$G_{r,p,n}$ tantalizers

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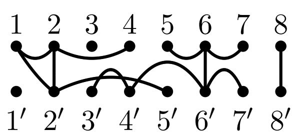
1 Partition algebras

For $k \in \mathbb{Z}_{>0}$, let

$$A_k = \left\{\text{set partitions of } \{1, 2, \ldots, k, 1', 2', \ldots, k'\}\right\}, \qquad \text{and}$$
$$A_{k+\frac{1}{2}} = \left\{d \in A_{k+1} \;\middle|\; (k+1) \text{ and } (k+1)' \text{ are in the same block}\right\}.$$

For convenience, represent a set partition $d \in A_k$ by a graph with k vertices in the top row, labeled $1, \ldots, k$ left to right, and k vertices in the bottom row, labeled $1', \ldots, k'$ left to right, with vertex i and vertex j connected by a path if i and j are in the same block of the set partition d. For example,



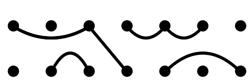
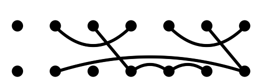
represents $\{\{1,2,4,2',5'\},\{3\},\{5,6,7,3',4',6',7'\},\{8,8'\},\{1'\}\}$,

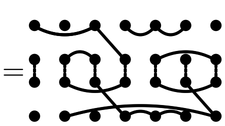
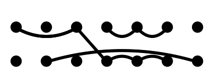
and has propagating number 3. The graph representing d is not unique.

The composition $d_1 \circ d_2$ of partition diagrams $d_1, d_2 \in A_k$ is the set partition obtained by placing d_1 above d_2 and identifying the bottom dots of d_1 with the top dots of d_2, removing any connected components that live entirely in the middle row. For $k \in \frac{1}{2}\mathbb{Z}_{>0}$ and $n \in \mathbb{C}$, the *partition algebra* $\mathbb{C}A_k(n)$ is the associative algebra over $\mathbb{C}$ with basis A_k,

$$\mathbb{C}A_k(n) = \mathbb{C}\text{span-}\{d \in A_k\}, \qquad \text{and multiplication defined by} \qquad d_1 d_2 = n^{\ell}(d_1 \circ d_2),$$

where ℓ is the number of blocks removed from the the middle row when constructing the composition $d_1 \circ d_2$. For example,

if $d_1 =$  and $d_2 =$  then

$d_1 d_2 =$  $= n^2$ , (1.1)

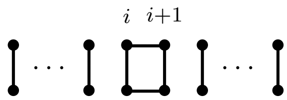
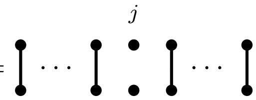
since two blocks are removed from the middle row.

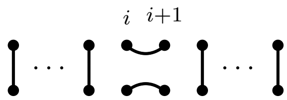
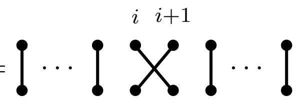
Another basis of $\mathbb{C}A_k(n)$ is

$$\{x_d \mid d \in A_k\} \qquad \text{given by} \qquad d = \sum_{d' \le d} x_d, \qquad \text{where} \tag{1.2}$$

$d' \le d$ if the set partition d' is coarser than the set partition d.

Let $k \in \mathbb{Z}_{>0}$. For $1 \le i \le k-1$ and $1 \le j \le k$, define

$p_{i+\frac{1}{2}} =$ , $\qquad p_j =$ ,

$e_i =$ , $\qquad s_i =$ . (1.3)

These elements satisfy relations

$$e_i^2 = e_i, \qquad e_i e_{i\pm 1} e_i = e_i, \qquad \text{and} \qquad e_i e_j = e_j e_i, \quad \text{for } |i-j| > 1.$$

$$p_i^2 = p_i, \qquad p_i p_{i\pm\frac{1}{2}} p_i = p_i, \qquad \text{and} \qquad p_i p_j = p_j p_i, \quad \text{for } |i-j| > 1/2.$$

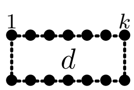
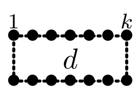
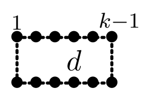
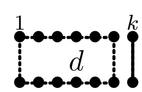
$$s_i^2 = 1, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad \text{and} \qquad s_i s_j = s_j s_i, \quad \text{for } |i-j| > 1.$$

and

$$\begin{array}{lll} s_i p_i p_{i+1} = p_i p_{i+1} s_i = p_i p_{i+1}, & s_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} s_i = p_{i+\frac{1}{2}}, & s_i p_i s_i = p_{i+1}, \\ s_i s_{i+1} p_{i+\frac{1}{2}} s_{i+1} s_i = p_{i+\frac{3}{2}}, & \text{and} \quad s_i p_j = p_j s_i, & \text{for } j \ne i - \frac{1}{2}, i, i+\frac{1}{2}, i+1, i+\frac{3}{2}. \end{array}$$

There are inclusions of algebras given by

$\mathbb{C}A_{k-\frac{1}{2}}(n) \hookrightarrow \mathbb{C}A_k(n)$ $\qquad$ and $\qquad$ $\mathbb{C}A_{k-1}(n) \hookrightarrow \mathbb{C}A_{k-\frac{1}{2}}(n)$

 $\mapsto$  $\qquad$ and $\qquad$  $\mapsto$ . (1.4)

giving

$$A_0(n) \subseteq A_{\frac{1}{2}}(n) \subseteq A_1(n) \subseteq A_{\frac{3}{2}}(n) \subseteq \cdots.$$

The *propagating number* of $d \in A_k$ is

$$pn(d) = \begin{pmatrix} \text{the number of blocks in } d \text{ that contain both an element} \\ \text{of } \{1, 2, \ldots, k\} \text{ and an element of } \{1', 2', \ldots, k'\} \end{pmatrix}. \tag{1.5}$$

The propagating number satisfies $pn(d_1 \circ d_2) \le \min(pn(d_1), pn(d_2))$ and so there is a chain of ideals in $A_k(r, p, n)$,

$$I_1 \subseteq I_1 \subseteq \cdots \subseteq I_{\lfloor k \rfloor}, \qquad \text{given by} \qquad I_\ell = \mathrm{span}\{d \mid d \in A_k(r,p,n), pn(d) \le \ell\}.$$

The maps

$$\begin{array}{ccc} \mathbb{C}A_{k-\frac{1}{2}}(n) \otimes_{\mathbb{C}A_{k-1}(n)} \mathbb{C}A_{k-\frac{1}{2}}(n) & \longrightarrow & \mathbb{C}I_k(n) \\ b_1 \otimes b_2 & \longmapsto & b_1 p_k b_2 \end{array} \tag{1.6}$$

are vector space isomorphisms.

