Generalized matrix algebra structure

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1 Generalized matrix algebras

Let A be an algebra and fix $a \in A$. The *homotope algebra* $A(a)$ is the algebra A with a new multiplication given by

$$x \cdot y = xay, \qquad \text{for } x, y \in A.$$

If p, q are invertible elements of A then the map

$$\begin{matrix} A(paq) & \longrightarrow & A(a) \\ x & \longmapsto & qxp \end{matrix} \quad \text{is an algebra isomorphism.}$$

1.1 The radical of a homotope algebra

Let R be a PID and let $A = M_n(R)$ and let $\varepsilon \in A$. The Smith normal form says that there exist

$$p, q \in GL_n(R) \qquad \text{such that} \quad p\varepsilon q = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, 0, 0, \ldots, 0), \quad \text{with} \quad \varepsilon_1 | \varepsilon_2 | \cdots | \varepsilon_k.$$

Thus,

$$M_n(\varepsilon) \cong M_n(\delta), \qquad \text{where} \qquad \delta = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, 0, 0, \ldots, 0),$$

and

$$\begin{aligned} \mathrm{Rad}(M_n(\delta)) &= \{x \in M_n \mid \text{if } x_{ST} \neq 0 \text{ then } S > k \text{ or } T > k\}, \\ \mathrm{Rad}^2(M_n(\delta)) &= \{x \in M_n \mid \text{if } x_{ST} \neq 0 \text{ then } S > k \text{ and } T > k\}, \qquad \text{and} \\ \mathrm{Rad}^3(M_n(\delta)) &= 0. \end{aligned}$$

Proposition 1.1. $\mathrm{Rad}(A(a)) = \{x \in A \mid axa \in \mathrm{Rad}(A)\}$ *and* $\mathrm{Rad}^3(A(a)) \subseteq \mathrm{Rad}(A)$.

Proof. The set $I = \{x \in A \mid axa \in \mathrm{Rad}(A)\}$ is an ideal in A since, if $y \in A$ then $a(x \cdot y)a = axay \in \mathrm{Rad}(A)$. If $x, y, z \in I$ then $x \cdot y \cdot z = xayaz \in \mathrm{Rad}(A)$. Thus I is a nilpotent ideal and $I \subseteq \mathrm{Rad}(A(a))$. Why and when is $I = \mathrm{Rad}(A)$??? Or do I care? $\square$

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1.2 $A \subseteq B$, both split semisimple

Assume $A \subseteq B$ is an inclusion of algebras and that A and B are split semisimple. Let

$\hat{A}$ be an index set for the irreducible A-modules A^μ,

$\hat{B}$ be an index set for the irreducible B-modules B^λ, and let

$\hat{A}^\mu = \{ \, {\scriptstyle P\to\mu} \, \}$ be an index set for a basis of the simple A-module A^μ,

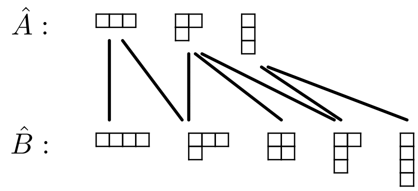
for each $\mu \in \hat{A}$ (the composite ${\scriptstyle P\to\mu}$ is viewed as a single symbol). Let Γ be the two level graph

$$\begin{array}{ll} \text{vertices on level A:} & \hat{A}, \\ \text{vertices on level B:} & \hat{B}, \qquad \text{and} \\ & \\ m_\mu^\lambda \text{ edges } \mu \to \lambda \text{ if } A^\mu \text{ appears with multiplicity } m_\mu^\lambda \text{ in } \mathrm{Res}_A^B(B^\lambda). & \end{array} \tag{1.1}$$

If $\lambda \in \hat{B}$ then

$$\hat{B}^\lambda = \{ {\scriptstyle P\to\mu} \to \lambda \mid \mu \in \hat{A},\ {\scriptstyle P\to\mu} \in \hat{A}^\mu \text{ and } \mu \to \lambda \text{ is an edge in } \Gamma \} \tag{1.2}$$

is an index set for a basis of the irreducible B-module B^λ. We think of $\hat{B}^\lambda$ as the set of paths to λ and $\hat{A}^\mu$ as the set of "paths to μ" in the graph Γ. For example, the graph Γ for the symmetric group algebras $A = \mathbb{C}S_3$ and $B = \mathbb{C}S_4$ is



