Schur functions

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1 Schur functions

The *group algebra of* P is the ring

$$\mathbb{Z}[P] \quad \text{with basis} \quad \{X^\lambda \mid \lambda \in P\} \qquad \text{and product} \qquad X^\lambda X^\mu = X^{\lambda+\mu},$$

for $\lambda, \mu \in P$. The group W acts on $\mathbb{C}[P]$ by

$$wX^\lambda = X^{w\lambda}, \qquad \text{for } w \in W,\ \lambda \in P.$$

The ring of *symmetric functions* and *Fock space* are

$$\begin{aligned} \mathbb{Z}[P]^W &= \{f \in \mathbb{Z}[P] \mid wf = f \text{ for all } w \in W\} \qquad \text{and} \\ \mathbb{Z}[P]^{\det} &= \{f \in \mathbb{Z}[P] \mid wf = \det(w)f, \text{ for all } w \in W\}, \end{aligned} \tag{1.1}$$

respectively. For $\lambda \in P$ define

$$m_\lambda = \sum_{\gamma \in W\lambda} X^\gamma \qquad \text{and} \qquad a_\lambda = \sum_{w \in W} \det(w^{-1}) X^{w\lambda}. \tag{1.2}$$

The straightening laws for these elements are

$$m_{w\lambda} = m_\lambda \qquad \text{and} \qquad a_{w\lambda} = \det(w) a_\lambda, \qquad \text{for } w \in W \text{ and } \lambda \in P. \tag{1.3}$$

The second relation implies that $a_\lambda = 0$ if there exists $w \in W_\lambda$ with $\det(w) \neq 1$, and it follows from the straightening laws that

$$\begin{aligned} \mathbb{Z}[P]^W \quad &\text{has basis} \quad \{m_\lambda \mid \lambda \in P^+\}, \qquad \text{and} \\ \mathbb{Z}[P]^{\det} \quad &\text{has basis} \quad \{a_{\lambda+\rho} \mid \lambda \in P^+\}. \end{aligned} \tag{1.4}$$

Then

The *Weyl characters* or *Schur functions* are defined by

$$s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho}, \qquad \text{for } \lambda \in P. \tag{1.5}$$

The following theorem shows that the s_λ are elements of $\mathbb{Z}[P]$ and that

$$\mathbb{Z}[P]^W \qquad \text{has basis} \qquad \{s_\lambda \mid \lambda \in P^+\}. \tag{1.6}$$

Theorem 1.1. *Fock space $\mathbb{Z}[P]^{\det}$ is a free $\mathbb{Z}[P]^W$ module with generator*

$$a_\rho = x^\rho \prod_{\alpha\in R^+}(1-x^{-\alpha}) \qquad \textit{and the map} \qquad \begin{matrix} \mathbb{Z}[P]^W & \longrightarrow & \mathbb{Z}[P]^{\det} \\ f & \longmapsto & a_\rho f \\ s_\lambda & \longmapsto & a_{\lambda+\rho} \end{matrix} \tag{1.7}$$

is a $\mathbb{Z}[P]^W$ module isomorphism.

Proof. Let $f \in \mathbb{Z}[P]^{\det}$ and let $\alpha \in R^+$. If f_γ is the coefficient of x^γ in f then

$$\sum_{\gamma\in P} f_\gamma x^\gamma = f = -s_\alpha f = \sum_{\gamma\in P} -f_\gamma x^{s_\alpha\gamma}, \qquad \text{and so} \qquad f = \sum_{\substack{\gamma\in P \\ \langle\gamma,\alpha^\vee\rangle\geq 0}} f_\gamma(x^\gamma - x^{s_\alpha\gamma}),$$

since $f_{s_\alpha\gamma} = -f_\gamma$. Since each term $x^\gamma - x^{s_\alpha\gamma}$ is divisible $1-x^{-\alpha}$, f is divisible by $1-x^{-\alpha}$, and thus

$$\text{each} \quad f\in\mathbb{Z}[P]^{\det} \quad \text{is divisible by} \quad x^\rho \prod_{\alpha\in R^+}(1-x^{-\alpha}) \tag{1.8}$$

since the polynomials $1-x^{-\alpha}$, $\alpha\in R^+$ are coprime in $\mathbb{Z}[P]$ (and x^ρ is a unit in $\mathbb{Z}[P]$). Comparing coefficients of the maximal terms in a_ρ and $x^\rho\prod_{\alpha\in R^+}(1-x^{-\alpha})$ shows that

$$a_\rho = x^\rho \prod_{\alpha\in R^+}(1-x^{-\alpha}).$$

Thus each $f \in \mathbb{Z}[P]^\varepsilon$ is divisible by a_ρ and so the inverse of the multiplication by a_ρ is well defined. □

The *dot action* of S_n on P is given by

$$w\circ\mu = w(\mu+\rho)-\rho, \qquad \text{for } w\in S_n,\ \mu\in P. \tag{1.9}$$

The straightening law

$$s_{w\circ\mu} = \det(w)s_\mu, \qquad \text{for } \mu\in P,\ w\in W. \tag{1.10}$$

for the Schur functions follows from the straightening law for the a_μ in (1.3).

