Some homological algebra

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1 Restriction and Induction

An *exact sequence*

$$\cdots \to M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+i} \to \cdots$$

is a sequence of left R-modules and R-module homomorphisms $f_i \colon M_i \to M_{i+1}$, such that

$$\ker f_i = \mathrm{im} f_{i-1} \ \text{for all} \ \ i.$$

A *short exact sequence* is an exact sequence of the form

$$(0) \to N \xrightarrow{g} M \xrightarrow{f} P \to (0).$$

A *split short exact sequence* is a short exact sequence

$$(0) \to N \xrightarrow{g} M \xrightarrow{f} P \to (0)$$

if there are are submodules N' and P' of M such that $N' = \mathrm{im} g$ and $M = N' \oplus P'$.

1.1 Adjoint functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $f_* \colon \mathcal{C} \to \mathcal{D}$ and $f^* \colon \mathcal{D} \to \mathcal{C}$ be functors. Then f_* is a *right adjoint* to f^* and f^* is a *left adjoint* to f_* if, for each $D \in \mathcal{D}$ and $C \in \mathcal{C}$ there is a natural vector space isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(f^* D, C) \xrightarrow{\Phi} \mathrm{Hom}_{\mathcal{D}}(D, f_* C).$$

For $C \in \mathcal{C}$ and $D \in \mathcal{D}$ define

$$\tau_C = \Phi^{-1}(\mathrm{id}_{f_* C}) \in \mathrm{Hom}_{\mathcal{C}}(f^* f_* C, C) \qquad \text{and} \qquad \varphi_D = \Phi(\mathrm{id}_{f^* D}) \in \mathrm{Hom}_{\mathcal{D}}(D, f_* f^* D).$$

In general τ_C and ϕ_D are neither injective nor surjective, see the examples below in ??? and ???.

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1.2 The functors $\mathrm{Hom}_B(M,\cdot)$ and $M\otimes_Z\cdot$

Let B and Z be algebras and let M be a left B-module and a right Z-module. If N is a left Z -module then $\mathrm{Hom}_B(M,N)$ is a left Z-module with Z-action given by

$$(z\phi)(m)=\phi(mz),\qquad \text{for } \phi\in \mathrm{Hom}_A(M,N),\ m\in M, z\in Z.$$

If P is a left Z-module then $M\otimes_Z P$ is the left B-module which as a $\mathbb{Z}$-module is given by generators $m\otimes p$, $\quad m\in M$, $p\in P$, and relations

$$\begin{aligned}
(m_1+m_2)\otimes p &= m_1\otimes p+m_2\otimes p, && \text{for } m_1,m_2\in M,\ p\in P,\\
m\otimes(p_1+p_2) &= m\otimes p_1+m\otimes p_2, && \text{for } m\in M,\ p_1,p_2\in P,\\
rm\otimes p &= m\otimes rp=r(m\otimes p), && \text{for } r\in\mathbb{Z},\ m\in M,\ p\in P,
\end{aligned}$$

and which has B-action given by

$$b(m\otimes p)=bm\otimes p,\qquad \text{for } m\in M,\ p\in P, \text{ and } b\in B.$$

The covariant functors

$$\begin{array}{rccc}
\mathrm{Hom}_B(M,\cdot)\colon & \{\text{left } B\text{-modules}\} & \longrightarrow & \{\text{left } Z\text{-modules}\}\\
M\otimes_Z\cdot\colon & \{\text{left } Z\text{-modules}\} & \longrightarrow & \{\text{left } B\text{-modules}\}
\end{array}$$

are adjoint since the map

$$\mathrm{Hom}_Z(P,\mathrm{Hom}_B(M,N)) \xrightarrow{\Phi} \mathrm{Hom}_B(M\otimes_Z P,N)$$

$$\begin{array}{rcccc}
\psi\colon & P & \to & \mathrm{Hom}_B(M,N) \\
& p & \mapsto & \psi_p\colon \begin{array}{ccc} M & \to & N\\ m & \mapsto & \psi_p(m)\end{array}
\end{array}
\longmapsto
\begin{array}{rccc}
\Phi(\psi)\colon & M\otimes_Z P & \to & N\\
& m\otimes p & \mapsto & \phi_p(m)
\end{array}$$

$$\begin{array}{rcccc}
\Phi^{-1}(\phi)\colon & P & \to & \mathrm{Hom}_B(M,N) \\
& p & \mapsto & \phi_p\colon \begin{array}{ccc} M & \to & N\\ m & \mapsto & \phi(m\otimes p)\end{array}
\end{array}
\longmapsto
\begin{array}{rccc}
\phi\colon & M\otimes_Z P & \to & N\\
& m\otimes p & \mapsto & \phi(m\otimes p)
\end{array}$$

is a $\mathbb{Z}$-module isomorphism.

