Algebras with Kazhdan-Lusztig theories

Arun Ram
Department of Mathematics
University of Wisconsin
Madison, WI 53706
ram@math.wisc.edu

February 17, 2005

1 Convolution algebras

1.1 Cellular algebras

A *cellular algebra* is an algebra A with

a basis	$\{a^\lambda_{ST} \mid \lambda \in \hat{A},\ S,T \in \hat{A}^\lambda\}$	
an involutive antihomomrphism	${}^*\colon A \to A,$	and
a partial order	$\le$ on $\hat{A}$	

such that

(a) $(a^\lambda_{ST})^* = a^\lambda_{TS}$,

(b) If $A(<\lambda) = \text{span-}\{a^\mu_{ST} \mid \mu < \lambda\}$

then

$$aa^\lambda_{ST} = \sum_{Q \in \hat{A}^\lambda} A^\lambda(a)_{QT} a^\lambda_{QT} \mod A(<\lambda), \qquad \text{for all } a \in A.$$

Applying the involution * to (b) and using (a) gives that

$$a^\lambda_{TS} a^* = \sum_{Q \in \hat{A}^\lambda} A^\lambda(a)_{QS} a^\lambda_{TQ} \mod A(<\lambda), \qquad \text{for all } a \in A.$$

1.2 The decomposition theorem

The concept of a cellular algebra is not really the "right" one. The "right" one comes from the structure of a convolution algebra whenever the decomposition theorem holds [CG, 8.6.9].

Let M be a smooth G-variety and let N be a G-variety with finitely many G-orbits such that the orbit decomposition is an algebraic stratification of N,

$$N = \bigsqcup_{\varphi} Gx_\varphi, \qquad \text{and} \qquad \mu\colon M \longrightarrow N$$

Research supported in part by NSF Grant ????.

is a G-equivariant projective morphism. Let $\mathcal{C}_M$ be the constant perverse sheaf on M. The decomposition theorem [CG, 8.4.12] says that

$$\mu_*\mathcal{C}_M = \bigoplus_{\substack{i\in\mathbb{Z}\\ \lambda=(\varphi,\chi)\in\hat{M}}} L(\lambda,i)\otimes IC^\lambda[i] \doteq \bigoplus_{\lambda\in\hat{M}} L(\lambda)\otimes IC^\lambda, \qquad \text{where} \qquad L(\lambda)=\bigoplus_{i\in\mathbb{Z}} L(\lambda,i),$$

μ_* is the derived functor of sheaf theoretic direct image, λ runs over the indexes of the intersection cohomology complexes IC^λ, $L(\lambda)$ are finite dimensional vector spaces, and $\doteq$ indicates an equality up to shifts in the derived category.

Let $x \in N$ and define

$$Z = M\times_N M = \{(m_1,m_2)\in M\times M \mid \mu(m_1)=\mu(m_2)\} \qquad \text{and} \qquad M_x = \mu^{-1}(x).$$

There are commutative diagrams

$$\begin{array}{ccc} Z = M\times_N M & \xrightarrow{\iota} & M\times M \\ \big\downarrow{\scriptstyle \mu_{12}} & & \big\downarrow{\scriptstyle \mu_1\times\mu_2} \\ N = N_\Delta & \xrightarrow{\Delta} & N\times N \end{array} \qquad \text{and} \qquad \begin{array}{ccc} M_x & \xrightarrow{\iota} & M \\ \big\downarrow{\scriptstyle \mu} & & \big\downarrow{\scriptstyle \mu} \\ \{x\} & \xrightarrow{i_x} & N \end{array}$$