Lemma 1.2. *Let $f\in\mathbb{Z}[P]^W$ and write $f=\sum_\gamma f_\gamma x^\gamma$ so that f_γ is the coefficient of x^γ in f. Then*

$$f = \sum_{\mu\in P^+} f_\mu m_\mu = \sum_{\lambda\in P^+} \eta^\lambda s_\lambda, \qquad \textit{where} \qquad \eta^\lambda = \sum_{w\in W}\det(w^{-1}) f_{\lambda+\rho-w\rho}.$$

Proof. The first equality is immediate from the definition of m_μ. Since $f\in\mathbb{Z}[P]^W$ and the s_λ, $\lambda\in P^+$, are a basis of $\mathbb{Z}[P]^W$, the element f can be written as a linear combination of s_λ. Then

$$\begin{aligned} \eta_\lambda &= (\text{coefficient of } s_\lambda \text{ in } f) = (\text{coefficient of } a_{\lambda+\rho} \text{ in } f a_\rho) \\ &= \left(\text{coefficient of } e^{\lambda+\rho} \text{ in } \sum_{\mu\in P}\sum_{w\in W}(-1)^{\ell(w)} f_\mu e^{\mu+w\rho}.\right) \end{aligned}$$

□

Proposition 1.3. *If $\nu \in \mathfrak{h}^*_{\mathbb{R}}$ and $f = \sum_{\mu\in P} f_\mu e^\mu \in \mathbb{Z}[P]$ define $f(e^\nu) = \sum_{\mu\in P} f_\mu e^{\langle \mu,\nu\rangle}$. Let $\lambda \in P^+$ and let $q = e^t$. Then*

$$s_\lambda(q^\rho) = \prod_{\alpha\in R^+} \frac{[\langle \lambda+\rho, \alpha^\vee\rangle]}{[\langle \rho, \alpha^\vee\rangle]} \qquad \text{and} \qquad s_\lambda(1) = \prod_{\alpha\in R^+} \frac{\langle \lambda+\rho, \alpha^\vee\rangle}{\langle \rho, \alpha^\vee\rangle}$$

where $[k] = (q^k-1)/(q-1)$ for an integer $k \neq 0$.

Proof.

$$s_\lambda(q^\rho) = s_\lambda(e^{t\rho}) = \frac{a_{\lambda+\rho}(e^{t\rho})}{a_\rho(e^{t\rho})} = \frac{\sum_{w\in W} (-1)^{\ell(w)} e^{\langle w(\lambda+\rho), t\rho\rangle}}{a_\rho(e^{t\rho})} = \frac{\sum_{w\in W} (-1)^{\ell(w)} e^{\langle \rho, wt(\lambda+\rho)\rangle}}{a_\rho(e^{t\rho})}$$

$$= \frac{a_\rho(e^{t(\lambda+\rho)})}{a_\rho(e^{t\rho})} = \frac{e^{\langle \rho, t(\lambda+\rho)\rangle}}{e^{\langle \rho, t\rho\rangle}} \prod_{\alpha\in R^+} \frac{1-e^{\langle -\alpha, t(\lambda+\rho)\rangle}}{1-e^{\langle -\alpha, t\rho\rangle}}$$

$$= \frac{e^{-\langle \rho, t(\lambda+\rho)\rangle}}{e^{-\langle \rho, t\rho\rangle}} \prod_{\alpha\in R^+} \frac{e^{\langle \alpha, t(\lambda+\rho)\rangle}-1}{e^{\langle \alpha, t\rho\rangle}-1} = q^{-\langle\lambda,\rho\rangle} \prod_{\alpha\in R^+} \frac{q^{\langle \lambda+\rho,\alpha\rangle}-1}{q^{\langle\rho,\alpha\rangle}-1}$$

Then

$$s_\lambda(1) = \lim_{q\to 1} s_\lambda(q^\rho) = \prod_{\alpha\in R^+} \frac{\langle \lambda+\rho, \alpha/2\rangle}{\langle \rho, \alpha/2\rangle}$$

$\square$

The *weight multiplicities* are the integers $K_{\lambda\gamma}$, $\lambda\in P^+$, $\gamma \in P$, defined by the equations

$$s_\lambda = \sum_{\gamma\in P} K_{\lambda\gamma} x^\gamma = \sum_{\mu\in P^+} K_{\lambda\mu} m_\mu. \tag{1.11}$$

The *tensor product multiplicities* are the integers $c^\lambda_{\mu\nu}$, $\mu,\nu,\lambda \in P^+$, defined by the equations

$$s_\mu s_\nu = \sum_{\lambda\in P^+} c^\lambda_{\mu\nu} s_\lambda. \tag{1.12}$$

The *partition function* is the function $p\colon P \to \mathbb{Z}_{\geq 0}$ defined by the equation

$$\prod_{\alpha\in R^+} \frac{1}{1-x^{-\alpha}} = \sum_{\gamma\in P} p(\gamma) x^{-\gamma}. \tag{1.13}$$

Proposition 1.4. *Let $\lambda,\mu,\nu \in P^+$.*

(a) $K_{\lambda\lambda} = 1$, $K_{\lambda,w\mu} = K_{\lambda\mu}$, *for $w \in W$, and* $K_{\lambda\mu} = 0$ *unless $\mu \leq \lambda$.*

(b) $K_{\lambda\mu} = \sum_{w\in W} \det(w) p(w(\lambda+\rho) - (\mu+\rho))$.

