The path model 12.01.05

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1 Crystals

1.1 Paths

Let $\lambda \in P$. The *straight line* path to λ is the map

$$p_\lambda \colon [0,1] \to \mathfrak{h}_{\mathbb{R}}^* \quad \text{given by} \quad p_\lambda(t) = \lambda t.$$

Let $\ell_1, \ell_2 \in \mathbb{R}_{\geq 0}$. The *concatenation* of maps $p_1 \colon [0,\ell_1] \to \mathfrak{h}_{\mathbb{R}}^*$ and $p_2 \colon [0,1] \to \mathfrak{h}_{\mathbb{R}}^*$ is the map $p_1 \otimes p_2 \colon [0, \ell_1+\ell_2] \to \mathfrak{h}_{\mathbb{R}}^*$ given by

$$(p_1 \otimes p_2)(t) = \begin{cases} p_1(t), & \text{for } t \in [0,\ell_1], \\ p_1(\ell_1) + p_2(t-\ell_1), & \text{for } t \in [\ell_1, \ell_1+\ell_2]. \end{cases}$$

Let $r, \ell \in \mathbb{R}_{\geq 0}$. The *r-stretch* of a map $p \colon [0,\ell] \to \mathfrak{h}_{\mathbb{R}}^*$ is the map $rp \colon [0,\ell] \to \mathfrak{h}_{\mathbb{R}}^*$ given by

$$(rp)(t) = r \cdot p(t/r).$$

The *reverse* of a map $p \colon [0,\ell] \to \mathfrak{h}_{\mathbb{R}}^*$ is the map $p^* \colon [0,\ell] \to \mathfrak{h}_{\mathbb{R}}^*$ given by

$$p^*(t) = p(\ell - t) - p(\ell).$$

The *weight* of a map $p \colon [0,\ell] \to \mathfrak{h}_{\mathbb{R}}^*$ is the endpoint of p,

$$\mathrm{wt}(p) = p(\ell).$$

Let B_{univ} be the set of maps generated by the straight line paths under concatenation, reversing and stretching. A *path* is an element of B_{univ}. A *crystal* is a set of paths B that is closed under the action of the *root operators*

$$\tilde{e}_i \colon B_{\mathrm{univ}} \longrightarrow B_{\mathrm{univ}} \cup \{0\} \qquad \text{and} \qquad \tilde{f}_i \colon B_{\mathrm{univ}} \longrightarrow B_{\mathrm{univ}} \cup \{0\}, \qquad 1 \leq i \leq n,$$

which are defined and constructed below, in Theorem ??? and in ???, respectively.

Let B be a set of paths (a subset of B_{univ}). The *crystal graph* of B is the graph with

vertices B

edges $p' \overset{i}{\longleftarrow} p$ if $p' = \tilde{f}_i p$.

The *character* of B is

$$\mathrm{char}(B) = \sum_{p \in B} X^{\mathrm{wt}(p)}.$$

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1.2 i-strings

Let B be a crystal. Let $p \in B$ and fix i $(1 \le i \le n)$. The *i-string* of p is the set of paths $S_i(p)$ generated from p by applications of the operators $\tilde{e}_i$ and $\tilde{f}_i$. The *head* of $S_i(p)$ is $h \in S_i(p)$ such that $\tilde{e}_i h = 0$. The *tail* of $S_i(p)$ is $t \in S_i(p)$ such that $\tilde{f}_i t = 0$. The crystal graph of $S_i(p)$ is

$$t \overset{i}{\longleftarrow} \tilde{e}_i t \overset{i}{\longleftarrow} \cdots \overset{i}{\longleftarrow} \tilde{f}_i p \overset{i}{\longleftarrow} p \overset{i}{\longleftarrow} \tilde{e}_i p \overset{i}{\longleftarrow} \cdots \overset{i}{\longleftarrow} \tilde{f}_i h \overset{i}{\longleftarrow} h$$

and the weights of the paths in $S_i(p)$ are

$$\mathrm{wt}(t) = s_i \mathrm{wt}(h) = \mathrm{wt}(h) - \langle \mathrm{wt}(h), \alpha_i^\vee \rangle \alpha_i, \quad \ldots \quad , \quad \mathrm{wt}(h) - 2\alpha_i, \quad \mathrm{wt}(h) - \alpha_i, \quad \mathrm{wt}(h).$$

Let

$$d_+(p_{\alpha_i}) = (\text{distance from } h \text{ to } p) \qquad \text{and} \qquad d_-(p_{\alpha_i}) = (\text{distance from } p \text{ to } t),$$

where distance is measured in number of edges.

1.3 Highest weight paths

A *highest weight path* is a path p such that

$$\tilde{e}_i p = 0, \qquad \text{for all } 1 \le i \le n.$$

A highest weight path is a path p such that for each $1 \le i \le n$, p is the head of the i-string $S_i(p)$. Thus $\langle p(t), \alpha_i^\vee \rangle > -1$ for all t and all $1 \le i \le n$ (CAN WE PUT $\ge$ HERE?). So a path p is a highest weight path if and only if

$$p \subseteq C - \rho, \qquad \text{where} \quad C - \rho = \{\mu - \rho \mid \mu \in C\}.$$

PICTURES

If p is a highest weight path with $\mathrm{wt}(p) \in P$ then, necessarily, $\mathrm{wt}(p) \in P^+$. The following theorem gives an expression for the character of a crystal in terms of the basis $\{s_\lambda \mid \lambda \in P^+\}$ of $\mathbb{C}[P]^W$.