Since A and B are split semisimple there exist (DOES THIS need proof?) be sets of matrix units in the algebras A and B,

$$\{ a_{\substack{PQ\\ \mu}} \mid \mu \in \hat{A}, {\scriptstyle P\to\mu}, {\scriptstyle Q\to\mu} \in \hat{A}^\mu \} \qquad \text{and} \qquad \{ b_{\substack{PQ\\ \mu\,\nu\\ \lambda}} \mid \lambda \in \hat{B}, {\scriptstyle P\to\mu} \to \lambda, {\scriptstyle Q\to\nu} \to \lambda \in \hat{B}^\lambda \}, \tag{1.3}$$

respectively, so that

$$a_{\substack{PQ\\ \mu}} a_{\substack{ST\\ \nu}} = \delta_{\mu\nu}\delta_{QS} a_{\substack{PT\\ \mu}} \qquad \text{and} \qquad b_{\substack{PQ\\ \mu\,\gamma\\ \lambda}} b_{\substack{ST\\ \tau\,\nu\\ \sigma}} = \delta_{\lambda\sigma}\delta_{QS}\delta_{\gamma\tau} b_{\substack{PT\\ \mu\,\nu\\ \lambda}}, \tag{1.4}$$

and such that

$$a_{\substack{PQ\\ \mu}} b_{\substack{ST\\ \sigma\,\tau\\ \lambda}} = \delta_{\substack{QS\\ \mu\,\sigma}} b_{\substack{PT\\ \mu\,\tau\\ \lambda}} \qquad \text{and} \qquad b_{\substack{ST\\ \sigma\,\tau\\ \lambda}} a_{\substack{PQ\\ \mu}} = \delta_{\substack{TP\\ \tau\,\mu}} b_{\substack{SQ\\ \sigma\,\mu\\ \lambda}}. \tag{1.5}$$

Then

$$1 = \sum b_{\substack{PP\\ \mu\,\mu\\ \lambda}} \tag{1.6}$$

and

$$a_{\substack{PQ\\ \mu}} = 1 \cdot a_{\substack{PQ\\ \mu}} \cdot 1 = \left(\sum b_{\substack{RR\\ \rho\,\rho\\ \lambda}} \right) a_{\substack{PQ\\ \mu}} \left(\sum b_{\substack{SS\\ \sigma\,\sigma\\ \gamma}} \right) = \sum b_{\substack{PQ\\ \mu\,\mu\\ \lambda}}. \tag{1.7}$$

where the sum is over all edges $\mu \to \lambda$ in the graph Γ.

Now assume that B is a subalgebra of an algebra C and there is an element $e \in C$ such that, for all $b \in B$,

(a) $ebe = \varepsilon_1(b)e$, with $\varepsilon_1(b) \in A$, and

(b) $\varepsilon_1(a_1 b a_2) = a_1 \varepsilon_1(b) a_2$, for all $a_1 a_2, \in A$, and

(c) $ea = ae$, for all $a \in A$.

Note that the map

$$\begin{array}{cccc} \varepsilon: & B \otimes B & \longrightarrow & A \\ & b_1 \otimes b_2 & \longmapsto & \varepsilon(b_1 b_2) \end{array} \qquad \text{is an } (A, A) \text{ bimodule homomorphism.}$$

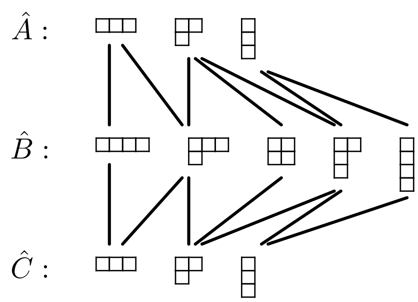
Though is not necessary for the following it is conceptually helpful to let $C = BeB$, let $\hat{C} = \hat{A}$ and extend the graph Γ to a graph $\hat{\Gamma}$ with three levels, so that the edges between level B and level C are the reflections of the edges between level A and level B. In other words, $\hat{\Gamma}$ has

$$\begin{array}{l} \text{vertices on level } C\text{:} \qquad \hat{C}, \qquad \text{and} \\ \text{an edge } \lambda \to \mu,\ \lambda \in \hat{B},\ \mu \in \hat{C}, \text{ for each edge } \mu \to \lambda,\ \mu \in \hat{A},\ \lambda \in \hat{B}. \end{array} \tag{1.8}$$

For each $\nu \in \hat{C}$ define

$$\hat{C}^\nu = \left\{ {\scriptstyle P} {\to} \mu \to \lambda \to \nu \;\middle|\; \begin{array}{c} \mu \in \hat{A},\ \lambda \in \hat{B},\ \nu \in \hat{C},\ {\scriptstyle P}{\to}\mu \in \hat{A}^\mu \text{ and} \\ \mu \to \lambda \text{ and } \lambda \to \nu \text{ are edges in } \hat{\Gamma} \end{array} \right\}, \tag{1.9}$$

so that $\hat{C}^\nu$ is the set of paths to ν in the graph $\hat{\Gamma}$. In the previous example $\hat{\Gamma}$ is