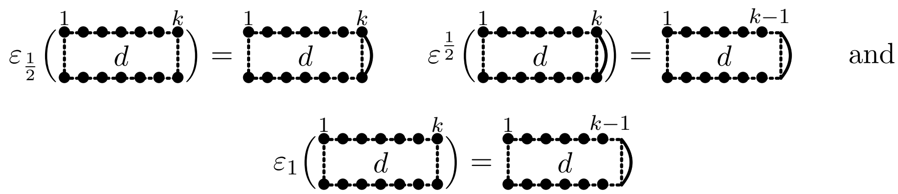
Let $k \in \mathbb{Z}_{>0}$. Define linear maps

$$\varepsilon_{\frac{1}{2}} : \mathbb{C}A_k \to \mathbb{C}A_{k-\frac{1}{2}}, \qquad \varepsilon^{\frac{1}{2}} : \mathbb{C}A_{k-\frac{1}{2}} \to \mathbb{C}A_{k-1} \qquad \text{and} \qquad \varepsilon_1 : \mathbb{C}A_k \to \mathbb{C}A_{k-1}$$

by the equations

$$p_{k+\frac{1}{2}} b p_{k+\frac{1}{2}} = \varepsilon_{\frac{1}{2}}(b) p_{k+\frac{1}{2}}, \qquad p_k b p_k = \varepsilon^{\frac{1}{2}}(b) p_k, \qquad \text{and} \qquad e_k b e_k = \varepsilon_1(b) e_k. \tag{1.7}$$

Pictorially



so that $\varepsilon_1 = \varepsilon^{\frac{1}{2}} \circ \varepsilon_{\frac{1}{2}}$, $\varepsilon_{\frac{1}{2}}(d)$ is the same as d except that the block containing k and the block containing k' are combined, and $\varepsilon^{\frac{1}{2}}(d)$ has the same blocks as d except with k and k' removed. There is a factor of n in $\varepsilon^{\frac{1}{2}}(d)$ if the removal of k and k' reduces the number of blocks by 1.

Pulling the product in the ideal I_k back across the isomorphism in (???) gives

$$\text{a product on} \qquad \mathbb{C}A_{k-\frac{1}{2}}(n) \otimes_{\mathbb{C}A_{k-1}(n)} \mathbb{C}A_{k-\frac{1}{2}}(n) \qquad \text{given by} \tag{1.8}$$

$$(b_1 \otimes b_2)(b_3 \otimes b_4) = b_1 \otimes \varepsilon_{\frac{1}{2}}(b_2 b_3) b_4, \qquad \text{and} \qquad (c_1 \otimes c_2)(c_3 \otimes c_4) = c_1 \otimes \varepsilon^{\frac{1}{2}}(c_2 c_3) c_4,$$

for $b_1, b_2, b_3, b_4 \in \mathbb{C}A_k(n)$, and for $c_1, c_2, c_3, c_4 \in \mathbb{C}A_{k-\frac{1}{2}}(n)$, $k \in \mathbb{Z}_{>0}$.

Define $tr_k \colon \mathbb{C}A_k \to \mathbb{C}$ and $tr_{k-\frac{1}{2}} \colon \mathbb{C}A_{k-\frac{1}{2}} \to \mathbb{C}$ by the equations

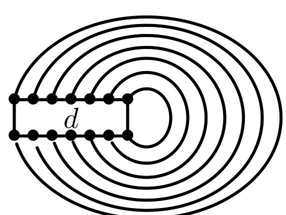
$$\mathrm{tr}_k(b) = \mathrm{tr}_{k-\frac{1}{2}}(\varepsilon_{\frac{1}{2}}(b)), \qquad \text{and} \qquad tr_{k-\frac{1}{2}}(b) = \mathrm{tr}_{k-1}(\varepsilon^{\frac{1}{2}}(b)), \tag{1.9}$$

so that

$$tr_k(b) = \varepsilon_1^k(b), \qquad \text{and} \qquad tr_{k-\frac{1}{2}}(b) = \varepsilon_1^{k-1}\varepsilon^{\frac{1}{2}}(b). \tag{1.10}$$

Pictorially $tr_k(d) = n^c$ where c is the number of connected components in the closure of the diagram d,

$$tr_k(d) = $$



$$, \qquad \text{for } d \in A_k. \tag{1.11}$$

Martin's Morita equivalence

The H colored partition algebra: color the vertices of each set partition by elements of a group H and define two colorings of a set partition to be equivalent if, for each block, the product of the colors on that block coincide. Define the product in the usual way except that the colors multiply when they hit and propagate to the end of the edge and each floating block turns into a factor of $n|H|$. Let $W = \mathrm{Ind}_{G_{H,1,n-1}}^{G_{H,1,n}}(\mathbf{1})$ so that $\dim(W) = n|H|$. Then

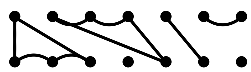
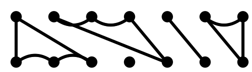
$$A_{k,H,1}(n) \cong \mathrm{End}_{G_{H,1,n}}(W^{\otimes k}).$$

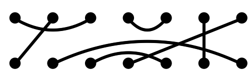
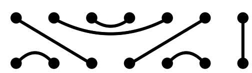
2 Subalgebras of the partition algebra

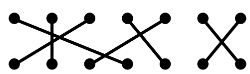
A set partition is *planar* [Jo] if it can be represented as a graph without edge crossings inside of the rectangle formed by its vertices. For each $k \in \frac{1}{2}\mathbb{Z}_{>0}$, the following are subalgebras of the partition algebra $\mathbb{C}A_k(n)$:

$$\begin{aligned} \mathbb{C}S_k &= \text{span}\{d \in A_k \mid pn(d) = k\}, \\ \mathbb{C}P_k(n) &= \text{span}\{d \in A_k \mid d \text{ is planar}\}, \\ \mathbb{C}B_k(n) &= \text{span}\{d \in A_k \mid \text{all blocks of } d \text{ have size } 2\}, \quad \text{and} \\ \mathbb{C}T_k(n) &= \text{span}\{d \in A_k \mid d \text{ is planar and all blocks of } d \text{ have size } 2\}. \end{aligned} \tag{2.1}$$

The algebra $\mathbb{C}S_k$ is the *group algebra of the symmetric group*, $\mathbb{C}P_k(n)$ is the *planar partition algebra*, $\mathbb{C}B_k(n)$ is the *Brauer algebra*, and $\mathbb{C}T_k(n)$ is the *Temperley-Lieb algebra*. Examples of set partitions in these algebras are

 $\in P_7$,  $\in P_{6+\frac{1}{2}}$,

 $\in B_7$,  $\in T_7$,

 $\in S_7$.

