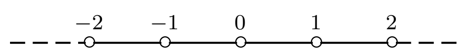
Multisegments

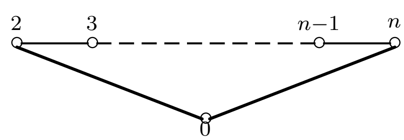
Arun Ram
Department of Mathematics
University of Wisconsin
Madison, WI 53706
ram@math.wisc.edu

July 2, 2005

1 Multisegments

A_{n-1} 

A_∞ 

$A^{(1)}_{n-1}$ 

Let

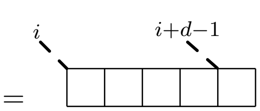
$$I = \begin{cases} \{1, 2, \ldots, \ell - 1\}, & \text{in type } A_{\ell-1}, \\ \mathbb{Z}, & i\text{in type } A_\infty, \\ \mathbb{Z}/\ell\mathbb{Z}, & \text{in type } A^{(1)}_{\ell-1}. \end{cases}$$

The elements of I index the (isomorphism classes) of simple representations of the quiver.

Consider a sheet of graph paper with diagonals indexed by $\mathbb{Z}$. The *content* $c(b)$ of a box b on this sheet of graph paper is

$$c(b) = \text{the diagonal number of the box } b$$

Let

$$[i; d) = [i, i + d - 1] = (d; i + d - 1] =$$ 

denote a sequence of boxes in a row which has length d with the leftmost box if content i and the rightmost box of content $i+d-1$. The set of segments is

$$R^+ = \begin{cases} \{[i;d) \mid i \in I, 1 \le d \le \ell - i\}, & \text{in type } A_{\ell-1}, \\ \{[i;d) \mid i \in I, d \in \mathbb{Z}_{\ge 0}\} & \text{in type } A_\infty, \\ \{[i;d) \mid i \in I, d \in \mathbb{Z}/\ell\mathbb{Z}\} & \text{in type } A_{\ell-1}^{(1)}, \end{cases}$$

The elements of R^+ index the (isomorphism classes) of indecomposable (nilpotent) representations of the quiver.

A *multisegment* is a (unordered) collection of segments, i.e. an element of

$$\tilde{B}(\infty) = \sum_{\alpha \in R^+} \mathbb{Z}_{\ge 0}\alpha.$$

For example

$$\begin{array}{cccccc} \boxed{3} & \boxed{4} & \boxed{5} & \boxed{6} & \boxed{7} & \\ & \boxed{3} & \boxed{4} & \boxed{5} & \boxed{6} & \boxed{7} \\ & & & \boxed{5} & \boxed{6} & \boxed{7} \\ & \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{5} \\ & & & \boxed{3} & \boxed{4} & \boxed{5} \end{array} = 3[3;5) + [5;3) + [1;5) + [3;3). \tag{1.1}$$

(the numbers in the boxes in the picture are the contents of the boxes).

A multisegment is *aperiodic* if it does not contain

$$[0;d) + [1;d) + \cdots + [\ell-1;d), \qquad \text{for any } d \in \mathbb{Z}_{>0}.$$

Pictorially, a multisegment is aperiodic if it does not contain a box of height ℓ. Let

$$B(\infty) = \{\text{aperiodic multisegments}\}.$$

In types $A_{\ell-1}$ and A_∞, $B(\infty) = \tilde{B}(\infty)$. The elements of $B(\infty)$ index the isomorphism classes of nilpotent representations of the quiver.

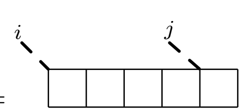
1.1 The partial order

Consider an (infinite) sheet of graph paper which has its diagonals labeled consecutively by $\ldots, -2, -1, 0, 1, 2, \ldots$. The *content* $c(b)$ of a box b on this sheet of graph paper is

$$c(b) = \text{the diagonal number of the box } b$$

A *segment* is a row of boxes on a sheet of graph paper with diagonals indexed $\mathbb{Z}$.

Consider graph paper with diagonals indexed by $\mathbb{Z}$. A *segment* is a sequence of boxes

$[i, j] =$ 

in a row with the leftmost box of content i and the rightmost box of content j. A *multisegment* is a (unordered) collection of segments. For example

$$\begin{array}{ccccccc} \boxed{3} & \boxed{4} & \boxed{5} & \boxed{6} & \boxed{7} & & \\ & \boxed{3} & \boxed{4} & \boxed{5} & \boxed{6} & \boxed{7} & \\ & & & & \boxed{5} & \boxed{6} & \boxed{7} \\ & \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{5} & \\ & & & & \boxed{3} & \boxed{4} & \boxed{5} \end{array} \qquad \text{has} \quad \lambda([3,7]) = 2, \lambda([5,7]) = 1, \lambda([1,5]) = 1, \lambda([3,5]) = 1,$$

and $\lambda([i,j]) = 0$ for all other segments $[i,j]$ (the numbers in the boxes in the picture are the contents of the boxes). Alternatively a multisegment λ can be viewed as a function

$$\lambda \colon \{\textit{segments}\} \longrightarrow \mathbb{Z}_{\geq 0} \qquad \text{where} \qquad \lambda([i,j]) = (\#\text{ of rows } [i,j] \text{ in } \lambda).$$

The set of segments is ordered by inclusion. Define

$$\lambda(\supseteq [i,j]) = \sum_{[r,s] \supseteq [i,j]} \lambda([r,s]). \tag{1.2}$$