The element of A given by

$$\varepsilon_1\left(b_{\substack{PQ \\ \mu\,\tau \\ \lambda}} \right) = \varepsilon_1\left(a_{\substack{PP \\ \mu}}\, b_{\substack{PQ \\ \mu\,\tau \\ \lambda}}\, a_{\substack{QQ \\ \tau}} \right) = a_{\substack{PP \\ \mu}} \varepsilon_1\left(b_{\substack{PQ \\ \mu\,\tau \\ \lambda}} \right) a_{\substack{QQ \\ \tau}}$$

equals 0 unless $\mu = \tau$ and

$$\varepsilon_1\left(b_{\substack{PQ \\ \mu\,\mu \\ \lambda}} \right) = \varepsilon_1\left(a_{\substack{PR \\ \mu}}\, b_{\substack{RR \\ \mu\,\mu \\ \lambda}}\, a_{\substack{RQ \\ \mu}} \right) = a_{\substack{PR \\ \mu}} \varepsilon_1\left(b_{\substack{RR \\ \mu\,\mu \\ \lambda}} \right) a_{\substack{RQ \\ \mu}} = \varepsilon_\mu^\lambda\, a_{\substack{PQ \\ \mu}}. \tag{1.10}$$

for some constant ε_μ^λ which does not depend on P or Q (since it depends only on R which can be chosen freely). The element of C given by

$$b_{\substack{PR \\ \mu\,\rho \\ \lambda}}\, e\, b_{\substack{TQ \\ \tau\,\nu \\ \sigma}} = b_{\substack{PR \\ \mu\,\rho \\ \lambda}}\, a_{\substack{RR \\ \rho}}\, e\, b_{\substack{TQ \\ \tau\,\nu \\ \sigma}} = b_{\substack{PR \\ \mu\,\rho \\ \lambda}}\, e\, a_{\substack{RR \\ \rho}}\, b_{\substack{TQ \\ \tau\,\nu \\ \sigma}}$$

is zero unless $R = T$ and $\rho = \tau$ and

$$b_{\substack{PR \\ \mu\,\rho \\ \lambda}}\, e\, b_{\substack{RQ \\ \rho\,\nu \\ \sigma}} = b_{\substack{PS \\ \mu\,\rho \\ \lambda}}\, a_{\substack{SR \\ \rho}}\, e\, b_{\substack{RQ \\ \rho\,\nu \\ \sigma}} = b_{\substack{PS \\ \mu\,\rho \\ \lambda}}\, e\, a_{\substack{SR \\ \rho}}\, b_{\substack{RQ \\ \rho\,\nu \\ \sigma}} = b_{\substack{PS \\ \mu\,\rho \\ \lambda}}\, e\, b_{\substack{SQ \\ \rho\,\nu \\ \sigma}}$$

does not depend on the choice of R. If

$$c_{\substack{PQ\\ \mu\,\nu\\ \lambda\,\sigma\\ \gamma}} = b_{\substack{PT\\ \mu\,\gamma\\ \lambda}}\, e\, b_{\substack{TQ\\ \gamma\,\nu\\ \sigma}} \tag{1.11}$$

then

$$\begin{aligned} c_{\substack{PQ\\ \mu\,\nu\\ \lambda\,\sigma\\ \gamma}}\, c_{\substack{RS\\ \tau\,\xi\\ \rho\,\eta\\ \pi}} &= \Big(b_{\substack{PT\\ \mu\,\gamma\\ \lambda}}\, e\, b_{\substack{TQ\\ \gamma\,\nu\\ \sigma}}\Big)\Big(b_{\substack{RX\\ \tau\,\pi\\ \rho}}\, e\, b_{\substack{XS\\ \pi\,\xi\\ \eta}}\Big) = \delta_{\substack{QR\\ \nu\,\tau\\ \sigma\,\rho}}\, b_{\substack{PT\\ \mu\,\gamma\\ \lambda}}\, \varepsilon_1 \Big(b_{\substack{TX\\ \gamma\,\pi\\ \sigma}}\Big) e\, b_{\substack{XS\\ \pi\,\xi\\ \eta}} \\ &= \delta_{\substack{QR\\ \nu\,\tau\\ \sigma\,\rho}}\, b_{\substack{PT\\ \mu\,\gamma\\ \lambda}}\, \delta_{\gamma\pi}\, \varepsilon^\sigma_\gamma\, a_{\substack{TX\\ \gamma}}\, e\, b_{\substack{XS\\ \pi\,\xi\\ \eta}} = \delta_{\substack{QR\\ \nu\,\tau\\ \sigma\,\rho}}\, \delta_{\gamma\pi}\, \varepsilon^\sigma_\gamma\, b_{\substack{PX\\ \mu\,\gamma\\ \lambda}}\, e\, b_{\substack{XS\\ \gamma\,\xi\\ \eta}} = \delta_{\substack{QR\\ \nu\,\tau\\ \sigma\,\rho}}\, \delta_{\gamma\pi}\, \varepsilon^\sigma_\gamma\, c_{\substack{PS\\ \mu\,\xi\\ \lambda\,\eta\\ \gamma}} \end{aligned}$$