The functor $\mathrm{Hom}_B(M,-)$ is very different from the functor $\mathrm{Hom}_B(-,M)$. There is always a canonical isomorphism

$$\begin{array}{ccc}
\mathrm{Hom}_B(B,M) & \longrightarrow & M\\
\phi & \longmapsto & \phi(1),
\end{array}
\qquad \text{but} \qquad \text{the } \textit{dual module} \text{ to } M,\quad \mathrm{Hom}_B(M,B),$$

can, in general, be much larger than M (take $B=\mathbb{C}$ and M an infinite dimensional vector space over $\mathbb{C}$).

The functor $\mathrm{Hom}_B(M,-)$ is left exact and the functor $M\otimes_Z -$ is right exact, i.e. if $0\to P'\to P\to P''$ is exact then

$$0\to\mathrm{Hom}_B(M,P')\to\mathrm{Hom}_B(M,P)\to\mathrm{Hom}_B(M,P'') \text{ is exact,}$$

and if $N'\to N\to N''\to 0$ is exact then

$$M\otimes_Z N'\to M\otimes_Z N\to M\otimes N''\to 0 \text{ is exact.}$$

A left A-module M is *projective* if the functor $\mathrm{Hom}_A(M,\cdot)$ is exact and a right Z-module M is *flat* if the functor $M\otimes_Z\cdot$ is exact.

Proposition 1.1. *(a) M is projective if and only if for any surjective $\beta\colon N' \to N \to 0$ and homomorphism $\alpha\colon P \to M$ there exists a map $\gamma\colon P \to N'$ such that $\beta \circ \gamma = \alpha$.*

(b) M is projective as a B-module if and only if there is a B-module M' such that $M \oplus M' = B^{\oplus I}$, i.e. $M \oplus M'$ is a free B-module.

(c) M is projective if and only if 1 is in the image of the centralizer map so that there exist $b_i \in M$ and $b_i^* \in M^*$ such that

$$\text{if } m \in M \quad \text{then} \quad m = \sum_i b_i^*(m) b_i.$$

Proof. $\Longrightarrow$: Let I be a set of generators of M and let $\phi\colon B^{\oplus I} \to M$ be the canonical map. Let $K = \ker \phi$. The exact sequence

$$0 \longrightarrow K \longrightarrow B^{\oplus I} \xrightarrow{\phi} M \longrightarrow 0 \tag{1.1}$$

gives an exact sequence

$$0 \longrightarrow \mathrm{Hom}_B(M, K) \longrightarrow \mathrm{Hom}_B(M, B^{\oplus I}) \xrightarrow{\phi^*} \mathrm{Hom}_B(M, M) \longrightarrow 0.$$

Since ϕ^* is surjective there is a map $\psi\colon M \to B^{\oplus I}$ such that $\phi^*(\psi) = \mathrm{id}_M$. So $\phi \circ \psi = \mathrm{id}_M$. So the first exact sequence splits,

$$0 \longrightarrow K \longrightarrow B^{\oplus I} \underset{\phi}{\overset{\psi}{\leftrightarrows}} M \longrightarrow 0.$$

So $B^{\oplus I} = \mathrm{im}\psi \oplus K \cong M \oplus K$.

