Reflection groups

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1 Definition of a reflection group

Let V be a finite dimensional complex vector space of dimension $n > 0$. Let $\langle , \rangle \colon V \times V \to \mathbb{C}$ be a Hermitian form on V, i.e. such that

$$\langle x, y\rangle = \overline{\langle y, x\rangle}, \qquad \text{for all } x, y \in V,$$

$$\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle \qquad \langle x, ay + bz\angle = \bar{a}\langle x, y\rangle + \bar{b}\langle x, z\rangle,$$

where $\bar{a}$ is the complex conjugate of a.

Let $\alpha \in V$ and let $s_{\alpha,\lambda} \colon V \to V$ be the reflection in the hyperplane

$$H_\alpha = \{x \in V \mid \langle x, \alpha\rangle = 0\}$$

with eigenvalue λ. Then

(1) $s_{\alpha,\lambda}(x) = x + (\lambda - 1)\dfrac{x, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha$

(2) $s_{\alpha,\lambda}s_{\alpha,\mu} = s_{\alpha,\lambda\mu}$,

(3) $\langle s_{\alpha,\lambda}x, s_{\alpha,\lambda}y\rangle = \langle x, y\rangle$ for all $x, y \in V$ if and only if $\lambda\bar{\lambda} = 1$.

(4) Let $f \colon V \to V$ be such that $\langle fx, fy\rangle = \langle x, y\rangle$ for all $x, y \in V$. Then

$$s_{f(\alpha),\lambda} = f s_{\alpha,\lambda} f^{-1}.$$

Proof. (1) Write $x = x_1\alpha + x_2$ with $x_1 \in \mathbb{C}$ and $x_2 \in (\mathbb{C}\alpha)^\perp$. Then

$$s_{\alpha,\lambda}(x) = \lambda x_1 + x_2$$

and

$$x + (\lambda - 1)\frac{\langle x, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha = x_1\alpha + x_2 + (\lambda - 1)\frac{\langle x, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha = x_1\alpha + x_2 + \lambda x_1\alpha - x_1\alpha = \lambda x_1\alpha + x_2.$$

(2) $s_{\alpha,\lambda}s_{\alpha,\mu}(x) = s_{\alpha,\lambda}(\mu x_1\alpha + x_2) = \lambda\mu x_1\alpha + x_2 = s_{\alpha,\lambda\mu}(x)$.

(3) $\langle \lambda x_1\alpha + x_2, \lambda y_1\alpha + y_2\rangle = \lambda\bar{\lambda}\langle \alpha, \alpha\rangle x_1 y_1 + \langle x_2, y_2\rangle$ and $\langle x_1\alpha + x_2, y_1\alpha + y_2\rangle = x_1 y_1\langle \alpha, \alpha\rangle + \langle x_2, y_2\rangle$. If $x_2 = y_2 = 0$ and $x_1 y_1 = 1$ we get $\lambda\bar{\lambda} = 1$.

(4) $f s_{\alpha,\lambda} f^{-1} = f\big(f^{-1}(x) + (\lambda - 1)\frac{f^{-1}x, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha\big) = x + (\lambda - 1)\frac{f^{-1}x, f^{-1}f\alpha\rangle}{\langle f\alpha, f\alpha\rangle} f\alpha = s_{f\alpha,\lambda}$. $\square$

Let $\mathfrak{h}^*$ be a vector space over a field $\mathbb{F}$ and let $n = \dim(\mathfrak{h}^*)$. A *reflection* is an element $s_\alpha \in GL(\mathfrak{h}^*)$ such that

$$\dim\big((\mathfrak{h}^*)^{s_\alpha}\big) = n-1, \qquad \text{where} \quad (\mathfrak{h}^*)^{s_\alpha} = \{x \in \mathfrak{h}^* \mid s_\alpha x = x\}.$$

A *reflection group* is a finite subgroup W of $GL(\mathfrak{h}^*)$ generated by reflections. If W is a reflection group the set

$$\mathcal{A} = \{H_\alpha \mid s_\alpha \text{ is a reflection in } W\}$$

is the *hyperplane arrangement* corresponding to G. Since H_s is codimension 1

$$H_\alpha = \ker\alpha \qquad \text{where} \qquad \alpha\colon V \to \mathbb{C}$$

is a linear form on V. The form $\alpha \in \mathbb{C}^*$ is determined up to constant multiples.