Define

$$e_{\substack{PQ\\ \mu\,\nu\\ \lambda\,\sigma\\ \gamma}} = \left(\frac{1}{\varepsilon^\sigma_\gamma}\right) c_{\substack{PQ\\ \mu\,\nu\\ \lambda\,\sigma\\ \gamma}}, \qquad \text{whenever } \varepsilon^\sigma_\gamma \neq 0. \qquad \text{Then} \quad e_{\substack{PQ\\ \mu\,\nu\\ \lambda\,\sigma\\ \gamma}}\, e_{\substack{RS\\ \tau\,\xi\\ \rho\,\eta\\ \pi}} = \delta_{\substack{QR\\ \nu\,\tau\\ \sigma\,\rho}}\, \delta_{\gamma\pi}\, e_{\substack{PS\\ \mu\,\xi\\ \lambda\,\eta\\ \gamma}} \tag{1.12}$$

so that these are matrix units. Furthermore

$$b_{\substack{PQ\\ \mu\,\nu\\ \lambda}}\, e_{\substack{RS\\ \tau\,\xi\\ \rho\,\eta\\ \pi}} = \delta_{\substack{QR\\ \nu\,\tau\\ \lambda\,\rho}}\, e_{\substack{PS\\ \mu\,\xi\\ \lambda\,\eta\\ \pi}} \qquad \text{and} \qquad e_{\substack{PQ\\ \mu\,\nu\\ \lambda\,\sigma\\ \gamma}}\, b_{\substack{RS\\ \tau\,\xi\\ \rho}} = \delta_{\substack{QR\\ \nu\,\tau\\ \sigma\,\rho}}\, \delta_{\gamma\pi}\, e_{\substack{PS\\ \mu\,\xi\\ \lambda\,\rho\\ \gamma}} \tag{1.13}$$

so that the bs are related to the es in the same way that the as are related to the bs. Then

$$e = 1\cdot e\cdot 1 = \left(\sum b_{\substack{RR\\ \rho\,\rho\\ \lambda}}\right) e \left(\sum b_{\substack{SS\\ \sigma\,\sigma\\ \gamma}}\right) = \sum b_{\substack{RR\\ \rho\,\rho\\ \lambda}}\, e\, b_{\substack{RR\\ \rho\,\rho\\ \gamma}} = \sum c_{\substack{RR\\ \rho\,\rho\\ \lambda\,\gamma\\ \rho}} = \sum \varepsilon^\gamma_\rho\, e_{\substack{RR\\ \rho\,\rho\\ \lambda\,\gamma\\ \rho}} \tag{1.14}$$

In summary

$$e\, b_{\substack{PQ\\ \mu\,\mu\\ \lambda}}\, e = \delta_{\mu\tau} \varepsilon^\lambda_\mu\, a_{\substack{PQ\\ \mu}}.$$

$$e_{\substack{PQ\\ \mu\,\nu\\ \lambda\,\sigma\\ \gamma}} = \left(\frac{1}{\varepsilon^\sigma_\gamma}\right) b_{\substack{PT\\ \mu\,\gamma\\ \lambda}}\, e\, b_{\substack{TQ\\ \gamma\,\nu\\ \sigma}}$$

$$b_{\substack{PQ\\ \mu\,\nu\\ \lambda}} = \sum_{\lambda\to\gamma} e_{\substack{PQ\\ \mu\,\nu\\ \lambda\,\lambda\\ \gamma}},$$

and

$$e\, e_{\substack{PQ\\ \mu\,\nu\\ \lambda\,\sigma\\ \gamma}} = \delta_{\mu\gamma} \varepsilon^\lambda_\gamma \sum_{\gamma\to\tau\to\gamma} e_{\substack{PQ\\ \gamma\,\nu\\ \tau\,\sigma\\ \gamma}}.$$

