Springer correspondences for reflection groups

Cathy Kriloff
Department of Mathematics
Idaho State University
Pocatello, ID 83209-8085
krilcath@math.wisc.edu

and

Arun Ram
Department of Mathematics
University of Wisconsin
Madison, WI 53706
ram@math.wisc.edu

Abstract

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1 Introduction

2 Preliminaries:

Let $\mathfrak{h}^*$ be a vector space over a field $\mathbb{F}$ and let $n = \dim(\mathfrak{h}^*)$. A *reflection* is an element $s_\alpha \in GL(\mathfrak{h}^*)$ such that

$$\dim\big((\mathfrak{h}^*)^{s_\alpha}\big) = n - 1, \qquad \text{where} \quad (\mathfrak{h}^*)^{S_n} = \{x \in \mathfrak{h}^* \mid s_\alpha x = x\}.$$

A *reflection group* is a finite subgroup W of $GL(\mathfrak{h}^*)$ generated by reflections.

Theorem 2.1. *Let $\mathfrak{h}^*$ be a vector space and let W be a finite subgroup of $GL(\mathfrak{h}^*)$. The following are equivalent*

(a) W is a reflection group, $\quad W = \langle s_\alpha \mid s_\alpha \in W \text{ is a reflection}\rangle$.

(b) $S(\mathfrak{h}^*)^W$ is a polynomial ring, $\quad S(\mathfrak{h}^*)^W = \mathbb{C}[f_1, f_2, \ldots, f_n]$.

(c) $S(\mathfrak{h}^*)$ is a free $S(\mathfrak{h}^*)^W$-module.

MSC 2000: 20C08 (05E10)
Keywords:
Research of the first author supported in part by National Security Agency grant MDA904-03-1-0093.
Research of the second author supported in part by National Security Agency grant MDA904-01-1-0032 and NSF Grant ????.

3 The symmetric groups S_n

Let

$$\mathfrak{h}^* = \text{span}\{x_1, \ldots, x_n\} \qquad \text{and} \qquad W = S_n,$$

acting by permuting $x_1, x_2, \ldots, x_n$. The reflections in S_n are the transpositions $s_{ij} = (i,j)$, $1 \le i < j \le n$, and

$$S(\mathfrak{h}^*)^W = \mathbb{C}[x_1, \ldots, x_n]^{S_n} = \mathbb{C}[e_1, \ldots, e_n], \qquad \text{where} \qquad \prod_{i=1}^{n}(1 + tx_i) = 1 + \sum_{i=1}^{n} t^r e_r.$$

A *partition* is a collection of boxes in a corner,

$$\lambda = PICTURE = (\lambda_1, \ldots, \lambda_n), \qquad \text{with } \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0,$$

where $\lambda_i =$ (number of boxes in row i).

The group

$$G = GL_n(\mathbb{C}) \quad \text{acts on} \quad \mathfrak{g} = M_n(\mathbb{C}) \quad \text{by conjugation.}$$

The *nilpotent cone* is

$$\mathcal{N} = \{x \in \mathfrak{g} \mid x \text{ is nilpotent}\} = \bigsqcup_{\lambda} G \cdot x_\lambda,$$

where the orbits $G \cdot x_\lambda$ are indexed by the partitions with n boxes. Define

$$\mu \le \lambda \qquad \text{if} \qquad G \cdot x_\mu \subseteq \overline{G \cdot x_\lambda}.$$

Let

$$M^\lambda = \text{Ind}_{S_\lambda}^{S_n}, \qquad \text{where } S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_n},$$

and define

$$K_{\lambda\mu} \in \mathbb{Z}_{\ge 0} \qquad \text{by} \qquad M^\lambda = \sum_{\mu} K_{\lambda\mu} S_n^\mu,$$

where S_n^μ are the simple S_n-modules. Then the matrix $K = (K_{\lambda\mu})$ is

(a) square,

(b) upper triangular,

(c) has diagonal entries 1.

