Column strict tableaux

Arun Ram
Department of Mathematics
University of Wisconsin
Madison, WI 53706
ram@math.wisc.edu

July 18, 2005

1 Column strict tableaux

A *letter* is an element of $B(\varepsilon_1) = \{\varepsilon_1, \ldots, \varepsilon_n\}$ and a *word of length* k is an element of

$$B(\varepsilon_1)^{\otimes k} = \{\varepsilon_{i_1} \otimes \cdots \otimes \varepsilon_{i_k} \mid 1 \le i_1, \ldots, i_k \le n\}.$$

For $1 \le i \le n-1$ define

$$\tilde{f}_i \colon B(\varepsilon_1)^{\otimes k} \longrightarrow B(\varepsilon_1)^{\otimes k} \cup \{0\} \quad \text{and} \quad \tilde{e}_i \colon B(\varepsilon_1)^{\otimes k} \longrightarrow B(\varepsilon_1)^{\otimes k} \cup \{0\}$$

as follows. For $b \in B(\varepsilon_1)^{\otimes k}$,

> place $+1$ under each ε_i in b,
>
> place -1 under each ε_{i+1} in b, and
>
> place 0 under each ε_j, $j \ne i, i+1$.

Ignoring 0s, successively pair adjacent $(-1, +1)$ pairs to obtain a sequence of unpaired $+1$s and -1s

$$+1 \ \ +1 \ \ +1 \ \ +1 \ \ +1 \ \ +1 \ \ +1 \ \ -1 \ \ -1 \ \ -1 \ \ -1$$

(after pairing and ignoring 0s). Then

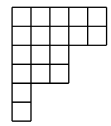
$\tilde{f}_i b =$ same as b except the letter corresponding to the rightmost unpaired $+1$ is changed to ε_{i+1},
$\tilde{e}_i b =$ same as b except the letter corresponding to the leftmost unpaired -1 is changed to ε_i.

If there is no unpaired $+1$ after pairing then $\tilde{f}_i b = 0$.
If there is no unpaired -1 after pairing then $\tilde{e}_i b = 0$.

A *partition* is a collection μ of boxes in a corner where the convention is that gravity goes up and to the left. As for matrices, the rows and columns of μ are indexed from top to bottom and left to right, respectively.

$$\begin{array}{ll}
\text{The } \textit{parts} \text{ of } \mu \text{ are} & \mu_i = (\text{the number of boxes in row } i \text{ of } \mu), \\
\text{the } \textit{length} \text{ of } \mu \text{ is} & \ell(\mu) = (\text{the number of rows of } \mu), \\
\text{the } \textit{size} \text{ of } \mu \text{ is} & |\mu| = \mu_1 + \cdots + \mu_{\ell(\mu)} = (\text{the number of boxes of } \mu).
\end{array} \tag{1.1}$$

Then μ is determined by (and identified with) the sequence $\mu = (\mu_1, \dots, \mu_\ell)$ of positive integers such that $\mu_1 \geq \mu_2 \geq \dots \geq \mu_\ell > 0$, where $\ell = \ell(\mu)$. For example,

$$(5,5,3,3,1,1) =$$  .

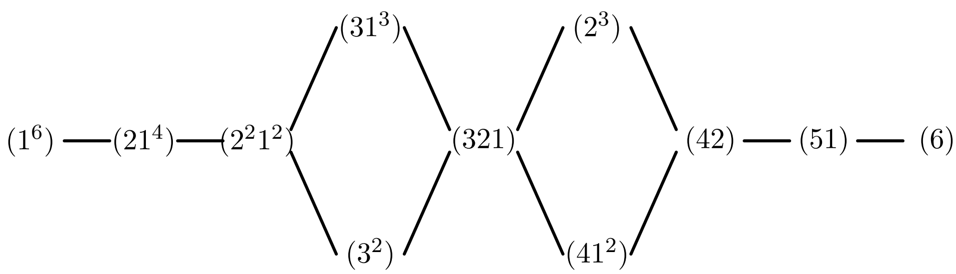
A *partition of* k is a partition λ with k boxes. Write $\lambda \vdash k$ if λ is a partition of k. Make the convention that $\lambda_i = 0$ if $i > \ell(\lambda)$. The *dominance order* is the partial order on the set of partitions of k,

$$P^+(k) = \{\text{partitions of } k\} = \{\lambda = (\lambda_1, \dots, \lambda_\ell) \mid \lambda_1 \geq \dots \geq \lambda_\ell > 0,\ \lambda_1 + \ldots + \lambda_\ell = k\},$$

given by

$$\lambda \geq \mu \qquad \text{if} \qquad \lambda_1 + \lambda_2 + \dots + \lambda_i \geq \mu_1 + \mu_2 + \dots + \mu_i \quad \text{for all } 1 \leq i \leq \max\{\ell(\lambda), \ell(\mu)\}.$$

For example, for $k = 6$ the Hasse diagram of the dominance order is



Let λ be a partition and let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$ be a sequence of nonnegative integers. A *column strict tableau of shape* λ *and weight* μ is a filling of the boxes of λ with μ_1 1s, μ_2 2s, ..., μ_n ns, such that

(a) the rows are weakly increasing from left to right,

(b) the columns are strictly increasing from top to bottom.