1.3 $R \otimes_A L$, for A semisimple

Fix isomorphisms

$$\bar{L} \cong \bigoplus_{\mu\in\hat{A}} \overrightarrow{A}^\mu \otimes L^\mu \qquad \text{and} \qquad \bar{R} \cong \bigoplus_{\mu\in\hat{A}} R^\mu \otimes \overleftarrow{A}^\mu \tag{1.15}$$

where $\overrightarrow{A}^\mu$, $\mu \in \hat{A}$, are the simple left $\bar{A}$-modules, $\overleftarrow{A}^\mu$, $\mu \in \hat{A}$, are the simple right $\bar{A}$-modules, and L^μ, R^μ, $\mu \in \hat{A}$ are vector spaces. In other words, if A has matrix units

$$\{a_{\underset{\mu}{PQ}} \mid \mu \in \hat{A}, {\scriptstyle P\to\mu, Q\to\mu} \in \hat{A}^\mu\} \qquad \text{then} \qquad \begin{matrix} L & \text{has a basis} & \{\ell_{PX} \mid P \in \hat{A}^\mu, X \in \hat{L}^\mu\} \\ R & \text{has a basis} & \{r_{YQ} \mid Q \in \hat{A}^\mu, Y \in \hat{L}^\mu\} \end{matrix}$$

such that

$$a_{\underset{\mu}{PQ}}\ell_{\underset{\nu}{RX}} = \delta_{\mu\nu}\delta_{QR}\ell_{\underset{\mu}{PX}} \qquad \text{and} \qquad r_{\underset{\nu}{YS}}a_{\underset{\mu}{PQ}} = \delta_{\mu\nu}\delta_{SP}\ell_{\underset{\mu}{YQ}}. \tag{1.16}$$

The map $\varepsilon : \bar{L} \otimes_{\mathbb{F}} \bar{R} \to \bar{A}$ is determined by the constants $\varepsilon^\mu_{XY} \in \mathbb{F}$ given by

$$\varepsilon(\ell_{\underset{\mu}{QX}} \otimes r_{\underset{\mu}{YP}}) = \varepsilon^\mu_{XY} a_{\underset{\mu}{QP}} \tag{1.17}$$

and ε^μ_{XY} does not depend on Q and P since

$$\begin{aligned} \varepsilon(\ell_{\underset{\lambda}{SX}} \otimes r_{\underset{\mu}{YT}}) &= \varepsilon(a_{\underset{\lambda}{SQ}}\ell_{\underset{\lambda}{QX}} \otimes r_{\underset{\mu}{YP}}a_{\underset{\mu}{PT}}) = a_{\underset{\lambda}{SQ}}\varepsilon(\ell_{\underset{\lambda}{QX}} \otimes r_{\underset{\mu}{YP}})a_{\underset{\mu}{PT}} \\ &= \delta_{\lambda\mu}a_{\underset{\mu}{SQ}}\varepsilon^\mu_{XY}a_{\underset{\mu}{QP}}a_{\underset{\mu}{PT}} = \varepsilon^\mu_{XY}a_{\underset{\mu}{ST}}. \end{aligned}$$

For each $\mu \in \hat{A}$ construct a matrix

$$\mathcal{E}^\mu = (\varepsilon^\mu_{XY}) \tag{1.18}$$

and use row reduction (Smith normal form) to find invertible matrices

$$D^\mu = (D^\mu_{ST}) \quad \text{and} \quad C^\mu = (C^\mu_{ZW}) \qquad \text{such that} \qquad D^\mu\mathcal{E}^\mu C^\mu = \mathrm{diag}(\varepsilon^\mu_X). \tag{1.19}$$

is a diagonal matrix with diagonal entries denoted ε^μ_X, The ε^μ_P are the *invariant factors* of the matrix $\mathcal{E}^\mu$.

Theorem 1.2. *Using notation as in (???) define elements of $R \otimes_A L$ by*

$$m_{\underset{\mu}{XY}} = r_{\underset{\mu}{XP}} \otimes \ell_{\underset{\mu}{PY}}, \qquad \textit{and} \qquad n_{\underset{\mu}{XY}} = \sum_{Q_1,Q_2} C^\mu_{Q_1X}D^\mu_{YQ_2}m_{\underset{\mu}{Q_1Q_2}}. \tag{1.20}$$

where $\mu \in \hat{A}, X \in \hat{R}^\mu, Y \in \hat{L}^\mu$.

(a) The sets

$$\{m_{\underset{\mu}{XY}} \mid \mu \in \hat{A}, X \in \hat{R}^\mu, Y \in \hat{L}^\mu\} \qquad \textit{and} \qquad \{n_{\underset{\mu}{XY}} \mid \mu \in \hat{A}, X \in \hat{R}^\mu, Y \in \hat{L}^\mu\}$$

are bases of $\bar{R} \otimes_{\bar{A}} \bar{L}$, which satisfy

$$m_{\underset{\lambda}{ST}}m_{\underset{\mu}{QP}} = \delta_{\lambda\mu}\varepsilon^\mu_{TQ}m_{\underset{\mu}{SP}} \qquad \textit{and} \qquad n_{\underset{\lambda}{ST}}n_{\underset{\mu}{QP}} = \delta_{\lambda\mu}\delta_{TQ}\varepsilon^\mu_T n_{\underset{\mu}{SP}},$$

where ε^μ_{TQ} and ε^μ_T are as defined in (4.12) and (4.15).

