Crystals

Arun Ram
Department of Mathematics
University of Wisconsin
Madison, WI 53706
ram@math.wisc.edu

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1 The path model

1.1 Paths

Let $\lambda \in P$. The *straight line* path to λ is the map

$$p_\lambda\colon [0,1] \to \mathfrak{h}^*_{\mathbb{R}} \quad \text{given by} \quad p_\lambda(t) = \lambda t. \tag{1.1}$$

Let $\ell_1, \ell_2 \in \mathbb{R}_{\geq 0}$. The *concatenation* of maps $p_1\colon [0,\ell_1] \to \mathfrak{h}^*_{\mathbb{R}}$ and $p_2\colon [0,\ell_2] \to \mathfrak{h}^*_{\mathbb{R}}$ is the map $p_1 \otimes p_2\colon [0,\ell_1+\ell_2] \to \mathfrak{h}^*_{\mathbb{R}}$ given by

$$(p_1 \otimes p_2)(t) = \begin{cases} p_1(t), & \text{for } t \in [0,\ell_1], \\ p_1(\ell_1) + p_2(t-\ell_1), & \text{for } t \in [\ell_1, \ell_1+\ell_2]. \end{cases} \tag{1.2}$$

Let $r, \ell \in \mathbb{R}_{\geq 0}$. The *$r$-stretch* of a map $p\colon [0,\ell] \to \mathfrak{h}^*_{\mathbb{R}}$ is the map $rp\colon [0,r\ell] \to \mathfrak{h}^*_{\mathbb{R}}$ given by

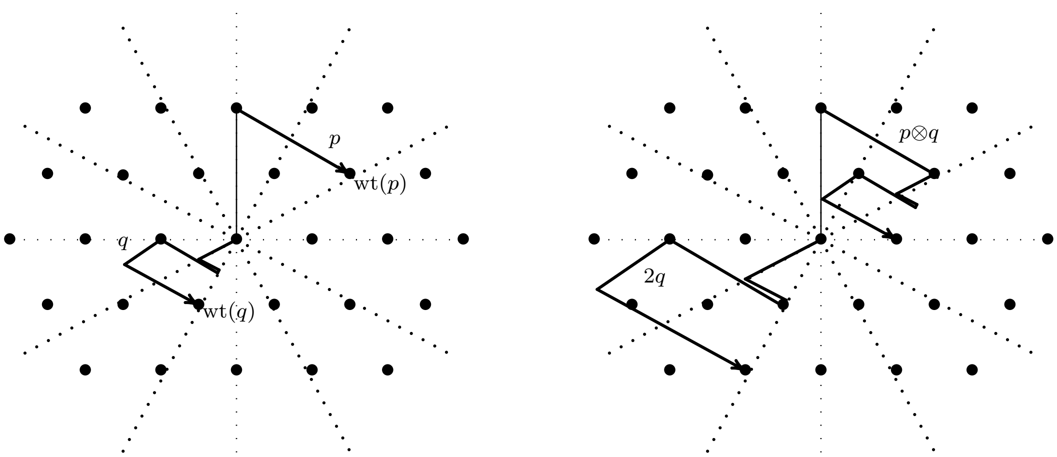
$$(rp)(t) = r \cdot p(t/r). \tag{1.3}$$

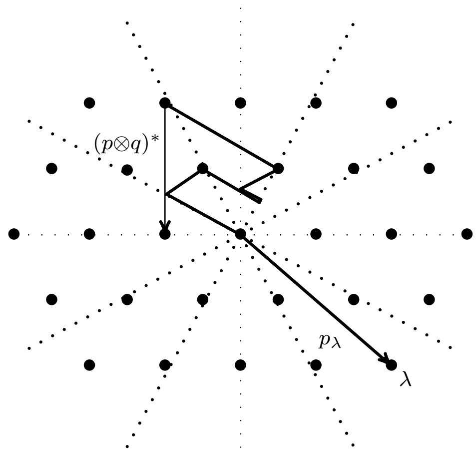
The *reverse* of a map $p\colon [0,\ell] \to \mathfrak{h}^*_{\mathbb{R}}$ is the map $p^*\colon [0,\ell] \to \mathfrak{h}^*_{\mathbb{R}}$ given by

$$p^*(t) = p(\ell - t) - p(\ell). \tag{1.4}$$

The *weight* of a map $p\colon [0,\ell] \to \mathfrak{h}^*_{\mathbb{R}}$ is the endpoint of p,

$$\mathrm{wt}(p) = p(\ell). \tag{1.5}$$





Let

B_{univ} be the set of maps generated by the straight line paths by operations of concatenation, reversing and stretching.

