Reflection group examples

Arun Ram
Department of Mathematics
University of Wisconsin
Madison, WI 53706
ram@math.wisc.edu

1 The symmetric groups S_n

Let

$$\mathfrak{h}^* = \mathrm{span}\{x_1, \ldots, x_n\} \qquad \text{and} \qquad W = S_n,$$

acting by permuting $x_1, x_2, \ldots, x_n$. The reflections in S_n are the transpositions $s_{ij} = (i, j)$, $1 \le i < j \le n$, and

$$S(\mathfrak{h}^*)^W = \mathbb{C}[x_1, \ldots, x_n]^{S_n} = \mathbb{C}[e_1, \ldots, e_n], \qquad \text{where} \qquad \prod_{i=1}^{n}(1 + tx_i) = 1 + \sum_{i=1}^{n} t^r e_r.$$

A *partition* is a collection of boxes in a corner,

$$\lambda = PICTURE = (\lambda_1, \ldots, \lambda_n), \qquad \text{with } \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0,$$

where $\lambda_i =$ (number of boxes in row i).

The group

$$G = GL_n(\mathbb{C}) \quad \text{acts on} \quad \mathfrak{g} = M_n(\mathbb{C}) \quad \text{by conjugation.}$$

The *nilpotent cone* is

$$\mathcal{N} = \{x \in \mathfrak{g} \mid x \text{ is nilpotent}\} = \bigsqcup_{\lambda} G \cdot x_\lambda,$$

where the orbits $G \cdot x_\lambda$ are indexed by the partitions with n boxes. Define

$$\mu \le \lambda \qquad \text{if} \qquad G \cdot x_\mu \subseteq \overline{G \cdot x_\lambda}.$$

Let

$$M^\lambda = \mathrm{Ind}_{S_\lambda}^{S_n}, \qquad \text{where } S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_n},$$

and define

$$K_{\lambda\mu} \in \mathbb{Z}_{\ge 0} \qquad \text{by} \qquad M^\lambda = \sum_{\mu} K_{\lambda\mu} S_n^\mu,$$

where S_n^μ are the simple S_n-modules. Then the matrix $K = (K_{\lambda\mu})$ is

(a) square,

(b) upper triangular,

(c) has diagonal entries 1.

Let $\mu = (\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n)$ be a partition and let $\mu' = (\mu'_1 \leq \mu'_2 \leq \cdots)$ be the conjugate partition. Let

$$d_k(\mu) = \mu'_1 + \mu'_2 + \cdots + \mu'_k, \qquad \text{for } 1 \leq k \leq n.$$

and define

$$I_\mu = \langle e_r(S) \mid k - d_k(\mu) < r \leq k,\ S \subseteq \{x_1, \ldots, x_n\},\ \mathrm{Card}(S) = k\rangle.$$

Then, as a graded ring,

$$H^*(\mathcal{B}_\mu) \cong \frac{\mathbb{C}[x_1, \ldots, x_n]}{I_\mu}.$$

2 The groups $G_{r,p,n}$

The group $G_{r,p,n}$ is the group of $n \times n$ matrices with

(a) exactly one non zero entry in each row and each column,

(b) r^{th} roots of unity as nonzero entries,

(c) $\Big(\prod_{\text{nonzero entries}} a_{ij}\Big) = 1.$

The exact sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_{r,p,n} & \longrightarrow & G_{r,1,n} & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & 1 \\ & & & & t^\lambda w & \longmapsto & |\lambda| & & \end{array}$$

where

$$t^\lambda w = t_1^{\lambda_1} \cdots t_n^{\lambda_n} w, \qquad \text{for } w \in S_n,\ t_i = \mathrm{diag}(1, \ldots, 1, e^{2\pi i/r}, 1, \ldots, 1)$$

is conceptually helpful since it shows that $G_{r,p,n}$ is a normal subgroup of index p in $G_{r,1,n}$.