which (via base change) provide isomorphisms

$$\begin{aligned} H_*(Z) &= \mathrm{Hom}_{D^b(Z_{12})}(\mathbb{C}_{Z_{12}}, (\mathbb{C}_{Z_{12}}[*])^\vee) \\ &= \mathrm{Hom}_{D^b(Z_{12})}(\mu_{12}^*\mathbb{C}_N, \iota^!\mathcal{C}_{M_1\times M_2}[m_1+m_2][-*]) \\ &= \mathrm{Hom}_{D^b(N)}(\mathbb{C}_N, (\mu_{12})_*\iota^!\mathcal{C}_{M_1\times M_2}[m_1+m_2-*]) \\ &= \mathrm{Hom}_{D^b(N)}(\mathbb{C}_N, \Delta^!(\mu_1\times\mu_2)_*(\mathcal{C}_{M_1}\boxtimes\mathcal{C}_{M_2})[m_1+m_2-*]) \\ &= \mathrm{Hom}_{D^b(N)}(\mathbb{C}_N, \Delta^!((\mu_1)_*\mathcal{C}_{M_1}\boxtimes(\mu_2)_*\mathcal{C}_{M_2})[m_1+m_2-*]) \\ &= \mathrm{Ext}^{m_1+m_2-*}_{D^b(N)}((\mu_1)_*\mathcal{C}_{M_1}, (\mu_2)_*\mathcal{C}_{M_2}), \end{aligned}$$

$$\begin{aligned} H_*(M_x) &= \mathrm{Hom}_{D^b(M_x)}(\mathbb{C}_{M_x}, (\mathbb{C}_{M_x}[*])^\vee) = \mathrm{Hom}_{D^b(M_x)}(\mu^*\mathbb{C}_{\{x\}}, ((\iota^*\mathbb{C}_M)[*])^\vee) \\ &= \mathrm{Hom}_{D^b(\{x\})}(\mathbb{C}_{\{x\}}, \mu_*(\iota^!\mathbb{C}_M[2m])[-*]) = \mathrm{Hom}_{D^b(\{x\})}(\mathbb{C}_{\{x\}}, i_x^!\mu_*\mathcal{C}_M[m-*]) \\ &= H^{m-*}(i_x^!\mu_*\mathcal{C}_M), \end{aligned}$$

and

$$\begin{aligned} H^*(M_x) &= \mathrm{Hom}_{D^b(M_x)}(\mathbb{C}_{M_x}, \mathbb{C}_{M_x}[*]) = \mathrm{Hom}_{D^b(M_x)}(\mu^*\mathbb{C}_{\{x\}}, \mathbb{C}_{M_x}[*]) \\ &= \mathrm{Hom}_{D^b(\{x\})}(\mathbb{C}_{\{x\}}, \mu_*\mathbb{C}_{M_x}[*]) = \mathrm{Hom}_{D^b(\{x\})}(\mathbb{C}_{\{x\}}, \mu_!\iota^*\mathbb{C}_M[*]) \\ &= \mathrm{Hom}_{D^b(\{x\})}(\mathbb{C}_{\{x\}}, i_x^*\mu_!\mathbb{C}_M[*]) = \mathrm{Hom}_{D^b(\{x\})}(\mathbb{C}_{\{x\}}, i_x^*\mu_*\mathcal{C}_M[*-m]) \\ &= H^{*-m}(i_x^*\mu_*\mathcal{C}_M). \end{aligned}$$

1.3 Convolution algebras

Let $\mu\colon M\to N$ be a proper map. The *convolution algebra* is

$$A = \mathrm{Ext}^*_{D^b(N)}(\mu_*\mathcal{C}_M, \mu_*\mathcal{C}_M) = \bigoplus_{k\in\mathbb{Z}} \mathrm{Ext}^k(\mu_*\mathcal{C}_M, \mu_*\mathcal{C}_M),$$

where

$$\mathrm{Ext}^k_{D^b(X)}(A,B) = \mathrm{Hom}_{D^b(X)}(A, B[k]),$$

with product given by the *Yoneda product*

$$\mathrm{Ext}^p_{D^b(N)}(A_1,A_2) \otimes \mathrm{Ext}^q_{D^b(N)}(A_2,A_3) \longrightarrow \mathrm{Ext}^{p+q}_{D^b(N)}(A_1,A_3)$$

which arises from the composition map

$$\mathrm{Hom}_{D^b(N)}(A_1,A_2[p]) \otimes \mathrm{Hom}_{D^b(N)}(A_2[p],A_3[p+q]) \longrightarrow \mathrm{Hom}_{D^b(N)}(A_1,A_3[p+q])$$

and the identification

$$\mathrm{Hom}_{D^b(N)}(A_2,A_3[q]) \cong \mathrm{Hom}_{D^b(N)}(A_2[p],A_3[p+q]).$$