A *path* is an element $p\colon [0,\ell] \to \mathfrak{h}^*_{\mathbb{R}}$ in B_{univ}. concatenation, reversing and stretching. Let B be a set of paths (a subset of B_{univ}). The *character* of B is the element of $\mathbb{C}[P]$ given by

$$\text{char}(B) = \sum_{p\in B} X^{\text{wt}(p)}. \tag{1.6}$$

A *crystal* is a set of paths B that is closed under the action of the *root operators*

$$\tilde{e}_i\colon B_{\text{univ}} \longrightarrow B_{\text{univ}} \cup \{0\} \qquad \text{and} \qquad \tilde{f}_i\colon B_{\text{univ}} \longrightarrow B_{\text{univ}} \cup \{0\}, \qquad 1 \le i \le n,$$

which are defined and constructed below, in Proposition 1.3 and Theorem 1.4. The *crystal graph* of B is the graph with

$$\text{vertices } B \qquad \text{and} \qquad \text{labeled edges} \quad p' \xleftarrow{i} p \quad \text{if } p' = \tilde{f}_i p.$$

1.2 i-strings

Let B be a crystal. Let $p \in B$ and fix i $(1 \le i \le n)$. The *i-string* of p is the set of paths $S_i(p)$ generated from p by applications of the operators $\tilde{e}_i$ and $\tilde{f}_i$.

The *head* of $S_i(p)$ is $h \in S_i(p)$ such that $\tilde{e}_i h = 0$.

The *tail* of $S_i(p)$ is $t \in S_i(p)$ such that $\tilde{f}_i t = 0$.

The weights of the paths in $S_i(p)$ are

$$\text{wt}(t) = s_i\text{wt}(h) = \text{wt}(h) - \langle \text{wt}(h), \alpha_i^\vee\rangle \alpha_i, \quad \ldots \quad , \quad \text{wt}(h) - 2\alpha_i, \quad \text{wt}(h) - \alpha_i, \quad \text{wt}(h),$$

and if

$$d_+(p_{\alpha_i}) = (\text{distance from } h \text{ to } p) \qquad \text{and} \qquad d_-(p_{\alpha_i}) = (\text{distance from } p \text{ to } t),$$

so that

$$\tilde{e}_i^{d_+(p_{\alpha_i})} p = h \qquad \text{and} \qquad \tilde{f}_i^{d_-(p_{\alpha_i})} p = t,$$

the crystal graph of $S_i(p)$ is

$$|\longleftarrow \qquad d_-(p_{\alpha_i}) \qquad \longrightarrow|$$

$$t \xleftarrow{\ i\ } \tilde{e}_i t \xleftarrow{\ i\ } \quad \cdots \quad \xleftarrow{\ i\ } \tilde{f}_i p \xleftarrow{\ i\ } p \xleftarrow{\ i\ } \tilde{e}_i p \xleftarrow{\ i\ } \quad \cdots \quad \xleftarrow{\ i\ } \tilde{f}_i h \xleftarrow{\ i\ } h$$

$$|\longleftarrow \qquad d_+(p_{\alpha_i}) \qquad \longrightarrow|$$

1.3 Highest weight paths

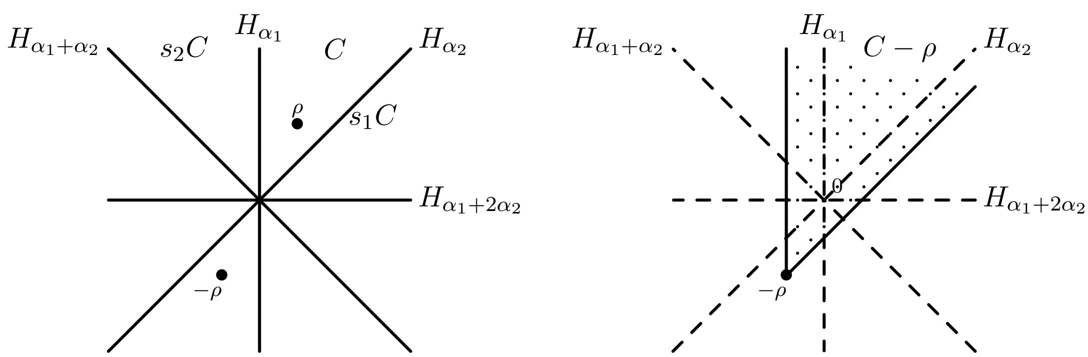
A *highest weight path* is a path p such that