A linear form $\alpha\colon V \to \mathbb{C}$ determines a hyperplane

$$H_\alpha = \{v \in V \mid \alpha(v) = 0\}$$

and a reflection $s_\alpha\colon V \to V$ by

$$s_{\alpha,\xi}(v) = v + (\xi - 1)\frac{\langle v, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha^\vee$$

Let V be a complex vector space of dimension n. A *reflection* is an element $s \in GL(V)$ such that

$$\operatorname{codim}(V^s) = 1.$$

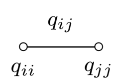
Let $M = (q_{ij})$ be such that

(i) $q_{ij} \in \mathbb{Z}_{\geq 2}$,

(ii) M is symmetric,

(iii) if q_{ij} is odd then $q_{ii} = q_{jj}$.

Let D be the graph with vertices indexed by $1, 2, \ldots, n$ with edges labeled q_{ij} and the label q_{ii} are vertex i.



If $q_{ij} = 2$ we do not draw the edge between vertex i and vertex j. The *Cartan matrix* $A = (a_{ij}$ is given by setting

$$a_{ii} = \sin\big(\frac{\pi}{q_{ii}}\big), \qquad a_{ij} = 0 \text{ if } q_{ij} = 2,$$

$$a_{ij} = -\sqrt{\cos^2(\frac{\pi}{q_{ij}}) - \sin^2\big(\frac{\pi}{2q_{ii}} - \frac{\pi}{2q_{ij}}\big)}, \qquad \text{if } q_{ij} > 2.$$

Since $\cos^2(\pi/3) = 3/2$ and $\sin^2(\pi/4) = 1/2$ we have that

$$\cos^2\big(\frac{\pi}{q_{ij}}\big) - \sin^2\big(\frac{\pi}{2q_{ii}} - \frac{\pi}{2q_{ij}}\big) \geq \frac{1}{4} \qquad \text{if } q_{ij} > 2.$$

So A is a real symmetric matrix. The matrix A is *irreducible* if we cannot partition the index set $\{1, 2, \ldots, n\}$ into two proper subsets I and J such that $a_{ij} = 0$ for all $i \in I$, $j \in J$,

$$\begin{matrix} I \\ J \end{matrix} \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

(also figure out how to get TeX to label the columns I and J). A is irreducible if and only if D is connected. The matrix A is of *affine type* if there is a vector of positive real numbers

$$m = \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} \qquad \text{such that} \quad Am = 0.$$

A *subdiagram* of D is any diagram obtained from D by

(a) deleting some vertices (and the edges issuing from them),

(b) decreasing the labels on some of the edges ($q_{ij} \mapsto q'_{ij}$).

(c) decreasing the labels on some of the vertices ($q_{ii} \mapsto q'_{ii}$) such that

$$\text{if} \quad q'_{ij} > 2 \quad \text{then} \quad \left|\frac{1}{q_{ii}} - \frac{1}{q_{jj}}\right| \geq \left|\frac{1}{q'_{ii}} - \frac{1}{q'_{jj}}\right|.$$

The graph D is of *affine type* if the corresponding Cartan matrix is of affine type.

Lemma 1.1. *Let d be a diagram of affine type. Then no proper subdiagram of D is of affine type.*

Proof. (a) Reducing numbers on edges of D corresponds to increasing off diagonal entries a_{ij} of A (decreasing (a_{ij})).
(b) Reducing numbers in vertices of D corresponds to increasing diagonal entries a_{ii} of A.
(c) Deleting vertices of D corresponds to passing to a principal submatrix.
Hence the Cartan matrix of D' is of the form B_I where $B \geq A$ (take $B = A$ outside I). By Lemma 2, B_I is nonsingular, hence D' is *not* of affine type. □