Theorem 1.1. *Let B be a crystal. Then*

$$\mathrm{char}(B) = \sum_{\substack{p \in B \\ p \subseteq C - \rho}} s_{\mathrm{wt}(p)},$$

where the sum is over highest weight paths $p \in B$.

Proof. Fix i, $1 \le i \le n$. If $p \in B$ let $s_i p$ be the element of the i-string of p which satisfies

$$\mathrm{wt}(s_i p) = s_i \mathrm{wt}(p).$$

PICTURE

Then $s_i(s_i p) = p$ and

$$s_i \mathrm{char}(B) = \sum_{p \in B} X^{s_i \mathrm{wt}(p)} = \sum_{p \in B} X^{\mathrm{wt}(s_i p)} = \mathrm{char}(B).$$

Hence

$$\mathrm{char}(B) \in \mathbb{C}[P]^W.$$

Let

$$\varepsilon = \sum_{w\in W} \det(w) w, \quad \text{so that} \quad a_\mu = \varepsilon(X^\mu), \quad \text{for } \mu \in P.$$

Since $\mathrm{char}(B) \in \mathbb{C}[P]^W$,

$$\mathrm{char}(B) a_\rho = \mathrm{char}(B)\varepsilon(X^\rho) = \varepsilon(\mathrm{char}(B)X^\rho)$$

and

$$\begin{aligned}\mathrm{char}(B) &= \frac{1}{a_\rho}\mathrm{char}(B)a_\rho = \frac{\varepsilon(\mathrm{char}(B)X^\rho)}{a_\rho} \\ &= \sum_{p\in B} \frac{\varepsilon(X^{\mathrm{wt}(p)+\rho})}{a_\rho} = \sum_{p\in B} \frac{a_{\mathrm{wt}(p)+\rho}}{a_\rho} \\ &= \sum_{p\in B} s_{\mathrm{wt}(p)}.\end{aligned}$$

There is some cancellation which can occur in this sum. If $p \in B$ such that $p \not\subseteq C - \rho$ let t be the first time that p leaves the cone $C - \rho$. In other words let $t \in \mathbb{R}_{>0}$ be minimal such that there exists an i with

$$p(t) \in H_{\alpha_i - \delta} \qquad (\langle p(t), \alpha_i^\vee\rangle = -1).$$

Let i be the minimal index such that the point $p(t) \in H_{\alpha_i-\delta}$ and let $s_i \circ p$ be the element of the i-string of p such that

$$\mathrm{wt}(s_i \circ p) = s_i \circ p.$$

PICTURE

Note that since $\langle p_i(t), \alpha_i^\vee\rangle = -1$, p is not the head of its i-string and $s_i \circ p$ is well defined. If $q = s_i \circ p$ then the first time t that q leaves the cone $C - \rho$ is the same as the first time that p leaves the cone $C - \rho$ and $p(t) = q(t)$. Thus $s_i \circ q = p$ and $s_i \circ (s_i \circ p)) = p$. Since

$$s_{\mathrm{wt}(s_i\circ p)} = s_{s_i\circ \mathrm{wt}(p)} = -s_{\mathrm{wt}(p)}$$

the terms $s_{\mathrm{wt}(s_i\circ p)}$ and $s_{wt(p)}$ cancel in the sum (*) and so

$$\mathrm{char}(B) = \sum_{\substack{p\in B \\ p\subseteq C-\rho}} s_{\mathrm{wt}(p)}.$$

□

If $\lambda \in P^+$ let $B(\lambda)$ be the crystal generated by the straight line path p_λ with endpoint λ.

Corollary 1.2. $s_\lambda = \mathrm{char}(B(\lambda))$.

Proof. The path p_λ is the unique highest weight path in $B(\lambda)$. Thus, by Theorem ???, $\mathrm{char}(B(\lambda)) = s_\lambda$. □

Corollary 1.3. *Let $\mu, \nu \in P^+$. Then*

$$s_\mu s_\nu = \sum_{\substack{q\in B(\nu) \\ p_\mu\otimes q\subseteq C-\rho}} s_{\mu+\mathrm{wt}(q)}.$$

Proof. By (???) the set

$$B(\mu)\otimes B(\nu) = \{p\otimes q \mid p\in B(\mu), q\in B(\nu)\}$$

is a crystal. Since $\mathrm{wt}(p\otimes q) = \mathrm{wt}(p)+\mathrm{wt}(q)$,

$$\begin{aligned} s_\mu s_\nu &= \mathrm{char}(B(\mu))\mathrm{char}(B(\nu)) = \mathrm{char}((B(\mu)\otimes B(\nu)) \\ &= \sum_{\substack{p\otimes q\in B(\mu)\times B(\nu)\\ p\otimes q\subseteq C-\rho}} s_{\mathrm{wt}(p)+\mathrm{wt}(q)} = \sum_{\substack{q\in B(\nu)\\ p_\mu\otimes q\subseteq C-\rho}} s_{\mu+\mathrm{wt}(q)}, \end{aligned}$$

where the third equality is from Theorem ??? and the last equality is because the path p_μ has $\mathrm{wt}(p_\mu)=\mu$ and is the only highest weight path in $B(\mu)$. □