Let $\mu = (\mu_1 \le \mu_2 \le \ldots \le \mu_n)$ be a partition and let $\mu' = (\mu'_1 \le \mu'_2 \le \cdots)$ be the conjugate partition. Let

$$d_k(\mu) = \mu'_1 + \mu'_2 + \cdots + \mu'_k, \qquad \text{for } 1 \le k \le n.$$

and define

$$I_\mu = \langle e_r(S) \mid k - d_k(\mu) < r \le k,\ S \subseteq \{x_1, \ldots, x_n\},\ \text{Card}(S) = k \rangle.$$

Then, as a graded ring,

$$H^*(\mathcal{B}_\mu) \cong \frac{\mathbb{C}[x_1, \ldots, x_n]}{I_\mu}.$$

4 The reflection groups $G(r, 1, n)$

The invariants are given by

$$f_i = e_i(x_1^r, \ldots, x_n^r), \qquad \text{for } 1 \le i \le n.$$

5 Type A_1: the group $G(2, 1, 1)$

In this case $W = \langle s_1 \rangle$ with $s_1^2 = 1$ and $\mathfrak{h}^* = \mathbb{C}\text{-span}\{x_1\}$ with $s_1 x_1 = -x_1$. Then

$$\mathbb{H} = \mathbb{C}W \otimes \mathbb{C}[x_1].$$

The module

$$M(\gamma) \quad \text{has basis} \quad \{v_\gamma, t_{s_1} v_\gamma\}$$

with

$$x_1 v_\gamma = \gamma v_\gamma.$$

Then

$$x_1 t_{s_1} v_\gamma = -t_{s_1} \gamma v_\gamma + 2c v_\gamma.$$

Thus the matrix of x_1 has eigenvalues γ and $-\gamma$ and satisfies the equation $x_1^2 - \gamma^2 = 0$.

Let

$$v^+ = \frac{1}{2}(v_\gamma + t_{s_1} v_\gamma) \quad \text{and} \quad v^- = \frac{1}{2}(v_\gamma - t_{s_1} v_\gamma).$$

Then

$$t_{s_1} v^\pm = \pm v^\pm \qquad \text{and} \qquad x_1 v^+ = \frac{1}{2}(2cv^+ + (2c + 2\gamma)v^-), \qquad x_1 v^- = \frac{1}{2}((2\gamma - 2c)v^+ - 2cv^-).$$

Hence v^- spans a submodule if $2\gamma = 2c$ and v^+ spans a submodule if $2\gamma = -2c$. If $\gamma = -c$ then

$$x_1 v^+ = -\gamma v^+ = cv^+, \qquad \text{and, in the quotient} \qquad x_1 v^- = \gamma v^- = -cv^-,$$

so that $(x_1 - \gamma)v^- = (x_1 + c)v^- = 0$. On the quotient, all eigenvalues of x_1 are negative.

If $\gamma = 0$ then

$$x_1 v^+ = c(v^+ + v^-) \quad \text{and} \quad x_1 v^- = -c(v^+ + v^-),$$

so that $x^2 v^- = 0$ and all eigenvalues of x_1 are 0.

6 The reflection groups $G(r, p, n)$

The invariants are

$$f_i = e_i(x_1^r, \ldots, x_n^r), \quad \text{for } 1 \le i \le n-1, \qquad \text{and} \qquad f_n = e_n(x_1^{r/p}, \ldots, x_n^{r/p}).$$

7 The group G_4

The complex reflection group G_4 is the subgroup of $GL_2(\mathbb{C})$ generated by the elements

$$S = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad \text{and} \quad T = \frac{-\omega^2}{\sqrt{2}} \begin{pmatrix} \varepsilon & \varepsilon^3 \\ \varepsilon & \varepsilon^7 \end{pmatrix}, \qquad \text{where} \qquad \varepsilon = e^{2\pi i/8} \quad \text{and} \quad \omega = e^{2\pi i/3}.$$

These elements satisfy the relations

$$S^2 = -1, \quad T^3 = 1, \quad (ST)^3 = 1.$$

The invariants are

$$f = x_1^4 + 2i\sqrt{3}\, x_1^2 x_2^2 + x_2^4 \qquad \text{and} \qquad x_1 x_2 (x_1^4 - x_2^4).$$

The group G_4 is order 24 and the elements are

$$G_4 = \left\{ \begin{matrix} 1, & -1, & T, & -T, & T^2, & -T^2 \\ S, & -S, & ST, & -ST, & ST^2, & -ST^2 \\ TS, & -TS, & T^2S, & -T^2S, & T^2ST, & -T^2ST \\ TST & -TST & STS & -STS & TST^2 & -TST^2 \end{matrix} \right\}.$$

It is useful to use the following additional relations

$$\begin{matrix} TSTST = -S, & TSTS = -ST^2, & TST = ST^2S, \\ STSTS = T^2, & STST = -T^2S, & STS = -T^2ST^2. \end{matrix}$$