If p is a column strict tableau write $\mathrm{shp}(p)$ and $\mathrm{wt}(p)$ for the shape and the weight of p so that

$$\begin{aligned} \mathrm{shp}(p) &= (\lambda_1, \dots, \lambda_n), &\quad \text{where} \quad \lambda_i &= \text{number of boxes in row } i \text{ of } p, \quad \text{and} \\ \mathrm{wt}(p) &= (\mu_1, \dots, \mu_n), &\quad \text{where} \quad \mu_i &= \text{number of } i\,\text{s in } p. \end{aligned}$$

For example,

$$p = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 2 & 2 & 3 & 3 & 4 \\ \cline{1-7} 3 & 3 & 3 & 4 & 4 & 4 & 5 \\ \cline{1-7} 4 & 5 & 5 & 6 \\ \cline{1-4} 6 & 7 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} \qquad \text{has} \quad \begin{array}{l} \mathrm{shp}(p) = (9,7,7,4,2,1,0) \quad \text{and} \\ \mathrm{wt}(p) = (7,6,5,5,3,2,2). \end{array}$$

For a partition λ and a sequence $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}$ of nonnegative integers write

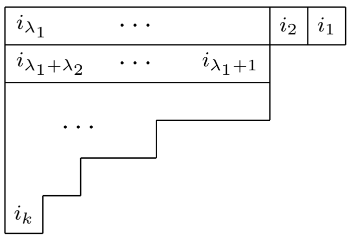
$$
\begin{aligned}
B(\lambda) &= \{\text{column strict tableaux } p \mid \operatorname{shp}(p) = \lambda\}, \\
B(\lambda)_\mu &= \{\text{column strict tableaux } p \mid \operatorname{shp}(p) = \lambda \text{ and } \operatorname{wt}(p) = \mu\},
\end{aligned}
\tag{1.2}
$$

Let λ be a partition with k boxes and let

$$B(\lambda) = \{\text{column strict tableaux of shape } \lambda\}.$$

The set $B(\lambda)$ is a subset of $B(\varepsilon_1)^{\otimes k}$ via the injection

$$
\begin{array}{ccc}
B(\lambda) & \hookrightarrow & B(\varepsilon_1)^{\otimes k} \\
p & \longmapsto & (\text{the arabic reading of } p)
\end{array}
$$



$$\longmapsto \qquad \varepsilon_{i_1} \otimes \varepsilon_{i_2} \otimes \cdots \otimes \varepsilon_{i_k}$$

where the arabic reading of p is $\varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_k}$ if the entries of p are $i_1, i_2, \ldots, i_k$ read right to left by rows with the rows read in sequence beginning with the first row.

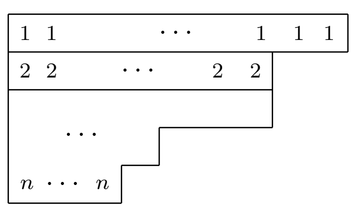
Proposition 1.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition with k boxes. Then $B(\lambda)$ is the subset of $B(\varepsilon_1)^{\otimes k}$ generated by*

$$p_\lambda = \underbrace{\varepsilon_1 \otimes \varepsilon_1 \otimes \cdots \otimes \varepsilon_1}_{\lambda_1 \text{ factors}} \otimes \underbrace{\varepsilon_2 \otimes \varepsilon_2 \otimes \cdots \otimes \varepsilon_2}_{\lambda_2 \text{ factors}} \otimes \cdots \otimes \underbrace{\varepsilon_n \otimes \varepsilon_n \otimes \cdots \otimes \varepsilon_n}_{\lambda_n \text{ factors}}$$

under the action of the operators $\tilde{e}_i$, $\tilde{f}_i$, $1 \leq i \leq n$.

Proof. If $P = P(b)$ is a filling of the shape λ then $b_{i_1} \otimes \cdots b_{i_k} = b$ is obtained from P by reading the entries of P in arabic reading order (right to left across rows and from top to bottom down the page). The tableau

$$P_\lambda = P(p_\lambda) =$$



is the column strict tableau of shape λ with 1s in the first row, 2s in the second row, and so on. Define operators $\tilde{e}_i$ and $\tilde{f}_i$ on a filling of λ by

$$\tilde{e}_i P = P(\tilde{e}_i p) \quad \text{and} \quad \tilde{f}_i P = P(\tilde{f}_i b), \qquad \text{if } P = P(b).$$

To prove the proposition we shall show that if P is a column strict tableau of shape λ then