Fix $J\subseteq\{1,2,\ldots,n\}$. The subgroup of W generated by the reflections in the hyperplanes H_{α_j} $(j\in J)$,

$$W_J = \langle s_j \mid j\in J\rangle, \quad \text{acts on } \mathfrak{h}^*_{\mathbb{R}}, \qquad \text{with} \qquad C_J = \{\mu\in\mathfrak{h}^*_{\mathbb{R}} \mid \langle\mu,\alpha_j^\vee\rangle\ge 0 \text{ for } j\in J\}$$

as a fundamental chamber. The group W_J acts on P and

$$\mathbb{C}[P]^{W_J} = \{f\in\mathbb{C}[P] \mid wf=f \text{ for } w\in W_J\}$$

is a subalgebra of $\mathbb{C}[P]$ which contains $\mathbb{C}[P]^W$. If

$$P_J^+ = P\cap\overline{C_J}, \qquad \rho_J = \sum_{j\in J}\omega_j,$$

and

$$a_\mu^J = \sum_{w\in W_J}\det(w)wX^\mu, \quad \text{for } \mu\in P, \qquad \text{and} \qquad s_\lambda^J = \frac{a^J_{\lambda+\rho_J}}{a^J_{\rho_J}}, \quad \text{for } \lambda\in P,$$

then

$$\{s_\lambda^J \mid \lambda\in P_J^+\} \text{ is a basis of } \mathbb{C}[P]^{W_J}.$$

A *J-crystal* is a set of paths B which is closed under the operators $\tilde{e}_j$, $\tilde{f}_j$, for $j\in J$.

Corollary 1.4. *Let* $\lambda\in P^+$. *Then*

$$s_\lambda = \sum_{\substack{p\in B(\lambda)\\ p\subseteq C_J-\rho_J}} s^J_{\mathrm{wt}(p}.$$

Proof. Since $s_\lambda = \mathrm{char}(B(\lambda))$ this corollary follows by applying Theorem ??? to $B(\lambda)$ viewed as a J-crystal. □

1.4 The root operators

If $p\colon[0.\ell]\to\mathfrak{h}^*_{\mathbb{R}}$ and α_i is a simple root define

$$p_{\alpha_i}\colon[0.\ell]\to\mathbb{R} \qquad \text{given by} \qquad p_{\alpha_i}(t) = \langle p(t),\alpha_i^\vee\rangle.$$

Theorem 1.5. *There are unique operators* $\tilde{e}_i$ *and* $\tilde{f}_i$ *such that*

(1) If $p\in B_{\mathrm{univ}}$ *and* $\tilde{f}_ip\neq 0$ *then* $\tilde{e}_i\tilde{f}_ip = p$.

If $p \in B_{\mathrm{univ}}$ *and* $\tilde{e}_i p \neq 0$ *then* $\tilde{f}_i \tilde{e}_i p = p$.

(2) If $\lambda \in P$ *and* $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{>0}$ *then*

$$\tilde{f}_i^{\langle \lambda, \alpha_i^\vee \rangle} p_\lambda = p_{s_i \lambda}.$$

(3) If $p, q \in B_{\mathrm{univ}}$ *then*

$$\tilde{f}_i(p \otimes q) = \begin{cases} \tilde{f}_i p \otimes q, & \textit{if } d_-(p_{\alpha_i}) > d_+(q_{\alpha_i}), \\ p \otimes \tilde{f}_i q, & \textit{if } d_-(p_{\alpha_i}) \leq d_+(q_{\alpha_i}), \end{cases} \qquad \textit{and}$$

$$\tilde{e}_i(p \otimes q) = \begin{cases} \tilde{e}_i p \otimes q, & \textit{if } d_-(p_{\alpha_i}) \geq d_+(q_{\alpha_i}), \\ p \otimes \tilde{e}_i q, & \textit{if } d_-(p_{\alpha_i}) < d_+(q_{\alpha_i}). \end{cases}$$

(4) If $p \in B_{\mathrm{univ}}$ *and* $r \in \mathbb{Z}_{\geq 0}$ *then*

$$\tilde{f}_i^r(rp) = r(\tilde{f}_i p) \qquad \textit{and} \qquad \tilde{e}_i^r(rp) = r(\tilde{e}_i p).$$

(5) If $p \in B_{\mathrm{univ}}$ *then*

$$\tilde{f}_i(p^*) = (\tilde{e}_i p)^* \qquad \textit{and} \qquad \tilde{e}_i(p^*) = (\tilde{f}_i p)^*.$$

(6) If $p \in B_{\mathrm{univ}}$ *and* $\tilde{f}_i p \neq 0$ *then* $\mathrm{wt}(\tilde{f}_i p) = \mathrm{wt}(p) - \alpha_i$.

If $p \in B_{\mathrm{univ}}$ *and* $\tilde{e}_i p \neq 0$ *then* $\mathrm{wt}(\tilde{e}_i p) = \mathrm{wt}(p) + \alpha_i$.