$\Longleftarrow$: If $M \oplus M' \cong B^{\oplus I}$ and

$$0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0$$

is an exact sequence of B-modules then

$$0 \longrightarrow \mathrm{Hom}_B(B^{\oplus I}, P') \longrightarrow \mathrm{Hom}_B(B^{\oplus I}, P) \longrightarrow \mathrm{Hom}_B(B^{\oplus I}, P'') \longrightarrow 0$$

is the same as

$$0 \longrightarrow \coprod_{i \in I} \mathrm{Hom}_B(B, P') \longrightarrow \coprod_{i \in I} \mathrm{Hom}_B(B, P) \longrightarrow \coprod_{i \in I} \mathrm{Hom}_B(B, P'') \longrightarrow 0$$

which is the same as

$$0 \longrightarrow \coprod_{i \in I} P' \longrightarrow \coprod_{i \in I} P \longrightarrow \coprod_{i \in I} P'' \longrightarrow 0$$

which is exact since the first sequence is. So

$$0 \longrightarrow \mathrm{Hom}_B(M \oplus M', P') \longrightarrow \mathrm{Hom}_B(M \oplus M', P) \longrightarrow \mathrm{Hom}_B(M \oplus M', P'') \longrightarrow 0$$

which is the same as

$$0 \longrightarrow \begin{matrix} \mathrm{Hom}_B(M, P') \\ \oplus \\ \mathrm{Hom}_B(M', P') \end{matrix} \longrightarrow \begin{matrix} \mathrm{Hom}_B(M, P) \\ \oplus \\ \mathrm{Hom}_B(M', P) \end{matrix} \longrightarrow \begin{matrix} \mathrm{Hom}_B(M, P'') \\ \oplus \\ \mathrm{Hom}_B(M', P'') \end{matrix} \longrightarrow 0$$

is exact. This forces that the sequences

$$0 \longrightarrow \mathrm{Hom}_B(M, P') \longrightarrow \mathrm{Hom}_B(M, P) \longrightarrow \mathrm{Hom}_B(M, P'') \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Hom}_B(M', P') \longrightarrow \mathrm{Hom}_B(M', P) \longrightarrow \mathrm{Hom}_B(M', P'') \longrightarrow 0$$

are *both* exact. So both M and M' are projective. $\square$

Let L be a C module and let

$$Z = \mathrm{End}_C(L)$$

so that L is a (C, Z) bimodule. The *dual module* to L is the (Z, C) bimodule

$$L^* = \mathrm{Hom}_C(L, C).$$

The *evaluation map* is the (C, C) bimodule homomorphism

$$\begin{array}{rccc} \mathrm{ev}\colon & L \otimes_Z L^* & \longrightarrow & C \\ & \ell \otimes \lambda & \longmapsto & \lambda(\ell) \end{array}$$

and the *centralizer map* is the (Z, Z) bimodule homomorphism

$$\begin{array}{rccc} \xi\colon & L^* \otimes L & \longrightarrow & Z \\ & \lambda \otimes \ell & \longmapsto & \begin{array}{rccc} z_{\lambda,\ell}\colon & L & \to & L \\ & m & \mapsto & \lambda(m)\ell \end{array} \end{array}$$

The module L is projective if and only if there exist $b_i \in L$ and $b_i^* \in L^*$ such that

$$\text{if } \ell \in L \quad \text{then} \quad \ell = \sum_i b_i^*(\ell) b_i,$$

so that

$$\xi^{-1}(1) = \sum_i b_i^* \otimes b_i.$$

Define Ext and Tor here.

1.3 Duals and Projectives

Let L be a C-module and let

$$Z = \mathrm{End}_C(L)$$

so that L is a (C, Z) bimodule. The *dual module* to L is the (Z, C) bimodule

$$L^* = \mathrm{Hom}_C(L, C).$$

The *evaluation map* is the (C, C) bimodule homomorphism

$$\begin{array}{rccc} \mathrm{ev}\colon & L \otimes_Z L^* & \longrightarrow & C \\ & \ell \otimes \lambda & \longmapsto & \lambda(\ell) \end{array}$$

and the *centralizer map* is the (Z, Z) bimodule homomorphism

$$\begin{array}{rccc} \xi\colon & L^* \otimes_C L & \longrightarrow & Z \\ & \lambda \otimes \ell & \longmapsto & \begin{array}{rccc} z_{\lambda,\ell}\colon & L & \to & L \\ & m & \mapsto & \lambda(m)\ell \end{array} \end{array}$$

Recall that [Bou, Alg. II §4.2 Cor.]

(a) L is a projective C-module if and only if $1 \in \operatorname{im} \xi$,

(b) If L is a projective C-module then ξ is injective,

(c) If L is a finitely generated projective C-module then ξ is bijective,

(d) If L is a finitely generated free module then

$$\xi^{-1}(z) = \sum_i b_i^* \otimes z(b_i),$$

where $\{b_1, \ldots, b_d\}$ is a basis of L and $\{b_1^*, \ldots, b_d^*\}$ is the dual basis in M^*.