The conjugacy classes are

$$\mathcal{C}_1 = \{1\}, \qquad \mathcal{C}_T = \{T, TS, ST, -STS\}, \qquad \mathcal{C}_{T^2} = \{T^2, -TST, -T^2S, -ST^2\},$$

$$\mathcal{C}_{-1} = \{-1\}, \qquad \mathcal{C}_{-T} = \{-T, -TS, -ST, STS\}, \qquad \mathcal{C}_{-T^2} = \{-T^2, TST, T^2S, ST^2\}.$$

$$\mathcal{C}_S = \{S, TST^2, T^2ST, -TST^2, -T^2ST, -S\},$$

where $\mathcal{C}_T$ and $\mathcal{C}_{T^2}$ are the conjugacy classes of reflections. The character table of G_4 is

	$\mathcal{C}_1$	$\mathcal{C}_{-1}$	$\mathcal{C}_S$	$\mathcal{C}_T$	$\mathcal{C}_{-T}$	$\mathcal{C}_{T^2}$	$\mathcal{C}_{-T^2}$
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	ω^2	ω^2	ω	ω
χ_3	1	1	1	ω	ω	ω^2	ω^2
χ_4	2	-2	0	$-\omega^2$	ω^2	$-\omega$	ω
χ_5	2	-2	0	-1	1	-1	1
χ_6	3	3	-1	0	0	0	0
χ_7	2	-2	0	$-\omega$	ω	$-\omega^2$	ω^3

The ring

$$\frac{S(\mathfrak{h}^*)}{\langle f, t\rangle} \quad \text{has basis} \quad \begin{matrix} 1, \\ x_1, x_2, \\ x_1^2, x_1x_2, x_2^2, \\ x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, \\ x_1^4, x_1^3x_2, x_1^2x_2^2, x_1x_2^3, \\ x_1^5, x_1^4x_2, x_1^3x_2^2, x_1^2x_2^3, \\ x_1^6, x_1^5x_2, x_1^4x_2, \\ x_1^7, x_1^6x_2, \\ x_1^8, \end{matrix}$$

and relations

$$x_2^4 = -x_1^4 - Nx_1^2x_2^2$$

$$x_1x_2^4 = -x_1^5 - Nx_1^3x_2^2, \quad x_2^5 = -x_1^4x_2 - Nx_1^2x_2^3,$$

$$x_1^3x_2^3 = \tfrac{-2}{N}x_1^5x_2, \quad x_1^2x_2^4 = -x_1^6 - Nx_1^4x_2^2, \quad x_1x_2^5 = x_1^5x_2, \quad x_2^6 = Nx_1^6 - 13x_1^4x_2^2,$$

$$x_1^5x_2^2 = \tfrac{N}{2}x_1^7, \quad x_1^4x_2^3 = \tfrac{-2}{N}x_1^6x_2, \quad x_1^3x_2^4 = \tfrac{-1}{7}x_1^7, \quad x_1^2x_2^5 = x_1^6x_2, \quad x_1x_2^6 = \tfrac{N}{14}x_1^7, \quad x_2^7 = \tfrac{14}{N}x_1^6x_2,$$

$$x_1^7x_2 = 0, \quad x_1^6x_2^2 = \tfrac{N}{14}x_1^8, \quad x_1^5x_2^3 = 0, \quad x_1^4x_2^4 = \tfrac{-1}{7}x_1^8, \quad x_1^3x_2^6 = \tfrac{N}{14}x_1^8, \quad x_1x_2^7 = 0, \quad x_2^8 = x_1^8.$$

The graded character table of this module is

	$\mathcal{C}_1$	$\mathcal{C}_{-1}$	$\mathcal{C}_S$	$\mathcal{C}_T$	$\mathcal{C}_{-T}$	$\mathcal{C}_{T^2}$	$\mathcal{C}_{-T^2}$
η_0	1	1	1	1	1	1	1
η_1	2	-2	0	$-\omega^2$	ω^2	$-\omega$	ω
η_2	3	3	-1	0	0	0	0
η_3	4	-4	0	1	-1	1	-1
η_4	4	4	0	ω	ω	ω^2	ω^2
η_5	4	-4	0	ω^2	$-\omega^2$	ω	$-\omega$
η_6	3	3	-1	0	0	0	0
η_7	2	-2	0	-1	1	-1	1
η_8	1	1	1	ω^2	ω^2	ω	ω

which shows that

$$\begin{aligned} \eta_0 &= \chi_1, \\ \eta_1 &= \chi_4, \\ \eta_2 &= \chi_6, \\ \eta_3 &= \chi_4 + \chi_7, \\ \eta_4 &= \chi_3 + \chi_6 \\ \eta_5 &= \chi_5 + \chi_7, \\ \eta_6 &= \chi_6, \\ \eta_7 &= \chi_5, \\ \eta_8 &= \chi_2. \end{aligned}$$