1.5 The rank 1 case

Let

$$B^{\otimes k} = \{b_1 \otimes \cdots \otimes b_k \mid b_i \in B\}, \qquad \text{where} \qquad B = \{+1, -1, 0\}.$$

Define

$$\tilde{f}\colon B^{\otimes k} \to B^{\otimes k} \cup \{0\} \quad \text{and} \quad \tilde{e}\colon B^{\otimes k} \to B^{\otimes k} \cup \{0\}$$

as follows. Let $b = b_1 \otimes \cdots \otimes b_k \in B^{\otimes k}$. Ignoring 0s successively pair adjacent unpaired $(-1, +1)$ pairs to obtain a sequence of unpaired $+1$s and -1s

$$+1 \ +1 \ +1 \ +1 \ +1 \ +1 \ +1 \ -1 \ -1 \ -1 \ -1$$

(after pairing and ignoring 0s). Then

$\tilde{f}b =$ same as b except the rightmost unpaired $+1$ is changed to -1, $\tilde{e}b =$ same as b except the leftmost unpaired

If there is no unpaired $+1$ after pairing then $\tilde{f}b = 0$. If there is no unpaired -1 after pairing then $\tilde{e}b = 0$.

Let $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}$. By identifying $+1$, -1, 0 with the straight line paths

$$\begin{array}{rccc} p_1\colon & [0,1] & \to & \mathfrak{h}_{\mathbb{R}}^* \\ & t & \mapsto & t, \end{array} \qquad \begin{array}{rccc} p_{-1}\colon & [0,1] & \to & \mathfrak{h}_{\mathbb{R}}^* \\ & t & \mapsto & -t, \end{array} \qquad \begin{array}{rccc} p_0\colon & [0,1] & \to & \mathfrak{h}_{\mathbb{R}}^* \\ & t & \mapsto & 0, \end{array}$$

respectively, the set $B^{\otimes k}$ is viewed as a set of maps $p\colon [0,k] \to \mathfrak{h}_{\mathbb{R}}^*$. Let $B^{\otimes 0} = \{\phi\}$ with $\tilde f\phi = 0$ and $\tilde e\phi = 0$. Then

$$T_{\mathbb{Z}}(B) = \bigsqcup_{k\in\mathbb{Z}_{\geq 0}} B^{\otimes k}$$

is a set of paths closed under products, reverses and r-stretches ($r \in \mathbb{Z}_{\geq 0}$) and the action of $\tilde e$ and $\tilde f$. For $p \in B$ let

$d_+(p) =$ (number of unpaired $+1$s after pairing), $d_-(p) \;\; =$ (number of unpaired -1s after pairing),

These are the nonnegative integers such that

$$\begin{matrix} \tilde f^{d_+(p)}p \neq 0 & \text{and} & \tilde f^{d_+(p)+1}p = 0, & \text{and} \\ \tilde e^{d_-(p)}p \neq 0 & \text{and} & \tilde e^{d_-(p)+1}p = 0. \end{matrix}$$

Proposition 1.6. *(1) If $p \in B$ and $\tilde fp \neq 0$ then $\tilde e\tilde fp = p$.*

If $p \in B$ and $\tilde ep \neq 0$ then $\tilde e\tilde ep = p$.

(2) If $p \in B$ and $r \in \mathbb{Z}_{\geq 0}$ then

$$\tilde f^r(rp) = r(\tilde fp) \qquad \text{and} \qquad \tilde e^r(rp) = r(\tilde ep).$$

(3) If $p, q \in B$ then

$$\tilde f(p\otimes q) = \begin{cases} \tilde fp\otimes q, & \text{if } d_-(p) > d_+(q), \\ p\otimes \tilde fq, & \text{if } d_-(p) \leq d_+(q), \end{cases} \qquad \text{and} \qquad \tilde e(p\otimes q) = \begin{cases} \tilde ep\otimes q, & \text{if } d_-(p) \geq d_+(q), \\ p\otimes \tilde eq, & \text{if } d_-(p) < d_+(q). \end{cases}$$

(4) If $p \in B$ and $r \in \mathbb{Z}_{\geq 0}$ then

$$\tilde f(p^*)(\tilde p)^* \qquad \text{and} \qquad \tilde e(p^*)(\tilde fp)^*.$$

Proof. (1), (2) and (4) are direct consequences of the definition of the operators $\tilde e$ and $\tilde f$ and the definitions of r-stretching and reversing.
(3) View p and q as paths. After pairing, p and q have the form

$$p = PICTURE \qquad \text{and} \qquad q = PICTURE$$

where the left traveling portion of the path corresponds to -1s and the right traveling portion of the path corresponds to $+1$s. Then

$$\tilde f(p\otimes q) = \begin{cases} \tilde fp\otimes q, & \text{if } p\otimes q = PICTURE, \text{ i.e. } d_-(p) > d_+(q), \\ p\otimes \tilde fq, & \text{if } p\otimes q = PICTURE, \text{ i.e. } d_-(p) \leq d_+(q), \end{cases}$$

since, in the first case the leftmost unpaired $+1$ is from p and, in the second case it is from q. □

Use property (2) in Proposition ??? to extend the operators $\tilde e$ and $\tilde f$ to operators on $T_{\mathbb{Q}}(B)$, the set of maps $p\colon [0,\ell] \to \mathbb{R}$ generated by B under the operations of concatentation, reversing and r-stretching ($r \in \mathbb{Q}_{\geq 0}$). Then, by completion, the operators $\tilde e$ and $\tilde f$ extend to operators on

$T_{\mathbb{R}}(B)$, the set of maps $p\colon [0,\ell] \to \mathbb{R}$ generated by B under concatenation, reversing and $r$stretching ($r \in \mathbb{R}_{\geq 0}$).