Statement (a) says that L is projective if and only if there exist $b_i \in L$ and $b_i^* \in L^*$ such that

$$\text{if } \ell \in L \quad \text{then} \quad \ell = \sum_i b_i^*(\ell) b_i, \qquad \text{so that} \quad \xi\Big(\sum_i b_i^* \otimes b_i\Big) = 1.$$

2 Homological algebra

2.1 Categories

A *category* is a collection $\mathrm{Ob}\mathcal{C}$ of *objects* and *morphisms* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, for each pair of objects $X, Y \in \mathcal{C}$ with a composition law $\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ such that

(a) composition is associative,

(b) $\mathrm{Hom}(X, X)$ contains an *identity morphism* id_X such that $\mathrm{id}_X \circ \varphi = \varphi$ and $\psi \circ \mathrm{id}_X = \psi$.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ is a map such that

$$F(\phi \circ \psi) = F(\phi) \circ F(\psi) \qquad \text{and} \qquad F(\mathrm{id}_X) = \mathrm{id}_{F(X)}.$$

Let $f\colon \mathcal{C} \to \mathcal{D}$ and $g\colon \mathcal{C} \to \mathcal{D}$ be functors. A *morphism* of functors $f\colon F \to G$ is a family $f_X\colon FX \to GX$ of morphisms in $\mathcal{D}$ such that

$$\begin{array}{ccc} FX & \xrightarrow{f_X} & GX \\ \big\downarrow{\scriptstyle F\varphi} & & \big\downarrow{\scriptstyle G\varphi} \\ GX & \xrightarrow{f_Y} & GY \end{array} \quad \text{is commutative.}$$

Two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there exist morphisms $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ such that $GF \cong \mathrm{id}_{\mathcal{C}}$ and $FG \cong \mathrm{id}_{\mathcal{D}}$.

A *full subcategory* of $\mathcal{D}$ is a subcategory $\mathcal{C}$ such that if $X, Y \in \mathcal{C}$ then $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathcal{D}}(X, Y)$. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *full* if the maps $F\colon \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY)$ are injective, and *faithful* if the maps $F\colon \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY)$ are surjective. *Fully faithful* functors correspond to embeddings of full subcategories.

An *initial object* is an object B such that $\mathrm{Hom}(B, X)$ is only one element. A *final object* is an object C such that $\mathrm{Hom}(X, C)$ is only one element.

A *generator* of a category $\mathcal{C}$ is an object X such that the morphism $\mathrm{Hom}(X, -)$ is a faithful functor from $\mathcal{C}$ to the category of sets. Let A and B be rings, and let A-mod be the category of left A-modules and mod-A the category of right A-modules??? Morita's theorem says that the following are equivalent:

(a) A-mod and B-mod are equivalent.

(b) mod-A and mod-B are equivalent.

(c) There exists a finitely generated projective generator P of mod-A and a ring isomorphism $B \cong \mathrm{End}_A(P)$.

An *additive category* is a category $\mathcal{A}$ such that

(a) The sets $\mathrm{Hom}(X,Y)$ are abelian groups and composition is bilinear,

(b) There exists a 0 object in $\mathcal{A}$ such that $\mathrm{Hom}(0,0)=\{0\}$,

(c) If $X_1, X_2 \in \mathcal{A}$ then there exists

$$X_1 \underset{i_1}{\overset{p_1}{\leftrightarrows}} Y \underset{i_2}{\overset{p_2}{\leftrightarrows}} X_2,$$

such that

$$p_1 i_1 = \mathrm{id}_{X_1}, \quad p_2 i_2 = \mathrm{id}_{X_2}, \quad i_1 p_1 + i_2 p_2 = \mathrm{id}_Y, \quad p_2 i_1 = p_1 i_2 = 0.$$

The third axiom implies that Y is both the direct sum and the direct product of X_1 and X_2, i.e. that the diagrams

$$\begin{tikzcd} Y \arrow[r, "p_1"] \arrow[d, "p_2"] & X_1 \arrow[d] \\ X_2 \arrow[r] & 0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y \arrow[d, "p_2"] & X_1 \arrow[l, "p_1"'] \arrow[d] \\ X_2 & 0 \arrow[l] \end{tikzcd} \quad \text{commute.}$$

Let $\mathcal{A}$ be an additive category and let $\varphi\colon X \to Y$ be a morphism. The *kernel* of φ is an object $K \in \mathcal{A}$ and a morphism k,

$$K \xrightarrow{k} X \xrightarrow{\varphi} Y, \qquad \text{such that} \qquad \phi \circ k = 0,$$

and, for any morphism $k'\colon K' \to X$ such that $\varphi \circ k' = 0$ there exists a unique morphism $h\colon K' \to K$ with $k' = k \circ h$,

PICTURE??