Let

$$N = 2i\sqrt{3}, \quad \text{so that} \quad N^2 = -\sqrt{12}.$$

Then, in the basis above, the representations in each degree of the coinvariant algebra are given by:

$$\rho_1(S) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

$$\rho_2(S) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rho_3(S) = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

$$\rho_4(S) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\rho_5(S) = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

$$\rho_6(S) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\rho_7(S) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\rho_8(S) = (1)$$

$$\rho_1(T) = \frac{-\omega^2\varepsilon}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

$$\rho_2(T) = \frac{\omega\varepsilon^2}{2} \begin{pmatrix} 1 & i & -1 \\ 2 & 0 & 2 \\ 1 & -i & -1 \end{pmatrix},$$

$$\rho_3(T) = \frac{\varepsilon}{2\sqrt{2}} \begin{pmatrix} -i & 1 & i & -1 \\ -3i & 1 & -i & 3 \\ -3i & -1 & -i & -3 \\ -3i & -1 & -i & -3 \\ -i & -1 & i & 1 \end{pmatrix}$$

$$\rho_4(T) = \frac{\omega^2}{4} \begin{pmatrix} 0 & -2i & 0 & 2i \\ -4 & -2i & 0 & -2i \\ -6+N & -Ni & -2-N & Ni \\ -4 & 2i & 0 & 2i \end{pmatrix}$$

$$\rho_5(T) = \frac{\omega\varepsilon}{4\sqrt{2}} \begin{pmatrix} -4 & 4i & 0 & 0 \\ 4 & 4i & 0 & 0 \\ 10-5N & (2+3N)i & 2+N & (2+N)i \\ 10-N & (-2+N)i & 2+N & (-2-N)i \end{pmatrix}$$

$$\rho_6(T) = \frac{1}{8} \begin{pmatrix} (-14+N)(-i) & 6-N & (-2-N)(-i) \\ (-\frac{40}{N}+12)(-i) & 0 & (-4-\frac{8}{N})(-i) \\ (2-15N)(-i) & 18+5N & (14-N)(-i) \end{pmatrix}$$

$$\rho_7(T) = \begin{pmatrix} \frac{\omega^2\varepsilon^3}{8\sqrt{2}} - 4 + 2N & (-\frac{12}{7} - \frac{2}{7}N) \\ 28 - \frac{56}{N} & -4i - \frac{24}{N}i \end{pmatrix}$$

$$\rho_8(T) = \frac{\omega}{16}(-8+4N)$$

$$\rho_2(T^2) = \frac{\omega^2}{4}\begin{pmatrix} -2i & -2i & -2i \\ -4 & 0 & 4 \\ 2i & -2i & 2i \end{pmatrix}$$

$$\rho_3(T^2) = \frac{1}{8}\begin{pmatrix} 2+2i & 2+2i & 2+2i & 2+2i \\ 2-2i & 2-2i & -2+2i & -2+2i \\ -2-2i & 2+2i & 2+2i & 2-2i \\ -2+2i & 2-2i & -2+2i & 2-2i \end{pmatrix}$$

$$\rho_4(T^2) = \frac{\omega^2}{16}\begin{pmatrix} 0 & -8 & 0 & -8 \\ 4i & 8i & 0 & -8i \\ 4+4N & -4N & -8+4N & -4N \\ -4i & 8i & 0 & -8i \end{pmatrix}$$

$$\rho_5(T^2) = \frac{\omega^2}{32}\begin{pmatrix} -16(1-i) & 16(1-i) & 0 & 0 \\ 16(1+i)-16(1+i) & 0 & 0 & \\ (-40-20N)(1-i) & (-8+12N)(1-i) & (8-4N)(1-i) & (8-4N)(1-i) \\ (40+4N)(1+i) & (-8-8N)(1+i) & (-8+4N)(1+i) & (8-4N)(1+i) \end{pmatrix}$$