We can use the graph D (or the matrix M) to define a group W by generators $r_1, \ldots, r_n$ and relations

$$r_i^{q_{ii}} = 1, \qquad \underbrace{r_i r_j r_i \cdots}_{q_{ij}\ mathrm factors} = \underbrace{r_j r_i r_j \cdots}_{q_{ij}\ mathrm factors}.$$

We will define a representation of W on a space V by reflections. Let

$$V = \mathrm{span}\{\alpha_1, \ldots, \alpha_n\}$$

so that the symbols $\alpha_1, \ldots, \alpha_n$ are a basis of V. Define a Hermitian form on V by

$$\langle \alpha_i, \alpha_i \rangle = a_{ii} = \sin(\frac{\pi}{q_{ii}}),$$
$$\langle \alpha_i, \alpha_j \rangle = a_{ij} = -\sqrt{\cos^2(\frac{\pi}{q_{ij}}) - \sin^2\left(\frac{\pi}{2q_{ii}} - \frac{\pi}{2q_{ij}}\right)}$$

In general the formula $s_\alpha \colon V \to V$,

$$s_{\alpha,\lambda}(x) = x + (\lambda - 1)\frac{\langle x, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha, \qquad \alpha \in V, \lambda \in \mathbb{C}^*,$$

defines the reflection in the hyperplane

$$H_\alpha = \{x \in V \mid \langle x, \alpha\rangle = 0\}$$

with eigenvalue λ. The endomorphism $s_{\alpha,\lambda}$ is an isometry if and only if $\lambda\bar{\lambda} = 1$.

Define a representation of W on V by

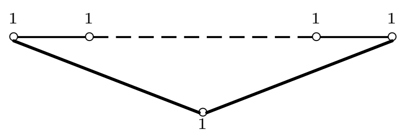
$$\begin{array}{rccc} \Phi\colon & W & \longrightarrow & GL(V) \\ & r_j & \longmapsto & s_{\alpha_j,\lambda_j} \end{array} \qquad \text{where} \quad \lambda_j = e^{2\pi i/q_{jj}}.$$

This representation has $\langle , \rangle$ as a W-invariant form and it is faithful if $\Phi(W)$ is finite. This happens exactly when the form $\langle , \rangle$ is positive definite. (The proof of this in Koster refers to Coxeter's classifications and presentations for one direction. This is unpleasant.)

1.1 Classification of diagrams of affine type

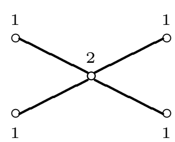
Let D be an affine diagram.

(1) Suppose D contains a cycle (with ≥ 3 vertices). Then D has a subdiagram of the form



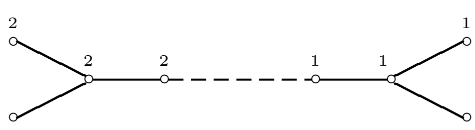
(A_n) $(n + 1$ vertices, $n \geq 2)$,

(2) D has a branch of order ≥ 4. Then D has a subdiagram of the form



(D_4)

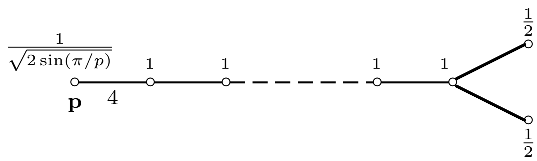
(3) Suppose that D has 2 or more branch points of order 3. The D has a subdiagram of the form



$(D_n, n \geq 5)$

(4) Suppose that D has one branch point of order 3 and at least one multiple bond. Then D

has a subdiagram of the form

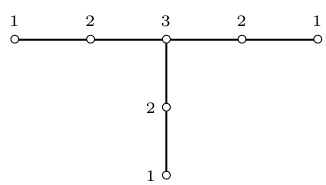


$$A = \begin{pmatrix} \sin(\pi/p) & a & & & & & & & \\ -a & 1 & -\frac{1}{2} & & & & & & \\ & -\frac{1}{2} & 1 & -\frac{1}{2} & & & & & \\ & & -\frac{1}{2} & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 \\ & & & & & -\frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} \\ & & & & & 0 & -\frac{1}{2} & 1 & 0 \\ & & & & & 0 & -\frac{1}{2} & 0 & 1 \end{pmatrix} \qquad (\tilde{B}_n^p, n \geq 3)$$