(b) The radical of the algebra $R \otimes_A L$ is

$$\mathrm{Rad}(R \otimes_A L) = \mathbb{F}\text{-}\textit{span}\{n_{\underset{\mu}{YT}} \mid \varepsilon^\mu_Y = 0 \textit{ or } \varepsilon^\mu_T = 0\}$$

and the images of the elements

$$e_{\underset{\mu}{YT}} = \frac{1}{\varepsilon^\mu_T}n_{\underset{\mu}{YT}}, \qquad \textit{for } \varepsilon^\mu_Y \neq 0 \textit{ and } \varepsilon^\mu_T \neq 0,$$

are a set of matrix units in $(R \otimes_A L)/\mathrm{Rad}(R \otimes_A L)$.

Proof. Since

$$(r_{\underset{\lambda}{SW}} \otimes \ell_{\underset{\mu}{ZT}}) = (r_{\underset{\lambda}{SP}} a_{\underset{\lambda}{PW}} \otimes \ell_{\underset{\mu}{ZT}}) = (r_{\underset{\lambda}{SP}} \otimes a_{\underset{\lambda}{PW}} \ell_{\underset{\mu}{ZT}}) = \delta_{\lambda\mu}\delta_{WZ}(r_{\underset{\lambda}{SP}} \otimes \ell_{\underset{\lambda}{PT}})$$

the element $m_{\underset{\mu}{XY}}$ does not depend on P and

$$\{m_{\underset{\mu}{XY}} \mid \mu \in \hat{A}, X \in \hat{R}^\mu, Y \in \hat{L}^\mu\} \qquad \text{is a basis of} \qquad R \otimes_A L. \tag{1.21}$$

and hence $R \otimes_A L$ is a direct sum of generalized matrix algebras. If $(C^{-1})^\mu$ and $(D^{-1})^\mu$ are the inverses of the matrices C^μ and D^μ then

$$\begin{aligned}\sum_{X,Y} (C^{-1})^\mu_{XS}(D^{-1})^\mu_{TY} n_{\underset{\mu}{XY}} &= \sum_{X,Y,Q_1,Q_2} (C^{-1})^\mu_{XS} C^\mu_{Q_1X} m_{\underset{\mu}{Q_1Q_2}} D^\mu_{YQ_2}(D^{-1})^\mu_{TY} \\ &= \sum_{Q_1,Q_2} \delta_{SQ_1}\delta_{Q_2T} m_{\underset{\mu}{Q_1Q_2}} = m_{\underset{\mu}{ST}},\end{aligned}$$

and so the elements $m_{\underset{\mu}{ST}}$ can be written as linear combinations of the $n_{\underset{\mu}{XY}}$. Thus

$$\{n_{\underset{\mu}{XY}} \mid \mu \in \hat{A}, X \in \hat{R}^\mu, Y \in \hat{L}^\mu\} \qquad \text{is a basis of} \qquad R \otimes_A L. \tag{1.22}$$

By direct computation,

$$\begin{aligned} m_{\underset{\lambda}{ST}} m_{\underset{\mu}{QP}} &= (r_{\underset{\lambda}{SW}} \otimes \ell_{\underset{\lambda}{WT}})(r_{\underset{\mu}{QZ}} \otimes \ell_{\underset{\mu}{ZP}}) = r_{\underset{\lambda}{SW}} \otimes \varepsilon(\ell_{\underset{\lambda}{WT}} \otimes r_{\underset{\mu}{QZ}})\ell_{\underset{\mu}{ZP}} \\ &= \delta_{\lambda\mu}(r_{\underset{\lambda}{SW}} \otimes \varepsilon^\lambda_{TQ} a_{\underset{\lambda}{WZ}} \ell_{\underset{\lambda}{ZP}}) = \delta_{\lambda\mu}\varepsilon^\lambda_{TQ}(r_{\underset{\lambda}{SW}} \otimes \ell_{\underset{\lambda}{WP}}) = \delta_{\lambda\mu}\varepsilon^\lambda_{TQ} m_{\underset{\lambda}{SP}}, \end{aligned}$$

and

$$\begin{aligned} n_{\underset{\lambda}{ST}} n_{\underset{\mu}{UV}} &= \sum_{Q_1,Q_2,Q_3,Q_4} C^\lambda_{Q_1S} D^\lambda_{TQ_2} m_{\underset{\lambda}{Q_1Q_2}} C^\mu_{Q_3U} D^\mu_{VQ_4} m_{\underset{\mu}{Q_3Q_4}} \\ &= \sum_{Q_1,Q_2,Q_3,Q_4} \delta_{\lambda\mu} C^\lambda_{Q_1S} D^\lambda_{TQ_2} \varepsilon^\mu_{Q_2Q_3} C^\mu_{Q_3U} D^\mu_{VQ_4} m_{\underset{\mu}{Q_1Q_4}} \\ &= \delta_{\lambda\mu} \sum_{Q_1,Q_4} \delta_{TU}\varepsilon^\mu_T C^\mu_{Q_1S} D^\mu_{VQ_4} m_{\underset{\mu}{Q_1Q_4}} = \delta_{\lambda\mu}\delta_{TU}\varepsilon^\mu_T n_{\underset{\mu}{SV}}. \end{aligned} \tag{1.23}$$