Then

$$\lambda([i,j]) = \lambda(\supseteq [i-1,j+1]) - \lambda(\supseteq [i-1,j]) - \lambda(\supseteq [i,j+1]) + \lambda(\supseteq [i,j]).$$

PICTURE

and so the multisegment λ can be specified by the numbers $\lambda(\supseteq [i,j])$. Note that

$$\lambda(\supseteq [i]) = (\#\text{ of boxes in } \lambda \text{ in diagonal } i).$$

Define a partial order on multisegments by

$$\lambda \geq \mu \qquad \text{if} \qquad \lambda(\supseteq [i,j]) \geq \mu(\supseteq [i,j]) \quad \text{for all segments } [i.j].$$

If $[b,c] \subseteq [a,d]$ are segments define a *degeneration*

$$R_{[b,c],[a,d]} \colon \{\text{multisegments}\} \longrightarrow \{\text{multisegments}\}$$

by

$$\begin{aligned} R_{[b,c],[a,d]}\lambda([a,d]) &= \lambda([a,d]) - 1, \\ R_{[b,c],[a,d]}\lambda([b,d]) &= \lambda([b,d]) + 1, \\ R_{[b,c],[a,d]}\lambda([a,c]) &= \lambda([a,c]) + 1, \\ R_{[b,c],[a,d]}\lambda([b,c]) &= \lambda([b,c]) - 1, \\ R_{[b,c],[a,d]}\lambda([i,j]) &= \lambda([i,j]), \qquad \text{if } [i,j] \neq [a,d], [a,c], [b,d], [b,c]. \end{aligned}$$

The degeneration $R_{[b,c],[a,d]}\lambda$ is *elementary* if

$$\lambda([i,j]) = 0 \quad \text{for all } [b,c] \subseteq [i,j] \subseteq [a,d] \text{ except } [i,j] = [b,c], [a,c], [b,d] \text{ or } [a,d].$$

Pictorially a degeneration takes

$$\textit{PICTURE} \longrightarrow \textit{PICTURE}$$

and

$$PICTURE \longrightarrow PICTURE \qquad \text{for } c = b - 1,$$

or, equivalently,

$$PICTURE \longrightarrow PICTURE.$$

Let A_∞ be the quiver (I, Ω^+) with

$$I = \mathbb{Z}, \qquad \Omega^+ = \{i \to i+1 \mid i \in \mathbb{Z}\}.$$

Fix an I-graded vector space

$$V = \bigoplus_{i \in I} V_i,$$

and let

$$E_V = \bigoplus_{i \to i+1} \mathrm{Hom}(V_i, V_{i+1}),$$

$$GL_V = \prod_{i \in \mathbb{Z}} GL(V_i), \qquad \text{which acts on } E_V\text{, and}$$

$$\mathcal{N}_V = \{x \in E_V \mid x \text{ is a nilpotent element of } \mathrm{Hom}(V, V)\}.$$

The map

$$\begin{array}{ccc} \mathcal{N}_V & \longrightarrow & \{\text{multisegments}\} \\ x & \longmapsto & \lambda_x \end{array} \qquad \text{given by}$$

$$\lambda(\supseteq [i]) = \dim(V_i) \qquad \text{and} \qquad \lambda(\supseteq [i,j]) = \mathrm{rank}(\lambda\colon V_i \to \cdots \to V_j).$$

provides a bijection

$$\{\text{multisegments } \lambda \mid \lambda(\supseteq [i]) = \dim(V_i)\} \longleftrightarrow \{GL_V \text{ orbits in } \mathcal{N}_V\}$$

Theorem 1.1. *Let λ and μ be multisegments and let $\mathbb{O}_\lambda$ and $\mathbb{O}_\mu$ be the corresponding orbits in $\mathcal{N}_V/GL_V$. Then the following are equivalent*

(1) $\lambda \geq \mu$,

(2) $\overline{\mathbb{O}_\lambda} \supseteq \mathbb{O}_\mu$,

(3) $\lambda = R_{i_1} \cdots R_{i_r} \mu$ *for some sequence of elementary degenerations* $R_{i_1}, \ldots, R_{i_r}$.

Proof. (1) $\Longrightarrow$ (2):

$$PICTURE + \varepsilon PICTURE \cong PICTURE,$$

and so

$$\mathbb{O}_{PICTURE} \subseteq \overline{\mathbb{O}_{PICTURE}}.$$

(2) $\Longrightarrow$ (3): If $\mathbb{O}_\mu \subseteq \overline{\mathbb{O}_\lambda}$ then

$$\mu(\supseteq [i,j]) = \mathrm{rank}(\mu\colon V_i \to \cdots \to V_j) \leq \mathrm{rank}(\lambda\colon V_i \to \cdots \to V_j) = \lambda(\supseteq [i,j]).$$

(3) $\Longrightarrow$ (1): Assume $\lambda(\supseteq [i,j]) \geq \mu(\supseteq [i,j])$ for all segments $[i,j]$. Find (THIS STILL NEEDS DOING) a sequence $R_{i_1} \cdots R_{i_r}$ of elementary degenerations which takes μ to λ, i.e.

$$R_{i_1} \cdots R_{i_r} \mu = \lambda.$$

□

1.2 Hecke algebra representations

Let $\tilde{H}_k$ be the affine Hecke algebra at an ℓth root of unity so that $q^\ell = 1$ (allow $\ell = \infty$ if desired). For each $b \in \tilde{B}(\infty)$ let

$$b = \sum_j [s_j, n_j) \qquad \text{and define the } \textit{standard module} \qquad M(b) = \mathrm{Ind}_{\tilde{H}_\nu}^{\tilde{H}_k}(\mathbb{C}_s),$$

where $\nu = (n_1, \ldots, n_r)$ and $k = n_1 + \cdots + n_r$. The simple $\tilde{H}_k$-modules are indexed by $b \in B(\infty)$ and are determined by the equations

$$[M(b)] = [(L(b)] + \sum_{\substack{b' > b \\ b' \in B(\infty)}} d_{b'b}[L(b')], \qquad b \in B(\infty), \quad d_{b'b} \in \mathbb{Z}_{\geq 0},$$

in the Grothendieck group of $\tilde{H}_k(q)$-modules.