A *rank 1 path* is an element of $T_{\mathbb{R}}(B)$.

1.6 The general case

Let

B_{univ} be the set of maps generated by the straight line paths under concatenation, reversing and stretching.

A *path* is an element $p\colon [0,\ell] \to \mathfrak{h}^*_{\mathbb{R}}$ in B_{univ}.

Let $p\colon [0,\ell] \to \mathbb{R}$ be a path and let $\alpha \in R^+$ be a positive root. The map

$$p_\alpha\colon [0,\ell] \to \mathbb{R} \quad \text{given by} \quad p_\alpha(t) = \langle p(t), \alpha^\vee \rangle$$

is a rank 1 path. Define operators

$$\tilde{e}_\alpha\colon B_{\text{univ}} \to B_{\text{univ}} \cup \{0\} \quad \text{and} \quad \tilde{f}_\alpha\colon B_{\text{univ}} \to B_{\text{univ}} \cup \{0\}$$

by

$$\tilde{e}_\alpha p = p + \frac{1}{2}(\tilde{e}p_\alpha - p_\alpha)\alpha \qquad \text{and} \qquad \tilde{f}_\alpha p = p - \frac{1}{2}(p_\alpha - \tilde{f}p_\alpha)\alpha,$$

and set

$$\tilde{e}_i = \tilde{e}_{\alpha_i} \quad \text{and} \quad \tilde{f}_i = \tilde{f}_{\alpha_i}, \qquad \text{for } 1 \le i \le n.$$

1.7 Tableaux

A *letter* is an element of $B(\varepsilon_1) = \{\varepsilon_1, \ldots, \varepsilon_n\}$ and a *word of length* k is an element of

$$B(\varepsilon_1)^{\otimes k} = \{\varepsilon_{i_1} \otimes \cdots \varepsilon_{i_k} \mid 1 \le i_1, \ldots, i_k \le n\}.$$

For $1 \le i \le n-1$ define

$$\tilde{f}_i\colon B(\varepsilon_1)^{\otimes k} \longrightarrow B(\varepsilon_1)^{\otimes k} \cup \{0\} \quad \text{and} \quad \tilde{e}_i\colon B(\varepsilon_1)^{\otimes k} \longrightarrow B(\varepsilon_1)^{\otimes k} \cup \{0\}$$

as follows. For $b \in B(\varepsilon_1)^{\otimes k}$,

place $+1$ under each ε_i in b,

place -1 under each ε_{i+1} in b, and

place 0 under each ε_j, $j \ne i, i+1$.

Ignoring 0s, successively pair adjacent $(-1,+1)$ pairs to obtain a sequence of unpaired $+1$s and -1s

$$+1 \ +1 \ +1 \ +1 \ +1 \ +1 \ +1 \ -1 \ -1 \ -1 \ -1$$

(after pairing and ignoring 0s). Then

$\tilde{f}_i b =$ same as b except the letter corresponding to the rightmost unpaired $+1$ is changed to ε_{i+1},
$\tilde{e}_i b =$ same as b except the letter corresponding to the leftmost unpaired -1 is changed to ε_i.

If there is no unpaired $+1$ after pairing then $\tilde{f}_i b = 0$. If there is no unpaired -1 after pairing then $\tilde{e}_i b = 0$.

Let λ be a partition with k bokes and let

$$B(\lambda) = \{\text{column strict tableaux of shape } \lambda\}.$$

The set $B(\lambda)$ is a subset of $B(\varepsilon_1)^{\otimes k}$ via the injection

$$\begin{array}{ccc} B(\lambda) & \hookrightarrow & B(\varepsilon_1)^{\otimes k} \\ p & \longmapsto & (\text{the arabic reading of } p) \\ PICTURE & \longmapsto & \varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_k} \end{array}$$

where the arabic reading of p is $\varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_k}$ if the entries of p are $i_1, i_2, \ldots, i_k$ read right to left by rows with the rows read in sequence beginning with the first row.

Proposition 1.7. *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition with k boxes. Then $B(\lambda)$ is the subset of $B(\varepsilon_1)^{\otimes k}$ generated by*

$$p_\lambda = \underbrace{\varepsilon_1\varepsilon_1\cdots\varepsilon_1}_{\lambda_1 \text{ factors}}\underbrace{\varepsilon_2\varepsilon_2\cdots\varepsilon_2}_{\lambda_2 \text{ factors}}\cdots\underbrace{\varepsilon_n\varepsilon_n\cdots\varepsilon_n}_{\lambda_n \text{ factors}}$$

under the action of the operators $\tilde{e}_i$, $\tilde{f}_i$, $1 \le i \le n$.