Then the decomposition theorem for $\mu_*\mathcal{C}_M$ induces a decomposition of A. Since the intersection cohomology complexes IC_ϕ are the simple objects in the category of perverse sheaves,

$$\mathrm{Ext}^0_{D^b(N)}(IC^\lambda, IC^\mu) = \delta_{\lambda\mu}\mathbb{C}, \qquad \text{and} \qquad \mathrm{Ext}^k_{D^b(N)}(IC^\lambda, IC^\mu) = 0, \qquad \text{for } k \in \mathbb{Z}_{<0},$$

and the decomposition of A simplifies to

$$A = \bigoplus_{\lambda\in\hat{M}} \mathrm{End}_{\mathbb{C}}(L(\lambda)) \bigoplus \left(\bigoplus_{\lambda,\mu\in\hat{M}} \mathrm{Hom}_{\mathbb{C}}(L(\lambda),L(\mu)) \otimes \Big(\bigoplus_{k\in\mathbb{Z}_{>0}} \mathrm{Ext}^k_{D^b(N)}(IC^\lambda, IC^\mu) \Big) \right).$$

In this context there is a good theory of projective, standard and simple modules, and their decomposition matrices satisfy a BGG reciprocity. View elements of A as sums

$$\sum_{\lambda,\mu} \sum_{P\in\hat{L}(\lambda),Q\in\hat{L}(\mu)} c^{\lambda\mu}_{PQ} a^{\lambda\mu}_{PQ} \quad \text{where} \quad c^{\lambda\lambda}_{PQ} \in \mathbb{C}, \quad \text{and} \quad a^{\lambda\mu}_{PQ} \in \bigoplus_{k>0} \mathrm{Ext}^k_{D^b(N)}(IC^\lambda, IC^\mu).$$

The algebra A is completely controlled by the multiplication in

$$\mathrm{Ext}^*(IC, IC) \qquad \text{where} \qquad IC = \bigoplus_{\lambda\in\hat{M}} IC^\lambda.$$

an algebra which has all one dimensional simple modules. The radical of A is

$$\mathrm{Rad}(A) = \bigoplus_{\lambda,\mu\in\hat{M}} \mathrm{Hom}_{\mathbb{C}}(L(\lambda),L(\mu)) \otimes \Big(\bigoplus_{k\in\mathbb{Z}_{>0}} \mathrm{Ext}^k_{D^b(N)}(IC^\lambda, IC^\mu) \Big)$$

and the nonzero

$$L(\lambda) \text{ are the simple } A\text{-modules.}$$

1.4 Projective modules

Let e^λ be a minimal idempotent in $\bigoplus_\mu \mathrm{End}(L(\mu))$. Then

$$P(\lambda) = Ae^\lambda = L(\lambda) \bigoplus \left(\bigoplus_{\substack{k>0\\ \mu}} L(\mu) \otimes \mathrm{Ext}^k_{D^b(N)}(IC^\mu, IC^\lambda) \right)$$

is the projective cover of the simple A-module $L(\lambda)$. Define an A-module filtration

$$P(\lambda) \supseteq P(\lambda)^{(1)} \supseteq P(\lambda)^{(2)} \supseteq \cdots$$

by

$$P(\lambda)^{(m)} = \bigoplus_{\substack{k\geq m\\ \mu}} L(\mu) \otimes \mathrm{Ext}^k_{D^b(N)}(IC^\mu, IC^\lambda).$$

Then

$$L(\lambda) = P(\lambda)/P(\lambda)^{(1)} \qquad \text{and} \qquad \mathrm{gr}\big(P(\lambda)\big) \qquad \text{is a semisimple } A\text{-module.}$$

Thus the multiplicity of the simple A-module $L(\mu)$ in a composition series of $P(\lambda)$ is

$$[P(\lambda) : L(\mu)] = \dim\big(\mathrm{Ext}^*(IC_{\mathbb{O},\chi}, IC_{\mathbb{O}',\chi'})\big) = \sum_{k\geq 0} \dim\big(\mathrm{Ext}^k_{D^b(N)}(IC^\mu, IC^\lambda)\big).$$

