(n,m)z,$$

then

$$[d_n, d_m] = -[d_m, d_n] \qquad \text{forces} \qquad c(n,m) = -c(m,n),$$

and the Jacobi identity forces

$$c(n+m, \ell) + c(\ell+n, m) + c(m+\ell, n) = 0.$$

The Virasoro algebra has triangular structure and skew linear ($\theta(\xi x) = \bar{\xi}\theta(x)$, for $\xi \in \mathbb{C}$ and $x \in \mathrm{Vir}$) *Cartan involution* given by

$$\begin{array}{l} \mathrm{Vir}_{<0} = \mathrm{span}\{d_i \mid i \in \mathbb{Z}_{<0}\}, \\ \mathrm{Vir}_0 = \mathrm{span}\{c, d_0\}, \\ \mathrm{Vir}_{>0} = \mathrm{span}\{d_i \mid i \in \mathbb{Z}_{>0}\}, \end{array} \qquad \text{with} \qquad \begin{array}{rccc} \theta\colon & \mathrm{Vir} & \longrightarrow & \mathrm{Vir} \\ & d_n & \longmapsto & d_{-n} \\ & c & \longmapsto & c \end{array}$$

Let U be the universal enveloping algebra of Vir. The action of $\mathfrak{h} = \mathrm{Vir}_0$ on $U_{<0}$ gives $U_{<0}$ a $\mathbb{Z}_{<0}$ grading such that

$$U_{-n} \qquad \text{has basis} \qquad \{d_{-\lambda} \mid \lambda \text{ is a partition of } n\} \qquad \text{where} \qquad d_{PICTURE} = d_{-\lambda} = d_{-\lambda_1} \cdots d_{-\lambda_\ell},$$

if $\lambda = (\lambda_1, \ldots, \lambda_\ell)$. This is the Poincaré-Birkhoff-Witt basis of $U_{<0}$.

1.1 The action on admissible $\hat{\mathfrak{g}}$ modules

Because the Witt algebra is the space of derivations of $\mathbb{C}[t, t^{-1}]$ the Witt algebra acts on the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, and the Virasoro algebra also acts on $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ by

$$[\tilde{d}_k, t^{\otimes} x] = t^{k+1}\frac{d}{dt}(t^n \otimes x) = nt^{n+k} \otimes x$$

and c acting by 0?? We can "extend" this action to an action admissible $\hat{\mathfrak{g}}$ modules.

Let h be the Coxeter number of $\mathfrak{g}$ and let

$$T_k = \frac{1}{2}\sum_{j\in\mathbb{Z}}\sum_i : u_i(-j)u^i(j+k) :$$

where the *normal ordering* is

$$: u_i(-j)u^i(j+k) := \begin{cases} u_i(-j)u^i(j+k), & \text{if } -j \le j+k, \\ u^i(j+k)u_i(-j), & \text{if } -j > j+k. \end{cases}$$

Proposition 1.1. *If V is a restricted $\hat{\mathfrak{g}}$-module of level ℓ and $\ell \neq -h$ then*

$$d_k \longmapsto \frac{1}{\ell+h}T_k \qquad \text{and} \qquad z \longmapsto \frac{\ell}{\ell+h}\dim(\mathfrak{g})$$

define an action of Vir *on V.*

Let $\mathfrak{g} = \mathfrak{sl}_2$ and use the imbedding

$$\begin{array}{cccc} \iota\colon & \mathfrak{sl}_2 & \longrightarrow & \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \\ & x & \longmapsto & (x,x) \end{array}$$

to define an action of Vir on $L(\xi) \otimes L(m\xi + \frac{n}{2}\alpha)$ by

$$d_k \longmapsto \frac{1}{\ell+h}(T_k \otimes 1 + 1 \otimes T_k) - \frac{1}{\ell+h}\iota(T_k).$$