2 The nil-affine Hecke algebra

The *nil-affine Hecke algebra* is the algebra K_T given by

$$\text{generators} \qquad T_w, \quad w \in W, \quad \text{and} \quad X^\lambda, \quad \lambda \in P$$

and relations

$$\begin{aligned} T_{s_i}T_w &= \begin{cases} T_{s_iw}, & \text{if } s_iw > w, \\ T_w, & \text{if } s_iw < w, \end{cases} \\ X^\lambda X^\mu &= X^{\lambda+\mu}, \\ X^\lambda T_{s_i} &= T_{s_i}X^{s_i\lambda} + \frac{X^\lambda - X^{s_i\lambda}}{1 - X^{-\alpha_i}}. \end{aligned}$$

Make a "change of variable" and let

$$\tilde{T}_{s_i} = q^{-1}T_{s_i}$$

in the affine Hecke algebra $\tilde{H}$. For $w \in W$ let $\tilde{T}_w = q^{-\ell(w)}T_w$ so that $\tilde{T}_w = \tilde{T}_{s_{i_1}}\tilde{T}_{s_{i_2}}\cdots\tilde{T}_{s_{i_p}}$ for a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_p}$. In terms of the generators

$$T_w, \quad w \in W, \qquad \text{and} \qquad X^\lambda, \quad \lambda \in P,$$

the affine Hecke algebra $\tilde{H}$ is given by the relations

$$\begin{aligned} \tilde{T}_{s_i}\tilde{T}_w &= \begin{cases} \tilde{T}_{s_iw}, & \text{if } s_iw > w, \\ q^{-2}\tilde{T}_{s_iw} + (1 - q^{-2})\tilde{T}_w, & \text{if } s_iw < w, \end{cases} \\ X^\lambda X^\mu &= X^{\lambda+\mu}, \\ X^\lambda \tilde{T}_{s_i} &= \tilde{T}_{s_i}X^{s_i\lambda} + (1 - q^{-2})\frac{X^\lambda - X^{s_i\lambda}}{1 - X^{-\alpha_i}}. \end{aligned}$$

Thus the nil-affine Hecke algebra K_T is the algebra $\tilde{H}$ at $q^{-2} = 0$.

2.1 The commutation formula

Let S_i be an i-string with head h and tail t. The initial and final directions of the paths in the string S_i satisfy

(1) Either $s_i\iota(h) = \iota(t) = \iota(\tilde{e}_i t) = \cdots = \iota(\tilde{f}_i h) > \iota(h)$

or $s_i\iota(h) > \iota(t) = \iota(\tilde{e}_i t) = \cdots = \iota(\tilde{f}_i h) = \iota(h)$.

(2) Either $s_i\varphi(h) = \varphi(t) > \varphi(\tilde{e}_i t) = \cdots = \varphi(h)$

or $\varphi(t) = \varphi(\tilde{e}_i t) = \cdots = \varphi(h) > s_i\varphi(h)$.

(3) $\varphi(p) = \iota(p^*)w_0$.

(4) If p is a highest weight path then $iota(p) = 1$ and $\varphi(p) = 1$.

Theorem 2.1. *Let $\lambda \in P^+$ and $w \in W$. Then*

$$X^\lambda T_w = \sum_{\substack{p\in B(\lambda)\\ \iota(p)\le w}} T_{\varphi(p)^{-1}} X^{\mathrm{wt}(p)}.$$

Proof. The proof is by induction on $\ell(w)$. Let $w = vs_i$ with $\ell(v) = \ell(w) - 1$. *Case 1:* $s_i\varphi(h) = \varphi(t) > \varphi(\tilde{e}_i t) = \cdots = \varphi(h)$.

$$\begin{aligned}\left(\sum_{p\in S_i(h)} T_{\varphi(p)} X^{\mathrm{wt}(p)}\right) T_{s_i} &= T_{\varphi(h)}\big(T_{s_i}X^{\mathrm{wt}(t)} + (X^{\mathrm{wt}(\tilde{e}_i t)} + \cdots + X^{\mathrm{wt}(h)})\big)\\ &= T_{\varphi(h)}X^{\mathrm{wt}(h)}T_{s_i}T_{s_i} = T_{\varphi(h)}X^{\mathrm{wt}(h)}T_{s_i}\\ &= \sum_{p\in S_i(h)} T_{\varphi(p)}X^{\mathrm{wt}(p)}\end{aligned}$$

Case 2: $\varphi(t) = \cdots \varphi(\tilde{f}_i h) = \varphi(h) > s_i\varphi(h)$

$$\begin{aligned}\left(\sum_{p\in S_i(h)} T_{\varphi(p)^{-1}} X^{\mathrm{wt}(p)}\right) T_{s_i} &= T_{\varphi(h)^{-1}}(X^{\mathrm{wt}(t)} + \cdots + X^{\mathrm{wt}(h)})T_{s_i}\\ &= T_{\varphi(h)^{-1}}T_{s_i}(X^{\mathrm{wt}(t)} + \cdots + X^{\mathrm{wt}(h)})\\ &= T_{\varphi(h)^{-1}}(X^{\mathrm{wt}(t)} + \cdots + X^{\mathrm{wt}(h)})\\ &=\sim_{p\in S_i(h)} T_{\varphi(p)^{-1}}X^{\mathrm{wt}(p)}.\end{aligned}$$