Let $x_1, \ldots, x_n$ be the orthonormal basis of V where x_i is the column vector with 1 in the ith spot and 0 elsewhere. Then, if $W = G_{r,p,n}$

$$S(V)^W = \mathbb{C}[p_i(x_1^r, \ldots, x_n^r), (x_1 x_2 \cdots x_n)^{r/p} \mid 1 \leq i \leq n-1],$$

so that the generators of $S(V)^W$ are

$$p_i(x_1^r, \ldots, x_n^r), \quad 1 \leq i \leq n-1, \qquad \text{and} \qquad (x_1 x_2 \cdots x_n)^{r/p},$$

and the degrees of W are

$$r, 2r, \ldots, (n-1)r, n(\frac{r}{p}).$$

3 Type A_1: the group $G(2,1,1)$

In this case $W = \langle s_1 \rangle$ with $s_1^2 = 1$ and $\mathfrak{h}^* = \mathbb{C}\text{-span}\{x_1\}$ with $s_1 x_1 = -x_1$. Then

$$\mathbb{H} = \mathbb{C}W \otimes \mathbb{C}[x_1].$$

The module

$$M(\gamma) \quad \text{has basis} \quad \{v_\gamma, t_{s_1} v_\gamma\}$$

with

$$x_1 v_\gamma = \gamma v_\gamma.$$

Then

$$x_1 t_{s_1} v_\gamma = -t_{s_1}\gamma v_\gamma + 2c v_\gamma.$$

Thus the matrix of x_1 has eigenvalues γ and $-\gamma$ and satisfies the equation $x_1^2 - \gamma^2 = 0$.

Let

$$v^+ = \frac{1}{2}(v_\gamma + t_{s_1} v_\gamma) \quad \text{and} \quad v^- = \frac{1}{2}(v_\gamma - t_{s_1} v_\gamma).$$

Then

$$t_{s_1} v^\pm = \pm v^\pm \qquad \text{and} \qquad x_1 v^+ = \frac{1}{2}(2cv^+ + (2c+2\gamma)v^-), \qquad x_1 v^- = \frac{1}{2}((2\gamma - 2c)v^+ - 2cv^-).$$

Hence v^- spans a submodule if $2\gamma = 2c$ and v^+ spans a submodule if $2\gamma = -2c$. If $\gamma = -c$ then

$$x_1 v^+ = -\gamma v^+ = cv^+, \qquad \text{and, in the quotient} \qquad x_1 v^- = \gamma v^- = -cv^-,$$

so that $(x_1 - \gamma)v^- = (x_1 + c)v^- = 0$. On the quotient, all eigenvalues of x_1 are negative.

If $\gamma = 0$ then

$$x_1 v^+ = c(v^+ + v^-) \quad \text{and} \quad x_1 v^- = -c(v^+ + v^-),$$

so that $x^2 v^- = 0$ and all eigenvalues of x_1 are 0.

4 The groups T, O and I

The rank 2 exceptional complex reflection groups, $G_4, \ldots, G_{22}$ in the list of Shephard and Todd, are all built from the 4 basic groups,

$$I_2(4) = \text{the dihedral group of order 8}$$

The *tetrahedral group*

$$T = \text{the tetrahedral group (order 24)} \cong S_4$$

is generated by the matrices

$$S_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad T_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \epsilon & \epsilon^3 \\ \epsilon & \epsilon^7 \end{pmatrix}, \qquad S_1T_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \epsilon^3 & \epsilon^5 \\ \epsilon^7 & \epsilon^5 \end{pmatrix}$$

where $i = e^{2\pi i/4}$ and $\epsilon = e^{2\pi i/8}$. These matrices satisfy the relations

$$S_1^2 = T_1^3 = -1, \qquad (S_1T_1)^3 = 1.$$

The *octahedral group*,

$$O = \text{the octahedral group} \quad \text{(order 48)} \cong WB_3,$$

is generated by the matrices

$$S_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix}, \qquad T_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \epsilon & \epsilon \\ \epsilon^3 & \epsilon^7 \end{pmatrix}, \qquad S_1T_1 = \begin{pmatrix} \epsilon^3 & 0 \\ 0 & \epsilon^5 \end{pmatrix}$$

where $i = e^{2\pi i/4}$ and $\epsilon = e^{2\pi i/8}$. These matrices satisfy the relations