The *cokernel* of φ is a morphism $c\colon Y \to K'$ such that

for any object Z the sequence $0 \to \mathrm{Hom}(K',Z) \to \mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z)$ is exact,

or equivalently,

$c \circ \varphi = 0$ and for any morphism $c_1\colon Y \to K_1'$ such that $c_1 \circ \varphi = 0$ there exists a unique morphism $h\colon K' \to K_1'$ with $c_1 = h \circ c$,

PICTURE

An *additive category* is an abelian category $\mathcal{A}$ such that if $\varphi\colon X \to Y$ is a morphism then there exists a sequence

$$\ker\varphi \xrightarrow{k} X \xrightarrow{i} \ker k = \mathrm{coker} c \xrightarrow{j} Y \longrightarrow \mathrm{coker}\varphi \qquad \text{such that} \quad j \circ i = \varphi.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories. An *additive functor* is a functor $F\colon \mathcal{A} \to \mathcal{B}$ such that the maps $F\colon \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(FX,FY)$ are homomorphisms of abelian groups.

2.2 Complexes

Let $\mathcal{A}$ be an abelian category. The category $\mathrm{Kom}(\mathcal{A})$ is the category of *complexes* C,

$$\cdots \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} \cdots, \qquad \text{such that} \quad d^n \circ d^{n-1} = 0,$$

with morphisms $f^\bullet \colon B \to C$,

$$\begin{array}{ccccccc} \cdots & \xrightarrow{d^{n-1}} & B^n & \xrightarrow{d^n} & B^{n+1} & \xrightarrow{d^{n+1}} & \cdots \\ & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\ \cdots & \xrightarrow{d^{n-1}} & C^n & \xrightarrow{d^n} & C^{n+1} & \xrightarrow{d^{n+1}} & \cdots \end{array}, \qquad \text{such that} \qquad d^n \circ f^n = f^{n+1} \circ d^n.$$

The *shift functor* is $[n] \colon \mathrm{Kom}(\mathcal{A}) \to \mathrm{Kom}(\mathcal{A})$ shifts all complexes by n,

$$(C^\bullet[n])^i = C^{i+n} \qquad \text{and} \qquad (d[n])^i = d^{i+n}.$$

The *cone* of a morphism $f \colon K^\bullet \to L^\bullet$ is the complex $C(f)$ given by

$$C(f) = K[1] \oplus L \qquad \text{with} \qquad d\begin{pmatrix} k^{i+1} \\ \ell^i \end{pmatrix} = \begin{pmatrix} -d_k k^{i+1} \\ f(k^{i+1} + d\ell^i \end{pmatrix}.$$

The *cylinder* of a morphism $f \colon K \to L$ is the complex $\mathrm{Cyl}(f)$ given by

$$\mathrm{Cyl}(f) = K \oplus K[1] \oplus L \qquad \text{with} \qquad d\begin{pmatrix} k^i \\ k^{i+1} \\ \ell^i \end{pmatrix} = \begin{pmatrix} dk^i - k^{i+1} \\ -d_k k^{i+1} \\ f(k^{i+1} + d\ell^i \end{pmatrix}.$$

There is a canonical diagram

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & L & \xrightarrow{\iota_2} & C(f) & \xrightarrow{\pi_1} & K[1] & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \iota_3} & & \downarrow{\scriptstyle =} & & & & \\ 0 & \longrightarrow & K & \xrightarrow{\iota_1} & \mathrm{Cyl}(f) & \xrightarrow{\pi_{23}} & C(f) & \longrightarrow & 0 & & \\ & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \beta} & & & & & & \\ & & K & \xrightarrow{f} & L & & & & & & \end{array}$$

where π, α, and β are defined by

$$\beta\begin{pmatrix} k^i \\ k^{i+1} \\ \ell^i \end{pmatrix} = f(k^i) + \ell^i.$$

2.3 Cohomology

Let $\mathcal{A}$ be an abelian category and let C be a complex. The *cohomology* of C is

$$H^n(C) = \frac{\ker d^n}{\operatorname{im} d^{n-1}}.$$

A complex C is *acylic* at n if $H^n(C) = 0$.
A complex C is *exact* if all $H^n(C) = 0$ for all n.