1.5 Standard and costandard modules

Let $\lambda = (\varphi, \chi)$,

$$x \in \mathbb{O}^\varphi, \qquad \text{and let} \qquad i_x\colon \{x\} \hookrightarrow N \quad \text{be the injection.}$$

Then $i_x^! \mu_* \mathcal{C}_M$ is the *stalk* of $\mu_*\mathcal{C}_M$ at x and the Yoneda product makes

$$\Delta^\varphi = H^*(i_x^! \mathcal{C}_M) = \mathrm{Hom}_{D^b(\{x\})}(\mathbb{C}, i_x^!\mu_*\mathcal{C}_M[*]) = \mathrm{Hom}_{D^b(N)}((i_x)_!\mathbb{C}[-*], \mu_*\mathcal{C}_M), \quad \text{and}$$
$$\nabla^\varphi = H^*(i_x^* \mathcal{C}_M) = H^*(\{x\}, i_x^*\mu_*\mathcal{C}_M) = \mathrm{Hom}_{D^b(\{x\})}(\mathbb{D}, i_x^!\mu_*\mathcal{C}_M[*]) = \mathrm{Hom}_{D^b(N)}((i_x)_!\mathbb{C}[-*], \mu_*\mathcal{C}_M),$$

into right A-modules. The action of an element $a \in \mathrm{Ext}^k(\mu_*\mathcal{C}_M, \mu_*\mathcal{C}_M) = \mathrm{Hom}_{D^b(N)}(\mu_*\mathcal{C}_M, \mu_*\mathcal{C}_M[k])$ sends

$$H^*(\{x\}, i_x^!\mu_*\mathcal{C}_M) \longrightarrow H^{*+k}(\{x\}, i_x^!\mu_*\mathcal{C}_M).$$

A G-equivariant *local system* is a G-equivariant locally constant sheaf. The orbit $\mathbb{O}^\varphi$ can be identified with G/G_x where G_x is the stabilizer of x. $\pi_0(\mathbb{O}^\varphi, x) = G_x/G_x^\circ$ where G_x° is the connected component of the identity in G_x). There is a homomorphism $\pi_1(\mathbb{O}^\varphi, x) \to \pi_0(\mathbb{O}^\varphi, x) = G_x/G_x^\circ$ and the representations of $\pi_1(\mathbb{O}^\varphi, x)$ on the fibers $\mathcal{L}_x$ of G-equivariant local systems $\mathcal{L}$ are exactly the pullbacks of finite dimensional representations of $C = G_x/G_x^\circ$ to $\pi_1(\mathbb{O}^\varphi, x)$. In this way the irreducible G-equivariant local systems on $\mathbb{O}^\varphi$ can be indexed by (some of the) irreducible representations of G_x/G_x° [CG, Lemma 8.4.11]. There is an action of $C = G_x/G_x^\circ$ on Δ^φ which commutes with the action of A. Similar arguments apply to ∇^φ. As (A, C) bimodules,

$$\Delta^\varphi = \bigoplus_{\chi\in\hat{C}} \Delta(\varphi,\chi) \otimes \chi \qquad \text{and} \qquad \nabla^\varphi = \bigoplus_{\chi\in\hat{C}} \nabla(\varphi,\chi) \otimes \chi,$$

and the *standard* and *costandard* A-modules are

$$\Delta(\lambda) = \Delta(\varphi,\chi) \qquad \text{and} \qquad \nabla(\lambda) = \nabla(\varphi,\chi).$$

Using the decomposition theorem

$$\Delta(\lambda) = H^*(i_x^!\mathcal{C}_M)_\chi = \bigoplus_{\substack{k\in\mathbb{Z}\\ \mu}} L(\mu) \otimes H^k(i_x^! IC^\mu)_\chi,$$

where the subscript χ denotes the χ-isotypic component. Define a filtration

$$\Delta(\lambda) \supseteq \Delta(\lambda)^{(1)} \supseteq \Delta(\lambda)^{(2)} \supseteq \cdots \qquad \text{by} \qquad \Delta(\lambda)^{(m)} = \bigoplus_{j\geq m} \bigoplus_{\phi} L(\mu) \otimes H^j(i_x^! IC^\mu)_\chi.$$