$$S_1^2 = T_1^3 = (S_1T_1)^4 = -1.$$

The *icosahedral group*,

$$I = \text{the icosahedral group} \quad \text{(order 120)},$$

is generated by the matrices

$$S_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} \eta^4-\eta & \eta^2-\eta^3 \\ \eta^2-\eta^3 & \eta-\eta^4 \end{pmatrix}, \qquad T_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} \eta^2-\eta^4 & \eta^4-1 \\ 1-\eta & \eta^3-\eta \end{pmatrix}, \qquad S_1T_1 = \begin{pmatrix} -\eta^3 & 0 \\ 0 & -\eta^2 \end{pmatrix}$$

where $\eta = e^{2\pi i/5}$. These matrices satisfy the relations

$$S_1^2 = -1, \quad T_1^3 = 1, \quad (S_1T_1)^5 = -1.$$

It is useful to note that

(a) As given, these all consist of *unitary* matrices (please check) so that they are subgroups of $U_2(\mathbb{C})$. This means that they preserve the usual hermitian inner product on V and so we can take x_1, x_2 as an orthonormal basis of V.

$I \cong WH_3$ is a twofold cover of the alternating group A_5 and

$$I_2(4) \triangleleft T \triangleleft O.$$

Apparently the generating invariants for T, O and I were given by F. Klein around 1900, I think they can be found in the book of Orlik and Terao. Each of T, O and I have three basic invariants

$$f, \qquad h = \text{Hessian of } f, \qquad t = \text{Jacobian of } f \text{ and } h.$$

which have degrees

case T	case O	case I
4	6	12
4	8	20
6	12	30

In terms of these three invariants of T, O, or I, we can specify the generating invariants of $G_4, \ldots, G_{22}$:

Case T:

Group	Generating invariants	degrees
G_4	f, t	$4, 6$
G_5	f^3, t	$12, 6$
G_6	f, t^2	$4, 12$
G_7	f^3, t^2	$12, 12$

Case O:

Group	Generating invariants	degrees
G_8	h, t	$8, 12$
G_9	h, t^2	$8, 24$
G_{10}	h^3, t	$24, 12$
G_{11}	h^3, t^2	$24, 24$
G_{12}	f, h	$6, 8$
G_{13}	f^2, h	$12, 8$
G_{14}	f, t^2	$6, 24$
G_{15}	f^2, t^2	$12, 24$

Case I:

Group	Generating invariants	degrees
G_{16}	h, t	$20, 30$
G_{17}	h, t^2	$20, 60$
G_{18}	h^3, t	$60, 30$
G_{19}	h^3, t^2	$60, 60$
G_{20}	f, t	$12, 30$
G_{21}	f, t^2	$12, 60$
G_{22}	f, h	$12, 20$

The groups which are exceptional *real reflection groups* are

Group	degrees
$G_{23} = WH_3$	$2, 6, 10$
$G_{28} = WF_4$	$2, 6, 8, 12$
$G_{30} = WH_4$	$2, 12, 20, 30$
$G_{35} = WE_6$	$2, 5, 6, 8, 9, 12$
$G_{36} = WE_7$	$2, 6, 8, 10, 12, 14, 18$
$G_{37} = WE_8$	$2, 8, 12, 14, 18, 20, 24, 30$

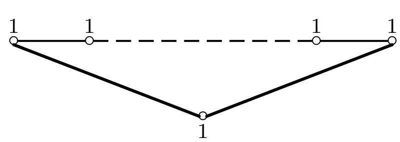
Shephard-Todd refer to Coxeter, Duke Math. J, **18** (1951), 765-782, for the invariants. Are these in Orlik-Terao? It would be good to use orthonormal bases for V as in Bourbaki Chapt. 4-6. (Group G_{37} is the last group in the Shephard-Todd list).

The McKay correspondence works for finite subgroups G of $SL_2(\mathbb{C})$ so we should work with these to start (the rank 2 complex reflection groups are semidirect products of these by cyclic groups). One associates a simply laced affine Dynkin diagram to each of these by making the graph with vertices indexed by the simple G-modules and an edge from L_i to L_j if L_j appears in $L_i \otimes V$ (note that this is a special kind of translation functor) where V is the 2 dimensional representation of G obtained from the fact that G is a subgroup of $SL_2(\mathbb{C})$. If one labels the nodes by the dimension of the irreducible then this gives the coefficients of the highest root of the root system in terms of the simple roots.