$$\rho_6(T^2) = \frac{1}{64}\begin{pmatrix} -112-8Ni & 48i+8Ni & 16i-8Ni \\ 96+\frac{360}{N} & 0 & 32-\frac{66}{N} \\ -16-120Ni & -144i+40Ni & 112i+8Ni \end{pmatrix}$$

$$\rho_7(T^2) = \frac{\omega}{128}\begin{pmatrix} 32-48i+16N+16Ni & -\frac{116}{7}-\frac{116}{7}i+\frac{16N}{7}+\frac{16N}{7}i \\ -224+224i-\frac{448}{N}+\frac{448}{N}i & -60+60i+\frac{122N}{3}-\frac{122N}{3}i \end{pmatrix}$$

$$\rho_8(T^2) = \omega$$

8 The dihedral groups $G(r,r,2)$

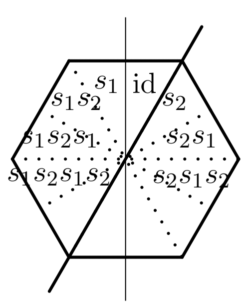
Let r be a positive integer and let

$$\theta = \pi/r \qquad \text{and} \qquad \xi = e^{i2\theta}.$$

With respect to the orthonormal basis $\{\varepsilon_1, \varepsilon_2\}$ of $\mathbb{C}^2$ the *dihedral group* $G(r,r,2)$ is the group of 2×2 matrices given by

$$G(r,r,2) = \left\{ \begin{pmatrix} -\cos 2k\theta & sin 2k\theta \\ \sin 2k\theta & \cos 2k\theta \end{pmatrix}, \begin{pmatrix} \cos 2k\theta & -\sin 2k\theta \\ \sin 2k\theta & \cos 2k\theta \end{pmatrix} \mid k = 0,1,\ldots,r-1 \right\}.$$

In this form $G(r,r,2)$ is the group of symmetries of a regular r-gon (embedded in $\mathbb{R}^2$ with its center at the origin),



with s_1 being the reflection in H_{α_1} and s_2 the reflection in H_{α_2}.

Let x_1 and x_2 be given by

$$\varepsilon_1 = \frac{1}{\sqrt{2}}(\xi^{1/2}x_1 - \xi^{-1/2}x_2), \qquad\qquad x_1 = \frac{\xi^{-1/2}}{\sqrt{2}}(\varepsilon_1 - i\varepsilon_2),$$
$$\text{and}$$
$$\varepsilon_2 = \frac{1}{\sqrt{2}}(\xi^{1/2}x_1 + \xi^{-1/2}x_2), \qquad\qquad x_2 = -\frac{\xi^{1/2}}{\sqrt{2}}(\varepsilon_1 + i\varepsilon_2).$$

Then, with respect to the basis $\{x_1, x_2\}$,

$$G(r,r,2) = \left\{ \begin{pmatrix} \xi^k & 0 \\ 0 & \xi^{-k} \end{pmatrix}, \ \begin{pmatrix} 0 & \xi^k \\ \xi^{-k} & 0 \end{pmatrix} \ \middle|\ k = 0, 1, \ldots, r-1 \right\}.$$

The roots are

$$\beta_k = \cos(k\theta)\varepsilon_1 + \sin(k\theta)\varepsilon_2 = \frac{1}{\sin\theta}\big(\sin((k+1)\theta)\alpha_1 + \sin(k\theta)\alpha_2\big), \qquad 0 \le k \le 2r-1,$$

and if the positive roots are

$$R^+ = \{\beta_k \mid 0 \le k \le r-1\} \qquad \text{then} \qquad \begin{aligned} \alpha_1 &= \beta_0 = \varepsilon_1 = \tfrac{1}{\sqrt{2}}(\xi^{1/2}x_1 - \xi^{-1/2}x_2), \\ \alpha_2 &= \beta_{r-1} = -\cos\theta\varepsilon_1 + \sin\theta\varepsilon_2 = \tfrac{1}{\sqrt{2}}(x_1 - x_2), \end{aligned}$$

are the simple roots with

$$\beta_k = \frac{1}{\sin\theta}\big(\sin((k+1)\theta)\alpha_1 + \sin(k\theta)\alpha_2\big), \qquad 0 \le k \le 2r-1.$$