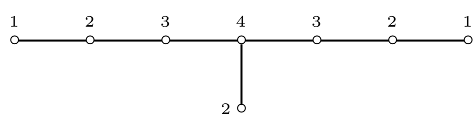
where

$$a = \sqrt{\frac{\sin(\pi/p)}{2}}.$$

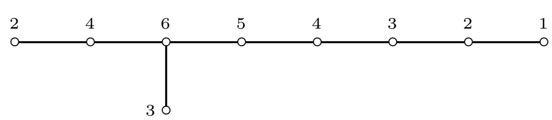
(5) Suppose D has one branch point or order 3 and no mulitple bonds. Then D has a subdiagram of the form



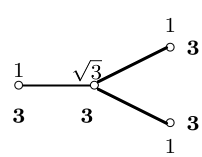
E_6



E_7

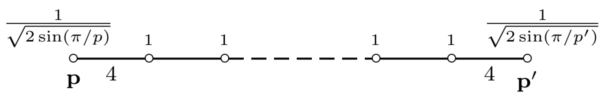


E_8



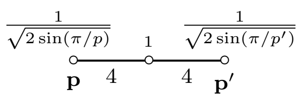
(6) We have exhausted the possibilities where D has a branch point. Assume now that D is a

chain. If D has at least two multiple bonds it will contain one of the following diagrams



$$A = \begin{pmatrix} \sin(\pi/p) & a & & & & & & & \\ -a & 1 & -\frac{1}{2} & & & & & & \\ & -\frac{1}{2} & 1 & -\frac{1}{2} & & & & & \\ & & -\frac{1}{2} & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & -\frac{1}{2} & & \\ & & & & & -\frac{1}{2} & 1 & -\frac{1}{2} & \\ & & & & & & -\frac{1}{2} & 1 & -b \\ & & & & & & & -b & \sin(\pi/p') \end{pmatrix} \qquad (C_n^{p,p'}, n \geq 2)$$

where $a = \sqrt{\frac{\sin(\pi/p)}{2}}$ and $b = \sqrt{\frac{\sin(\pi/p')}{2}}$.

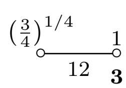


$$A = \begin{pmatrix} 1 & -a & 0 \\ -a & \sin(\pi/p) & -a \\ 0 & -a & 1 \end{pmatrix} \qquad (C_2^p, p \geq 3)$$

where $a = -\sqrt{\frac{\sin(\pi/p)}{2}}$.

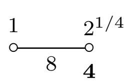
(7) Assume now that D is a chain with just one multiple bond.

(a) strength 6



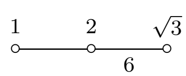
$$A = \begin{pmatrix} 1 & -\frac{3^{1/4}}{2^{1/2}} \\ -\frac{3^{1/4}}{2^{1/2}} & \frac{3^{1/2}}{2} \end{pmatrix}$$

(b) strength 4

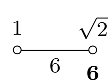


$$A = \begin{pmatrix} 1 & -2^{-1/4} \\ -2^{-1/4} & 2^{-1/2} \end{pmatrix}$$

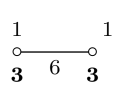
(c) strength 3



$$A = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & 1 \end{pmatrix}$$

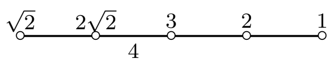


$$A = \begin{pmatrix} 1 & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{1}{2} \end{pmatrix}$$

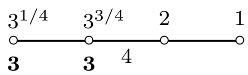


$$A = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

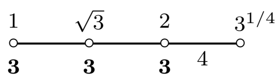
(d) strength 2



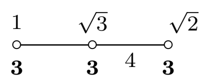
$$A = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} & 1 \end{pmatrix} \qquad (F_4)$$



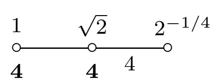
$$A = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{3^{1/4}}{2} & 0 \\ 0 & -\frac{3^{1/4}}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 1 \end{pmatrix}$$



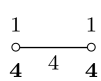
$$A = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & -\frac{3^{1/4}}{2} \\ 0 & 0 & -\frac{3^{1/4}}{2} & 1 \end{pmatrix}$$