2 The Fock space representation of $U_v\hat{\mathfrak{sl}}_\ell$

2.1 The crystal graph

Let

$$\lambda = \begin{bmatrix} (\lambda+\rho)_1 & (\lambda+\rho)_2 & \cdots & (\lambda+\rho)_n \\ (\mu+\rho)_1 & (\mu+\rho)_2 & \cdots & (\mu+\rho)_n \end{bmatrix} = \begin{pmatrix} (\lambda+\rho)_1 & (\lambda+\rho)_2 & \cdots & (\lambda+\rho)_n \\ d_1 & d_2 & \cdots & d_n \end{bmatrix}$$

be a multisegment and assume that it is ordered so that

(a) $(\lambda+\rho)_i \geq (\lambda+\rho)_{i+1}$,

(b) $(\mu+\rho)_i \leq (\mu+\rho)_{i+1}$ if $(\lambda+\rho)_i = (\lambda+\rho)_{i+1}$,

These conditions are equivalent to saying that

(a′) The $\mathfrak{gl}(n)$-weight λ is integrally dominant,

(b′) $\mu = w \circ \nu$ where ν is integrally dominant and w is longest in its coset $W_{\lambda+\rho} w W_{\mu+\rho}$.

Place

-1 above each $(\lambda+\rho)_j = i$,

$+1$ above each $(\lambda+\rho)_j = i-1$,

0 above each $(\lambda+\rho)_j \neq i, i+1$.

Then, ignoring 0s, read the sequence of $+1$s, -1s left to right and successively cancel adjacent $(-1, +1)$ pairs to get a sequence of the form

$$\underbrace{+1 \ +1 \ \ldots + \overset{\substack{\text{cogood} \\ \downarrow}}{1}}_{\text{conormal nodes}} \quad \underbrace{\overset{\substack{\text{good} \\ \downarrow}}{-1} \ -1 \ \ldots -1}_{\text{normal nodes}}$$

The -1s in this sequence are the *normal nodes* and the $+1$s are the *conormal nodes*. The *good node* is the leftmost normal node and the *cogood node* is the right most conormal node.

Define

$$\mathrm{wt}(\lambda) = \sum_{i \in I} -(\text{number of boxes of content } i \text{ in } \lambda)\, \alpha_i, \qquad \text{and}$$

$$\varepsilon_i(\lambda) = (\text{number of normal nodes}), \qquad \varphi_i(\lambda) = (\text{number of conormal nodes}),$$

$$\tilde{e}_i\lambda = \big(\text{same as } \lambda \text{ but with the good node } (\lambda+\rho)_j = i \text{ changed to } i-1\big),$$
$$\tilde{f}_i\lambda = \big(\text{same as } \lambda \text{ but with the cogood node } (\lambda+\rho)_j = i-1 \text{ changed to } i\big),$$

for each $i \in I$.

Remark. If this algorithm is being executed where $I = \mathbb{Z}/\ell\mathbb{Z}$ then take

$(\lambda+\rho)_j = \ell$, when $i = 0$ and $(\lambda+\rho)_j \equiv 0 \bmod \ell$, and

$(\lambda+\rho)_j = 0$, when $i = 1$ and $(\lambda+\rho)_j \equiv 0 \bmod \ell$.

Theorem 2.1.

(a) In type $A_{\ell-1}^{(1)}$, $B(\infty)$ is the connected component of $\emptyset$ in the crystal graph $\tilde{B}(\infty)$.

(b) $B(\infty)$ is the crystal graph of $U_v^-\mathfrak{g}$.

2.2 The crystals $B(\Lambda)$

Type $A_{\ell-1}$: Let

$$\lambda = \sum_{i=1}^{\ell} \lambda_i \epsilon_i = \sum_{i \in I} \gamma_i \omega_i \in P^+,$$

and identify λ with the partition which has λ_i boxes in row i. Let

$$B(\lambda) = \{\text{column strict tableaux of shape } \lambda\}$$

and define an imbedding

$$\begin{array}{ccc} B(\lambda) & \longrightarrow & B(\infty) \\ P & \longmapsto & \begin{bmatrix} 1 & 1 & \cdots & 1 & 2 & 2 & \cdots & 2 & \cdots & n \\ n & \cdots & n & & & & & & & \\ i_1 & i_2 & \cdots & i_{\lambda_1} & i_{\lambda_1+1} & & \cdots & i_{\lambda_1+\lambda_2} & \cdots & \\ & \cdots & i_k & & & & & & & \end{bmatrix} \end{array}$$

where the entries $i_1 i_2 \cdots i_k$ are the entries of P read in Arabic reading order.