If B is a block of a set partition d define

$$\kappa(B) = \big|\, (\#\text{ of top vertices in } B) - (\#\text{ of bottom vertices in B}) \,\big|$$

and let

$$A_{k,r,p} = \bigsqcup_{\ell=0}^{(r/p)-1} \{d \in A_k \mid \text{ for all blocks } B \text{ of } d,\ \kappa(B) = \ell(r/p) \bmod r\}$$

Then

$$\mathbb{C}A_{k,r,p}(n) = \text{span}\{x_d \mid d \in A_{k,r,p}\}$$

is a subalgebra of $\mathbb{C}A_k(n)$. Then

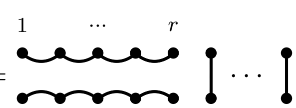
$$\mathbb{C}A_{k,r,p}(n) \supseteq \mathbb{C}A_{k,r,1}(n), \qquad \mathbb{C}A_{k,1,1}(n) = \mathbb{C}A_k(n),$$

and

$$\mathbb{C}A_{k,\infty,1} = \text{span}\{d \in A_k \mid \kappa(B) = 0 \text{ for all blocks } B \text{ of } d\}$$

does not depend on the parameter n.

Let

$$f_r = p_{1/2}p_{3/2}\cdots p_{(r-1)/2}p_1p_2\cdots p_rp_{1/2}p_{3/2}\cdots p_{(r-1)/2} =$$ 

,

The algebra $A_{k,r,1}(n)$ is generated by $s_1, \ldots, s_{k-1}$, $p_{\frac{3}{2}}$ and f_r.

Is $CCA_k(r,p,n) = \mathbb{C}A_k(r,p,n) \times \mathbb{Z}/p\mathbb{Z}$?

3 Schur-Weyl dualities

Let $n \in \mathbb{Z}_{>0}$ and let V be a vector space with basis $v_1, \ldots, v_n$. Then the tensor product

$$V^{\otimes k} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{k \text{ factors}} \qquad \text{has basis} \qquad \{\, v_{i_1} \otimes \cdots \otimes v_{i_k} \mid 1 \le i_1, \ldots, i_k \le n \,\}.$$

For $k \in \mathbb{Z}_{>0}$ let

$$V^{\otimes(k+\frac{1}{2})} = V^{\otimes k} \otimes v_n, \qquad \text{a subspace of } V^{\otimes(k+1)}$$

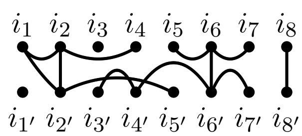
(which is isomorphic, as a vector space, to $V^{\otimes k}$). If $b \in \mathrm{End}(V^{\otimes k})$ let $b^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}} \in \mathbb{C}$ be the coefficients in the expansion

$$b(v_{i_1} \otimes \cdots \otimes v_{i_k}) = \sum_{1 \le i_{1'},\ldots i_{k'} \le n} b^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}} v_{i_{1'}} \otimes \cdots \otimes v_{i_{k'}}. \tag{3.1}$$

For $d \in A_k$ and values $i_1, \ldots, i_k, i_{1'}, \ldots, i_{k'} \in \{1, \ldots, n\}$ define

$$(d)^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}} = \begin{cases} 1, & \text{if } i_r = i_s \text{ when } r \text{ and } s \text{ are in the same block of } d, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

For example, viewing $(d)^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}}$ as the diagram d with vertices labeled by the values $i_1, \ldots, i_k$ and $i_{1'}, \ldots, i_{k'}$, we have



$$= \delta_{i_1 i_2} \delta_{i_1 i_4} \delta_{i_1 i_{2'}} \delta_{i_1 i_{5'}} \delta_{i_5 i_6} \delta_{i_5 i_7} \delta_{i_5 i_{3'}} \delta_{i_5 i_{4'}} \delta_{i_5 i_{6'}} \delta_{i_5 i_{7'}} \delta_{i_8 i_{8'}}.$$

It follows from (???) and (???) that for all $d \in A_k$,

$$(\Phi_k(x_d))^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}} = \begin{cases} 1, & \text{if } i_r = i_s \text{ if and only if } r \text{ and } s \text{ are in the same block of } d, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

The group $GL_n(\mathbb{C})$ acts on the vector spaces V and $V^{\otimes k}$ by

$$gv_i = \sum_{j=1}^{n} g_{ji} v_j, \qquad \text{and} \qquad g(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = gv_{i_1} \otimes gv_{i_2} \otimes \cdots \otimes gv_{i_k}, \tag{3.4}$$

for $g = (g_{ij}) \in GL_n(\mathbb{C})$. For any subgroup $G \subseteq GL_n(\mathbb{C})$,

$$\mathrm{End}_G(V^{\otimes k}) = \left\{ b \in \mathrm{End}(V^{\otimes k}) \mid bgv = gbv \text{ for all } g \in G \text{ and } v \in V^{\otimes k} \right\}.$$

Theorem 3.1. *Let $n \in \mathbb{Z}_{>0}$ and let $\{x_d \mid d \in A_k\}$ be the basis of $\mathbb{C}A_k(n)$ defined in (???). Then the notation in (???) and (???) defines algebra homomorphisms*

$$\Phi_k : \mathbb{C}A_k(n) \longrightarrow \mathrm{End}(V^{\otimes k}) \qquad \textit{for } k \in \tfrac{1}{2}\mathbb{Z}_{>0}. \tag{3.5}$$

giving a right action of the partition algebra $\mathbb{C}A_k(n)$ on $V^{\otimes k}$. View the symmetric group S_n as the subgroup of $GL_n(\mathbb{C})$ of permutation matrices.

(a) $\Phi_k : \mathbb{C}A_k(n) \to \mathrm{End}(V^{\otimes k})$ *has*

$$\mathrm{im}\, \Phi_k = \mathrm{End}_{S_n}(V^{\otimes k}) \qquad \textit{and} \qquad \ker \Phi_k = \mathbb{C}\text{-}\textit{span}\{x_d \mid d \textit{ has more than } n \textit{ blocks}\}, \qquad \textit{and}$$

(b) $\Phi_{k+\frac{1}{2}} : \mathbb{C}A_{k+\frac{1}{2}}(n) \to \mathrm{End}(V^{\otimes k})$ *has*

$$\mathrm{im}\, \Phi_{k+\frac{1}{2}} = \mathrm{End}_{S_{n-1}}(V^{\otimes k}) \qquad \textit{and} \qquad \ker \Phi_{k+\frac{1}{2}} = \mathbb{C}\text{-}\textit{span}\{x_d \mid d \textit{ has more than } n \textit{ blocks}\}.$$

Proof. (a) As a subgroup of $GL_n(\mathbb{C})$, S_n acts on V via its permutation representation and S_n acts on $V^{\otimes k}$ by

$$\sigma(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = v_{\sigma(i_1)} \otimes v_{\sigma(i_2)} \otimes \cdots \otimes v_{\sigma(i_k)}. \tag{3.6}$$

Then $b \in \mathrm{End}_{S_n}(V^{\otimes k})$ if and only if $\sigma^{-1} b \sigma = b$ (as endomorphisms on $V^{\otimes k}$) for all $\sigma \in S_n$. Thus, using the notation of (???), $b \in \mathrm{End}_{S_n}(V^{\otimes k})$ if and only if

$$b^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}} = (\sigma^{-1} b \sigma)^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}} = b^{\sigma(i_1),\ldots,\sigma(i_k)}_{\sigma(i_{1'}),\ldots,\sigma(i_{k'})}, \qquad \text{for all } \sigma \in S_n.$$

It follows that the matrix entries of b are constant on the S_n-orbits of its matrix coordinates. These orbits decompose $\{1, \ldots, k, 1', \ldots, k'\}$ into subsets and thus correspond to set partitions $d \in A_k$. Thus $\Phi_k(x_d)$ has 1s in the matrix positions corresponding to d and 0s elsewhere, and so b is a linear combination of $\Phi_k(x_d), d \in A_k$. Since $x_d, d \in A_k$, form a basis of $\mathbb{C}A_k$, im $\Phi_k = \mathrm{End}_{S_n}(V^{\otimes k})$.