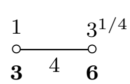
$$A = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & -\sqrt{\frac{1}{2}} \\ 0 & -\sqrt{\frac{1}{2}} & \frac{\sqrt{3}}{2} \end{pmatrix}$$



$$A = \begin{pmatrix} \sqrt{\frac{1}{2}} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{2^{1/4}}{2} \\ 0 & -\frac{2^{1/4}}{2} & 1 \end{pmatrix}$$

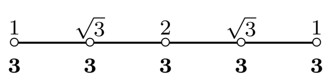


$$A = \begin{pmatrix} \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{pmatrix}$$

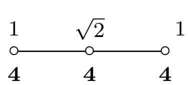


$$A = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{3^{1/4}}{2} \\ -\frac{3^{1/4}}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

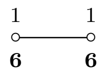
(8) Assume that D has *no* multiple bonds. Then D has a subdiagram of the form



$$A = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 & \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & \\ 0 & 0 & - & -\frac{1}{2} & \frac{\sqrt{3}}{2} & \end{pmatrix}$$



$$A = \begin{pmatrix} \sin(\pi/4) & -\cos(\pi/3) & 0 \\ -\cos(\pi/3) & \sin(\pi/4) & -\cos(\pi/3) \\ 0 & -\cos(\pi/3) & \sin(\pi/4) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \sqrt{\frac{1}{2}} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$$



$$A = \begin{pmatrix} \sin(\pi/6) & -\cos(\pi/3) \\ -\cos(\pi/3) & \sin(\pi/6) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(note that all the numbers on the vertices must be equal in this case).

2 The Chevalley-Shephard-Todd theorem

Theorem 2.1. *Let $\mathfrak{h}^*$ be a vector space and let W be a finite subgroup of $GL(\mathfrak{h}^*)$. The following are equivalent*

(a) W is a reflection group, $\quad W = \langle s_\alpha \mid s_\alpha \in W \text{ is a reflection}\rangle$.

(b) $S(\mathfrak{h}^*)^W$ is a polynomial ring, $\quad S(\mathfrak{h}^*)^W = \mathbb{C}[f_1, f_2, \ldots, f_n]$.

(c) $S(\mathfrak{h}^*)$ is a free $S(\mathfrak{h}^*)^W$-module.

Let R be a local regular ring,

$$fm \text{ a maximal ideal,} \qquad \text{and} \qquad K = R/\mathfrak{m} \text{ the residue field.}$$

The R^G is a local ring with maximal ideal

$$\mathfrak{m}^G = \mathfrak{m} \cap R^G.$$

Assume that

(a) R^G is noetherian and R is a finite type R^G module, and

(b) The composition $R^G \hookrightarrow R \to k$ is surjective.

Define

$$V = \mathfrak{m}/\mathfrak{m}^2 \qquad \text{(a k vector space: the tangent space).}$$

The action of G on R define a homomorphism $\varepsilon \colon G \to GL(V)$.

(a) Let $\mathfrak{p}$ be a prime ideal of height 1 in R and let $s \in G$ be such that $s(\mathfrak{p}) = \mathfrak{p}$ and s operates trivially on $R/\mathfrak{p}$. Show that $\varepsilon(s)$ is a pseudoreflection in V. (Remark taht the image of $\mathfrak{p}$ in $\mathfrak{m}/\mathfrak{m}^2$ is of dimension 0 or 1.

2.1 Structure theorems

Theorem 2.2.

(a) (Chevalley, Shephard-Todd) A finite group $W \subseteq \mathrm{GL}(\mathfrak{h}^*)$ is generated by reflections if and only if

$$S(\mathfrak{h}^*)^W = \mathbb{C}[I_1, \ldots, I_r]$$

where $I_1, \ldots, I_r$ are algebraically independent and homogeneous.

(b) (Solomon) Let W be a finite reflection group. Then

$$(S(\mathfrak{h}^*) \otimes \Lambda(\mathfrak{h}))^W = \mathbb{C}[I_1, \ldots, I_r] \otimes \Lambda(dI_1, \ldots, dI_r)$$

(see Benson page 86).