(b) Let

$$I = \mathbb{F}\text{-span}\{n_{\underset{\mu}{YT}} \mid \varepsilon^\mu_Y = 0 \text{ or } \varepsilon^\mu_T = 0\}.$$

The multiplication rule for the $n_{\underset{\mu}{YT}}$ implies that I is an ideal of $R \otimes_A L$. If $n_{\underset{\mu}{Y_1T_1}}, n_{\underset{\mu}{Y_2T_2}}, n_{\underset{\mu}{Y_3T_3}} \in \{n_{\underset{\mu}{YT}} \mid \varepsilon^\mu_Y = 0 \text{ or } \varepsilon^\mu_T = 0\}$ then

$$n_{\underset{\mu}{Y_1T_1}} n_{\underset{\mu}{Y_2T_2}} n_{\underset{\mu}{Y_3T_3}} = \delta_{T_1Y_2}\varepsilon^\mu_{Y_2} n_{\underset{\mu}{Y_1T_2}} n_{Y_3}T_3\mu = \delta_{T_1Y_2}\delta_{T_2Y_3}\varepsilon^\mu_{Y_2}\varepsilon^\mu_{T_2} n_{\underset{\mu}{Y_1T_3}} = 0,$$

since $\varepsilon^\mu_{Y_2} = 0$ or $\varepsilon^\mu_{T_2} = 0$. Thus any product $n_{\underset{\mu}{Y_1T_1}} n_{\underset{\mu}{Y_2T_2}} n_{\underset{\mu}{Y_3T_3}}$ of three basis elements of I is 0. So I is an ideal of $R \otimes_A L$ consisting of nilpotent elements and so $I \subseteq \mathrm{Rad}(R \otimes_A L)$.

Since

$$e_{\underset{\lambda}{YT}} e_{\underset{\mu}{UV}} = \frac{1}{\varepsilon^\lambda_T}\frac{1}{\varepsilon^\mu_V} n_{\underset{\lambda}{YT}} n_{\underset{\mu}{UV}} = \delta_{\lambda\mu}\delta_{TU}\frac{1}{\varepsilon^\lambda_T\varepsilon^\lambda_V}\varepsilon^\lambda_T n_{\underset{\lambda}{YV}} = \delta_{\lambda\mu}\delta_{TU} e_{\underset{\lambda}{YV}} \qquad \mathrm{mod}\ I,$$

the images of the elements $e_{\underset{\lambda}{YT}}$ in (????) form a set of matrix units in the algebra $(R \otimes_A L)/I$. Thus $(R \otimes_A L)/I$ is a split semisimple algebra and so $I \supseteq \mathrm{Rad}(R \otimes_A L)$. □

2 Structure of $Z(\varepsilon)$

Let

$$\varepsilon\colon L\otimes_D R\longrightarrow C \qquad \text{be a } (C,C) \text{ bimodule homomorphism.}$$

Let *left radical* $L(\varepsilon)$ and the *right radical* $R(\varepsilon)$ of ε are defined by

$$\begin{aligned} L(\varepsilon) &= \{\ell\in L \mid \varepsilon(\ell\otimes r)\in \mathrm{Rad}(C), \text{ for all } r\in R\},\\ R(\varepsilon) &= \{r\in R \mid \varepsilon(\ell\otimes r)\in \mathrm{Rad}(C), \text{ for all } \ell\in L\}, \end{aligned}$$

The map ε is *nondegenerate* if $\mathrm{Rad}(C)=0$, $L(\varepsilon)=0$, and $R(\varepsilon)=0$. Let

$$\begin{array}{l} \overline{C}=C/\mathrm{Rad}(C),\\ \overline{L}=L/L(\varepsilon),\\ \overline{R}=R/R(\varepsilon), \end{array} \qquad \text{and} \qquad \begin{array}{rccc} \phi\colon & R\otimes_C L & \longrightarrow & \bar{R}\otimes_{\bar{C}}\bar{L}\\ & \bar{r}\otimes\bar{\ell} & \longmapsto & \overline{r\otimes\ell} \end{array}$$