2.3 The tensor product representation

The ℓ-dimensional simple $U_q\mathfrak{sl}_\ell$-module of highest weight ω_1 is given by

$$L(\omega_1) = \mathbb{C}\text{-span}\{v_0, \ldots, v_{\ell-1}\}$$

with $U_q\mathfrak{sl}_\ell$-action

$$e_i v_j = \begin{cases} v_{i-1}, & \text{if } j = i, \\ 0, & \text{if } j \neq i, \end{cases} \qquad f_i v_j = \begin{cases} v_i, & \text{if } j = i-1, \\ 0, & \text{if } j \neq i, \end{cases} \qquad k_i v_j = \begin{cases} q v_{i-1}, & \text{if } j = i, \\ q^{-1} v_i, & \text{if } j = i-1, \\ v_j, & \text{if } j \neq i, i-1. \end{cases}$$

Then

$$L(\omega_1)^{\otimes k} = \mathbb{C}\text{-span}\{v_{j_1} \otimes \cdots \otimes v_{j_k} \mid 1 \le j_1, j_2, \ldots, j_k \le \ell\}.$$

If $v = v_{j_1} \otimes \cdots \otimes v_{j_k}$ place

$$\begin{array}{l} +1 \text{ over each } v_{i-1} \text{ in } v, \\ -1 \text{ over each } v_i \text{ in } v, \\ 0 \text{ over each } v_j,\ j \ne i, i-1. \end{array}$$

Then the $U_q\mathfrak{sl}_\ell$-action on $L(\omega_1)^{\otimes k}$ is given by

$$e_i(v) = \textstyle\sum_{v^-} q^{-(\text{sum of } \pm 1\text{s before } v/v^-)} v^-, \qquad f_i(v) = \textstyle\sum_{v^+} q^{(\text{sum of } \pm 1\text{s after } v^+/v)} v^+,$$
$$k_i(v) = q^{(\text{sum of } \pm 1\text{s for } v)} v,$$

where the first sum is over all v^- which are obtained from v by changing a v_i to v_{i-1} and the second sum is over all v^+ which are obtained from v by changing a v_{i-1} to v_i.

2.4 The Fock space

Let $\mu \in \mathfrak{h}^*$ for $\mathfrak{gl}_n$. Define

$$\mathcal{F}_\mu = \mathbb{C}\text{-span}\{\text{multisegments } \lambda = \lambda/\mu\}.$$

Define an action of $U_v\widehat{\mathfrak{sl}}_\ell$ on $\mathcal{F}_\mu$ by

$$e_i\lambda = \sum_{c(\lambda/\lambda^-)\equiv i} q^{(\text{sum of } \pm 1\text{s before } \lambda/\lambda^-)} \lambda^-, \qquad f_i\lambda = \sum_{c(\lambda^+/\lambda)\equiv i} q^{(\text{sum of } \pm 1\text{s after } \lambda^+/\lambda)} \lambda^+,$$

$$k_i\lambda = q^{(\text{sum of the } \pm 1 \text{ sequence for } \lambda)} \lambda, \qquad D\lambda = q^{(\#\text{ofboxesofcontent0in } \lambda)} \lambda,$$

Theorem 2.2.

(a) These formulas make $\mathcal{F}_\mu$ into a $U_v\widehat{\mathfrak{sl}}_\ell$-module

(b) If $L_\mu = \mathbb{Z}[q, q^{-1}]$-span$\{$multisegments $\lambda = \lambda/\mu\}$ so that the multisegments form a $\mathbb{Z}[q, q^{-1}]$ basis of L_μ then

$$\tilde{e}_i[\lambda] = [\tilde{e}_i\lambda] \bmod qL_\mu \qquad \textit{and} \qquad \tilde{f}_i[\lambda] = [\tilde{f}_i\lambda] \bmod qL_\mu.$$

Proof. The permutations of the sequence $+1\ +1, \ldots, +1, -1, -1, \ldots, -1$ are indexed by the elements of $S_t/S_k \times S_{t-k}$ where t is the number of nodes after $(-1, +1)$ pairing. The group $(\mathbb{Z}/2\mathbb{Z})^p$ acts on the $(-1, +1)$ pairs by changing a pair $(-1, +1)$ to $(+1, -1)$. For each $1 \le k \le r$ define

$$u_k = \sum_{\sigma \in S_t/S_k \times S_{t-k}} \sum_{\tau \in (\mathbb{Z}/2\mathbb{Z})^t} q^{\ell(\sigma)} (-1)^{\ell(\tau)} (\sigma\tau\lambda[k]).$$

Then

$$u_k = \lambda[k] \bmod qL_\mu \qquad \text{and} \qquad e_i u_k = [k] u_{k-1}.$$