(a) $\tilde{e}_i P$ and $\tilde{f}_i P$ are column strict tableaux,

(b) P can be obtained by applying a sequence of $\tilde{f}_i$ to P_λ. Let $P^{(j)}$ be the column strict tableau formed by the entries of P which are $\le j$ and let $\lambda^{(j)} = \mathrm{shp}(P^{(j)})$. Identify P with the sequence

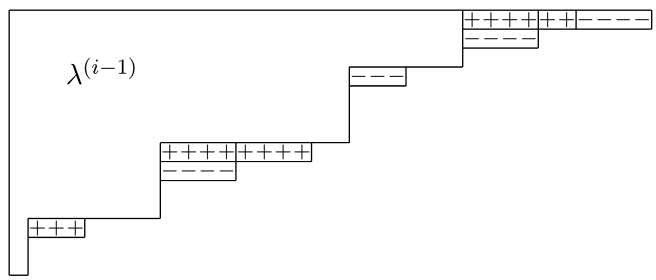
$$P = \left(\emptyset = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(n)} = \lambda\right), \qquad \text{where}$$
$$\lambda^{(i)}/\lambda^{(i-1)} \text{is a horizontal strip for each } 1 \le i \le n.$$

(a) Let us analyze the action of $\tilde{e}_i$ and $\tilde{f}_i$ on P. The sequence of $+1, -1, 0$ constructed in (???) is given by

placing $+1$ in each box of $\lambda^{(i)}/\lambda^{(i-1)}$,
placing -1 in each box of $\lambda^{(i+1)}/\lambda^{(i)}$,
placing 0 in each box of $\lambda^{(j)}/\lambda^{(j-1)}$, for $j \neq i, i+1$,

and reading the resulting filling in Arabic reading order, see (???). The process of removing $+1, -1$ pairs can be executed on the horizontal strips $\lambda^{(i+1)}/\lambda^{(i)}$ and $\lambda^{(i)}/\lambda^{(i-1)}$,

$\lambda^{(i+1)} =$



with the effect that the entries in any configuration of boxes of the form

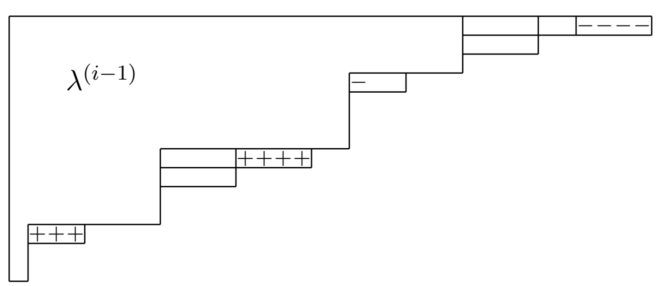
$+1$	$+1$	$\cdots$	$+1$
-1	-1	$\cdots$	-1

will be removed. Other $+1, -1$ pairs will also be removed and the final sequence

$$\underbrace{-1\ -1\ \cdots\ -1}_{d_+(p)}\underbrace{+1+1\ \cdots\ +1}_{d_-(p)} \tag{1.3}$$

will correspond to a configuration of the form

$\lambda^{(i+1)} =$



The rightmost -1 in the sequence (*) corresponds to a box in $\lambda^{(i+1)}/\lambda^{(i)}$ which is leftmost in its row and which does not cover a box of $\lambda^{(i)}/\lambda^{(i-1)}$. Similarly the leftmost $+1$ in the sequence (*) corrresponds to a box in $\lambda^{(i)}/\lambda^{(i-1)}$ which is rightmost in its row and which does not have a box of $\lambda^{(i+1)}/\lambda^{(i)}$ covering it. These conditions guarantee that $\tilde{e}_iP$ and $\tilde{f}_iP$ are column strict tableaux.

(b) Applying the operator

$$\tilde{f}_{n,i} = \tilde{f}_{n-1} \cdots \tilde{f}_{i+1}\tilde{f}_i \quad \text{to} \quad P_\lambda$$

will change the rightmost i in row i to n. A sequence of applications of

$$\tilde{f}_{n,i}, \quad \text{as } i \text{ decreases (weakly) from } n-1 \text{ to } 1,$$

can be used to produce a column strict tableau P_n in which

(1) the entries equal to n match the entries equal to n in P, and

(2) the subtableau of P_n containing the entries $\leq n-1$ is $P_{\lambda(n-1)}$.

Iterating the process and applying a sequence of operators

$$\tilde{f}_{n-1,i}, \quad \text{as } i \text{ decreases (weakly) from } n-2 \text{ to } 1,$$

to the tableau P_n can be used to produce a tableau P_{n-1} in which

(1) the entries equal to n and $n-1$ match the entries equal to n and $n-1$ in P, and

(2) the subtableau of P_{n-1} containing the entries $\leq n-2$ is $P_{\lambda(n-2)}$.

The tableau P is obtained after a total of n iterations of this process. $\square$