$$\tilde{e}_i p = 0, \qquad \text{for all } 1 \le i \le n.$$

A highest weight path is a path p such that, for each $1 \le i \le n$, p is the head of the i-string $S_i(p)$. Thus $\langle p(t), \alpha_i^\vee \rangle > -1$ for all t and all $1 \le i \le n$. So a path p is a highest weight path if and only if

$$p \subseteq C - \rho, \qquad \text{where} \quad C - \rho = \{\mu - \rho \mid \mu \in C\}.$$

For the root system of type C_2 the picture is



the chambers　　　　the region $C - \rho$

If p is a highest weight path with $\mathrm{wt}(p) \in P$ then, necessarily, $\mathrm{wt}(p) \in P^+$. The following theorem gives an expression for the character of a crystal in terms of the basis $\{s_\lambda \mid \lambda \in P^+\}$ of $\mathbb{C}[P]^W$.

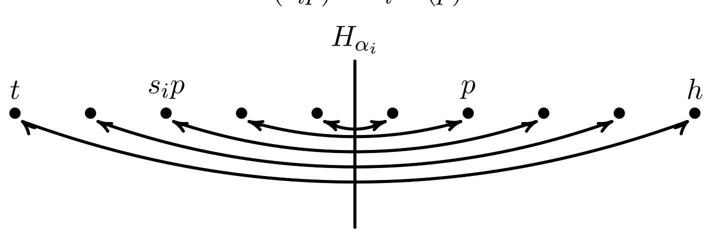
Theorem 1.1. *Let B be a crystal. Then*

$$\mathrm{char}(B) = \sum_{\substack{p \in B \\ p \subseteq C - \rho}} s_{\mathrm{wt}(p)},$$

where the sum is over highest weight paths $p \in B$.

Proof. Fix i, $1 \le i \le n$. If $p \in B$ let $s_i p$ be the element of the i-string of p which satisfies

$$\mathrm{wt}(s_i p) = s_i \mathrm{wt}(p).$$



Then $s_i(s_i p) = p$ and

$$s_i \mathrm{char}(B) = \sum_{p\in B} X^{s_i \mathrm{wt}(p)} = \sum_{p\in B} X^{\mathrm{wt}(s_i p)} = \mathrm{char}(B).$$

Hence $\mathrm{char}(B) \in \mathbb{C}[P]^W$.

Let

$$\varepsilon = \sum_{w\in W} \det(w) w, \quad \text{so that} \quad a_\mu = \varepsilon(X^\mu), \quad \text{for } \mu \in P.$$

Since $\mathrm{char}(B) \in \mathbb{C}[P]^W$,

$$\mathrm{char}(B) a_\rho = \mathrm{char}(B)\varepsilon(X^\rho) = \varepsilon(\mathrm{char}(B) X^\rho)$$

and

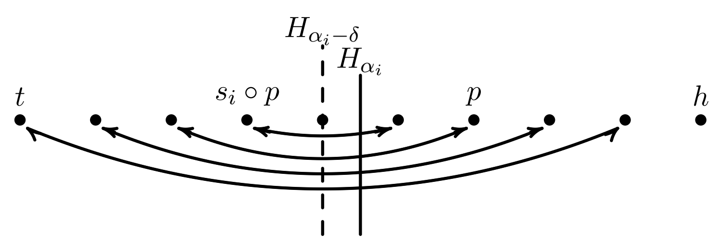
$$\begin{aligned} \mathrm{char}(B) &= \frac{1}{a_\rho}\mathrm{char}(B) a_\rho = \frac{\varepsilon(\mathrm{char}(B)X^\rho)}{a_\rho} \\ &= \sum_{p\in B} \frac{\varepsilon(X^{\mathrm{wt}(p)+\rho})}{a_\rho} = \sum_{p\in B} \frac{a_{\mathrm{wt}(p)+\rho}}{a_\rho} = \sum_{p\in B} s_{\mathrm{wt}(p)}. \end{aligned} \tag{1.7}$$