The first statement is clear. To obtain the second statement

$$\begin{aligned}
e_i u_k &= \sum_{\sigma\in S_t/S_k\times S_{t-k}} \sum_{\tau\in(\mathbb{Z}/2\mathbb{Z})^t} q^{\ell(\sigma)}(-1)^{\ell(\tau)}(e_i\sigma\tau\lambda[k]) \\
&= \sum_{e_i \text{ changes a pair}} \sum_{\sigma\in S_t/S_k\times S_{t-k}} \sum_{\tau\in(\mathbb{Z}/2\mathbb{Z})^t} q^{\ell(\sigma)}(-1)^{\ell(\tau)}(\sigma\tau\lambda[k])^- \\
&\qquad + \sum_{e_i \text{ changes a node}} \sum_{\sigma\in S_t/S_k\times S_{t-k}} \sum_{\tau\in(\mathbb{Z}/2\mathbb{Z})^t} q^{\ell(\sigma)}(-1)^{\ell(\tau)}(\sigma\tau\lambda[k])^- \\
&= 0 + \sum_{\tau\in(\mathbb{Z}/2\mathbb{Z})^t} \sum_{\sigma\in S_t/S_k\times S_{t-k}} \sum_{e_i \text{ changes a node}} q^{\ell(\sigma)}(-1)^{\ell(\tau)}(\sigma\tau\lambda[k])^- \\
&= \sum_{\tau\in(\mathbb{Z}/2\mathbb{Z})^t} [k] \left(\sum_{\sigma\in S_t/S_{k-1}\times S_{t-k+1}} q^{\ell(\sigma)}(-1)^{\ell(\tau)}(\sigma\tau\lambda[k])^- \right)
\end{aligned}$$

□

3 A Schur-Weyl duality connection to affine Hecke algebras

A *multisegment* is a collection of rows of boxes (segments) placed on graph paper. We can label this multisegment by a pair of weights $\lambda = \lambda_1\varepsilon_1 + \cdots \lambda_{n+1}\varepsilon_{n+1}$ and $\mu = \mu_1\varepsilon_1 + \cdots + \mu_{n+1}\varepsilon_{n+1}$ by setting

$$\begin{aligned}
(\lambda+\rho)_i &= \text{content of the last box in row } i, \qquad \text{and} \\
(\mu+\rho)_i &= (\text{content of the first box in row } i) - 1.
\end{aligned}$$

For example

$$\begin{array}{ccccccc}
\boxed{3}&\boxed{4}&\boxed{5}&\boxed{6}&\boxed{7}& & \\
 &\boxed{3}&\boxed{4}&\boxed{5}&\boxed{6}&\boxed{7}& \\
 & & & &\boxed{5}&\boxed{6}&\boxed{7} \\
 &\boxed{1}&\boxed{2}&\boxed{3}&\boxed{4}&\boxed{5}& \\
 & & & &\boxed{3}&\boxed{4}&\boxed{5}
\end{array}
\qquad \text{corresponds to} \qquad
\begin{array}{rl}
\lambda+\rho &= (7,\ 7,\ 7,\ 5,\ 5) \quad \text{and} \\
\mu+\rho &= (2,\ 2,\ 4,\ 0,\ 2)
\end{array}
\tag{3.1}$$

(the numbers in the boxes in the picture are the contents of the boxes). The construction forces the condition

(a) $(\lambda+\rho)_i - (\mu+\rho)_i \in \mathbb{Z}_{\geq 0}$.

and since we want to consider *unordered* collections of boxes it is natural to take the following pseudo-lexicographic ordering on the segments

(b) $(\lambda+\rho)_i \geq (\lambda+\rho)_{i+1}$,

(c) $(\mu+\rho)_i \leq (\mu+\rho)_{i+1}$ if $(\lambda+\rho)_i = (\lambda+\rho)_{i+1}$,

when we denote the multisegment λ/μ by a pair of weights λ, μ. In terms of weights the conditions (a), (b) and (c) can be restated as (note that in this case both λ and μ are integral)

(a′) $\lambda - \mu$ is a weight of $V^{\otimes k}$, where k is the number of boxes in λ/μ, (b′) λ is integrally dominant,

(c′) $\mu = w \circ \nu$ with ν integrally dominant and w maximal length in the coset $W_{\lambda+\rho} w W_{\nu+\rho}$,

Let λ/μ be a multisegment with k boxes and number the boxes of λ/μ from left to right (like a book). Define

$$\tilde{H}_{\lambda/\mu} = \text{ subalgebra of } \tilde{H}_k \text{ generated by } \{X^\lambda,\ T_j \mid \lambda \in L, \text{box}_j \text{ is not at the end of its row}\},$$

so that $\tilde{H}_{\lambda/\mu}$ is the "parabolic" subalgebra of $\tilde{H}_k$ corresponding to the multisegment λ/μ. Define a one-dimensional $\tilde{H}_{\lambda/\mu}$ module $\mathbb{C}_{\lambda/\mu} = \mathbb{C}v_{\lambda/\mu}$ by setting

$$X^{\varepsilon_i} v_{\lambda/\mu} = q^{2c(\text{box}_i)} v_{\lambda/\mu}, \qquad \text{and} \qquad T_j v_{\lambda/\mu} = q v_{\lambda/\mu}, \tag{3.2}$$

for $1 \le i \le k$ and j such that box_j is not at the end of its row.

Let $\mathfrak{g}$ be of type A_n and let F_λ be the functor $\text{Hom}_{U_h\mathfrak{g}}(M(\lambda), \cdot \otimes V^{\otimes k})$ where $V = L(\omega_1)$. The *standard module* for the affine Hecke algebra $\tilde{H}_k$ is

$$\mathcal{M}^{\lambda/\mu} = F_\lambda(M(\mu)) \tag{3.3}$$

as defined in (4.1). It follows from the above discussion that these modules are naturally indexed by multisegments λ/μ. The following proposition shows that this standard module coincides with the usual standard module for the affine Hecke algebra as considered by Zelevinsky [Ze2] (see also [Ar], [CG] and [KL]).