□

2.2 Demazure operators

The *Iwahori-Hecke algebra* is the subalgebra of $\tilde{H}$ given by

$$H = \mathrm{span}\{\tilde{T}_w \mid w \in W\}.$$

Let $\mathbf{1}_0$ be the element of H determined by the conditions

$$\mathbf{1}_0^2 = \mathbf{1}_0 \qquad \text{and} \qquad \tilde{T}_{s_i}\mathbf{1}_0 = \mathbf{1}_0, \quad \text{for } 1 \le i \le n.$$

A formula for $\mathbf{1}_0$ is

$$\mathbf{1}_0 = \frac{1}{P_W(q^2)} \sum_{w\in W} \tilde{T}_w, \qquad \text{where} \quad P_W(t) = \sum_{w\in W} t^{\ell(w)}$$

is the *Poincaré polynomial of* W. The map

$$\begin{array}{rccc} \Phi\colon & \mathbb{C}[P] & \xrightarrow{\sim} & \mathbf{1}_0\tilde{H} \\ & f & \longmapsto & \mathbf{1}_0 f \end{array}$$

is a vector space isomorphism.

Let $w \in W$. The *Demazure operator* $\tilde{\Delta}_w$ is the operator on $\mathbb{C}[P]$ corresponding to the operator on $\mathbf{1}_0\tilde{H}$ given by right multiplication by $\tilde{T}_{w^{-1}}$,

$$\tilde{\Delta}_w f = \Phi^{-1}(\Phi(f)\tilde{T}_{w^{-1}}), \qquad \text{for } f \in \mathbb{C}[P].$$

For a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_p}$,

$$\tilde{\Delta}_w = \tilde{\Delta}_{s_{i_1}}\cdots\tilde{\Delta}_{s_{i_p}}, \qquad \text{since} \quad \tilde{T}_{w^{-1}} = \tilde{T}_{s_{i_p}}\cdots\tilde{T}_{s_{i_1}}.$$

As operators on $\mathbb{C}[P]$,

$$\tilde{\Delta}_i = X^{-\rho}\Delta_i X^\rho - q^{-2}\Delta_i, \quad \text{where} \quad \begin{array}{rccc} \Delta_i\colon & \mathbb{C}[P] & \longrightarrow & \mathbb{C}[P] \\ & f & \longmapsto & \dfrac{f - s_i f}{1 - X^{-\alpha_i}}, \end{array}$$

because

$$\begin{aligned}
(\mathbf{1}_0 X^\lambda)\tilde{T}_{s_i} &= \mathbf{1}_0\left(\tilde{T}_{s_i}X^{s_i\lambda} + (1-q^{-2})\frac{X^\lambda - X^{s_i\lambda}}{1-X^{-\alpha_i}}\right) \\
&= \mathbf{1}_0\left(X^{s_i\lambda} + (1-q^{-2})\frac{X^\lambda - X^{s_i\lambda}}{1-X^{-\alpha_i}}\right) \\
&= \mathbf{1}_0\left(\frac{X^{s_i\lambda} - X^{s_i\lambda-\alpha_i} + X^\lambda - X^{s_i\lambda} - q^{-2}(X^\lambda - X^{s_i\lambda})}{1-X^{-\alpha_i}}\right) \\
&= \mathbf{1}_0\left(\frac{X^{-\rho}(X^{\lambda+\rho} - X^{s_i\lambda-\alpha_i+\rho}) - q^{-2}(X^\lambda - X^{s_i\lambda})}{1-X^{-\alpha_i}}\right) \\
&= \mathbf{1}_0\left(X^{-\rho}\frac{X^{\lambda+\rho} - X^{s_i(\lambda+\rho)}}{1-X^{-\alpha_i}} - q^{-2}\frac{X^\lambda - X^{s_i\lambda}}{1-X^{-\alpha_i}}\right) \\
&= \big((X^{-\rho}\Delta_i X^\rho - q^{-2}\Delta_i)(X^\lambda)\big).
\end{aligned}$$

When $q^{-2} = 0$, in the algebra K_T,

$\mathbf{1}_0 = T_{w_0}$, where w_0 is the longest element of W,

$\tilde{\Delta}_w = \tilde{\Delta}_{s_{i_1}}\cdots\tilde{\Delta}_{s_{i_p}}$, for a reduced word $w = s_{i_1}\cdots s_{i_p}$, and

$\tilde{\Delta}_{s_i} = X^{-\rho}\Delta_i X^\rho$, as operators on $\mathbb{C}[P]$.

Theorem 2.2. *Let* $\lambda \in P+$ *and* $w \in W$. *Then*

(a) $\displaystyle \tilde{\Delta}_w X^\lambda = \sum_{\substack{p\in B(\lambda) \\ \iota(p)\le w}} X^{\mathrm{wt}(p)}.$

(b) $s_\lambda = \tilde{\Delta}_{w_0} X^\lambda$,

where w_0 *is the longest element of* W.