The simple reflections are

$$s_1 = \begin{pmatrix} 0 & \xi \\ \xi^{-1} & 0 \end{pmatrix} \qquad \text{and} \qquad s_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \text{in the basis } \{x_1, x_2\},$$

and

$$s_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad s_2 = \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}, \qquad \text{in the basis } \{\varepsilon_1, \varepsilon_2\}.$$

Thus $\varepsilon_1, \varepsilon_2$ are the eigenvectors of s_1 and x_1, x_2 are the eigenvectors of $s_1 s_2$. Then

$$-\beta_k = \beta_{r+k}, \qquad s_1\beta_k = \beta_{r-k}, \qquad s_2\beta_k = \beta_{r-2-k}.$$

The elements s_1, s_2 satisfy

$$\underbrace{s_1 s_2 s_1 s_2 \cdots}_{r \text{ factors}} = \underbrace{s_2 s_1 s_2 s_1 \cdots}_{r \text{ factors}}, \qquad s_1^2 = 1, \qquad s_2^2 = 1,$$

and $t = s_1 s_2$ and $s = s_2$ satisfy

$$t^r = 1, \qquad s^2 = 1, \qquad st = t^{-1}s,$$

The invariants are given by

$$f_1 = x_1^r + x_2^r = \mathrm{Re}((\varepsilon_1 + i\varepsilon_2)^r), \qquad \text{and} \qquad f_2 = x_1 x_2 = -\frac{1}{2}(\varepsilon_1^2 + \varepsilon_2^2).$$

Another choice for the invariant of degree r is

$$f_1' = \prod_{i=0}^{r-1} (\cos(2k\theta)\varepsilon_1 + \sin(2k\theta)\varepsilon_2).$$

The Cartan matrix of $I_2(m)$ is

$$A = \begin{pmatrix} 2 & -2\cos(\pi/m) \\ -2\cos(\pi/m) & 2 \end{pmatrix} \qquad \text{and} \qquad A^{-1} = \frac{1}{2\sin^2(\pi/m)} \begin{pmatrix} 1 & \cos(\pi/m) \\ \cos(\pi/m) & 1 \end{pmatrix}.$$

9 Computation of the degree filtration for M_γ

In the graded Hecke algebra

$$x_1 = \xi^{-1} t_{s_1} x_2 t_{s_1} + \sqrt{2}\,\xi^{-1/2}\, t_{s_1} \qquad \text{and} \qquad x_1 = t_{s_2} x_2 t_{s_2} + \sqrt{2}\, t_{s_2}.$$

If Δ_1 and Δ_2 are the BGG operators,

$$\Delta_1 p = \frac{p - s_1 p}{\alpha_1} \qquad \text{and} \qquad \Delta_2 p = \frac{p - s_2 p}{\alpha_2},$$

then

$$\Delta_1 x_1^k = \frac{x_1^k - \xi^{-k} x_2^k}{(1/\sqrt{2})(\xi^{1/2} x_1 - \xi^{-1/2} x_2)} = \sqrt{2}\xi^{-k/2} \left(\frac{\xi^{k/2} x_1^k - \xi^{-(k/2)} x_2^k}{\xi^{1/2} x_1 - \xi^{-1/2} x_2} \right), \qquad \Delta_1 x_2^k = -\xi^k \Delta_1 x_1^k,$$

$$\Delta_2 x_1^k = \frac{x_1^k - x_2^k}{(1/\sqrt{2})(x_1 - x_2)} = \sqrt{2} \left(\frac{x_1^k - x_2^k}{x_1 - x_2} \right), \qquad \Delta_2 x_2^k = -\Delta_2 x_1^k,$$

Let v^- be the vector in M_γ such that

$$t_{s_1} v^- = -v^- \qquad \text{and} \qquad t_{s_2} v^- = -v^-.$$

Assume that

$$\dim(M_\gamma) = 2(\ell - 1) + 1 = 2\ell - 1.$$

Then

$$\{v^-, x_1 v^-, x_2 v^-, x_1^2 v^-, x_2^2 v^-, \ldots, x_1^{\ell-1} v^-, x_2^{\ell-1} v^-\} \quad \text{is a basis of } M_\gamma.$$