Then $\Delta(\lambda)^{(m)}$ is an A-module and $\mathrm{gr}\big(\Delta(\lambda))\big)$ is a semisimple A-module. This (and a similar argument for $\nabla(\lambda)$) show that the multiplicity of the simple A-module $L(\mu)$ in composition series of $\Delta(\lambda)$ and $\nabla(\lambda)$ are

$$[\Delta(\lambda) : L(\mu)] = \sum_k \dim\big(H^k(i_x^! IC^\mu)_\chi\big) \qquad \text{and} \qquad [\nabla(\lambda) : L(\mu)] = \sum_k \dim\big(H^k(i_x^* IC^\mu)_\chi\big).$$

Define the *standard KL-polynomial* and the *costandard KL-polynomial* of A to be

$$P^\Delta_{\lambda\mu}(\mathtt{t}) = \sum_k \mathtt{t}^k \dim\big(H^k(i_x^! IC^\mu)_\chi\big) \qquad \text{and} \qquad P^\nabla_{\lambda\mu}(\mathtt{t}) = \sum_k \mathtt{t}^k \dim\big(H^k(i_x^* IC^\mu)_\chi\big),$$

respectively. Then ??? says that

$$[\Delta(\lambda) : L(\mu)] = P^\Delta_{\lambda\mu}(1) \qquad \text{and} \qquad [\nabla(\lambda) : L(\mu)] = P^*_{\lambda\mu}(1).$$

These identities are analogues of the original Kazhdan-Lusztig conjecture describing the multiplicities of simple $\mathfrak{g}$-modules in Verma modules.

1.6 The contravariant form

Note that there is a canonical homomorphism

$$\Delta(\lambda) \xrightarrow{c_\lambda} \nabla(\lambda)$$

coming from applying the functor H^* to the composition

$$(i_x)_!(i_x)^! \mu_* \mathcal{C}_M \longrightarrow \mu_* \mathcal{C}_M \longrightarrow (i_x)_*(i_x)^* \mu_* \mathcal{C}_M,$$

where the two maps arise from the canonical adjoint functor maps. Use the map c_λ to define a bilinear form on $\Delta(\lambda)$ by

$$\begin{array}{rccc} \langle , \rangle : & \Delta(\lambda) \otimes \Delta(\lambda) & \longrightarrow & \mathbb{C} \\ & m_1 \otimes m_2 & \longmapsto & m_1 \cap c_\lambda(m_2) \end{array}$$

Then

$$L(\lambda) = \Delta(\lambda)/\mathrm{Rad}(\langle , \rangle).$$

1.7 Contragradient modules

There is an involutive antiautomorphism ${}^t \colon A \to A$ on A (coming from switching the two factors in $Z = M \times_N M$). If M is an A-module the *contragredient* module is

$$M^* = \mathrm{Hom}_\mathbb{C}(M, \mathbb{C}) \qquad \text{with} \qquad (a\psi)(m) = \psi(a^t(m)), \quad \text{for } a \in A,\ \psi \in M^*, \text{ and } m \in M.$$

Then

$$\nabla(\lambda) \cong \Delta(\lambda)^*.$$

1.8 Reciprocity

If $\lambda = (\varphi, \rho)$ define

$$d_\lambda = \dim_{\mathbb{C}}(\mathbb{O}^\varphi), \qquad \text{and assume that} \qquad \mathrm{Ext}^{d_\psi+d_\varphi+k}_{D^b(N)}(IC^\varphi, IC^\psi) = 0, \quad \text{for all odd } k.$$