If d has more than n blocks, then by (???) the matrix entry $(\Phi_k(x_d))^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}} = 0$ for all indices $i_1, \ldots, i_k, i_{1'}, \ldots, i_{k'}$, since we need a distinct $i_j \in \{1, \ldots, n\}$ for each block of d. Thus, $x_d \in \ker \Phi_k$. If d has $\leq n$ blocks, then we can find an index set $i_1, \ldots, i_k, i_{1'}, \ldots, i_{k'}$ with $(\Phi_k(x_d))^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}} = 1$ simply by choosing a distinct index from $\{1, \ldots, n\}$ for each block of d. Thus, if d has $\leq n$ blocks then $x_d \notin \ker \Phi_k$, and so $\ker \Phi_k = \mathbb{C}$-span$\{x_d | d$ has more than n blocks$\}$.

(b) The vector space $V^{\otimes k} \otimes v_n \subseteq V^{\otimes (k+1)}$ is a submodule both for $\mathbb{C}A_{k+\frac{1}{2}} \subseteq \mathbb{C}A_{k+1}$ and $\mathbb{C}S_{n-1} \subseteq \mathbb{C}S_n$. If $\sigma \in S_{n-1}$, then $\sigma(v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_n) = v_{\sigma(i_1)} \otimes \cdots \otimes v_{\sigma(i_k)} \otimes v_n$. Then as above $b \in \mathrm{End}_{S_{n-1}}(V^{\otimes k})$ if and only if

$$b^{i_1,\ldots,i_k,n}_{i_{1'},\ldots,i_{k'},n} = b^{\sigma(i_1),\ldots,\sigma(i_k),n}_{\sigma(i_{1'}),\ldots,\sigma(i_{k'}),n}, \qquad \text{for all } \sigma \in S_{n-1}.$$

The S_{n-1} orbits of the matrix coordinates of b correspond to set partitions $d \in A_{k+\frac{1}{2}}$; that is vertices i_{k+1} and $i_{(k+1)'}$ must be in the same block of d. The same argument as part (a) can be used to show that $\ker \Phi_{k+\frac{1}{2}}$ is the span of x_d with $d \in A_{k+\frac{1}{2}}$ having more than n blocks. We always choose the index n for the block containing $k+1$ and $(k+1)'$. □

The multiplication in $\mathbb{C}A_k(n)$, in terms of the basis $\{x_d\}$ is

$$x_{d_1} x_{d_2} = \begin{cases} 0, & \text{if } d_1 \circ d_2 \text{ don't exactly match in the middle,} \\ \displaystyle\sum_d c_d x_d, & \text{if } d_1 \circ d_2 \text{ exactly match in the middle,} \end{cases}$$

where the sum is over all coarsenings of $d_1 \circ d_2$ obtained by merging a top horizontal block and a bottom horizontal block and

$$c_d = (n - |d|)_{]d_1 \circ d_2]}, \qquad \text{where} \qquad (\ell)_r = \ell(\ell-1)\cdots(\ell-r+1),$$

$|d|$ is the number of blocks of d and $[d_1 \circ d_2]$ is the number of internal blocks of $d_1 \circ d_2$.

Proof. Let $n >> k$ so that $\mathbb{C}A_k(n) \cong \mathrm{End}_{S_n}(V^{\otimes k})$. Then

$$(v_{i_1} \otimes \cdots v_{i_k}) x_{d_1} x_{d_2} = \sum_{\substack{i_{1'},\ldots,i_{k'} \\ i_{1''},\ldots,i_{k''}}} (x_{d_1})^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}} (x_{d_2})^{i_{1'},\ldots,i_{k'}}_{i_{1''},\ldots,i_{k''}}$$

and

$$\sum_{i_{1'},\ldots,i_{k'}} (x_{d_1})^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}} (x_{d_2})^{i_{1'},\ldots,i_{k'}}_{i_{1''},\ldots,i_{k''}} = \sum_{\text{interior blocks}} 1 = (n-|d)(n-|d|+1)\cdots(n-|d|-(\ell-1),$$

where the sum is over labeled interior blocks of $d_1 \otimes d_2$ such that the labels on these blocks are distinct and do not lie in $\{i_1, \ldots, i_k, i_{1'}, \ldots, i_{k'}\}$. $\square$

For $k \in \mathbb{Z}_{>0}$ the following result is due to Tanabe [Ta, Lemma 2.1].

Theorem 3.2.

(a) Let T be the subgroup of $GL_n(\mathbb{C})$ of diagonal matrices in $GL_n(\mathbb{C})$ and let $N = G_{\mathbb{C}^*,1,n}$ be the normalizer of T in $GL_n(\mathbb{C})$. Then

$$\Phi_k(\mathbb{C}A_{k\infty,1}(n)) = \mathrm{End}_N(V^{\otimes k}) \qquad \textit{and} \qquad \ker \Phi_k \cap \mathbb{C}A_{k,\infty,1}(n) = ????$$

(b) Then

$$\Phi_k(\mathbb{C}S_k) = \mathrm{End}_{\mathfrak{gl}_n}(V^{\otimes k}) \qquad \textit{and} \qquad \ker \Phi_k \cap \mathbb{C}S_k = \textit{the ideal generated by} \sum_{w \in S_k} \det(w) w.$$

Proof. $\square$

Theorem 3.3. *Let $L_{n-1} = \big(G_{r,1,n} \times (\mathbb{Z}/r\mathbb{Z})\big) \cap G_{r,p,n}$. Then*

$$\begin{aligned} &\mathrm{im}\ \Phi_k = \mathrm{End}_{G_{r,p,n}}(V^{\otimes k}) \quad \textit{and} \quad \ker \Phi_k = \mathbb{C}\textit{-span}\{x_d \mid d \textit{ has more than } n \textit{ blocks}\}, \\ &\mathrm{im}\ \Phi_{k+\frac{1}{2}} = \mathrm{End}_{L_{n-1}}(V^{\otimes k}) \quad \textit{and} \quad \ker \Phi_{k+\frac{1}{2}} = \mathbb{C}\textit{-span}\{x_d \mid d \textit{ has more than } n \textit{ blocks}\}. \end{aligned}$$