There is some cancellation which can occur in this sum. Assume $p \in B$ such that $p \not\subseteq C - \rho$ let t be the first time that p leaves the cone $C - \rho$. In other words let $t \in \mathbb{R}_{>0}$ be minimal such that there exists an i with

$$p(t) \in H_{\alpha_i - \delta} \qquad \text{where} \qquad H_{\alpha_i-\delta} = \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \lambda, \alpha_i^\vee \rangle = -1\}.$$

Let i be the minimal index such that the point $p(t) \in H_{\alpha_i-\delta}$ and define $s_i \circ p$ to be the element of the i-string of p such that

$$\mathrm{wt}(s_i \circ p) = s_i \circ p.$$



Note that since $\langle p_i(t), \alpha_i^\vee \rangle = -1$, p is not the head of its i-string and $s_i \circ p$ is well defined. If $q = s_i \circ p$ then the first time t that q leaves the cone $C - \rho$ is the same as the first time that p leaves the cone $C - \rho$ and $p(t) = q(t)$. Thus $s_i \circ q = p$ and $s_i \circ (s_i \circ p) = p$. Since

$$s_{\mathrm{wt}(s_i \circ p)} = s_{s_i \circ \mathrm{wt}(p)} = -s_{\mathrm{wt}(p)}$$

the terms $s_{\mathrm{wt}(s_i\circ p)}$ and $s_{wt(p)}$ cancel in the sum (1.7). Thus

$$\mathrm{char}(B) = \sum_{\substack{p\in B \\ p \subseteq C-\rho}} s_{\mathrm{wt}(p)}.$$

□

Theorem 1.2. *Recall the notations from (???) and (???) in the section on Schur functions. For each* $\lambda \in P^+$ *fix a highest weight path* p_λ^+ *with endpoint* λ *and let*

$$B(\lambda) \text{ be the crystal generated by } p_\lambda^+,$$

Let $\lambda, \mu, \nu \in P^+$ *and let* $J \subseteq \{1, 2, \ldots, n\}$*. Then*

$$s_\lambda = \sum_{p \in B(\lambda)} X^{\mathrm{wt}(p)}, \qquad s_\mu s_\nu = \sum_{\substack{q \in B(\nu) \\ p_\mu^+ \otimes q \subseteq C - \rho}} s_{\mu + \mathrm{wt}(q)}, \qquad \textit{and} \qquad s_\lambda = \sum_{\substack{p \in B(\lambda) \\ p \subseteq C_J - \rho_J}} s^J_{\mathrm{wt}(p)}.$$

Proof. (a) The path p_λ^+ is the unique highest weight path in $B(\lambda)$. Thus, by Theorem 1.1, $\mathrm{char}(B(\lambda)) = s_\lambda$.

(b) By Theorem 1.4c the set

$$B(\mu) \otimes B(\nu) = \{p \otimes q \mid p \in B(\mu), q \in B(\nu)\}$$

is a crystal. Since $\mathrm{wt}(p \otimes q) = \mathrm{wt}(p) + \mathrm{wt}(q)$,

$$\begin{aligned} s_\mu s_\nu &= \mathrm{char}(B(\mu))\mathrm{char}(B(\nu)) = \mathrm{char}((B(\mu) \otimes B(\nu)) \\ &= \sum_{\substack{p \otimes q \in B(\mu) \otimes B(\nu) \\ p \otimes q \subseteq C - \rho}} s_{\mathrm{wt}(p) + \mathrm{wt}(q)} = \sum_{\substack{q \in B(\nu) \\ p_\mu^+ \otimes q \subseteq C - \rho}} s_{\mu + \mathrm{wt}(q)}, \end{aligned}$$

where the third equality is from Theorem 1.1 and the last equality is because the path p_μ^+ has $\mathrm{wt}(p_\mu^+) = \mu$ and is the only highest weight path in $B(\mu)$.