(c) $S(\mathfrak{h}^*) \cong \mathcal{H} \otimes S(\mathfrak{h}^*)^W$.

Some additional remarks:

(a)

$$\det\Big(\frac{\partial I_j}{\partial x_j}\Big) = \lambda p, \qquad \text{where} \quad p = \prod_{\alpha\in R^+} \alpha$$

and $\lambda \in \mathbb{R}$, $\lambda \neq 0$.

(b) $\mathcal{H}$ has basis $\{h_w = \Delta_w^*(1) \mid w \in W\}$ where Δ_w are the BGG operators and $\deg(h_w) = \ell(w)$.

Theorem 2.3. *(Molien theorems)*

(a)

$$P((S(\mathfrak{h}^*)\otimes\Lambda(\mathfrak{h}))^W; q, t) = \frac{1}{W}\sum_{w\in W}\frac{\det(1+wq)}{\det(1-wt)}$$

(b)

$$P(S(\mathfrak{h}^*)^W; t) = \frac{1}{W}\sum_{w\in W}\frac{1}{\det(1-wt)}$$

Proof.

$$\sum_{j\in\mathbb{Z}_{\geq 0}} q^j \operatorname{Tr}(w, \Lambda^j\mathfrak{h}) = \prod_{i=1}^{r}(1+\lambda_i^{-1}q) = \det(1+w^{-1}q, \mathfrak{h}^*).$$

$$\sum_{j\in\mathbb{Z}_{\geq 0}} t^j \operatorname{Tr}(w, S^j\mathfrak{h}) = \frac{1}{\det(1-wt, \mathfrak{h}^*)} = \prod_{i=1}^{r}\frac{1}{(1-\lambda_i t)}.$$

Now apply

$$\frac{1}{|W|}\sum_{w\in W} w$$

to $S(\mathfrak{h}^*)\otimes\Lambda(\mathfrak{h})$:

$$P((S(\mathfrak{h}^*)\otimes\Lambda(\mathfrak{h}))^W; q, t) = \frac{1}{|W|}\operatorname{Tr}_{q,t}\Big(\sum_{w\in W} w, S(\mathfrak{h}^*)\otimes\Lambda(\mathfrak{h})\Big)\Big) = \frac{1}{|W|}\sum_{w\in W}\operatorname{Tr}_{q,t}(w, S(\mathfrak{h}^*)\otimes\Lambda(\mathfrak{h})).$$

□

Theorem 2.4.

$$P(S(\mathfrak{h}^*); t) = \prod_{i=1}^{r}\frac{1}{1-t} \qquad P(S(\mathfrak{h}^*)^W; t) = \prod_{i=1}^{r}\frac{1}{1-t^{d_i}} \qquad P(\mathcal{H}; t) = \prod_{i=1}^{r}\frac{1-t^{d_i}}{1-t}$$

and

$$P((S(\mathfrak{h}^*)\otimes\Lambda(\mathfrak{h}))^W; q, t) = \prod_{i=1}^{r}\frac{1+qt^{d_i-1}}{1-t^{d_i}}.$$

Proof. (a) is clear. (b) follows from Chevalley's theorem. (c) follows from part (c) of the structure theorem since it implies

$$P(S(\mathfrak{h}^*); t) = P(\mathcal{H}; t)P(S(\mathfrak{h}^*)^W; t).$$

(d) follows from Solomon's theorem. □

Let

$$\begin{aligned} d(w) &= \dim(V^w) = \text{multiplicity of 1 as an eigenvalue of } w, \\ d_m(w) &= \text{multiplicity of } e^{2\pi i/m} \text{ as an eigenvalue of } w, \\ \chi(m|d_i) &= \begin{cases} 1, & \text{if } m \text{ divides } d_i, \\ 0, & \text{if } m \text{ does not divide } d_i. \end{cases} \end{aligned}$$

Corollary 2.5.

(a)

$$\sum_{w \in W} t^{d_m(w)} = \prod_{i=1}^{r} (t^{\chi(m|d_i)} + d_i - 1).$$

(b)

$$\sum_{w \in W} t^{d(w)} = \prod_{i=1}^{r} (t + d_i - 1).$$

(c) The number of reflections in W is $\sum_{i=1}^{r}(d_i - 1)$.