Proof. In the case $r = p = 1$ this is a result of Jones [Jo] and Martin [Ma] (see [HR, Theorem ???]). A direct computation (which we will not do here) shows that if $d \in A_k(r,p,n)$ then x_d commutes with each of the generators $t_1^p, s_1, s_2, \ldots, s_n$ of $G_{r,p,n}$. Thus $\mathrm{im}\ \Phi_k \subseteq \mathrm{End}_{G_{r,p,n}}(V^{\otimes k})$. If $a \in \mathrm{End}_{G_{r,p,n}}(V^{\otimes k})$ then $a \in \mathrm{End}_{S_n}(V^{\otimes k})$ and so by the Jones-Martin Theorem,

$$a = \sum_{d \in A_k} c_d x_d, \qquad \text{for some } c_d \in \mathbb{C}.$$

We shall show that if $d \notin A_k(r,p,n)$ then $c_d = 0$. Let $d \in A_k$ and let

$$v_t = v_{i_1} \otimes \cdots \otimes v_{i_k} \qquad \text{and} \qquad v_b = v_{i_{1'}} \otimes \cdots \otimes v_{i_{k'}}$$

be such that $i_r = i_s$ if and only if r and s are in the same block of d. Then

$$c_d = av_t\big|_{v_b},$$

the coefficient of the basis element v_b in the expansion of av_t. Choose a block B of d and let $\ell \in B$. Then

$$c_d = (av_t)\big|_{v_b} = (t_{i_\ell}^{-p} a t_{i_\ell}^p)\big|_{v_b} = \xi^{p\kappa(B)} c_d.$$

Hence $c_d = 0$ unless $\kappa(B) = 0 \bmod r/p$. Now choose a pair of distinct blocks B_1 and B_2 in d. Let $\ell \in B_1$ and $m \in B_2$. Then

$$c_d = (av_t)\big|_{v_b} = (t_{i_m} t_{i_\ell}^{-1} a t_{i_\ell} t_{i_m}^{-1} v_t)\big|_{v_b} = \xi^{\kappa(B_1)-\kappa(B_2)} c_d.$$

So $c_d = 0$ unless $\kappa(B_1) = \kappa(B_2) = 0 \bmod r$. The same argument with

$$v_t = v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_n \qquad \text{and} \qquad v_b = v_{i_{1'}} \otimes \cdots \otimes v_{i_{k'}} \otimes v_n$$

applies to establish case (b). □

If $d \in B_k$ is a diagram choose a labeling of the blocks of d from with $1, 2, \ldots, k$ by marking one vertex in each block. An element $\sigma \in S_k$ permutes the marked vertices to produce a new diagram $\sigma d \in B_k$.

$PICTUREEXAMPLEHERE$

Suppose $n + 1 \leq k$. For a given element $d \in B_k$, the element of $\mathbb{C}B_k(n)$ given by

$$\sum_{\sigma \in S_{n+1}} (-1)^{\ell(\sigma)} \sigma d,$$

depends on the choice of the labeling of d, but the set

$$\left\{ \sum_{\sigma \in S_{n+1}} (-1)^{\ell(\sigma)} \sigma d \,\middle|\, d \in B_k \right\}$$

does not???.

If $k > n$ let S_{n+1} be the subgroup of S_k whiich fixes $n+2, \ldots, k$.

Theorem 3.4. *(Schur-Weyl) Let $n \in \mathbb{Z}_{>0}$.*

(a) $\Phi_k : \mathbb{C}S_k \to \mathrm{End}(V^{\otimes k})$ *has*

$$\text{im } \Phi_k = \mathrm{End}_{GL_n(\mathbb{C})}(V^{\otimes k})$$

and $\ker \Phi_k$ *is the ideal of* $\mathbb{C}S_k$ *generated by*

$$\left\{ \sum_{\sigma \in S_{n+1}} (-1)^{\ell(\sigma)} \sigma d \;\middle|\; d \in S_k \textit{ has more than } n \textit{ blocks} \right\}.$$

(b) $\Phi_k : \mathbb{C}B_k \to \mathrm{End}(V^{\otimes k})$ *has*

$$\text{im}\Phi_k = \mathrm{End}_{O_n(\mathbb{C})}(V^{\otimes k})$$

and $\ker \Phi_k$ *is the ideal generated by*

$$\left\{ \sum_{\sigma \in S_{n+1}} (-1)^{\ell(\sigma)} \sigma d \,\middle|\, d \in B_k \textit{ has more than } n \textit{ blocks} \right\}$$

Need to define action of σ on d.

Define linear maps

$$\varepsilon_{\frac{1}{2}}: \mathrm{End}(V^{\otimes k}) \to \mathrm{End}(V^{\otimes k}) \qquad \varepsilon^{\frac{1}{2}}: \mathrm{End}(V^{\otimes k}) \to \mathrm{End}(V^{\otimes (k-1)}) \qquad \text{and}$$

$$\varepsilon_1: \mathrm{End}(V^{\otimes k}) \to \mathrm{End}(V^{\otimes (k-1)}) \tag{3.7}$$

by

$$\varepsilon_{\frac{1}{2}}(b)^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}} = b^{i_1,\ldots,i_k}_{i_{1'},\ldots,i_{k'}}\delta_{i_k i_{k'}} \qquad \varepsilon^{\frac{1}{2}}(b)^{i_1,\ldots,i_{k-1}}_{i_{1'},\ldots,i_{(k-1)'}} = \sum_{j,\ell=1}^{n} b^{i_1,\ldots,i_{k-1},j}_{i_{1'},\ldots,i_{(k-1)'},\ell}. \qquad \text{and}$$

$$\varepsilon_1(b)^{i_1,\ldots,i_{k-1}}_{i_{1'},\ldots,i_{(k-1)'}} = \sum_{j=1}^{n} b^{i_1,\ldots,i_{k-1},j}_{i_{1'},\ldots,i_{(k-1)'},j}, \tag{3.8}$$

Then

$$\varepsilon_1 = \varepsilon_{\frac{1}{2}} \circ \varepsilon^{\frac{1}{2}} \qquad \text{and} \qquad \mathrm{Tr}(b) = \varepsilon_1^k(b), \qquad \text{for } b \in \mathrm{End}(V^{\otimes k}). \tag{3.9}$$

The relation between the maps $\varepsilon^{\frac{1}{2}}$, $\varepsilon_{\frac{1}{2}}$ and ε_1 in (???) and the maps $\varepsilon^{\frac{1}{2}}$, $\varepsilon_{\frac{1}{2}}$, ε_1 in (???) is given by

$$\Phi_{k-\frac{1}{2}}(\varepsilon_{\frac{1}{2}}(b)) = \varepsilon_{\frac{1}{2}}(\Phi_k(b))\big|_{V^{\otimes(k-\frac{1}{2})}}, \qquad \Phi_{k-1}(\varepsilon^{\frac{1}{2}}(b)) = \tfrac{1}{n}\varepsilon^{\frac{1}{2}}(\Phi_k(b)), \qquad \text{and}$$

$$\Phi_{k-1}(\varepsilon_1(b)) = \varepsilon_1(\Phi_k(b)), \tag{3.10}$$

where, on the right hand side of the middle equality b is viewed as an element of $\mathbb{C}A_k$ via the natural inclusion $\mathbb{C}A_{k-\frac{1}{2}}(n) \subseteq \mathbb{C}A_k(n)$. Then, for $k \in \mathbb{Z}_{>0}$,

$$\mathrm{Tr}(\Phi_k(b)) = \mathrm{tr}_k(b), \qquad \text{and} \qquad \mathrm{Tr}(\Phi_{k-\frac{1}{2}}(b)) = \tfrac{1}{n}\,\mathrm{tr}_{k-\frac{1}{2}}(b). \tag{3.11}$$