For the cyclic group of order r

$$G = \left\{ \begin{pmatrix} \xi^j & 0 \\ 0 & \xi^{-j} \end{pmatrix} \;\middle|\; 0 \le j \le r-1 \right\} \qquad \text{where} \quad \xi = e^{2\pi i/r},$$

gives the affine Dynkin diagram of type $\tilde{A}_r$



(A_n) $(n+1$ vertices, $n \ge 2)$,

since it has irreducible representations L_j, $0 \le j \le r-1$, and $V = L_1 \oplus L_{-1}$ and the tensor product rule is $L_i \otimes L_j = L_{i+j}$ (where the indices are taken mod r. From this Dynkin diagram one builds the Bratelli diagram of the tantalizer. For r large the tantalizer

$$T_k = \mathrm{End}_G(V^{\otimes k})$$

has irreducible representations indexed by

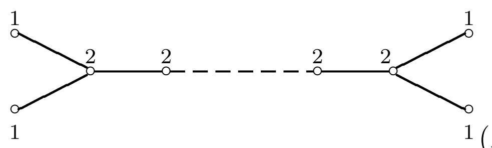
$$\hat{T}_k = \{k, k-2, \ldots, -(k-2), -k\} \qquad \text{with} \qquad \dim(T^{\pm(k-2j)}) = \binom{k}{j}, \quad \text{for } 0 \le j \le \lfloor \tfrac{k}{2} \rfloor.$$

This seems to be isomorphic to the algebra you were looking at (rook monoid algebra for GL_1?) at the AMS meeting in Chicago (and with your student?) For r not large one should take the piece of paper for the r large Bratelli diagram and roll it up so it becomes a cylinder with r columns. This is the equivalent of taking the labels on the L_i mod r. I haven't thought about what the dimension of the algebra is.

For the two fold cover of the dihedral group $I_2(r)$ of order 2r,

$$G = \left\{ \begin{pmatrix} \xi^j & 0 \\ 0 & \xi^{-j} \end{pmatrix}, \begin{pmatrix} 0 & \xi^j \\ \xi^{-j} & 0 \end{pmatrix} \;\middle|\; 0 \le j \le r-1 \right\} \qquad \text{where} \quad \xi = e^{2\pi i/r},$$

(did I get this right? it should be a group of order $4r$) we get the affine Dynkin diagram of type $\tilde{D}_r$



$(D_n, n \ge 5)$

since G has irreducible representations

???

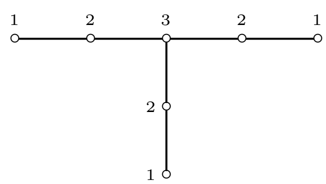
and

$$V = L_2^1$$

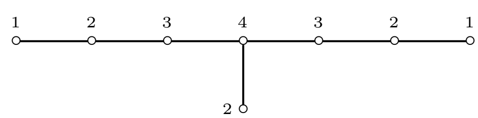
and the tensor product rule is

???

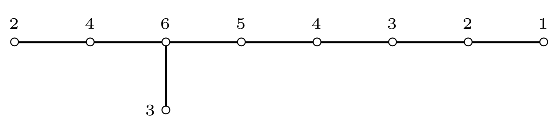
For $I_2(4)$, T and I we should get the affine E_6, E_7 and E_8 Dynkin diagrams respectively.



E_6



E_7



E_8

These McKay tantalizers should be the algebras which give the towers for type II_1 subfactors in Ocneanu's classification (is this in the book of Jones and Sunder on subfactors??)