Proposition 3.1. *Let λ/μ be a multisegment determined by a pair weights (λ, μ) with λ integrally dominant. Let $\mathbb{C}_{\lambda/\mu}$ be the one dimensional representation of the parabolic subalgebra $\tilde{H}_{\lambda/\mu}$ of the affine Hecke algebra $\tilde{H}_k$ defined in (???). Then*

$$\mathcal{M}^{\lambda/\mu} \cong \text{Ind}_{\tilde{H}_{\lambda/\mu}}^{\tilde{H}_k}(\mathbb{C}_{\lambda/\mu}).$$

Proof. To remove the constants that come from the difference between $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$ the affine braid group action in Theorem 6.17a should be normalized so that $\Phi_k(X^{\varepsilon_1}) = q^{2|\mu|/(n+1)} \check{R}_0^2$ and $\Phi_k(T_i) = q^{1/(n+1)} \check{R}_i$.

By Proposition 4.3a, $cM^{\lambda/\mu} \cong (V^{\otimes k})_{\lambda-\mu}$ as a vector space. Let $\{v_1, v_2, \ldots, v_{n+1}\}$ be the standard basis of $V = L(\omega_1)$ with $\text{wt}(v_i) = \varepsilon_i$. If we let the symmetric group S_k act on $V^{\otimes k}$ by permuting the tensor factors then

$$(V^{\otimes k})_{\lambda-\mu} = \text{span-}\{\pi \cdot v^{\otimes(\lambda-\mu)} \mid \pi \in S_k\} = \text{span-}\{\pi \cdot v^{\otimes(\lambda-\mu)} \mid \pi \in S_k/S_{\lambda-\mu}\}, \qquad \text{where}$$

$$v^{\otimes(\lambda-\mu)} = \underbrace{v_1 \otimes \cdots \otimes v_1}_{\lambda_1-\mu_1} \otimes \cdots \otimes \underbrace{v_n \otimes \cdots \otimes v_n}_{\lambda_n-\mu_n} \qquad \text{and} \qquad S_{\lambda-\mu} = S_{\lambda_1-\mu_1} \times \cdots \times S_{\lambda_n-\mu_n}$$

is the parabolic subgroup of S_k which stabilizes the vector $v^{\otimes(\lambda-\mu)} \in V^{\otimes k}$. This shows that, as vector spaces,

$$\mathcal{M}^{\lambda/\mu} \cong \text{Ind}_{\tilde{H}_{\lambda/\mu}}^{\tilde{H}_k}(\mathbb{C}_{\lambda/\mu}) = \text{span-}\{T_\pi \otimes v_{\lambda/\mu} \mid \pi \in S_k/S_{\lambda-\mu}\} \tag{3.4}$$

are isomorphic.

For notational purposes let

$$b_{\lambda/\mu} = v_\mu^+ \otimes v^{\otimes(\lambda-\mu)} = v_\mu^+ \otimes v_{i_1} \otimes \cdots \otimes v_{i_k}$$

and let $\bar{b}_{\lambda/\mu}$ be the image of $b_{\lambda/\mu}$ in $(M \otimes V^{\otimes k})^{[\lambda]}$. Since λ is integrally dominant and $\bar{b}_{\lambda/\mu}$ has weight λ it must be a highest weight vector. We will show that X^{ε_ℓ} acts on $\bar{b}_{\lambda/\mu}$ by the constant $q^{c(\mathrm{box}_\ell)}$, where $c(\mathrm{box}_\ell)$ is the content of the ℓth box of the multisegment λ/μ (read left to right and top to bottom like a book).

Consider the projections

$$pr_\ell \colon M(\mu) \otimes V^{\otimes k} \to (M(\mu) \otimes V^{\otimes \ell})^{[\lambda^{(\ell)}]} \otimes V^{\otimes (k-\ell)} \qquad \text{where} \qquad \lambda^{(\ell)} = \mu + \sum_{j \le \ell} \mathrm{wt}(v_{i_\ell})$$

and pr_i acts as the identity on the last $k-i$ factors of $M(\mu) \otimes V^{\otimes k}$. Then

$$\bar{b}_{\lambda/\mu} = pr_k pr_{k-1} \dots pr_1 b_{\lambda/\mu},$$

and for each $1 \le \ell \le k$, (the first ℓ components of) $pr_{\ell-1} \cdots pr_1(b_{\lambda/\mu})$ form a highest weight vector of weight $\lambda^{(\ell)}$ in $M \otimes V^{\otimes \ell}$. It is the "highest" highest weight vector of

$$((M(\mu) \otimes V^{\otimes(\ell-1)})^{[\lambda^{(\ell-1)}]} \otimes V)^{[\lambda^{(\ell)}]} \tag{3.5}$$

with respect to the ordering in Lemma 4.2 and thus it is deepest in the filtration constructed there. Note that the quantum Casimir element acts on the space in (6.29) as the constant $q^{\langle \lambda^{(\ell)}, \lambda^{(\ell)}+2\rho\rangle}$ times a unipotent transformation, and the unipotent transformation must preserve the filtration coming from Lemma 4.2. Since $pr_\ell(b_{\lambda/\mu})$ is the highest weight vector of the smallest submodule of this filtration (which is isomorphic to a Verma module by Lemma 4.2b) it is an eigenvector for the action of the quantum Casimir. Thus, by (2.11) and (2.13), X^{ε_ℓ} acts on $pr_\ell(b_{\lambda/\mu})$ by the constant

$$q^{\langle \lambda^{(\ell)}, \lambda^{(\ell)}+2\rho\rangle - \langle \lambda^{(\ell-1)}, \lambda^{(\ell-1)}+2\rho\rangle - \langle \omega_1, \omega_1+2\rho\rangle} = q^{2c(\mathrm{box}_\ell)}.$$

(see [LR] Since X^{ε_ℓ} commutes with pr_j for $j > \ell$ it this also specifies the action of X^{ε_ℓ} on $\bar{b}_{\lambda/\mu} = pr_\ell(b_{\lambda/\mu})$.