4 The tower $\hat{A}_k(r,p,n)$

Let $\mathbf{1}_n$ be the trivial representation of $G = G_{r,1,n}$ and let $V = \mathbb{C}\text{-span}\{v_1,\ldots,v_n\}$ be the reflection representation. Let

$$G = G_{r,p,n} = G_{r,1,n} \cap G_{r,p,n} \qquad \text{and} \qquad L_{n-1} = \big(G_{r,1,n-1} \times (\mathbb{Z}/r\mathbb{Z})\big) \cap G_{r,p,n}.$$

The $G_{r,1,n-1} \times (\mathbb{Z}/r\mathbb{Z})$ module

$$\chi = \mathbf{1}_{n-1} \otimes \chi_1, \qquad \text{where} \quad \begin{array}{cccc} \chi_1: & \mathbb{Z}/r\mathbb{Z} & \longrightarrow & \mathbb{C}^* \\ & \xi & \longmapsto & \xi \end{array}$$

is, by restriction, an L_{n-1} module and for any G-module M,

$$\mathrm{Ind}^G_{L_{n-1}}(\mathrm{Res}^G_{L_{n-1}}(M) \otimes \chi) \cong M \otimes \mathrm{Ind}^G_{L_{n-1}}(\chi) \cong M \otimes V,$$

where the first isomorphism comes from the "tensor identity,"

$$\begin{array}{ccc} \mathrm{Ind}^G_{L_{n-1}}(\mathrm{Res}^G_{L_{n-1}}(M) \otimes N) & \xrightarrow{\sim} & M \otimes \mathrm{Ind}^G_{L_{n-1}} N \\ g \otimes (m \otimes n) & \mapsto & gm \otimes (g \otimes n) \end{array}, \tag{4.1}$$

for $g \in G$, $m \in M$, $n \in N$, and the fact that $\mathrm{Ind}^G_{L_{n-1}}(W) = \mathbb{C}G \otimes_{L_{n-1}} W$. Iterating (???) it follows that

$$(\mathrm{Ind}^G_{L_{n-1}}(\mathrm{Res}^G_{L_{n-1}} \otimes \chi))^k(\mathbf{1}) \cong V^{\otimes k} \qquad \text{and} \qquad \mathrm{Res}^G_{L_{n-1}}(\mathrm{Ind}^G_{L_{n-1}}\mathrm{Res}^G_{L_{n-1}})^k(\mathbf{1}) \cong V^{\otimes k} \tag{4.2}$$

as G-modules and L_{n-1}-modules, respectively.

This analysis allows us to build the Bratteli diagram of $A_k(r,1,n)$. This graph is constructed inductively as follows:

$$\hat{A}_0(r,1,n) = \{((n), \emptyset, \ldots, \emptyset)\}$$

If $k \in \mathbb{Z}_{>0}$ then there are edges

$$\begin{array}{lcl} \hat{A}_k(r,1,n): & \lambda & \\ & \big\downarrow & \text{if } \mu \text{ is obtained from } \lambda \text{ by removing a box from } \lambda^{(i)}, \\ \hat{A}_{k+\frac{1}{2}}: & (\mu, \square_i) & \end{array}$$

and

$$\begin{array}{lcl} \hat{A}_{k+\frac{1}{2}}: & (\nu, \square_i) & \\ & \big\downarrow & \text{if } \gamma \text{ is obtained from } \nu \text{ by adding a box to } \nu^{(i+1)}, \\ \hat{A}_k(r,1,n): & \gamma & \end{array}$$

where we make the convention that $\mu^{(r)} = \mu^{(0)}$).

4.1 The Bratteli diagram of the algebra $\mathbb{C}A_{k,r,p}(n)$

Recall that the simple $G_{r,1,n}$ modules are given by

$$G^{\lambda}_{r,1,n} = \operatorname{span}\{v_T \mid T \text{ is a standard tableau of shape } \lambda\}$$

with action

$$t_i v_T = s(T(i))v_T \qquad \text{and} \qquad s_i v_T = (s_i)_{TT} v_T + (1 + (s_i)_{TT}) v_{s_i T}.$$

Then $\mathbb{Z}/p\mathbb{Z}$ acts on the $G_{r,1,n}$ modules by

$$\begin{array}{cccc} \sigma\colon & G^{\lambda}_{r,1,n} & \longrightarrow & G^{\lambda}_{r,1,n} \\ & v_T & \longmapsto & v_{\sigma T} \end{array}$$

and, this action lifts to an action of $\mathbb{Z}/p\mathbb{Z}$ on $\mathbb{C}G_{r,1,n}$ by automorphisms

$$\begin{array}{cccc} \sigma\colon & \mathbb{C}G_{r,1,n} & \longrightarrow & \mathbb{C}G_{r,1,n} \\ & t_\lambda w & \longmapsto & \zeta^{|\lambda|} t_\lambda w \end{array} \qquad \text{where } \zeta = e^{2\pi i/p}.$$

Then $V^{\otimes k}$ is a $G_{r,1,n}$ module and we can twist this action by any automorphism. So

$$t_i \circ v_j = \begin{cases} \xi\zeta v_i, & \text{if } j = i, \\ v_j, & \text{if } j \neq i. \end{cases}$$

Then $\sigma\colon V^{\otimes k} \to (\sigma^* V)^{\otimes k}$ as $G_{r,1,n}$ modules and this operation commutes with the $G_{r,p,n}$ action. So

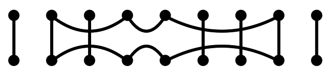
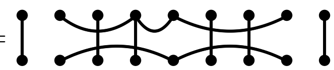
$$\sigma \in \operatorname{End}_{G_{r,p,n}}(V^{\otimes k}).$$

5 Murphy elements in $\mathbb{C}A_k(r,p,n)$

Let $S \subseteq \{1,2,\ldots,k\}$ and let $I \subseteq S \cup S'$. Define $b_S, d_I \in A_k$ by

$$b_S = \{S \cup S', \{\ell,\ell'\}_{\ell \notin S}\} \qquad \text{and} \qquad d_{I \subseteq S} = \{I, I^c, \{\ell,\ell'\}_{\ell \notin S}\}. \tag{5.1}$$

For example, in A_9, if $S = \{2,4,5,8\}$ and $I = \{2,4,4',5,8\}$ then

$b_S =$  and $d_I =$ .