5 The group G_4

The complex reflection group G_4 is the subgroup of $GL_2(\mathbb{C})$ generated by the elements

$$S = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad \text{and} \quad T = \frac{-\omega^2}{\sqrt{2}} \begin{pmatrix} \varepsilon & \varepsilon^3 \\ \varepsilon & \varepsilon^7 \end{pmatrix}, \qquad \text{where} \qquad \varepsilon = e^{2\pi i/8} \quad \text{and} \quad \omega = e^{2\pi i/3}.$$

These elements satisfy the relations

$$S^2 = -1, \quad T^3 = 1, \quad (ST)^3 = 1.$$

The invariants are

$$f = x_1^4 + 2i\sqrt{3}\, x_1^2 x_2^2 + x_2^4 \qquad \text{and} \qquad x_1 x_2 (x_1^4 - x_2^4).$$

The group G_4 is order 24 and the elements are

$$G_4 = \left\{ \begin{array}{llllll} 1, & -1, & T, & -T, & T^2, & -T^2 \\ S, & -S, & ST, & -ST, & ST^2, & -ST^2 \\ TS, & -TS, & T^2S, & -T^2S, & T^2ST, & -T^2ST \\ TST & -TST & STS & -STS & TST^2 & -TST^2 \end{array} \right\}.$$

It is useful to use the following additional relations

$$\begin{array}{lll} TSTST = -S, & TSTS = -ST^2, & TST = ST^2S, \\ STSTS = T^2, & STST = -T^2S, & STS = -T^2ST^2. \end{array}$$

The conjugacy classes are

$$\mathcal{C}_1 = \{1\}, \qquad \mathcal{C}_T = \{T, TS, ST, -STS\}, \qquad \mathcal{C}_{T^2} = \{T^2, -TST, -T^2S, -ST^2\},$$

$$\mathcal{C}_{-1} = \{-1\}, \qquad \mathcal{C}_{-T} = \{-T, -TS, -ST, STS\}, \qquad \mathcal{C}_{-T^2} = \{-T^2, TST, T^2S, ST^2\}.$$

$$\mathcal{C}_S = \{S, TST^2, T^2ST, -TST^2, -T^2ST, -S\},$$

where $\mathcal{C}_T$ and $\mathcal{C}_{T^2}$ are the conjugacy classes of reflections. The character table of G_4 is

	$\mathcal{C}_1$	$\mathcal{C}_{-1}$	$\mathcal{C}_S$	$\mathcal{C}_T$	$\mathcal{C}_{-T}$	$\mathcal{C}_{T^2}$	$\mathcal{C}_{-T^2}$
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	ω^2	ω^2	ω	ω
χ_3	1	1	1	ω	ω	ω^2	ω^2
χ_4	2	-2	0	$-\omega^2$	ω^2	$-\omega$	ω
χ_5	2	-2	0	-1	1	-1	1
χ_6	3	3	-1	0	0	0	0
χ_7	2	-2	0	$-\omega$	ω	$-\omega^2$	ω^3

The ring

$$\frac{S(\mathfrak{h}^*)}{\langle f, t\rangle} \quad \text{has basis} \quad \begin{array}{l} 1, \\ x_1, x_2, \\ x_1^2, x_1x_2, x_2^2, \\ x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, \\ x_1^4, x_1^3x_2, x_1^2x_2^2, x_1x_2^3, \\ x_1^5, x_1^4x_2, x_1^3x_2^2, x_1^2x_2^3, \\ x_1^6, x_1^5x_2, x_1^4x_2, \\ x_1^7, x_1^6x_2, \\ x_1^8, \end{array}$$

and relations

$$x_2^4 = -x_1^4 - Nx_1^2x_2^2$$

$$x_1x_2^4 = -x_1^5 - Nx_1^3x_2^2, \quad x_2^5 = -x_1^4x_2 - Nx_1^2x_2^3,$$

$$x_1^3x_2^3 = \tfrac{-2}{N}x_1^5x_2, \quad x_1^2x_2^4 = -x_1^6 - Nx_1^4x_2^2, \quad x_1x_2^5 = x_1^5x_2, \quad x_2^6 = Nx_1^6 - 13x_1^4x_2^2,$$

$$x_1^5x_2^2 = \tfrac{N}{2}x_1^7, \quad x_1^4x_2^3 = \tfrac{-2}{N}x_1^6x_2, \quad x_1^3x_2^4 = \tfrac{-1}{7}x_1^7, \quad x_1^2x_2^5 = x_1^6x_2, \quad x_1x_2^6 = \tfrac{N}{14}x_1^7, \quad x_2^7 = \tfrac{14}{N}x_1^6x_2,$$

$$x_1^7x_2 = 0, \quad x_1^6x_2^2 = \tfrac{N}{14}x_1^8, \quad x_1^5x_2^3 = 0, \quad x_1^4x_2^4 = \tfrac{-1}{7}x_1^8, \quad x_1^3x_2^6 = \tfrac{N}{14}x_1^8, \quad x_1x_2^7 = 0, \quad x_2^8 = x_1^8.$$