This action of Vir commutes with the action of $\hat{\mathfrak{sl}}_2'$. By a character computation

$$L(\xi) \otimes L(m\xi + \tfrac{n}{2}\alpha) \cong \bigoplus_{k \in I} L_{\hat{\mathfrak{sl}}_2'}(\xi + \lambda - k\alpha) \otimes U_{m,n,k}, \qquad \text{as } (\hat{\mathfrak{sl}}_2', \mathrm{Vir}) \text{ bimodules, and}$$

$$L(\xi) \otimes L(m\xi + \tfrac{n}{2}\alpha) \cong \bigoplus_{\substack{k \in I \\ j \geq k^2}} L_{\hat{\mathfrak{sl}}_2}(\xi + \lambda - k\alpha - j\delta) \otimes U^j_{m,n,k}, \qquad \text{as } \hat{\mathfrak{sl}}_2 \text{ modules,}$$

where

$$I = \{k \in \mathbb{Z} \mid \tfrac{n}{2} - \tfrac{m+1}{2} \leq k \leq \tfrac{n}{2}\}, \qquad \text{and}$$

$$\mathrm{char}(U_{m,n,k}) = \sum_{j \in \mathbb{Z}_{\geq 0}} \dim(U^j_{m,n,k}) q^j = (f_{m,n,k} - f_{m,n,n+1-k}) \prod_{j \in \mathbb{Z}_{\geq 0}} \frac{1}{1-q^j},$$

with

$$f_{m,n,k} = \sum_{j \in \mathbb{Z}} q^{(m+2)(m+3)j^2 + (n+1+2k(m+2))j + k^2}.$$

Then z acts on $U_{m,n,k}$ by the constant

$$c = 1 - \frac{6}{(m+2)(m+3)},$$

and d_0 acts on $U^j_{m,n,k}$ by the constant

$$\frac{n(n+2)}{4(m+2)} - \frac{(n-2k)(n-2k+2)}{4(m+3)} + j$$

and the minimum value of j for which $U^j_{m,n,k} \neq 0$ is

$$j = k^2 \quad \text{when } d_0 \text{ acts by} \quad h_{r,s} = \frac{((m+3)r - (m+2)s)^2 - 1}{4(m+2)(m+3)}$$

where

$$\begin{array}{lll} r = n+1, & s = n+1-2k, & \text{if } k \geq 0, \\ r = m-n+1, & s = m-n+2+2k, & \text{if } k < 0. \end{array}$$

1.2 The Shapovalov determinant

Lemma 1.2. *The highest power of h in* $\det(M(h,c)^{(h+n,c)}$ *is*

$$\sum_{\lambda\vdash n}\ell(\lambda),\qquad \textit{with coefficient}\qquad \prod_{\lambda\vdash n} z_{2\lambda},$$

where, for a partition λ of n, $n!/z_\lambda$ is the cardinality of the conjugacy class of the symmetric group S_n labeled by λ.

Proof. Let us first analyze the entries $\langle d_{-\mu}v^+, d_\lambda v^+\rangle$ in the matrix. Then

$$\langle d_{-\mu}v^+, d_{-\lambda}v^+\rangle = \langle v^+, d_{mu}d_{-\lambda}v^+\rangle = p_{0,0}(h.c),$$

where $p_{0,0}(d_0,z)$ is the polynomial in d_0 and z in the PBW basis expansion

$$d_\mu d_{-\lambda} = \sum_{\nu,\tau} d_{-\nu}p_{\nu,\tau}(d_0,z)d_\tau.$$

This expansion is obtained by using the relations

$$\begin{aligned} d_kd_j &= d_jd_k + (k-j)d_{j+k}, &&\text{if } j+k\neq 0,\\ d_kd_{-k} &= d_{-k}d_k + (2kd_0 + \frac{k^2(k-1)}{12}z), &&\text{for } k>0,\\ d_1d_{-1} &= d_{-1}d_1 + 2d_0, \\ d_0d_{-k} &= d_{-k}d_0 + kd_{-k} = d_{-k}(d_0+k), \\ zd_{-k} &= d_{-k}z, \end{aligned}$$

to put the d_i in increasing order. The first relation "combines" j and k into $j+k$. If $d_{-\nu}p_{\nu,\tau}(d_0,z)d_\tau$ is a term in the PBW expansion then the parts of $-\nu$ and τ are combinations of parts of μ and $-\lambda$ and the degree in d_0 of the polynomial $p_{\nu,\tau}(d_0,z)$ is the maximal number of 0 parts that can be obtained by combinations of the remaining parts of μ and $-\lambda$ (those that do not contribute to ν and $-\tau$).

Thus the degree (in d_0) of $p_{0,0}(d_0,z)$ is the maximal number of 0 parts that can be obtained by combinations of the parts of μ and $-\lambda$ and is at most $\ell(\mu)$ and at most $\ell(\lambda)$. Since both λ and μ are partitions of n, a term of degree $\ell(\lambda)$ is produced only when $\lambda=\mu$ and each part of λ is combined with a single part of $-\lambda$. Thus the maximal degree term in row λ of $A(h,c)^{(h+n,c)}$ appears in column λ, i.e. on the diagonal.