Proof. Since $\mathbf{1}_0 \tilde{T}_v = \mathbf{1}_0$, for $v \in W$, Theorem CF??? gives that

$$\mathbf{1}_0 X^\lambda T_{w^{-1}} = \mathbf{1}_0 \left(\sum_{\substack{p \in B(\lambda) \\ \iota(p) \le w}} T_{\phi(p)^{-1}} X^{\mathrm{wt}(p)} \right) = \mathbf{1}_0 \sum_{\substack{p \in B(\lambda) \\ \iota(p) \le w}} X^{\mathrm{wt}(p)},$$

which establishes (a). Combining (a) with Corollary BC??? and the fact that $B(\lambda) = \{p \in B(\lambda) \mid \iota(p) \le w_0\}$ gives (b). □

2.3 Example

The following example illustrates the connection between

(1) the affine Hecke algebra relation

$$T_{s_i} X^\lambda = X^{s_i\lambda} T_{s_i} + (q - q^{-1}) \frac{X^\lambda - X^{s_i\lambda}}{1 - X^{-\alpha_i}},$$

(2) the root operators $\tilde{f}_i$,

(3) paths, and

(4) tableaux.

Working in type A_2, if

$$\varepsilon_1 = \omega_1, \quad \varepsilon_2 = \omega_2 - \omega_1, \quad \varepsilon_3 = -\omega_2,$$

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then, in the affine Hecke algebra,

$$T_{s_1} X^{\varepsilon_1} = X^{\varepsilon_2} T_{s_1} + (q - q^{-1}) X^{\varepsilon_1}, \quad T_{s_1} X^{\varepsilon_2} = X^{\varepsilon_1} T_{s_1} - (q - q^{-1}) X^{\varepsilon_1}, \quad T_{s_1} X^{\varepsilon_3} = X^{\varepsilon_3} T_{s_1},$$

and

$$T_{s_2} X^{\varepsilon_1} = X^{\varepsilon_1} T_{s_2}, \quad T_{s_2} X^{\varepsilon_2} = X^{\varepsilon_3} T_{s_2} + (q - q^{-1}) X^{\varepsilon_2}, \quad T_{s_2} X^{\varepsilon_3} = X^{\varepsilon_2} T_{s_2} - (q - q^{-1}) X^{\varepsilon_2}.$$

Then $2\rho = 2(\omega_1 + \omega_2) = \varepsilon_1 + \varepsilon_1 + \varepsilon_1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_2$, and

$$\begin{aligned}
T_{s_1} X^{2\rho} &= T_{s_1} X^{\varepsilon_1} X^{\varepsilon_1} X^{\varepsilon_1} X^{\varepsilon_1} X^{\varepsilon_2} X^{\varepsilon_2} \\
&= X^{\varepsilon_2} T_{s_1} X^{\varepsilon_1} X^{\varepsilon_1} X^{\varepsilon_1} X^{\varepsilon_2} X^{\varepsilon_2} + (q - q^{-1}) X^{\varepsilon_1} X^{\varepsilon_1} X^{\varepsilon_1} X^{\varepsilon_1} X^{\varepsilon_2} X^{\varepsilon_2} \\
&= X^{\varepsilon_2} X^{\varepsilon_2} T_{s_1} X^{\varepsilon_1} X^{\varepsilon_1} X^{\varepsilon_2} X^{\varepsilon_2} + (q - q^{-1})(X^{\varepsilon_2} X^{\varepsilon_1} X^{\varepsilon_1} X^{\varepsilon_1} X^{\varepsilon_2} X^{\varepsilon_2} + X^{\varepsilon_1} X^{\varepsilon_1} X^{\varepsilon_1} X^{\varepsilon_1} X^{\varepsilon_2} X^{\varepsilon_2}) \\
&= X^{\varepsilon_2} X^{\varepsilon_2} X^{\varepsilon_1} X^{\varepsilon_1} X^{\varepsilon_2} X^{\varepsilon_2} T_{s_1} + (q - q^{-1})(X^{\varepsilon_2} X^{\varepsilon_1} X^{\varepsilon_1} X^{\varepsilon_1} X^{\varepsilon_2} X^{\varepsilon_2} + X^{\varepsilon_1} X^{\varepsilon_1} X^{\varepsilon_1} X^{\varepsilon_1} X^{\varepsilon_2} X^{\varepsilon_2})
\end{aligned}$$

since $T_{s_1} X^{\varepsilon_1} X^{\varepsilon_1} X^{\varepsilon_2} X^{\varepsilon_2} = X^{\varepsilon_1} X^{\varepsilon_1} X^{\varepsilon_2} X^{\varepsilon_2} T_{s_1}$. In terms of words,

$$\tilde{f}_1^3(\varepsilon_1\varepsilon_1\varepsilon_1\varepsilon_1\varepsilon_2\varepsilon_2) = \tilde{f}_1^2(\varepsilon_2\varepsilon_1\varepsilon_1\varepsilon_1\varepsilon_2\varepsilon_2) = \tilde{f}_1(\varepsilon_2\varepsilon_2\varepsilon_1\varepsilon_1\varepsilon_2\varepsilon_2) = 0;$$

in terms of paths

$$\tilde{f}_1^3\,(PICTURE) = \tilde{f}_1^2\,(PICTURE) = \tilde{f}_1\,(PICTURE) = 0;$$

in terms of tableaux,

$$\tilde{f}_1^3\,(PICTURE) = \tilde{f}_1^2\,(PICTURE) = \tilde{f}_1\,(PICTURE) = 0.$$

References

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