The graded character table of this module is

	$\mathcal{C}_1$	$\mathcal{C}_{-1}$	$\mathcal{C}_S$	$\mathcal{C}_T$	$\mathcal{C}_{-T}$	$\mathcal{C}_{T^2}$	$\mathcal{C}_{-T^2}$
η_0	1	1	1	1	1	1	1
η_1	2	-2	0	$-\omega^2$	ω^2	$-\omega$	ω
η_2	3	3	-1	0	0	0	0
η_3	4	-4	0	1	-1	1	-1
η_4	4	4	0	ω	ω	ω^2	ω^2
η_5	4	-4	0	ω^2	$-\omega^2$	ω	$-\omega$
η_6	3	3	-1	0	0	0	0
η_7	2	-2	0	-1	1	-1	1
η_8	1	1	1	ω^2	ω^2	ω	ω

which shows that

$$\begin{aligned}
\eta_0 &= \chi_1, \\
\eta_1 &= \chi_4, \\
\eta_2 &= \chi_6, \\
\eta_3 &= \chi_4 + \chi_7, \\
\eta_4 &= \chi_3 + \chi_6 \\
\eta_5 &= \chi_5 + \chi_7, \\
\eta_6 &= \chi_6, \\
\eta_7 &= \chi_5, \\
\eta_8 &= \chi_2.
\end{aligned}$$

Let

$$N = 2i\sqrt{3}, \quad \text{so that} \quad N^2 = -\sqrt{12}.$$

Then, in the basis above, the representations in each degree of the coinvariant algebra are given by:

$$\rho_1(S) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

$$\rho_2(S) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rho_3(S) = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

$$\rho_4(S) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\rho_5(S) = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

$$\rho_6(S) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\rho_7(S) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\rho_8(S) = (1)$$

$$\rho_1(T) = \frac{-\omega^2\varepsilon}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

$$\rho_2(T) = \frac{\omega\varepsilon^2}{2} \begin{pmatrix} 1 & i & -1 \\ 2 & 0 & 2 \\ 1 & -i & -1 \end{pmatrix},$$

$$\rho_3(T) = \frac{\varepsilon}{2\sqrt{2}} \begin{pmatrix} -i & 1 & i & -1 \\ -3i & 1 & -i & 3 \\ -3i & -1 & -i & -3 \\ -3i & -1 & -i & -3 \\ -i & -1 & i & 1 \end{pmatrix}$$

$$\rho_4(T) = \frac{\omega^2}{4} \begin{pmatrix} 0 & -2i & 0 & 2i \\ -4 & -2i & 0 & -2i \\ -6+N & -Ni & -2-N & Ni \\ -4 & 2i & 0 & 2i \end{pmatrix}$$

$$\rho_5(T) = \frac{\omega\varepsilon}{4\sqrt{2}} \begin{pmatrix} -4 & 4i & 0 & 0 \\ 4 & 4i & 0 & 0 \\ 10-5N & (2+3N)i & 2+N & (2+N)i \\ 10-N & (-2+N)i & 2+N & (-2-N)i \end{pmatrix}$$

$$\rho_6(T) = \frac{1}{8} \begin{pmatrix} (-14+N)(-i) & 6-N & (-2-N)(-i) \\ (-\frac{40}{N}+12)(-i) & 0 & (-4-\frac{8}{N})(-i) \\ (2-15N)(-i) & 18+5N & (14-N)(-i) \end{pmatrix}$$