An *exact functor* is an additive functor $F\colon \mathcal{C} \to \mathcal{D}$ such that

$$\begin{array}{lll} \text{if} & 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 & \text{is an exact sequence} \\ \text{then} & 0 \to FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \to 0 & \text{is an exact sequence.} \end{array}$$

Let $\mathcal{A}$ be an abelian category.

An object X in $\mathcal{A}$ is *projective* if the functor $\mathrm{Hom}_{\mathcal{A}}(X, -)$ is exact.

An object X in $\mathcal{A}$ is *injective* if the functor $\mathrm{Hom}_{\mathcal{A}}(-, X)$ is exact.

Let A be a ring. A left A-module X is *flat* if the functor $- \otimes_A X$ is exact.

If $f^\bullet\colon B \to C$ is a morphism in $\mathrm{Kom}(\mathcal{A})$ define

$$\begin{array}{rccc} H^n(f)\colon & H^n(B) & \longrightarrow & H^n(C) \\ & b & \longmapsto & f^n(\tilde{b}) \bmod \mathrm{im}\, d^{n-1} \end{array}$$

where $\tilde{b} \in \ker d^n$ is a representative of b.

A *quasi-isomorphism* is a morphism $f\colon B \to C$ such that $H^n(f)\colon H^n(B) \to H^n(C)$ is an isomorphism for all n.

The *derived category* of $\mathcal{A}$ is the category $D(\mathcal{A})$ with a functor $Q\colon \mathrm{Kom}(\mathcal{A}) \to D(A)$ such that

(a) If f is a quasi-isomorphism then $Q(f)$ is an isomorphism,

(b) If $\mathcal{D}$ is a category with a morphism $Q'\colon \mathrm{Kom}(\mathcal{A}) \to \mathcal{D}$ such that if f is a quasi-isomorphism then $Q'(f)$ is an isomorphism then there exists a unique functor $G\colon D(\mathcal{A}) \to \mathcal{D}$ such that $Q' = G \circ Q$.

The functor $\mathcal{A} \to D(\mathcal{A})$ that maps X to the complex with all 0s except X in degree 0, is a fully faithful functor with image equal to the full subcategory of $D(\mathcal{A})$ formed by complexes C with $H^i(C) = 0$ if $i \neq 0$.

Let

$$0 \longrightarrow A^\bullet \xrightarrow{\iota} B^\bullet \xrightarrow{p} C^\bullet \longrightarrow 0$$

be an exact sequence and define

$$\begin{array}{rccc} \delta^n(\iota^\bullet, p^\bullet)\colon & H^n(C^\bullet) & \longrightarrow & H^{n+1}(A^\bullet) \\ & c & \longmapsto & \tilde{a} \bmod \mathrm{im}\, d^{n+1} \end{array} \qquad \text{where} \qquad \begin{array}{l} \tilde{c} \in \ker d^n \text{ is a representative of } c, \\ \tilde{b} \text{ is such that } p^n(\tilde{b}) = \tilde{c}, \\ \tilde{a} \text{ is such that } d^n\tilde{b} = \iota^{n+1}(\tilde{a}).. \end{array}$$

Then the exact sequence

$$\cdots \longrightarrow H^n(A^\bullet) \xrightarrow{H^n(i^\bullet)} H^n(B^\bullet) \xrightarrow{H^n(p^\bullet)} H^n(C^\bullet) \xrightarrow{\delta^n(\iota^\bullet, p^\bullet)} H^{n+1}(A^\bullet) \longrightarrow \cdots$$

is the corresponding *long exact sequence*.