Suppose that

$$x_1^\ell v^- = \left(a_0 + \sum_{i=1}^{\ell-1} a_{1i} x_1^i + a_{2i} x_2^i \right) v^-, \qquad \text{and} \qquad x_2^\ell v^- = \left(b_0 + \sum_{i=1}^{\ell-1} b_{1i} x_1^i + b_{2i} x_2^i \right) v^-.$$

We want to solve for the constants a_0, b_0 and a_{1i}, a_{2i} and b_{1i}, b_{2i}. The polynomial $x_1 x_2$ acts by a constant z_2 on M_γ since it is in the center of $\mathbb{H}$. Also

$$-x_1(\Delta_1 x_1^{\ell-1}) - \sqrt{2}\xi^{-(2\ell-1)/2} x_2^{\ell-1} = -(\Delta_1 x_1^\ell).$$

As operators on v^-,

$$
\begin{aligned}
x_1^\ell &= x_1 x_1^{\ell-1} \\
&= (\xi^{-1} t_{s_1} x_2 t_{s_1} + \sqrt{2}\xi^{-1/2} t_{s_1}) x_1^{\ell-1} \\
&= (\xi^{-1} t_{s_1} x_2 + \sqrt{2}\xi^{-1/2})(\xi^{-(\ell-1)} x_2^{\ell-1} t_{s_1} + \Delta_1 x_1^{\ell-1}) \\
&= \xi^{-1} t_{s_1} x_2 + \sqrt{2}\xi^{-1/2})(-\xi^{-(\ell-1)} x_2^{\ell-1} + \Delta_1 x_1^{\ell-1} \\
&= \xi^{-\ell} t_{s_1} x_2^\ell + \xi^{-1}\xi x_1 (\Delta_1 x_1^{\ell-1}) t_{s_1} + \xi^{-1}(-\sqrt{2}\xi^{1/2})(\Delta_1 x_1^{\ell-1}) - \sqrt{2}\xi^{-(2\ell-1)/2} x_1^{\ell-1} + \sqrt{2}\xi^{-1/2}(\Delta_1 x_1^{\ell-1}) \\
&= -\xi^{-\ell} t_{s_1} \left(b_0 + \sum_{i=1}^{\ell-1} b_{1i} x_1^i + b_{2i} x_2^i \right) - (\Delta_1 x_1^\ell) \\
&= -\xi^{-\ell} \left(-b_0 - \sum_{i=1}^{\ell-1} b_{1i} \xi^{-i} x_2^i + b_{2i} \xi^i x_1^i \right) - \xi^{-\ell} \left(\sum_{i=1}^{\ell-1} b_{1i}(\Delta_1 x_1^i) + b_{2i}(\Delta_1 x_2^i) \right) - (\Delta_1 x_1^\ell) \\
&= \xi^{-\ell} b_0 + \sum_{i=1}^{\ell-1} b_{1i} \xi^{-(\ell+i)} x_2^i + b_{2i} \xi^{-(\ell-i)} x_1^i - \xi^{-\ell} \sum_{i=1}^{\ell-1} (b_{1i} - \xi^i b_{2i})(\Delta_1 x_1^i) - (\Delta_1 x_1^\ell) \\
&= \xi^{-\ell} b_0 + \sum_{i=1}^{\ell-1} b_{1i} \xi^{-(\ell+i)} x_2^i + b_{2i} \xi^{-(\ell-i)} x_1^i \\
&\quad + \sum_{i=1}^{\ell-1} \sqrt{2}(\xi^i b_{2i} - b_{1i})(\xi^{-(2\ell+1)/2} x_1^{i-1} + \xi^{-(2\ell+2i-1)/2} x_2^{i-1} + \xi^{-(2\ell+3)/2} z_2 x_1^{i-3} + \xi^{-(2\ell+2i-3)/2} z_2 x_2^{i-3} + \cdots) \\
&\quad - \sqrt{2}(\xi^{-1/2} x_1^{\ell-1} + \xi^{-(2\ell-1)/2} x_2^{\ell-1} + \xi^{-3/2} z_2 x_1^{\ell-3} + \xi^{-(2\ell-3)/2} z_2 x_2^{\ell-3} + \cdots)
\end{aligned}
$$