The explicit R-matrix $\check{R}_{VV} \colon V \otimes V \to V \otimes V$ for this case ($\mathfrak{g}$ of type A and $V = L(\omega_1)$) is well known (see, for example, the proof of [LR, Prop. 4.4]) and given by

$$(v_i \otimes v_j) q^{1/(n+1)} \check{R}_{VV} = \begin{cases} v_j \otimes v_i, & if \text{i¿j}, \\ (q-q^{-1}) v_i \otimes v_j + v_j \otimes v_i, & if \text{i¡j}, \\ q v_i \otimes v_j, & if \text{i=j}. \end{cases}$$

Since T_i acts by $\check{R}_{VV}$ on the ith and $(i+1)$st tensor factors of $V^{\otimes k}$ and commutes with the projection pr_λ it follows that $T_j(\bar{b}_{\lambda/\mu}) = q\, \bar{b}_{\lambda/\mu}$, if box_j is not a box at the end of a row of λ/μ. This analysis of the action of $\tilde{H}_{\lambda/\mu}$ on $\bar{b}_{\lambda/\mu}$ shows that there is an $\tilde{H}_k$-homomorphism

$$\begin{array}{ccc} \mathrm{Ind}_{\tilde{H}_{\lambda/\mu}}^{\tilde{H}_k}(\mathbb{C} v_{\lambda/\mu}) & \longrightarrow & \mathcal{M}^{\lambda/\mu} \\ v_{\lambda/\mu} & \longmapsto & \bar{b}_{\lambda/\mu}. \end{array}$$

This map is surjective since $\mathcal{M}^{\lambda/\mu}$ is generated by $\bar{b}_{\lambda/\mu}$ (the $\mathcal{B}_k$ action on $v^{\lambda-\mu}$ generates all of $(V^{\otimes k})_{\lambda-\mu}$). Finally, (6.28) guarantees that it is an isomorphism. $\square$

In the same way that each weight $\mu \in \mathfrak{h}^*$ has a *normal form*

$$\mu = w \circ \tilde{\mu}, \qquad \text{with} \qquad \begin{array}{l} \tilde{\mu} \text{ integrally dominant, and} \\ w \text{ maximal length in the coset } wW_{\tilde{\mu}+\rho}, \end{array}$$

every multisegment λ/μ has a *normal form*

$$\lambda/\mu = \nu/(w \circ \tilde{\nu}), \qquad \text{with} \qquad \begin{array}{l} \nu+\rho \text{ the sequence of contents of boxes of } \lambda/\mu, \\ \tilde{\nu} = \nu - (1,1,\ldots,1), \quad \text{and} \\ w \text{ maximal length in } W_{\nu+\rho} w W_{\nu+\rho}. \end{array}$$

The element w in the normal form $\nu/(w \circ \tilde{\nu})$ of λ/μ can be constructed combinatorially by the following scheme. We number (order) the boxes of λ/μ in two different ways.

First ordering: To each box b of λ/μ associate the following triple

$$\big(\text{content of the box to the left of } b, -(\text{content of } b), -(\text{row number of } b)\big)$$

where, if a box is the leftmost box in a row "the box to its left" is the rightmost box in the same row. The lexicographic ordering on these triples induces an ordering on the boxes of λ/μ.

Second ordering: To each box b of λ/μ associate the following pair

$$\big(\text{content of } b, -(\text{the number of box } b \text{ in the first ordering})\big)$$

The lexicographic ordering of these pairs induces a second ordering on the boxes of λ/μ.

If v is the permutation defined by these two numberings of the boxes then $w = w_0 v w_0$. For example, for the multisegment λ/μ displayed in (6.24) the numberings of the boxes are given by

$$\begin{array}{|c|c|c|c|c|cc} \hline 21 & 6 & 10 & 13 & 18 & & \\ \hline & 20 & 5 & 9 & 12 & 17 & \\ & & & & 19 & 11 & 16 \\ & 15 & 1 & 2 & 4 & 8 & \\ & & & & 14 & 3 & 7 \\ \end{array} \qquad \text{and} \qquad \begin{array}{ccccccc} 3 & 7 & 12 & 16 & 19 & & \\ & 4 & 8 & 13 & 17 & 20 & \\ & & & & 11 & 18 & 21 \\ & 1 & 2 & 6 & 9 & 14 & \\ & & & & 5 & 10 & 15 \end{array}$$

first ordering of boxes $\qquad\qquad$ second ordering of boxes

and the normal form of λ/μ is

$$\begin{aligned} \nu &= (7,7,7,6,6,6,5,5,5,5,5,4,4,4,4,3,3,3,3,2,1), \\ \tilde{\nu} &= (6,6,6,5,5,5,4,4,4,4,4,3,3,3,3,2,2,2,2,1,0), \ \text{ and } w = w_0 v w_0 \text{ where} \end{aligned}$$

$$v = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\ 21 \\ 15 & 1 & 21 & 20 & 14 & 2 & 6 & 5 & 4 & 3 \\ 19 & 10 & 9 & 8 & 7 & 13 & 12 & 11 & 18 & 17 \\ 16 \end{pmatrix}$$

Let $\mathfrak{g}$ be of type A_n and $V = L(\omega_1)$ and let

$$\mathcal{L}^{\lambda/\mu} = F_\lambda(L(\mu)), \tag{3.6}$$

as defined in (4.1). It is known (a consequence of Proposition 6.27 and Proposition 4.3c) that $\mathcal{L}^{\lambda/\mu}$ is always a simple $\tilde{H}_k$-module or 0. Furthermore, all simple $\tilde{H}_k$ modules are obtained by this construction. See [Su] for proofs of these statements. The following theorem is a reformulation of Proposition 4.12 in terms of the combinatorics of our present setting.