Then

$$\begin{aligned}
[P(\lambda) : L(\mu)] &= \sum_k \mathrm{dim}\mathrm{Ext}^k_{D^b(N)}(IC^\lambda, IC^\mu) \\
&= \sum_k \mathrm{dim}\mathrm{Ext}^{d_\lambda+d_\mu+k}_{D^b(N)}(IC^\lambda, IC^\mu) \\
&= \sum_k (-1)^k \mathrm{dim}\mathrm{Ext}^{d_\lambda+d_\mu+k}_{D^b(N)}(IC^\lambda, IC^\mu) \\
&= (-1)^{d_\phi+d_\psi} \sum_{\mathbb{O}} \chi(\mathbb{O}, i^!_{\mathbb{O}} IC^\vee_\phi \overset{!}{\otimes} i^!_{\mathbb{O}} IC_\psi) \\
&= (-1)^{d_\phi+d_\psi} \sum_{\mathbb{O}} \chi \left(\mathbb{O}, (-1)^{d_\phi} \sum_{\alpha,k} [\mathcal{H}^k i^!_{\mathbb{O}}(IC^\vee_\phi) : \alpha]\alpha \overset{!}{\otimes} (-1)^{d_\psi} \sum_{\beta,\ell} [\mathcal{H}^\ell i^!_{\mathbb{O}}(IC_\psi) : \beta]\beta \right) \\
&= \sum_{\mathbb{O},\alpha,\beta} \chi \left(\mathbb{O}, \sum_k [\mathcal{H}^k i^!_{\mathbb{O}}(IC_\phi) : \alpha^*]\alpha \overset{!}{\otimes} \sum_\ell [\mathcal{H}^\ell i^!_{\mathbb{O}}(IC_\psi) : \beta]\beta \right) \\
&= \sum_{\alpha,\beta} \sum_k \mathrm{dim}\mathcal{H}^k(i^!_\alpha IC_\phi) \left(\sum_{\mathbb{O}} \chi(\mathbb{O}, \alpha^* \overset{!}{\otimes} \beta) \right) \sum_\ell \mathrm{dim}\mathcal{H}^\ell(i^!_\beta IC_\psi) \\
&= \sum_{\alpha,\beta} [\mathcal{M}^!_\alpha : L_\phi] \left(\sum_{\mathbb{O}} \chi(\mathbb{O}, \alpha^* \otimes \beta) \right) [\mathcal{M}^!_\beta : L_\psi] \\
&= \sum_{\alpha,\beta} P_{\phi\alpha}(1) D_{\alpha\beta} P_{\psi\beta}(1) \\
&= (PDP^t)_{\phi\psi},
\end{aligned}$$

where

(1) the third equality follows from the vanishing of Ext groups in odd degrees,

(2) χ denotes the Euler characteristic,

(3) P is the matrix $(P_{\phi\alpha}(1))$, and

(4) D is the matrix $(\sum_{\mathbb{O}} \chi(\mathbb{O}, \alpha^* \otimes \beta))$.

This identity is the "BGG reciprocity" for the algebra A.

1.9 The category $D^b(N)$

The category $Comp^b(Sh(N))$ is the cateogry of all finite complexes

$$A = (0 \to A^{-m} \to A^{-m+1} \to \cdots \to A^{n-1} \to A^n \to 0), \qquad m, n \in \mathbb{Z}_{>>0},$$

of sheaves on N with morphisms being morphisms of complexes which commute with the differentials. The jth cohomology sheaf of A is

$$\mathcal{H}^j(A) = \frac{\ker(A^j \to A^{j+1})}{\mathrm{im}(A^{j-1} \to A^j)}.$$

A morphism in $Comp^b(Sh(N))$ is a *quasi-isomorphism* if it induces isomorphisms on cohomology. The category $D^b(Sh(N))$ is the category $Comp^b(Sh(N))$ with additional morphisms obtained by formally inverting all quasi-isomorphisms.

Assume that N is a G-variety with a finite number of orbits such that the G-orbit decomposition

$$N = \bigsqcup_{\varphi} \mathbb{O}^{\varphi} \qquad \text{is an algebraic stratification of } X.$$

A *constructible sheaf* is a sheaf that is locally constant on strata of N. A *constructible complex* is a complex such that all of its cohomology sheaves are constructible.

The *derived category of bounded constructible complexes of sheaves* on N is the full subcategory $D^b(N)$ of $D^b(Sh(N))$ consisting of constructible complexes. *Full* means that the morphisms in $D^b(N)$ are the same as those in $D^b(Sh(N))$.

The *shift functor* $[i]\colon D^b(N) \to D^b(N)$ is the functor that shifts all complexes by i.