Note that

$$d_I = d_{I^c}, \qquad d_{S \cup S'} = d_\emptyset = b_S, \qquad d_{\{\ell,\ell'\}} = d_{\{\ell,\ell'\}^c} = b_{S-\{\ell\}}.$$

For $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and $r \in \mathbb{Z}_{>0}$

$$Z_{k,r} = \binom{n}{2} + \sum_{|S|\geq 1} (-1)^{|S|}\Big((n-1)b_S + \sum_{\kappa(I)=0 \bmod r} (-1)^{\kappa(I)}(d_I - b_S)\Big),$$

where the outer sum is over $S \subseteq \{1,2,\ldots,k\}$ such that $S \neq \emptyset$ and the inner sum is over $I \subseteq S \cup S'$ such that $d_I \in A_k(r,1,n)$ and $d_I \neq b_S$.

Theorem 5.1.

(a) For $n, r \in \mathbb{Z}_{\geq 0}$,

$$\kappa_{r,n} = \frac{1}{r}\sum_{m=0}^{r-1} \sum_{1 \leq i < j \leq n} t_i^m t_j^{-m} s_{ij}$$

is a central element of $\mathbb{C}G_{r,p,n}$.

$$\kappa_{r,n} = \sum_{b \in \lambda} c(b), \qquad \textit{as operators on } G_{r,p,n}^{(\lambda,j)},$$

the irreducible $G_{r,p,n}$-module indexed by (λ, j), where $\lambda = (\lambda^{(0)}, \ldots, \lambda^{(r-1)})$ is a multipartition with n boxes.

(b) Let $n, k \in \mathbb{Z}_{\geq 0}$. Then,

$$\kappa_{r,n} = Z_{k,r} \quad \textit{and} \quad \kappa_{n-1} = Z_{k+\frac{1}{2},r}, \qquad \textit{as operators on } V^{\otimes k}.$$

(c) Let $n \in \mathbb{C}$, $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Then $Z_k(r)$ is a central element of $\mathbb{C}A_k(r,p,n)$, and, if $n \in \mathbb{C}$ is such that $\mathbb{C}A_k(r,p,n)$ is semisimple then

$$Z_k(r) = \sum_{b \in \lambda} c(b), \qquad \textit{as operators on } A_k^\lambda,$$

where A_k^λ is the irreducible $\mathbb{C}A_k(r,p,n)$-module indexed by λ.

Proof. (a) The element κ_n is the class sum corresponding to the conjugacy class of the element s_{12} in $G_{r,p,n}$ and thus $\kappa_{r,n}$ is a central element of $\mathbb{C}G_{r,p,n}$. The constant by which $\kappa_{r,n}$ acts on $G_{r,p,n}^{(\lambda,j)}$ follows from Theorem ???.

(c) The first statement follows from parts (a) and (b) and Theorems 3.6 and 3.22 as follows. By Theorem 3.6, $\mathbb{C}A_k(n) \cong \mathrm{End}_{S_n}(V^{\otimes k})$ if $n \geq 2k$. Thus, by Theorem 3.22, if $n \geq 2k$ then Z_k acts on the irreducible $\mathbb{C}A_k(n)$-module $A_k^\lambda(n)$ by the constant given in the statement. This means that Z_k is a central element of $\mathbb{C}A_k(n)$ for all $n \geq k$. Thus, for $n \geq 2k$, $dZ_k = Z_k d$ for all diagrams $d \in A_k$. Since the coefficients in dZ_k (in terms of the basis of diagrams) are polynomials in n, it follows that $dZ_k = Z_k d$ for all $n \in \mathbb{C}$.

If $n \in \mathbb{C}$ is such that $\mathbb{C}A_k(n)$ is semisimple let $\chi^\lambda_{\mathbb{C}A_k(n)}$ be the irreducible characters. Then Z_k acts on $A_k^\lambda(n)$ by the constant $\chi^\lambda_{\mathbb{C}A_k(n)}(Z_k)/\dim(A_k^\lambda(n))$. If $n \geq k$ this is the constant in the statement, and therefore it is a polynomial in n, determined by its values for $n \geq 2k$.

The proof of the second statement is completely analogous using $\mathbb{C}A_{k+\frac{1}{2}}$, S_{n-1}, and the second statement in part (b).

(b) Then

$$2\kappa_n(v_{i_1} \otimes \cdots \otimes v_{i_k}) = \frac{1}{r}\sum_{m=0}^{r-1}\sum_{\substack{i,j=1\\ i\neq j}}^{n} t_i^m t_j^{-m} s_{ij} v_{i_1} \otimes \cdots \otimes t_i^m t_j^{-m} s_{ij} v_{i_k}$$

$$= \frac{1}{r}\sum_{m=0}^{r-1}\sum_{\substack{i,j=1\\ i\neq j}}^{n} (1 - E_{ii} - E_{jj} + t_i^m E_{ij} + t_j^{-m} E_{ji}) v_{i_1} \otimes \cdots \otimes (1 - E_{ii} - E_{jj} + t_i^m E_{ij} + t_j^{-m} E_{ji}) v_{i_k}$$

Expanding this sum, let

$$c_{S,I} = \left(\prod_{\ell \in S^c} \delta_{i_\ell i_{\ell'}}\right)(-1)^{\#(\{\ell,\ell'\}\subseteq I)+\#(\{\ell,\ell'\}\subseteq I^c)}\xi^{m(\{\ell\in I,\ell'\in I^c\}-\#\{\ell'\in I,\ell\in I^c\})}\left(\prod_{\ell\in I}\delta_{i_\ell i}\right)\left(\prod_{\ell\in I^c}\delta_{i_\ell j}\right)$$

and

$$c'_{S,I} = \left(\prod_{\ell \in S^c} \delta_{i_\ell i_{\ell'}}\right)(-1)^{\#(\{\ell,\ell'\}\subseteq I)+\#(\{\ell,\ell'\}\subseteq I^c)}\xi^{m(\{\ell\in I,\ell'\in I^c\}-\#\{\ell'\in I,\ell\in I^c\})}\left(\prod_{\ell\in I}\delta_{i_\ell i}\right)\left(\prod_{\ell\in I^c}\delta_{i_\ell i}\right)$$

so that $2\kappa_n(v_{i_1} \otimes \cdots \otimes v_{i_k})$ is equal to

$$\frac{1}{r}\sum_{S\subseteq\{1,\ldots,k\}}\sum_{i_{1'},\ldots,i_{k'}}\sum_{m=0}^{r-1}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{I\subseteq S\cup S'} c_{S.I}(v_{i_{1'}} \otimes \cdots \otimes v_{i_{k'}})$$

$$= \frac{1}{r}\sum_{S\subseteq\{1,\ldots,k\}}\sum_{i_{1'},\ldots,i_{k'}}\sum_{m=0}^{r-1}\sum_{I\subseteq S\cup S'}\Big(\sum_{i,j=1}^{n} c_{S,I}(v_{i_{1'}} \otimes \cdots \otimes v_{i_{k'}}) - \sum_{i=1}^{n} c'_{S,I}(v_{i_{1'}} \otimes \cdots \otimes v_{i_{k'}})\Big).$$

Here $S^c \subseteq \{1,\ldots,k\}$ corresponds to the tensor positions where 1 is acting, $I \subseteq S \cup S'$ corresponds to the tensor positions that must equal i, and I^c corresponds to the tensor positions that must equal j.