Then $\ker\varphi$ is generated by $R\otimes_C L(\varepsilon)$ and $R(\varepsilon)\otimes_C L$, and we have that $\ker\varphi\cdot R\subseteq R(\varepsilon)$ and $L\cdot\ker\varphi\subseteq L(\varepsilon)$. Then

$$I=\mathrm{Rad}(C)+L(\varepsilon)+R(\varepsilon)+\ker\varphi \qquad \text{is a nilpotent ideal of } A(\varepsilon),$$

and

$$\frac{A(\varepsilon)}{I}\cong A(\bar{\varepsilon}) \qquad \text{where the map} \qquad \begin{array}{rccc} \bar{\varepsilon}\colon & \bar{L}\otimes_D\bar{R} & \longrightarrow & \bar{C}\\ & \ell\otimes r & \longmapsto & \bar{\ell}\otimes\bar{r} \end{array}$$

is a nondegenerate $(\bar{C},\bar{C})$ bimodule homomorphism.

If $\varepsilon\colon L\otimes_D R\to C$ is nondegenerate and R is a projective C-module then there is a (D,C) bimodule isomorphism

$$\begin{array}{rccc} \tau\colon & R & \xrightarrow{\sim} & L^*\\ & r & \mapsto & \begin{array}{rccc} \lambda_r\colon & L & \to & C\\ & \ell & \mapsto & \varepsilon(\ell\otimes r) \end{array} \end{array} \qquad \text{so that} \qquad \varepsilon=\mathrm{ev}\circ(\mathrm{id}\otimes\tau)$$

and

$$A(\varepsilon)\cong A(\mathrm{ev}_L).$$

If C,D,L,R are finite dimensional vector spaces over $\mathbb{F}$ and $D=\mathbb{F}$ then

$$\varepsilon=\varepsilon_0\oplus\mathrm{ev}_P\colon (L_0\oplus P^*)\otimes_D(R_0\oplus P)\longrightarrow C,$$

with P projective and $\mathrm{im}\varepsilon_0\subseteq\mathrm{Rad}(C)$.

If $\varepsilon=\varepsilon_0\oplus\mathrm{ev}_P$ with P finitely generated and projective then

$$\begin{array}{ccc} A(\varepsilon)\text{-mod} & \xrightarrow{\sim} & A(\varepsilon_0)\text{-mod}\\ M & \longmapsto & eM \end{array} \qquad \text{where} \qquad e=1-\sum_i p_i\otimes\alpha_i.$$

If $\mathrm{im}\,\varepsilon\subseteq\mathrm{Rad}(C)$ then

$$\mathrm{Rad}(A(\varepsilon_0))=I=\mathrm{Rad}(C)\oplus\mathrm{Rad}(D)\oplus L_0\oplus R_0\oplus R_0\otimes_C L_0$$

and

$$\frac{A(\varepsilon_0)}{\mathrm{Rad}(A(\varepsilon_0))}\cong\frac{C}{\mathrm{Rad}(C)}\oplus\frac{D}{\mathrm{Rad}(D)}.$$

3 Duals and Projectives

Let L be a C-module and let

$$Z = \mathrm{End}_C(L)$$

so that L is a (C, Z) bimodule. The *dual module* to L is the (Z, C) bimodule

$$L^* = \mathrm{Hom}_C(L, C).$$

The *evaluation map* is the (C, C) bimodule homomorphism

$$\begin{array}{rccc} \mathrm{ev}\colon & L \otimes_Z L^* & \longrightarrow & C \\ & \ell \otimes \lambda & \longmapsto & \lambda(\ell) \end{array}$$

and the *centralizer map* is the (Z, Z) bimodule homomorphism

$$\begin{array}{rccc} \xi\colon & L^* \otimes_C L & \longrightarrow & Z \\ & \lambda \otimes \ell & \longmapsto & \begin{array}{rccc} z_{\lambda,\ell}\colon & L & \to & L \\ & m & \mapsto & \lambda(m)\ell \end{array} \end{array}$$

Recall that [Bou, Alg. II §4.2 Cor.]

(a) L is a projective C-module if and only if $1 \in \mathrm{im}\,\xi$,

(b) If L is a projective C-module then ξ is injective,

(c) If L is a finitely generated projective C-module then ξ is bijective,

(d) If L is a finitely generated free module then

$$\xi^{-1}(z) = \sum_i b_i^* \otimes z(b_i),$$

where $\{b_1, \ldots, b_d\}$ is a basis of L and $\{b_1^*, \ldots, b_d^*\}$ is the dual basis in M^*.

Statement (a) says that L is projective if and only if there exist $b_i \in L$ and $b_i^* \in L^*$ such that

$$\text{if } \ell \in L \quad \text{then} \quad \ell = \sum_i b_i^*(\ell) b_i, \qquad\qquad \text{so that} \quad \xi\Big(\sum_i b_i^* \otimes b_i\Big) = 1.$$

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