(c) $c^\lambda_{\mu\nu} = \sum_{v,w\in W} \det(vw) p(v(\mu+\rho) + w(\nu+\rho) - (\lambda+\rho) - \rho)$.

Proof. (a) The equality $K_{\lambda,w\mu} = K_{\lambda\mu}$ follows from the definition. If $w \neq 1$ then $w(\lambda+\rho) < \lambda+\rho$ so that $w(\lambda+\rho) - \rho < \lambda$ and

$$s_\lambda = \left(\sum_{w\in W} \det(w) x^{w(\lambda+\rho)-\rho}\right) \cdot \prod_{\alpha\in R^+} \frac{1}{1-x^{-\alpha}} = x^\lambda + (\text{lower terms in dominance order}).$$

Thus $K_{\lambda\lambda} = 1$ and $K_{\lambda\mu} = 0$ unless $\mu \leq \lambda$.
(b) The coefficient of x^μ in

$$s_\lambda = \left(\sum_{w\in W} \det(w) x^{w(\lambda+\rho)-\rho}\right) \prod_{\alpha\in R^+} \frac{1}{1-x^{-\alpha}} = \sum_{\substack{w\in W\\ \gamma\in Q^+}} \det(w) p(\gamma) x^{w(\lambda+\rho)-\rho-\gamma},$$

has a contribution $\det(w)p(\gamma)$ when $w(\lambda+\rho) - \rho - \gamma = \mu$ so that $\gamma = w(\lambda+\rho) - (\mu+\rho)$.
(c) Since $c^\lambda_{\mu\nu}$ is the coefficient of $x^{\nu+\rho}$ in

$$\begin{aligned} s_\mu s_\nu a_\rho = \frac{\varepsilon(x^{\mu+\rho})\varepsilon(x^{\nu+\rho})}{a_\rho} &= \left(\sum_{v,w\in W} \det(vw) x^{v(\mu+\rho)+w(\nu+\rho)-\rho}\right)\left(\prod_{\alpha\in R^+}\frac{1}{1-x^{-\alpha}}\right) \\ &= \sum_{\substack{v,w\in W\\ \gamma\in Q^+}} \det(vw) p(\gamma) x^{v(\mu+\rho)+w(\nu+\rho)-\gamma-\rho}, \end{aligned}$$

there is a contribution $\det(vw)p(\gamma)$ to the coefficient $c^\lambda_{\mu\nu}$ when $\lambda+\rho = v(\mu+\rho)+w(\nu+\rho)-\gamma-\rho$ so that $\gamma = v(\mu+\rho) + w(\mu+\rho) - (\lambda+\rho) - \rho$. $\square$

Fix $J \subseteq \{1, 2, \ldots, n\}$. The subgroup of W generated by the reflections in the hyperplanes H_{α_j}, $j \in J$,

$$W_J = \langle s_j \mid j \in J\rangle, \quad \text{acts on } \mathfrak{h}^*_{\mathbb{R}}, \qquad \text{with} \qquad C_J = \{\mu \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \mu, \alpha_j^\vee\rangle \geq 0 \text{ for } j \in J\}$$

as a fundamental chamber. The group W_J acts on P and

$$\mathbb{C}[P]^{W_J} = \{f \in \mathbb{C}[P] \mid wf = f \text{ for } w \in W_J\}$$

is a subalgebra of $\mathbb{C}[P]$ which contains $\mathbb{C}[P]^W$. If

$$P_J^+ = P \cap \overline{C_J}, \qquad \rho_J = \sum_{j\in J} \omega_j,$$

$$a^J_\mu = \sum_{w\in W_J} \det(w) w X^\mu, \quad \text{for } \mu \in P, \qquad \text{and} \qquad s^J_\lambda = \frac{a^J_{\lambda+\rho_J}}{a^J_{\rho_J}}, \quad \text{for } \lambda \in P,$$

then

$$\{s^J_\lambda \mid \lambda \in P^+_J\} \text{ is a basis of } \mathbb{C}[P]^{W_J}.$$

The *restriction multiplicities* are the integers c^λ_ν given by

$$s_\lambda = \sum_{\nu\in P^+_J} c^\lambda_{J,\nu} s^J_\nu. \tag{1.14}$$

References

This theory was developed by H. Weyl [We]. The presentation here is analogous to [Mac] (Macdonald, Chapter 1). The element a_ρ is the *Weyl denominator*. Lemma 1.2 is a generalization of the *Jacobi-Trudi formula* and formulas in Proposition 1.3 are the *quantum dimension formula* and the *Weyl dimension formula*, respectively. The results in Proposition 1.4b and Proposition 1.4c are the *Kostant partition function formula* and the *Brauer-Klimyk formula*, respectively.