(c) A *J-crystal* is a set of paths B which is closed under the operators $\tilde{e}_j$, $\tilde{f}_j$, for $j \in J$. Since $s_\lambda = \mathrm{char}(B(\lambda))$ the statement follows by applying Theorem 1.1 to $B(\lambda)$ viewed as a J-crystal. □

1.4 Root operators for the rank 1 case

Let

$$B^{\otimes k} = \{b_1 \otimes \cdots \otimes b_k \mid b_i \in B\}, \qquad \text{where} \qquad B = \{+1, -1, 0\}.$$

Define

$$\tilde{f} \colon B^{\otimes k} \to B^{\otimes k} \cup \{0\} \quad \text{and} \quad \tilde{e} \colon B^{\otimes k} \to B^{\otimes k} \cup \{0\}$$

as follows. Let $b = b_1 \otimes \cdots \otimes b_k \in B^{\otimes k}$. Ignoring 0s successively pair adjacent unpaired $(-1, +1)$ pairs to obtain a sequence of unpaired +1s and −1s

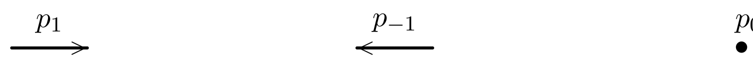
$$+1\ +1\ +1\ +1\ +1\ +1\ +1\ -1\ -1\ -1\ -1$$

(after pairing and ignoring 0s). Then

$$\begin{aligned} \tilde{f}b &= \text{same as } b \text{ except the rightmost unpaired } +1 \text{ is changed to } -1, \\ \tilde{e}b &= \text{same as } b \text{ except the leftmost unpaired } -1 \text{ is changed to } +1. \end{aligned}$$

If there is no unpaired +1 after pairing then $\tilde{f}b = 0$.
If there is no unpaired −1 after pairing then $\tilde{e}b = 0$.

Let $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}$. By identifying $+1$, -1, 0 with the straight line paths



$$p_1\colon \begin{array}{ccc} [0,1] & \to & \mathfrak{h}_{\mathbb{R}}^* \\ t & \mapsto & t, \end{array} \qquad p_{-1}\colon \begin{array}{ccc} [0,1] & \to & \mathfrak{h}_{\mathbb{R}}^* \\ t & \mapsto & -t, \end{array} \qquad p_0\colon \begin{array}{ccc} [0,1] & \to & \mathfrak{h}_{\mathbb{R}}^* \\ t & \mapsto & 0, \end{array}$$

respectively, the set $B^{\otimes k}$ is viewed as a set of maps $p\colon [0,k] \to \mathfrak{h}_{\mathbb{R}}^*$. Let $B^{\otimes 0} = \{\phi\}$ with $\tilde{f}\phi = 0$ and $\tilde{e}\phi = 0$. Then

$$T_{\mathbb{Z}}(B) = \bigsqcup_{k \in \mathbb{Z}_{\geq 0}} B^{\otimes k} \tag{1.8}$$

is a set of paths closed under products, reverses and r-stretches ($r \in \mathbb{Z}_{\geq 0}$) and the action of $\tilde{e}$ and $\tilde{f}$. For $p \in B$ let

$$\begin{aligned} d_+(p) &= (\text{number of unpaired } +1\text{s after pairing}), \\ d_-(p) &= (\text{number of unpaired } -1\text{s after pairing}), \end{aligned}$$

These are the nonnegative integers such that

$$\tilde{f}^{d_+(p)}p \neq 0 \quad \text{and} \quad \tilde{f}^{d_+(p)+1}p = 0, \qquad \text{and} \qquad \tilde{e}^{d_-(p)}p \neq 0 \quad \text{and} \quad \tilde{e}^{d_-(p)+1}p = 0.$$

See the picture in (1.9), below.

Proposition 1.3.

(a) If $p \in T_{\mathbb{Z}}(B)$ and $\tilde{f}p \neq 0$ then $\tilde{e}\tilde{f}p = p$.

If $p \in T_{\mathbb{Z}}(B)$ and $\tilde{e}p \neq 0$ then $\tilde{f}\tilde{e}p = p$.

(b) If $p \in T_{\mathbb{Z}}(B)$ and $r \in \mathbb{Z}_{\geq 0}$ then

$$\tilde{f}^r(rp) = r(\tilde{f}p) \qquad \text{and} \qquad \tilde{e}^r(rp) = r(\tilde{e}p).$$

(c) If $p, q \in T_{\mathbb{Z}}(B)$ then

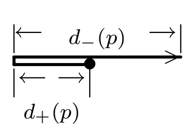
$$\tilde{f}(p \otimes q) = \begin{cases} \tilde{f}p \otimes q, & \text{if } d_-(p) > d_+(q), \\ p \otimes \tilde{f}q, & \text{if } d_-(p) \leq d_+(q), \end{cases} \qquad \text{and} \qquad \tilde{e}(p \otimes q) = \begin{cases} \tilde{e}p \otimes q, & \text{if } d_-(p) \geq d_+(q), \\ p \otimes \tilde{e}q, & \text{if } d_-(p) < d_+(q). \end{cases}$$