The identity

$$d_rd_{-r}^s = d_{-r}^sd_r + d_{-r}^{s-1}\Big(2rsd_0 + 2r^2\binom{s}{2} + s\big(\tfrac{r^3-r}{12}\big)z\Big),$$

is verified by induction on s, the induction step being

$$\begin{aligned} d_rd_{-r}^s &= d_{-r}d_rd_{-r}^{s-1} + \Big(2rd_0 + \big(\tfrac{r^3-r}{12}\big)z\Big)d_{-r}^{s-1}\\ &= d_{-r}\Big(d_{-r}^{s-1}d_r + d_{-r}^{s-2}\Big(2r(s-1)sd_0 + 2r^2\binom{s-1}{2} + (s-1)s\big(\tfrac{r^3-r}{12}\big)z\Big)\\ &\qquad + d_{-r}^{s-1}\Big(2r(d_0+r(s-1)) + \big(\tfrac{r^3-r}{12}\big)z\Big)\\ &= d_{-r}^sd_r + d_{-r}^{s-1}\Big(2rsd_0 + 2r^2\binom{s}{2} + s\big(\tfrac{r^3-r}{12}\big)z\Big). \end{aligned}$$

Suppose that

$$d_r^k d_{-r}^s = d_{-r}^s d_r^k + d_{-r}^{s-1} p_1^{k,s} d_r^{k-1} + d_{-r}^2 p_2^{k,s} d_r^{k-2} + \cdots + d_{-r}^{s-k} p_k^{k,s},$$

where $p_i^{k,s}$ are polynomials in d_0 and z. Then

$$\begin{aligned}
d_r^{k+1} d_{-r}^s &= d_r \sum_{j=0}^{k} d_{-r}^{s-j} p_j^{k,s}(d_0,z) d_r^{k-j} \\
&= \sum_{j=0}^{k} d_{-r}^{s-j} d_r p_j^{k,s}(d_0,z) d_r^{k-j} + d_{-r}^{s-j-1}\Big(2r(s-j)d_0 + 2r^2\binom{s-j}{2} + (s-j)\big(\tfrac{r^3-r}{12}\big)z\Big) p_j^{k,s}(d_0,z) d_r^{k-j} \\
&= \sum_{j=0}^{k} d_{-r}^{s-j} p_j^{k,s}(d_0-r,z) d_r^{k-j+1} + d_{-r}^{s-j-1}\Big(2r(s-j)d_0 + 2r^2\binom{s-j}{2} + (s-j)\big(\tfrac{r^3-r}{12}\big)z\Big) p_j^{k,s}(d_0,z) d_r^{k-j},
\end{aligned}$$

from which it follows that

$$p_\ell^{k+1.s}(d_0,z) = p_\ell^{k,s}(d_0-r.z) + \Big(2r(s-\ell+1)d_0 + 2r^2\binom{s-\ell+1}{2} + (s-\ell+1)\big(\tfrac{r^3-r}{12}\big)z\Big) p_{\ell+1}^{k,s}(d_0,z).$$

(I'm not quite sure if this calculation is exaclty right, I need to do some checks for $s=2$ and $s=3$ to make sure). In particular,

$$p_{k+1}^{k+1,s} = \prod_{j=1}^{k+1}\Big(2r(s-j)d_0 + 2r^2\binom{s-j}{2} + (s-j)\big(\tfrac{r^3-r}{12}\big)z\Big) = (2r)^{k+1}(k+1)!d_0^{k+1} + \text{lower degree terms in } d_0.$$

$\square$

There is a bijection

$$\begin{array}{ccccc}
\{(\lambda,i)\mid \lambda\vdash n, 1\le i\le \ell(\lambda)\} & \leftrightarrow & \{(\mu,(r^s))\mid r^s=\emptyset, |\mu|+rs=n\} & & \\
(\lambda,i) & \longrightarrow & (\lambda-(\lambda_i^{s_i}),(\lambda_i^{s_i})) & & PICTURE \\
(\mu\cup(r^s),j) & \longleftarrow & (\mu,(r^s)) & &
\end{array}$$

where $\lambda-(\lambda_i^{s_i})$ is the partition obtained by removing all rows of length λ_i which are in rows with number $\ge i$, s_i is the number of $j\ge i$ such that $\lambda_j=\lambda_i$ and $j-1$ is the row number of the largest part $\le r$ in the partition μ.