Theorem 3.2. *Let λ/μ and ρ/τ be multisegments with k boxes (with μ and τ assumed to be integral) and let*

$$\lambda/\mu = \nu/(w \circ \tilde{\nu}) \qquad \text{and} \qquad \rho/\tau = \gamma/(v \circ \tilde{\gamma})$$

be their normal forms. Then the multiplicities of $\mathcal{L}^{\rho/\tau}$ in a Jantzen filtration of $\mathcal{M}^{\lambda/\mu}$ are given by

$$\sum_{j\geq 0}\left[\frac{(\mathcal{M}^{\lambda/\mu})^{(j)}}{(\mathcal{M}^{\lambda/\mu})^{(j+1)}} : \mathcal{L}^{\rho/\tau}\right] \mathtt{v}^{\frac{1}{2}(\ell(y)-\ell(w)+j)} = \begin{cases} P_{wv}(\mathtt{v}), & \text{if } \nu = \gamma, \\ 0, & \text{if } \nu \neq \gamma, \end{cases}$$

where $P_{wv}(\mathtt{v})$ is the Kazhdan-Lusztig polynomial for the symmetric group S_k.

Theorem 6.31 says that every decomposition number for affine Hecke algebra representations is a Kazhdan-Lusztig polynomial. The following is a converse statement which says that every Kazhdan-Lusztig polynomial for the symmetric group is a decomposition number for affine Hecke algebra representations. This statement is interesting in that Polo [Po] has shown that *every* polynomial in $1 + \mathtt{v}\mathbb{Z}_{\geq 0}[\mathtt{v}]$ is a Kazhdan-Lusztig polynomial for some choice of n and permutations $v, w \in S_n$. Thus, the following proposition also shows that every polynomial arises as a generalized decomposition number for an appropriate pair of affine Hecke algebra modules.

Proposition 3.3. *Let $\lambda = (r, r, \ldots, r) = (r^r)$ and $\mu = (0, 0, \ldots, 0) = (0^r)$. Then, for each pair of permutations $v, w \in S_r$, the Kazhdan-Lusztig polynomial $P_{vw}(\mathtt{v})$ for the symmetric group S_r is equal to*

$$P_{vw}(\mathtt{v}) = \sum_{j\geq 0}\left[\frac{(\mathcal{M}^{\lambda/w\circ\mu})^{(j)}}{(\mathcal{M}^{\lambda/w\circ\mu})^{(j+1)}} : \mathcal{L}^{\lambda/v\circ\mu}\right] \mathtt{v}^{\frac{1}{2}(\ell(y)-\ell(w)+j)}.$$

Proof. Since $\mu + \rho$ and $\lambda + \rho$ are both regular, $W_{\lambda+\rho} = W_{\mu+\rho} = 1$ and the standard and irreducible modules $\mathcal{L}^{\lambda/(w\circ\mu)}$ and $\mathcal{M}^{\lambda/(v\circ\mu)}$ ranging over all $v, w \in S_k$. Thus, this statement is a corollary of Proposition 4.12. □

References

[GL] S. Gaussent, P. Littelmann, *LS galleries, the path model, and MV cycles*, Duke Math. J. **127** (2005), no. 1, 35–88.

[GR] S. Griffeth and A. Ram, *Affine Hecke algebras and the Schubert calculus*, European J. Combin. **25** (2004), no. 8, 1263–1283.

[Ka] J. Kamnitzer, *Mirkovic-Vilonen cycles and polytopes*, math.AG/050136.5

[KM] M. Kapovich and J.J. Millson, *A path model for geodesics in Euclidean buildings and its applications to representation theory*, math.RT/0411182.

[LP] C. Lenart and A. Postnikov, *Affine Weyl groups in K-theory and representation theory*, math.RT/0309207.

[PR] H. Pittie and A. Ram, *A Pieri-Chevalley formula in the K-theory of a G/B-bundle*, Electronic Research Announcements **5** (1999), 102–107, and *A Pieri-Chevalley formula for K(G/B)*, math.RT/0401332.

[1] A. Ram, *Affine Hecke algebras and generalized standard Young tableaux*, J. Algebra, **230** (2003), 367–415.

[2] A. Ram, *Skew shape representations are irreducible*, in *Combinatorial and Geometric representation theory*, S.-J. Kang and K.-H. Lee eds., Contemp. Math. **325** Amer. Math. Soc. 2003, 161–189, math.RT/0401326.

[3] RR A. Ram and J. Ramagge, *Affine Hecke algebras, cyclotomic Hecke algebras and Clifford theory*, in *A tribute to C.S. Seshadri: Perspectives in Geometry and Representation theory*, V. Lakshimibai et al eds., Hindustan Book Agency, New Delhi (2003), 428–466, math.RT/0401322.

[Sc] C. Schwer, Ph.D. Thesis, University of Wuppertal, 2006.