Generalized matrix algebras

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1 Generalized matrix algebras

The *algebra of $n \times n$ matrices*

$$M_n \qquad \text{has basis} \qquad \{E_{ST} \mid 1 \le S, T \le n\} \qquad \text{with} \qquad E_{ST}E_{UV} = \delta_{TU}E_{SV}. \tag{1.1}$$

Fix constants ε_{TU}, $1 \le T, U \le n$. The *generalized matrix algebra*

$$M_n(\varepsilon) \qquad \text{has basis} \qquad \{E_{ST} \mid 1 \le S, T \le n\} \qquad \text{with} \qquad E_{ST} \cdot E_{UV} = \varepsilon_{TU}E_{SV}. \tag{1.2}$$

If we view elements of $M_n(\varepsilon)$ as matrices then

$$x \cdot y = x\varepsilon y, \qquad \text{for} \quad x = \sum_{S,T} x_{ST}E_{ST}, \quad y = \sum_{U,V} y_{UV}E_{UV}, \quad \text{and} \quad \varepsilon = (\varepsilon_{TU}).$$

Let A be an algebra and fix $\varepsilon \in A$. The *homotope algebra* $A(\varepsilon)$ is the algebra A with a new multiplication given by

$$x \cdot y = x\varepsilon y, \qquad \text{for } x, y \in A. \tag{1.3}$$

Let p, q be invertible elements of A. Then

$$\begin{array}{ccc} A(paq) & \xrightarrow{\sim} & A(a) \\ x & \longmapsto & qxp \end{array} \qquad \text{is an algebra isomorphism.}$$

Rewrite the construction of $M_n(\varepsilon)$: Let $L = \mathbb{F}^n$ and $R = \mathbb{F}^n$ and identify M_n with $L \otimes R$.

$$\text{Identify} \quad \varepsilon \quad \text{with a map} \qquad \begin{array}{ccc} R \otimes L & \longrightarrow & \mathbb{F} \\ v_S \otimes v_T & \longmapsto & \varepsilon_{ST} \end{array}$$

Then

$$M_n(\varepsilon) = L \otimes R \qquad \text{with product} \qquad (\ell_1 \otimes r_1)(\ell_2 \otimes r_2) = \ell_1 \otimes \varepsilon(r_1 \otimes \ell_2)r_2.$$

Let A be an algebra and let L be a left A-module and R a right A-module. Let

$$\varepsilon : L \otimes_{\mathbb{F}} R \longrightarrow A, \qquad \text{be an } (A, A)\text{-bimodule homomorphism.} \tag{1.4}$$

The *basic construction* is the algebra

$$A(\varepsilon) = R \otimes_A L \qquad \text{with} \qquad (r_1 \otimes \ell_1)(r_2 \otimes \ell_2) = r_1 \otimes \varepsilon(\ell_1 \otimes r_2)\ell_2, \tag{1.5}$$

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for $r_1, r_2 \in R$ and $\ell_1, \ell_2 \in L$.

One case which is used often is when $A \subseteq B$ is an inclusion of algebras with a given (A, A) bimodule homormorphism $\varepsilon_1 \colon B \to A$, where A acts by left and right multiplication on B (and on A). Then

$$\begin{array}{rccc} \varepsilon: & B \otimes_{\mathbb{F}} B & \longrightarrow & A \\ & b_1 \otimes b_2 & \longmapsto & \varepsilon_1(b_1 b_2) \end{array} \tag{1.6}$$

is an (A, A) bimodule homomorphism and

$$A(\varepsilon) = B \otimes_A B \qquad \text{with} \qquad (b_1 \otimes b_2)(b_3 \otimes b_4) = b_1 \otimes \varepsilon(b_2 \otimes b_3) b_4, \tag{1.7}$$

for $b_1, b_2, b_3, b_4 \in B$.