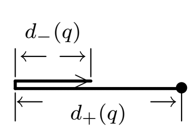
(d) If $p \in T_{\mathbb{Z}}(B)$ and $r \in \mathbb{Z}_{\geq 0}$ then

$$\tilde{f}(p^*) = (\tilde{e}p)^* \qquad \text{and} \qquad \tilde{e}(p^*) = (\tilde{f}p)^*.$$

Proof. (a), (b) and (d) are direct consequences of the definition of the operators $\tilde{e}$ and $\tilde{f}$ and the definitions of r-stretching and reversing.

(c) View p and q as paths. After pairing, p and q have the form

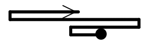
$p =$ 

and $q =$ 

(1.9)

where the left traveling portion of the path corresponds to -1s and the right traveling portion of the path corresponds to $+1$s. Then

$\tilde{f}(p\otimes q) =$

- $\tilde{f}p\otimes q$, if $p\otimes q =$  , i.e. $d_-(p) > d_+(q)$,
- $p\otimes \tilde{f}q$, if $p\otimes q =$  , i.e. $d_-(p) \le d_+(q)$,

since, in the first case, the leftmost unpaired $+1$ is from p and, in the second case, it is from q. □

Use property (b) in Proposition 1.3 to extend the operators $\tilde{e}$ and $\tilde{f}$ to operators on $T_{\mathbb{Q}}(B)$, the set of maps $p\colon [0,\ell]\to\mathbb{R}$ generated by B under the operations of concatentation, reversing and r-stretching ($r\in\mathbb{Q}_{\ge 0}$). Then, by completion, the operators $\tilde{e}$ and $\tilde{f}$ extend to operators on

$T_{\mathbb{R}}(B)$, the set of maps $p\colon [0,\ell]\to\mathbb{R}$ generated by B by operations of concatenation, reversing and r-stretching ($r\in\mathbb{R}_{\ge 0}$).

A *rank 1 path* is an element of $T_{\mathbb{R}}(B)$.

1.5 The root operators in the general case

Recall that

B_{univ} is the set of maps generated by the straight line paths by operations of concatenation, reversing and stretching.

and a *path* is an element $p\colon [0,\ell]\to\mathfrak{h}^*_{\mathbb{R}}$ in B_{univ}.

Let $p\colon [0,\ell]\to\mathbb{R}$ be a path and let $\alpha\in R^+$ be a positive root. The map

$$p_\alpha\colon [0,\ell]\to\mathbb{R}\quad\text{given by}\quad p_\alpha(t)=\langle p(t),\alpha^\vee\rangle$$

is a rank 1 path. Define operators

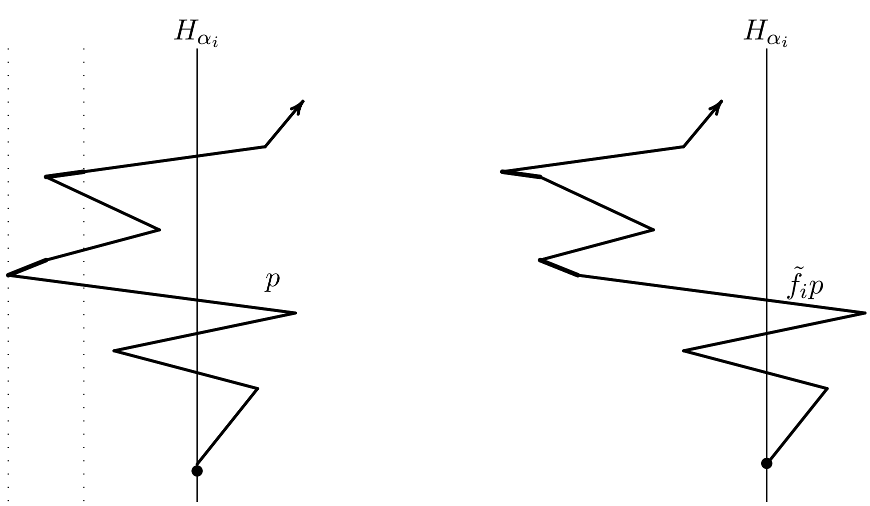
$$\tilde{e}_\alpha\colon B_{\mathrm{univ}}\to B_{\mathrm{univ}}\cup\{0\}\quad\text{and}\quad \tilde{f}_\alpha\colon B_{\mathrm{univ}}\to B_{\mathrm{univ}}\cup\{0\}$$