A similar computation using that

$$
-x_1(\Delta_2 x_1^{\ell-1}) - \sqrt{2} x_2^{\ell-1} = -(\Delta_2 x_1^\ell),
$$

gives that, as operators on v^-,

$$
\begin{aligned}
x_1^\ell &= x_1 x_1^{\ell-1} = (t_{s_2} x_2 t_{s_2} + \sqrt{2} t_{s_2}) x_1^{\ell-1} \\
&= b_0 + \sum_{i=1}^{\ell-1} b_{1i} x_2^i + b_{2i} x_1^i \\
&\quad + \sum_{i=1}^{\ell-1} \sqrt{2}(b_{2i} - b_{1i})(x_1^{i-1} + x_2^{i-1} + z_2 x_1^{i-3} + z_2 x_2^{i-3} + \cdots) \\
&\quad - \sqrt{2}(x_1^{\ell-1} + x_2^{\ell-1} + z_2 x_1^{\ell-3} + z_2 x_2^{\ell-3} + \cdots).
\end{aligned}
$$

Comparing coefficients of $x_1^{\ell-1}$ in these two expressions gives

$$
\xi^{-1} b_{2,\ell-1} - \sqrt{2}\,\xi^{-1/2} = b_{2,\ell-1} - \sqrt{2}, \qquad \text{so that} \qquad b_{2,\ell-1} = \frac{\sqrt{2}}{1+\xi^{-1/2}}.
$$

Comparing coefficients of $x_2^{\ell-1}$ in these two expressions gives

$$
\xi^{-(2\ell-1)} b_{1,\ell-1} - \sqrt{2}\,\xi^{-(2\ell-1)/2} = b_{1,\ell-1} - \sqrt{2}, \qquad \text{so that} \qquad b_{1,\ell-1} = \frac{\sqrt{2}}{1+\xi^{-(2\ell-1)/2}}.
$$

Comparing coefficients of $x_1^{\ell-2}$ gives

$$b_{2,\ell-2} + \sqrt{2}(b_{2,\ell-1} - b_{1,\ell-1}) = b_{2,\ell-2}\xi^{-2} + (\xi^{\ell-1}b_{2,\ell-1} - b_{1,\ell-1})\sqrt{2}\,\xi^{-(2\ell+1)/2},$$

and, solving for $b_{2,\ell-2}$ gives

$$b_{2,\ell-2} = \frac{2\xi^{-1/2}(1-\xi^{-\ell+1})}{(1+\xi^{-(2\ell-1)/2})(1+\xi^{-1/2})(1-\xi^{-1})}.$$

Comparing coefficients of $x_2^{\ell-2}$ gives

$$b_{1,\ell-2} + \sqrt{2}(b_{2,\ell-1} - b_{1,\ell-1}) = b_{2,\ell-2}\xi^{-2\ell-2} + \sqrt{2}(\xi^{-(2\ell-1)/2}b_{2,\ell-1} - \xi^{-(4\ell-3)/2}b_{1,\ell-1}),$$

and, solving for $b_{1,\ell-2}$ gives

$$b_{1,\ell-2} = \frac{2\xi^{-1/2}}{(1+\xi^{-(2\ell-1)/2})(1+\xi^{-1/2})}.$$

10 The graded Hecke algebra

The *graded Hecke algebra* is

$$\mathbb{H} = \mathbb{C}W \otimes S(\mathfrak{h}_{\mathbb{C}}^*)$$

with multiplication determined by the multiplication in $S(\mathfrak{h}_{\mathbb{C}}^*)$ and the multiplication in $\mathbb{C}W$ and the relations

$$xt_{s_i} = t_{s_i}s_i(x) + c_{\alpha_i}\langle x, \alpha_i^\vee\rangle\,, \qquad \text{for } x \in \mathfrak{h}_{\mathbb{C}}^*, \tag{10.1}$$

where $\alpha_1^\vee, \ldots, \alpha_n^\vee \in \mathfrak{h}_{\mathbb{R}}$ are the simple co-roots. More generally, it follows that for any $p \in S(\mathfrak{h}_{\mathbb{C}}^*)$,

$$pt_{s_i} = t_{s_i}(s_ip) + c_{\alpha_i}\Delta_i(p) \qquad \text{and} \qquad t_{s_i}p = (s_ip)t_{s_i} + c_{\alpha_i}\Delta_i(p),$$

where $\Delta_i : S(\mathfrak{h}_{\mathbb{C}}^*) \to S(\mathfrak{h}_{\mathbb{C}}^*)$ is the *BGG-operator* given by

$$\Delta_i(p) = \frac{p - s_ip}{\alpha_i}, \qquad \text{for } p \in S(\mathfrak{h}_{\mathbb{C}}^*).$$

References

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