The *homotopic category* $K(\mathcal{A})$ is the category given by

$$\mathrm{Ob}\, K(\mathcal{A}) = \mathrm{Ob}\, \mathrm{Kom}(\mathcal{A}) \qquad \text{and} \qquad \mathrm{Mor}\, K(\mathcal{A}) = (\mathrm{Mor}\, \mathrm{Kom}(\mathcal{A}) \text{ mod homotopic equivalence}).$$

A *triangle* is a sequence of morphisms

$$K \longrightarrow L \longrightarrow M \longrightarrow K[1].$$

A *distinguished triangle* is a triangle isomorphic to

$$K \xrightarrow{\bar{f}} \mathrm{Cyl}(f) \xrightarrow{\pi} C(f) \xrightarrow{\delta} K[1], \qquad \text{for some morphism } f\colon K \to L.$$

2.4 Ext

Define

$$\mathrm{Ext}^i_{\mathcal{A}}(X,Y) = \mathrm{Hom}_{D(\mathcal{A})}(X[0], Y[i]) \qquad \text{so that} \qquad \mathrm{Ext}^i(X,Y) = \mathrm{Hom}_{D(\mathcal{A})}(X, I_Y[i]),$$

for an injective resolution I_Y of Y.

$$\mathrm{Ext}^0_{\mathcal{A}}(X,Y) = \mathrm{Hom}_{\mathcal{A}}(X,Y) \qquad \text{and} \qquad \mathrm{Ext}^i_{\mathcal{A}}(X,Y) = 0, \quad \text{for } i < 0.$$

An acyclic complex

$$\begin{array}{ccccccc} \cdots \to 0 \to & K^{-i} & \to K^{-i+1} \to \cdots \to K^0 \to & K^1 & \to 0 \to \cdots \\ & \| & & \| & \\ & Y & & X & \end{array}$$

determines a left roof

$$X[0] \xleftarrow{s} \tilde{K} \xrightarrow{f} Y[i] \qquad \text{with} \qquad \begin{array}{l} \tilde{K}^1 = 0 \text{ and } \tilde{K}^\ell = K^\ell, \quad \text{for } \ell \neq 1, \\ s^0 = d^0_K, \quad f^{-i} = \mathrm{id}_Y, \end{array}$$

and a corresponding morphism $X[0] \to Y[i]$ in the derived category.

2.5 Derived functors

Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor. The *derived functor* is the extension $RF\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ of F to $D^+(\mathcal{A})$ determined by

$$RF(K)^i = F(K^i) \qquad \text{for a complex } K.$$

The *classical ith derived functor* is the functor $R^iF\colon \mathcal{A} \to \mathcal{B}$ given by

$$R^iF = H^0((RF)[i]) = H^i(RF).$$

Let $\mathcal{R}$ be a class of objects in $\mathcal{A}$.

The category $\mathcal{A}$ contains *sufficiently many objects from* $\mathcal{R}$ if every object from $\mathcal{A}$ is a subobject of an object from $\mathcal{R}$.

The class $\mathcal{R}$ is *adapted* to F if $\mathcal{A}$ contains sufficiently many objects from $\mathcal{R}$ and F maps any acyclic complex from $\mathrm{Kom}^+(\mathcal{R})$ into an acyclic complex.

Theorem 2.1. *(a) The derived functor RF exists if F admits an adapted class of objects $\mathcal{R}$.*

(b) If $\mathcal{A}$ contains sufficiently many injective objects then the injective objects form an adapted class for every left exact functor F.

(c) If $\mathcal{A}$ has sufficiently many injective objects then $\mathrm{Ext}^i(X,-) \cong R^i\mathrm{Hom}(X,-)$.

(d) If $\mathcal{A}$ has sufficiently many projective objects then $\mathrm{Ext}^i(-,X) \cong R^i\mathrm{Hom}(-,X)$.

A *projective resolution* of a complex A is a quasi-isomorphism $P \to A$ with P^i projective in $\mathcal{A}$ for all i.

A *K-projective complex* is a complex Q such that if A is an acyclic complex then $\mathrm{Hom}(P, A)$ is an acyclic complex.

A *K-projective resolution* of a complex A is a quasi-isomorphism $P \to A$ with a K-projective P.

If $\mathcal{A}$ has sufficiently many projective objects then any K-projective resolution is a projective resolution.

2.6 Tor

Let R be a ring with identity and N a right R module. The flat R-modules form an adapted class for the functor

$$- \otimes_R N, \qquad \text{the derived functor is denoted} \quad - \otimes^L N$$

(a left derived functor), and

$$\mathrm{Tor}_i^R(M, N) = H^{-i}(M \otimes^L N).$$

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