$$\rho_7(T) = \begin{pmatrix} \frac{\omega^2\varepsilon^3}{8\sqrt{2}} - 4 + 2N & (-\frac{12}{7} - \frac{2}{7}N) \\ 28 - \frac{56}{N} & -4i - \frac{24}{N}i \end{pmatrix}$$

$$\rho_8(T) = \frac{\omega}{16}(-8+4N)$$

$$\rho_2(T^2) = \frac{\omega^2}{4}\begin{pmatrix} -2i & -2i & -2i \\ -4 & 0 & 4 \\ 2i & -2i & 2i \end{pmatrix}$$

$$\rho_3(T^2) = \frac{1}{8}\begin{pmatrix} 2+2i & 2+2i & 2+2i & 2+2i \\ 2-2i & 2-2i & -2+2i & -2+2i \\ -2-2i & 2+2i & 2+2i & 2-2i \\ -2+2i & 2-2i & -2+2i & 2-2i \end{pmatrix}$$

$$\rho_4(T^2) = \frac{\omega^2}{16}\begin{pmatrix} 0 & -8 & 0 & -8 \\ 4i & 8i & 0 & -8i \\ 4+4N & -4N & -8+4N & -4N \\ -4i & 8i & 0 & -8i \end{pmatrix}$$

$$\rho_5(T^2) = \frac{\omega^2}{32}\begin{pmatrix} -16(1-i) & 16(1-i) & 0 & 0 \\ 16(1+i)-16(1+i) & 0 & 0 & \\ (-40-20N)(1-i) & (-8+12N)(1-i) & (8-4N)(1-i) & (8-4N)(1-i) \\ (40+4N)(1+i) & (-8-8N)(1+i) & (-8+4N)(1+i) & (8-4N)(1+i) \end{pmatrix}$$

$$\rho_6(T^2) = \frac{1}{64}\begin{pmatrix} -112-8Ni & 48i+8Ni & 16i-8Ni \\ 96+\frac{360}{N} & 0 & 32-\frac{66}{N} \\ -16-120Ni & -144i+40Ni & 112i+8Ni \end{pmatrix}$$

$$\rho_7(T^2) = \frac{\omega}{128}\begin{pmatrix} 32-48i+16N+16Ni & -\frac{116}{7}-\frac{116}{7}i+\frac{16N}{7}+\frac{16N}{7}i \\ -224+224i-\frac{448}{N}+\frac{448}{N}i & -60+60i+\frac{122N}{3}-\frac{122N}{3}i \end{pmatrix}$$

$$\rho_8(T^2) = \omega$$

6 The dihedral groups $G(r,r,2)$

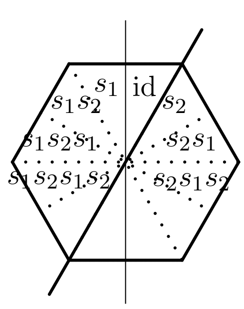
Let r be a positive integer and let

$$\theta = \pi/r \qquad \text{and} \qquad \xi = e^{i2\theta}.$$

With respect to the orthonormal basis $\{\varepsilon_1, \varepsilon_2\}$ of $\mathbb{C}^2$ the *dihedral group* $G(r,r,2)$ is the group of 2×2 matrices given by

$$G(r,r,2) = \left\{ \begin{pmatrix} -\cos 2k\theta & sin 2k\theta \\ \sin 2k\theta & \cos 2k\theta \end{pmatrix}, \begin{pmatrix} \cos 2k\theta & -\sin 2k\theta \\ \sin 2k\theta & \cos 2k\theta \end{pmatrix} \ \Big| \ k = 0, 1, \ldots, r-1 \right\}.$$

In this form $G(r,r,2)$ is the group of symmetries of a regular r-gon (embedded in $\mathbb{R}^2$ with its center at the origin),



with s_1 being the reflection in H_{α_1} and s_2 the reflection in H_{α_2}.