This bijection proves the identity

$$\sum_{\lambda\vdash n}\ell(\lambda)\sum_{\substack{(r^s)\ne\emptyset \\ n-rs\ge 0}} p(n-rs),$$

where $p(k)$ is the number of partitions with k boxes.

Lemma 1.3. *If $k<n$ and d_0-h divides the determinant* $\det(M_{+k})$ *then* $(d_0-h)^{p(n-k)}$ *divides the determinant* $\det(M_{+n})$.

Lemma 1.4. $C_{rs}(h,c)$ *divides the determinant* $\det(M_{+rs})$.

Proof. First proof:
Second proof: Define a Vir action on the space of semi-infinite forms

$$\mathcal{H}(\alpha,\beta) = \mathrm{span}\{\cdots \wedge f_{i_k} \wedge \cdots \wedge f_{i_1} \mid \text{with } i_1 < i_2 < \cdots \text{ and } i_k = -k \text{ for } k \text{ large}\},$$

by setting

$$d_n(f_j) = (j + \beta - (1-n)\alpha) f_{j-n}.$$

Then, for appropriate choice of α and β, the Vir module $\mathcal{H}(\alpha,\beta)$ becomes a highest weight module of highest weight (h,c). One can construct a number of highest weight vectors in $\mathcal{H}(\alpha,\beta)$, see ???. □

1.3 Blocks

Given (h,c) the equation

$$\mu + \frac{1}{\mu} = \frac{13-c}{6} \qquad \text{determines} \qquad \{\mu, 1/\mu\},$$

and for each choice of μ in this set,

$$y^2 = 4\mu\Big(\frac{1-c}{24} - h\Big) \qquad \text{determines} \qquad \{y, -y\}$$

giving 4 lines

$$s = \mu r + y, \qquad s = \mu r - y, \qquad s = \frac{1}{\mu} r - \frac{1}{\mu} y, \qquad s = \frac{1}{\mu} r + \frac{1}{\mu} y.$$

Conversely, given (μ, y) then

$$\frac{13-c}{6} = \mu + \frac{1}{\mu} \qquad \text{determines} \qquad c,$$

and

$$h = \frac{-y^2}{4\mu} + \frac{1-c}{24} \qquad \text{determines} \qquad h.$$

Define

$$\begin{aligned}
C_{rs}(h,c) &= \frac{1}{4^2}(s-\mu r + y)(s - \mu r - y)(s - \frac{1}{\mu} r - \frac{1}{\mu} y)(s - \frac{1}{\mu} r + \frac{1}{\mu} y) \\
&= \frac{1}{4^2}((s-\mu r)^2 - y^2)((s - \frac{1}{\mu} r)^2 - \frac{y^2}{\mu^2}) \\
&= \left((s-\mu r)\frac{(\frac{1}{\mu}s - r)}{4} - \frac{y^2}{4\mu}\right)\left((s - \frac{1}{\mu} r)\frac{(\mu s - r)}{4} - \frac{y^2}{4\mu}\right) \\
&= \left(\frac{\mu r^2 - 2rs + \frac{1}{\mu}s^2}{4} - \frac{y^2}{4\mu}\right)\left(\frac{\mu r^2 - 2rs + \mu s^2}{4} - \frac{y^2}{4\mu}\right) \\
&= \left(\frac{1}{4}\big(\mu r^2 + \frac{1}{\mu}s^2\big) - \frac{rs}{2} + h - \frac{1-c}{24}\right)\left(\frac{1}{4}\big(\frac{1}{\mu}r^2 + \mu s^2\big) - \frac{rs}{2} + h - \frac{1-c}{24}\right).
\end{aligned}$$

If

$$x = \frac{1}{2}\sqrt{\frac{25-c}{1-c}}$$

then

$$(x-\frac{1}{2})(x+\frac{1}{2}) = x^2 - \frac{1}{4} = \frac{1}{4}\left(\frac{25-c}{1-c}\right) - \frac{1}{4} = \frac{1}{4}\left(\frac{25-c-1+c}{1-c}\right)$$
$$= \frac{1}{4}(\frac{24}{1-c}) = \frac{6}{1-c}$$

so that

$$\frac{1-c}{6} = \frac{1}{(x-\frac{1}{2})(x+\frac{1}{2})} \qquad \text{and} \qquad \frac{13-c}{6} = 2 + \frac{1-c}{6} = 2 + \frac{1}{(x-\frac{1}{2})(x+\frac{1}{2})}.$$