(d) $|W| = \prod_{i=1}^{r} d_i$.

Proof. (b) follows from (a) by putting $m = 1$. (c) follows from taking the coefficient of t^{r-1} on both sides of the identity in (b). (d) follows by putting $t = 1$ in (b).
(a) (following Macdonald) Replace q and t with q/ξ and t/ξ, where $\xi = e^{2\pi i/m}$. Then

$$\frac{1}{|W|} \sum_{w \in W} \frac{\det(\xi + qw)}{\det(\xi - tw)} = \prod_{i=1}^{r} \left(\frac{\xi^{d_i} + qt^{d_i - 1}}{\xi^{d_i} - t^{d_i}} \right)$$

from

$$\frac{1}{|W|} \sum_{w \in W} \frac{\det(1 + qw)}{\det(1 - tw)} = \prod_{i=1}^{r} \left(\frac{1 + qt^{d_i - 1}}{1 - t^{d_i}} \right)$$

Now let $q = (1-t)X - 1$ Then

$$\frac{\det(\xi + qw)}{\det(\xi - tw)} = \prod_{i=1}^{r} \frac{(\xi + \lambda_i((1-t)X - 1))}{(\xi - \lambda_i t)}.$$

So, now take the limit as $t \to 1$. When we do this we get the result we want but we *will need*

$$\prod_{i=1}^{r} d_i = |W|.$$

To get this set $q = 0$ and multiply by $(1-t)^r$ in the Molien formula to get

$$\frac{1}{|W|} \sum_{w \in W} \frac{(1-t)^r}{\det(1 - tw)} = \prod_{i=1}^{r} \frac{1-t}{1-t^{d_i}}.$$

Now set $t = 1$. Then

$$LHS = \frac{1}{|W|}(1 + \sum_{w \neq 1} 0) = \frac{1}{|W|} \qquad \text{and} \qquad RHS = \prod_{i=1}^{r} \frac{1}{d_i}.$$

□

3 Nice formulas

3.1 Symmetric and determinantal functions

Let $S(\mathfrak{h}^*)$ be the symmetric algebra of $\mathfrak{h}^*$. Then

$$\begin{aligned} u^+S(\mathfrak{h}^*) &= \text{symmetric polynomials} \\ u^-S(\mathfrak{h}^*) &= \text{determinant symmetric polynomials} \end{aligned}$$

and, as vector spaces

$$\begin{array}{ccc} u^+S(\mathfrak{h}^*) & \xrightarrow{sim} & u^-S(\mathfrak{h}^*) \\ f & \longmapsto & fa_\rho \end{array}$$

3.2 Weyl denominators

Theorem 3.1.

$$\sum_{w\in W} w\left(\prod_{\alpha\in R^+}\frac{1-t_\alpha e^{-\alpha}}{1-e^{-\alpha}}\right) = \sum_{w\in W} t_{R(w)},$$

where $\{t_\alpha \mid \alpha\in R^+\}$ *are indeterminates*

$$t_E = \prod_{\alpha\in E} t_\alpha, \qquad \textit{for } E\subseteq R^+, \textit{ and}$$

$R(w) = \{\alpha\in R^+ \mid w\alpha\in R^-\}$.