Let x_1 and x_2 be given by

$$\varepsilon_1 = \frac{1}{\sqrt{2}}(\xi^{1/2}x_1 - \xi^{-1/2}x_2), \qquad x_1 = \frac{\xi^{-1/2}}{\sqrt{2}}(\varepsilon_1 - i\varepsilon_2),$$

and

$$\varepsilon_2 = \frac{1}{\sqrt{2}}(\xi^{1/2}x_1 + \xi^{-1/2}x_2), \qquad x_2 = -\frac{\xi^{1/2}}{\sqrt{2}}(\varepsilon_1 + i\varepsilon_2).$$

Then, with respect to the basis $\{x_1, x_2\}$,

$$G(r,r,2) = \left\{ \begin{pmatrix} \xi^k & 0 \\ 0 & \xi^{-k} \end{pmatrix}, \ \begin{pmatrix} 0 & \xi^k \\ \xi^{-k} & 0 \end{pmatrix} \ \Big| \ k = 0, 1, \ldots, r-1 \right\}.$$

The roots are

$$\beta_k = \cos(k\theta)\varepsilon_1 + \sin(k\theta)\varepsilon_2 = \frac{1}{\sin\theta}\big(\sin((k+1)\theta)\alpha_1 + \sin(k\theta)\alpha_2\big), \qquad 0 \le k \le 2r-1,$$

and if the positive roots are

$$R^+ = \{\beta_k \mid 0 \le k \le r-1\} \qquad \text{then} \qquad \begin{aligned} \alpha_1 &= \beta_0 = \varepsilon_1 = \tfrac{1}{\sqrt{2}}(\xi^{1/2}x_1 - \xi^{-1/2}x_2), \\ \alpha_2 &= \beta_{r-1} = -\cos\theta\varepsilon_1 + \sin\theta\varepsilon_2 = \tfrac{1}{\sqrt{2}}(x_1 - x_2), \end{aligned}$$

are the simple roots with

$$\beta_k = \frac{1}{\sin\theta}\big(\sin((k+1)\theta)\alpha_1 + \sin(k\theta)\alpha_2\big), \qquad 0 \le k \le 2r-1.$$

The simple reflections are

$$s_1 = \begin{pmatrix} 0 & \xi \\ \xi^{-1} & 0 \end{pmatrix} \qquad \text{and} \qquad s_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \text{in the basis } \{x_1, x_2\},$$

and

$$s_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad s_2 = \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}, \qquad \text{in the basis } \{\varepsilon_1, \varepsilon_2\}.$$

Thus $\varepsilon_1, \varepsilon_2$ are the eigenvectors of s_1 and x_1, x_2 are the eigenvectors of s_1s_2. Then

$$-\beta_k = \beta_{r+k}, \qquad s_1\beta_k = \beta_{r-k}, \qquad s_2\beta_k = \beta_{r-2-k}.$$

The elements s_1, s_2 satisfy

$$\underbrace{s_1s_2s_1s_2\cdots}_{r\ \text{factors}} = \underbrace{s_2s_1s_2s_1\cdots}_{r\ \text{factors}}, \qquad s_1^2 = 1, \qquad s_2^2 = 1,$$

and $t = s_1s_2$ and $s = s_2$ satisfy

$$t^r = 1, \qquad s^2 = 1, \qquad st = t^{-1}s,$$

The invariants are given by

$$f_1 = x_1^r + x_2^r = \mathrm{Re}((\varepsilon_1 + i\varepsilon_2)^r), \qquad \text{and} \qquad f_2 = x_1x_2 = -\frac{1}{2}(\varepsilon_1^2 + \varepsilon_2^2).$$

Another choice for the invariant of degree r is

$$f_1' = \prod_{i=0}^{r-1} (\cos(2k\theta)\varepsilon_1 + \sin(2k\theta)\varepsilon_2).$$

The Cartan matrix of $I_2(m)$ is

$$A = \begin{pmatrix} 2 & -2\cos(\pi/m) \\ -2\cos(\pi/m) & 2 \end{pmatrix} \qquad \text{and} \qquad A^{-1} = \frac{1}{2\sin^2(\pi/m)} \begin{pmatrix} 1 & \cos(\pi/m) \\ \cos(\pi/m) & 1 \end{pmatrix}.$$

References

[Dr1] .G. Drinfel'd, *A new realization of Yangians and quantized affine algebras*, Soviet Math. Dokl. **36** No, 2 (1998), 212–216.