When $|S| = 0$ the set I is empty and the sum in (???) is equal to

$$(n^2 - n)(v_{i_1} \otimes \cdots \otimes v_{i_k}) \qquad \text{since} \qquad c_{S,I} = c'_{S,I} = \Big(\prod_{\ell\in\{1,\ldots,k\}}\delta_{i_\ell i_{\ell'}}\Big).$$

Assume $|S| \geq 1$ and separate the sum according to the cardinality of I. Note that the sum for I is equal to the sum for I^c, since the whole sum is symmetric in i and j. When $I = S \cup S'$,

$$c_{S,I} = c'_{S,I} = \left(\prod_{\ell\in S^c}\delta_{i_\ell i_{\ell'}}\right)(-1)^{|S|}\left(\prod_{\ell\in S\cup S'}\delta_{i_\ell i}\right)$$

and the sum in (???) is equal to

$$\frac{1}{r}\sum_{m=0}^{r-1}\sum_{i_{1'},\dots,i_{k'}}(n-1)\sum_{i=1}^{n}c_{S,I}(v_{i_{1'}}\otimes\cdots\otimes v_{i_{k'}})=(n-1)(-1)^{|S|}b_S(v_{i_1}\otimes\cdots\otimes v_{i_k}).$$

We get a similar contribution from the sum of the terms with $I=\emptyset$.

For each of the remaining subsets $I\subseteq S\cup S'$ the sum in (???) contributes 0 when $\kappa(I)\neq 0 \bmod r$ and

$$(-1)^{\#(\{\ell,\ell'\}\subseteq I)+\#(\{\ell,\ell'\}\subseteq I^c)}(d_{I\subseteq S}-b_S)(v_{i_1}\otimes\cdots\otimes v_{i_k})=(-1)^{|S|+\kappa(I)}(d_{I\subseteq S}-b_S)(v_{i_1}\otimes\cdots\otimes v_{i_k}).$$

when $\kappa(I)=0 \bmod r$.

For the second statement, since $(1-E_{ii}-E_{jj}+E_{ii}E_{jj})v_n=\begin{cases}0, & \text{if } i=n \text{ or } j=n,\\ v_n, & \text{otherwise,}\end{cases}$

$$\begin{aligned}
2\kappa_{n-1}(v_{i_1}\otimes\cdots\otimes v_{i_k}\otimes v_n) &= \frac{1}{r}\left(\sum_{m=0}^{r-1}\sum_{\substack{i,j=1\\ i\neq j}}^{n-1}t_i^m t_j^{-m}s_{ij}\right)(v_{i_1}\otimes\cdots\otimes v_{i_k}\otimes v_n)\\
&= \frac{1}{r}\sum_{m=0}^{r-1}\sum_{\substack{i,j=1\\ i\neq j}}^{n}t_i^m t_j^{-m}s_{ij}v_{i_1}\otimes\cdots\otimes t_i^m t_j^{-m}s_{ij}v_{i_k}\otimes(1-E_{ii}-E_{jj}+E_{ii}E_{jj})v_n\\
&= \Big(\frac{1}{r}\sum_{m=0}^{r-1}\sum_{i\neq j}s_{ij}\Big)(v_{i_1}\otimes\cdots\otimes v_{i_k})\otimes v_n\\
&\quad+\frac{1}{r}\sum_{m=0}^{r-1}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(1-E_{ii}-E_{jj}+\xi^m E_{ij}+\xi^{-m}E_{ji})v_{i_1}\otimes\cdots\\
&\qquad\qquad\cdots\otimes(1-E_{ii}-E_{jj}+\xi^m E_{ij}+\xi^{-m}E_{ji})v_{i_k}\otimes(-E_{ii}-E_{jj})v_n\\
&\quad+\frac{1}{r}\sum_{m=0}^{r-1}\sum_{i,j=1}^{n}t_i^m t_j^{-m}s_{ij}v_{i_1}\otimes\cdots\otimes t_i^m t_j^{-m}s_{ij}v_{i_k}\otimes E_{ii}E_{jj}v_n
\end{aligned}$$

The first sum is known to equal $2\kappa_n(v_{i_1}\otimes\cdots\otimes v_{i_k})$ and is known by the computation proving the first statement, and the last sum is zero since $i\neq j$. The middle sum is treated exactly as in (???) except that now the sum is over S such that $k+1\in S$ and I such that $\{k+1,(k+1)'\}\subseteq I$ or $\{k+1,(k+1)'\}\subseteq I^c$. $\square$

$$
\begin{aligned}
\Big(\sum_{i=1}^{n} t_i^m\Big)(v_{i_1}\otimes\cdots\otimes v_{i_k}) &= \sum_{i=1}(t_i^m v_{i_1}\otimes\cdots\otimes t_i^m v_{i_k}) \\
&= \sum_{i=1}(1-E_{ii}+\xi^m E_{ii})v_{i_1}\otimes\cdots\otimes(1-E_{ii}+\xi^m E_{ii})v_{i_k}) \\
&= \sum_{i'_1,\ldots,i'_k}\ \sum_{I\subseteq\{1,\ldots,k\}}(\xi^m-1)^{|I|}\big(\prod_{\ell\in I^c}\delta_{i_\ell i_{\ell'}}\big)\big(\prod_{\ell\in I}\delta_{i_\ell i}\delta_{i_{\ell'} i}\big)(v_{i_{1'}}\otimes\cdots\otimes v_{i_{k'}}) \\
&= \sum_{I\subseteq\{1,2,\ldots,k\}}(\xi^m-1)^{|I|}b_I \\
&= n+k(\xi^m-1)+\sum_{|I|\geq 2}(\xi^m-1)^{|I|}b_I \\
&= n-k+k\xi^m+\sum_{|I|\geq 2}(\xi^m-1)^{|I|}b_I.
\end{aligned}
$$

References

[HR] T. Halverson and A. Ram, *Partition algebras*, European J. Combinatorics **26** (2005), 869–921.

[Ko1] M. Kosuda, *Irreducible representations of the party algebra*, preprint 2004.

[Ko2] M. Kosuda, *Characterization of the party algebras* Ryukyu Math. J. **13** (2003), 199–228.

[Ta] K. Tanabe, *On the centralizer algebra of the unitary reflection group $G(m,p,n)$*, Nagoya Math. J. **148** (1997), 113-126.

[Dr1] V.G. Drinfel'd, *A new realization of Yangians and quantized affine algebras*, Soviet Math. Dokl. **36** No, 2 (1998), 212–216.