Then the solutions to $\mu + 1/\mu = (13-c)/6$ are

$$\mu = \frac{x+\frac{1}{2}}{x-\frac{1}{2}} \qquad \text{and} \qquad \frac{1}{\mu} = \frac{x-\frac{1}{2}}{x+\frac{1}{2}},$$

since

$$\frac{x+\frac{1}{2}}{x-\frac{1}{2}} + \frac{x-\frac{1}{2}}{x+\frac{1}{2}} = \frac{x^2-x+\frac{1}{4}+x^2+x+\frac{1}{4}}{(x-\frac{1}{2})(x+\frac{1}{2})} = \frac{2x^2+\frac{1}{2}}{x^2-\frac{1}{4}} = \frac{2x^2-\frac{1}{2}+1}{x^2-\frac{1}{4}} = 2 + \frac{1}{(x-\frac{1}{2})(x+\frac{1}{2})}.$$

Then

$$\begin{aligned}
&\frac{1-c}{24} - \frac{1}{4}(\mu r^2 + \frac{1}{\mu}s^2) + \frac{rs}{2} \\
&\quad = \frac{1-c}{24} - \frac{1}{4}\left(\frac{x+\frac{1}{2}}{x-\frac{1}{2}}r^2 + \frac{x-\frac{1}{2}}{x+\frac{1}{2}}s^2\right) + \frac{rs}{2} \\
&\quad = \frac{1-c}{24} - \frac{1}{4}\left(\frac{(x^2+x+\frac{1}{4})r^2 + (x^2-x+\frac{1}{4})s^2}{(x-\frac{1}{2})(x+\frac{1}{2})}\right) + \frac{rs}{2} \\
&\quad = \frac{1-c}{24} - \frac{1}{4}\left(\frac{2x^2+\frac{1}{2}}{(x-\frac{1}{2})(x+\frac{1}{2})}(r^2+s^2) + \frac{x}{(x-\frac{1}{2})(x+\frac{1}{2})}(r^2-s^2)\right) + \frac{rs}{2} \\
&\quad = \frac{1-c}{24} - \frac{1}{4}\left(\left(\frac{13-c}{6}\right)(r^2+s^2) + \frac{1}{2}\sqrt{\frac{25-c}{1-c}}\left(\frac{1-c}{6}\right)(r^2-s^2)\right) + \frac{rs}{2} \\
&\quad = \frac{1-c}{24} - \frac{1}{4}\left(\left(\frac{13-c}{6}\right)(r^2+s^2) + \frac{1}{12}\sqrt{(25-c)(1-c)}(r^2-s^2)\right) + \frac{rs}{2}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1-c}{24} - \frac{1}{4}(\mu r^2 + \frac{1}{\mu}s^2) + \frac{rs}{2} \\
&\quad = \frac{1-c}{24} - \frac{1}{4}\frac{(s-\mu r)^2}{\mu} \\
&\quad = \frac{1-c}{24} - \frac{1}{4}\left(s - \frac{x+\frac{1}{2}}{x-\frac{1}{2}}r\right)^2 \frac{x-\frac{1}{2}}{x+\frac{1}{2}} \\
&\quad = \frac{1}{4}\left(\frac{1}{(x-\frac{1}{2})(x+\frac{1}{2})} - \frac{((x-\frac{1}{2})s-(x+\frac{1}{2})r)^2}{(x-\frac{1}{2})(x+\frac{1}{2})}\right) \qquad = \frac{((x+\frac{1}{2})r-(x-\frac{1}{2})s)^2-1}{-4(x-\frac{1}{2})(x+\frac{1}{2})}
\end{aligned}$$

Now put $m+\frac{5}{2} = x$ so that $x+\frac{1}{2} = m+3$ and $x-\frac{1}{2} = m+2$.

Theorem 1.5.

$$\det(A_{-n}) = \prod_{1\le r\le s\le n} \big((2r)^s s!\big)^{p(n-rs)-p(n-r(s+1))} \prod_{\substack{r,s\in\mathbb{Z}_{\ge 0}\\ rs\le n}} (h-h_{rs})^{p(n-rs)},$$

where

$$h_{rs} = \frac{1}{48}\Big((13-c)(r^2+s^2) + \sqrt{(c-1)(c-25)}(r^2-s^2) - 24rs - 2 + 2c\Big).$$

Then

$$C_{r,s}(h,c) = \begin{cases} (h-h_{rs})(h-h_{sr}), & \text{if } r\ne s,\\ h-h_{rr}, & \text{if } r=s.\end{cases}$$

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