The algebra $A(\varepsilon)$ does not usually have an identity. Let C and D be rings,

$$\begin{array}{l} L, \text{ a } (C,D) \text{ bimodule}, \\ R, \text{ a } (D,C) \text{ bimodule} \end{array} \qquad \text{and} \qquad \varepsilon \colon L \otimes_D R \to C,$$

a (C, C) bimodule homomorphism. Define an algebra

$$A = C \oplus D \oplus L \oplus R \oplus R \otimes_C L \tag{1.8}$$

and product determined by the multiplication in C and D, the module structure of R and L and the additional relations

$$cr = 0, \quad d\ell = 0, \quad rd = 0, \quad \ell c = 0, \quad \text{and} \quad (r_1 \otimes \ell_1)(r_2 \otimes \ell_2) = r_1 \otimes \varepsilon(\ell_1 \otimes r_2)\ell_2. \tag{1.9}$$

for $r, r_1, r_2 \in R$, $\ell, \ell_1, \ell_2 \in L$, $c \in C$ and $d \in D$. Let e_C be the image of the identity of C in A and e_D the image of the identity of D in A. Then

$$1 = e_C + e_D, \qquad \text{and if} \qquad \begin{array}{ll} C = e_C A e_C, & L = e_C A e_D, \\ R = e_D A e_C, & D' = e_D A e_D, \end{array}$$

then

$$A = \left\{ \begin{pmatrix} c & \ell \\ r & d' \end{pmatrix} \;\middle|\; c \in C, \ell \in L, r \in R, d' \in D' \right\}$$

with matrix multiplication. Then

$$\begin{array}{lll} e_D A e_D & = D + R \otimes_C L & \text{is a subring of } A, \qquad \text{and} \\ e_D A e_C A e_D & = R \otimes_C L & \text{is an ideal in } e_D A e_D. \end{array}$$

1.1 The module category of $Z(\varepsilon)$

Let $\mathcal{C}$ and $\mathcal{D}$ be categories

$$F \colon \mathcal{C} \to \mathcal{D} \quad \text{and} \quad G \colon \mathcal{C} \to \mathcal{D} \quad \text{be functors,} \quad \text{and} \quad F \xrightarrow{\varepsilon} G,$$

a natural transformation. Define a category $\mathcal{A}$ with

Objects: $(M, V; PICTURE)$, where $M \in \mathcal{C}$, $V \in \mathcal{D}$, and $m, n \in \mathrm{Mor}(\mathcal{D})$,

Morphisms: (f, g) with $f \in \mathrm{Mor}(\mathcal{C})$, $g \in \mathrm{Mor}(\mathcal{D})$ such that

$$PICTURE \qquad \text{commutes.}$$

A fundamental case is when $\mathcal{D}$ is the category of vector spaces over $\mathbb{F}$.

The equivalence between the category $\mathcal{A}$ and the module category of $Z(\varepsilon)$ is given by letting $\mathcal{C} = C$-mod and $\mathcal{D} = D$-mod and

$$F\colon \begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{D} \\ M & \longmapsto & R \otimes_C M \end{array} \qquad \text{and} \qquad G\colon \begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{D} \\ M & \longmapsto & \mathrm{Hom}_C(L, M) \end{array}$$

where the D-action on $\mathrm{Hom}_C(L, M)$ is given by

$$(d\phi)(\ell) = \phi(\ell d), \qquad \text{for } d \in D,\ \ell \in L, \text{ and } \phi \in \mathrm{Hom}_C(L, M).$$

Then let $\varepsilon\colon F \to G$ be the natural transformation given by

$$\varepsilon\colon \begin{array}{ccc} F & \longrightarrow & G \\ R \otimes_C M & \xrightarrow{\varepsilon_M} & \mathrm{Hom}_C(L, M) \\ r \otimes m & \longmapsto & \tau\colon \begin{array}{ccc} L & \to & M \\ \ell & \mapsto & \varepsilon(\ell \otimes r)m \end{array} \end{array}$$

Then

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\sim} & A\text{-mod} \\ (X, Y, \rho, \lambda) & \leftrightarrow & M \end{array} \qquad \text{where} \qquad X = eM, \quad Y = (1-e)M,$$

and the L-action and R-action on M define ρ and λ via

$$\ell y = (\lambda(y))(\ell) \qquad \text{and} \qquad rx = \rho(r \otimes x), \qquad \text{for } \ell \in L,\ r \in R,\ x \in X \text{ and } y \in Y.$$

Note that

$$\ell x = 0 \quad \text{and} \quad ry = 0, \qquad \text{for } \ell \in L,\ r \in R,\ x \in X,\ y \in Y,$$

and

$$PICTURE$$

commutes.

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