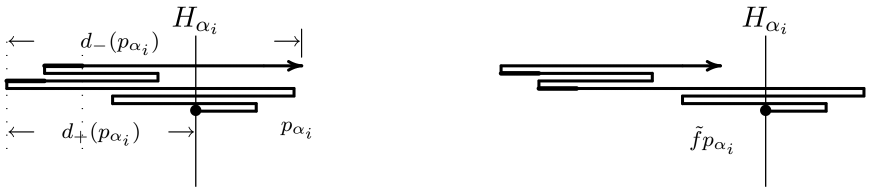
by

$$\tilde{e}_\alpha p = p+\tfrac{1}{2}(\tilde{e}p_\alpha - p_\alpha)\alpha \qquad\text{and}\qquad \tilde{f}_\alpha p = p-\tfrac{1}{2}(p_\alpha-\tilde{f}p_\alpha)\alpha, \tag{1.10}$$

and set

$$\tilde{e}_i=\tilde{e}_{\alpha_i}\quad\text{and}\quad \tilde{f}_i=\tilde{f}_{\alpha_i},\qquad \text{for } 1\le i\le n. \tag{1.11}$$





The dark parts of the path p are reflected (in a mirror parallel to H_{α_i}) to form the path f_ip. The left dotted line is the affine hyperplane parallel to H_{α_i} which intersects the path p at its leftmost (most negative) point (relative to H_{α_i}) and the distance between the dotted lines is exactly the distance between lines of lattice points in P parallel to H_{α_i}.

Theorem 1.4. *There are unique operators $\tilde{e}_i$ and $\tilde{f}_i$ such that*

(a) *If $p \in B_{\text{univ}}$ and $\tilde{f}_ip \neq 0$ then $\tilde{e}_i\tilde{f}_ip = p$.*

If $p \in B_{\text{univ}}$ and $\tilde{e}_ip \neq 0$ then $\tilde{f}_i\tilde{e}_ip = p$.

(b) *If $\lambda \in P$ and $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{>0}$ then*

$$\tilde{f}_i^{\langle \lambda, \alpha_i^\vee \rangle} p_\lambda = p_{s_i\lambda}.$$

(c) *If $p, q \in B_{\text{univ}}$ then*

$$\tilde{f}_i(p \otimes q) = \begin{cases} \tilde{f}_ip \otimes q, & \text{if } d_-(p_{\alpha_i}) > d_+(q_{\alpha_i}), \\ p \otimes \tilde{f}_iq, & \text{if } d_-(p_{\alpha_i}) \leq d_+(q_{\alpha_i}), \end{cases} \qquad \text{and}$$

$$\tilde{e}_i(p \otimes q) = \begin{cases} \tilde{e}_ip \otimes q, & \text{if } d_-(p_{\alpha_i}) \geq d_+(q_{\alpha_i}), \\ p \otimes \tilde{e}_iq, & \text{if } d_-(p_{\alpha_i}) < d_+(q_{\alpha_i}). \end{cases}$$

(d) *If $p \in B_{\text{univ}}$ and $r \in \mathbb{Z}_{\geq 0}$ then*

$$\tilde{f}_i^r(rp) = r(\tilde{f}_ip) \qquad \text{and} \qquad \tilde{e}_i^r(rp) = r(\tilde{e}_ip).$$

(e) *If $p \in B_{\text{univ}}$ then*

$$\tilde{f}_ip^* = (\tilde{e}_ip)^* \qquad \text{and} \qquad \tilde{e}_ip^* = (\tilde{f}_ip)^*.$$

(f) *If $p \in B_{\text{univ}}$ and $\tilde{f}_ip \neq 0$ then $\text{wt}(\tilde{f}_ip) = \text{wt}(p) - \alpha_i$.*

If $p \in B_{\text{univ}}$ and $\tilde{e}_ip \neq 0$ then $\text{wt}(\tilde{e}_ip) = \text{wt}(p) + \alpha_i$.

References

The path model, developed in [Li1-3], was motivated by [La] (Lakshmibai, Hyderabad) and [Ka] (Kashiwara, original crystal papers).