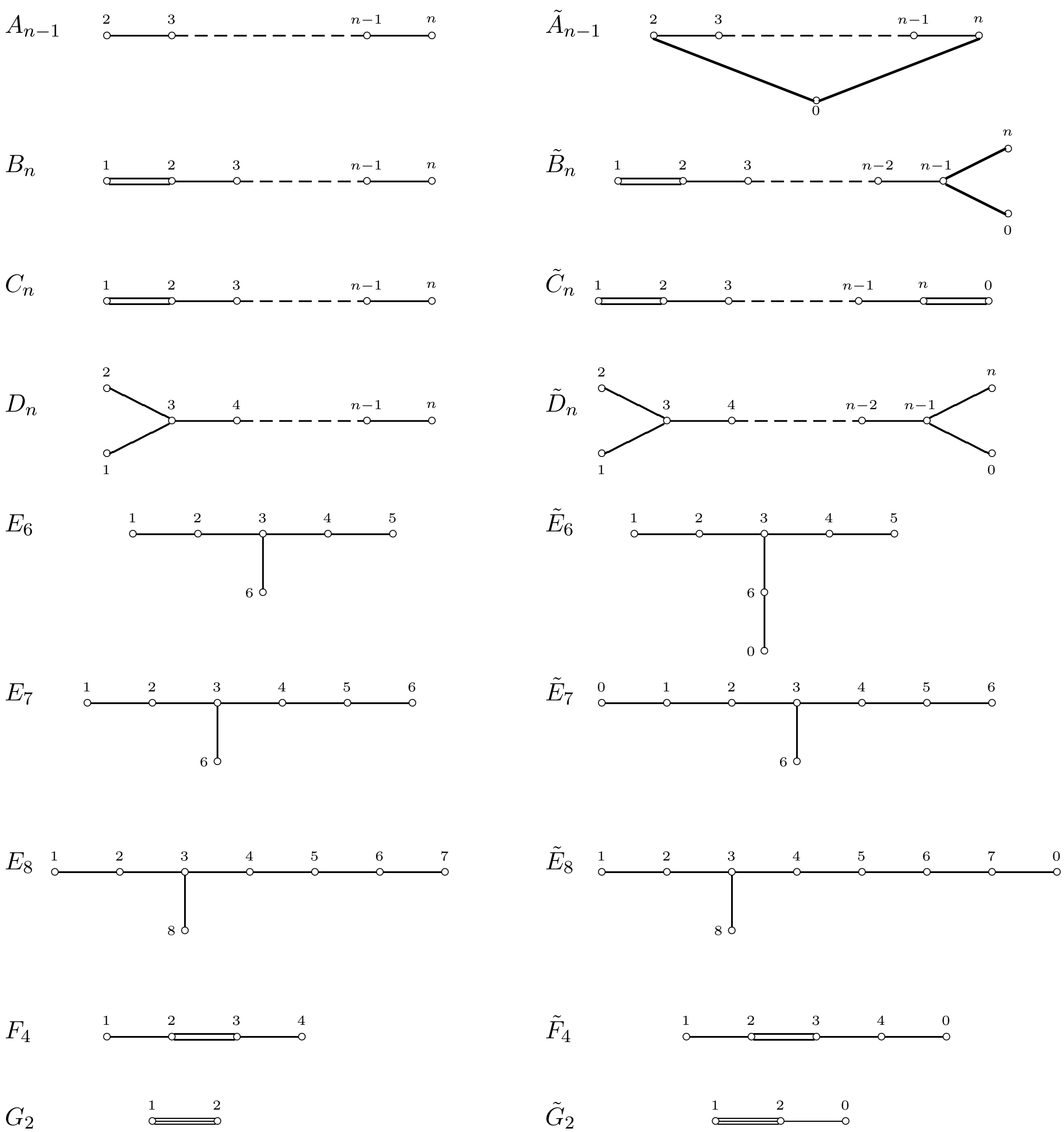
Corollary 3.2.

(1) $\displaystyle\sum_{w\in W} w\left(\prod_{\alpha\in R^+}\frac{1-te^{-\alpha}}{1-e^{-\alpha}}\right) = \sum_{w\in W} t^{\ell(w)}.$

(2) $\displaystyle\prod_{\alpha\in R^+}\frac{1-t^{1+\mathrm{ht}(\alpha)}}{1-t^{\mathrm{ht}(\alpha)}} = \sum_{w\in W} t^{\ell(w)}$ *where* $\mathrm{ht}(\alpha) = \langle\rho^\vee,\alpha\rangle$.

(3) If h_i *is the number of roots of height* i *then*

$$(h_1, h_2, \ldots) = (d_1-1, d_2-1, \ldots)^t.$$



References

[Dr1] .G. Drinfel'd, *A new realization of Yangians and quantized affine algebras*, Soviet Math. Dokl. **36** No, 2 (1998), 212–216.