The *Verdier duality functor* $^\vee\colon D^b(N) \to D^b(N)$ is defined by requiring

$\mathrm{Hom}_{D^b(N)}(A_1, A_2[i]) = \mathrm{Hom}_{D^b(N)}(\Delta^*(A_1 \boxtimes A_2^\vee)[-i], \mathbb{C}_N[2\dim_{\mathbb{C}} N]),$ for all $i \in \mathbb{Z}$, where

$\Delta\colon N \to N \times N$ is the diagonal map.

The Verdier duality functor satisfies the properties

$$(A^\vee)^\vee = A, \qquad (A[i\,])^\vee = A^\vee[-i\,], \qquad \text{and} \qquad \mathrm{Hom}_{D^b(N)}(A_1, A_2) = \mathrm{Hom}_{D^b(N)}(A_2^\vee, A_1^\vee).$$

Define

$$\begin{array}{ll}
\mathrm{Ext}^k_{D^b(X)}(A_1, A_2) = \mathrm{Hom}_{D^b(X)}(A_1, A_2[k]), & \\
H^k(A) = H^k(X, A) = \mathrm{Hom}_{D^b(X)}(\mathbb{C}_X, A[k]), & \text{the } \textit{hypercohomology} \text{ of } A \in D^b(N), \\
H^k(N) = \mathrm{Hom}_{D^b(N)}(\mathbb{C}_N, \mathbb{C}_N[k]), & \text{the } \textit{cohomology} \text{ of } N, \\
H_k(N) = \mathrm{Hom}_{D^b(N)}(\mathbb{C}_N, (\mathbb{C}_N[k])^\vee), & \text{the } \textit{Borel-Moore homology} \text{ of } N, \\
\mathbb{D}_X = \mathbb{C}_X^\vee, & \text{the } \textit{dualizing complex},
\end{array}$$

respectively. The *Yoneda product*

$$\mathrm{Ext}^p_{D^b(N)}(A_1, A_2) \times \mathrm{Ext}^q_{D^b(N)}(A_2, A_3) \longrightarrow \mathrm{Ext}^{p+q}_{D^b(N)}(A_1, A_3)$$

is given by

$$\mathrm{Hom}_{D^b(N)}(A_1, A_2[p]) \times \mathrm{Hom}_{D^b(N)}(A_2[p], A_3[p+q]) \longrightarrow \mathrm{Hom}_{D^b(N)}(A_1, A_3[p+q]),$$

using the canonical identification $\mathrm{Hom}_{D^b(N)}(A_2, A_3[q]) \cong \mathrm{Hom}_{D^b(N)}(A_2[p], A_3[p+q])$.

If $f\colon X \to Y$ is a morphism define

$$\begin{aligned}
f_* &= \text{derived functor of sheaf theoretic direct image,} \\
f^* &= \text{derived functor of sheaf theoretic inverse image,}
\end{aligned}$$

$$f^! A = (f^* A^\vee)^\vee, \text{ for } A \in D^b(Y), \qquad \text{and} \qquad f_! A = (f_* A^\vee)^\vee, \text{ for } A \in D^b(X).$$

Then

$$\begin{aligned} &\mathrm{Hom}_{D^b(X)}(f^* A_1, A_2) = \mathrm{Hom}_{D^b(Y)}(A_1, f_* A_2), \qquad \text{and} \\ &\mathrm{Hom}_{D^b(X)}(A_2, f^! A_1) = \mathrm{Hom}_{D^b(Y)}(f_! A_2, A_1). \end{aligned}$$

If $f\colon X \to Z$ and $g\colon Y \to Z$ define The *base change formula* is

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\pi_2} & Y \\ \big\downarrow{\scriptstyle \pi_1} & & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array} \qquad g^! f_* A = (\pi_2)_* \pi_1^! A, \qquad \text{for } A \in D^b(X),$$

where $X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$.

The category of *perverse sheaves* on X is a full subcategory of $D^b(X)$ which is abelian. The simple objects in the category of perverse sheaves are the *intersection cohomology complexes*

$$IC_\phi \qquad \text{indexed by pairs} \qquad \phi = (\mathbb{O}, \chi),$$

where $\mathbb{O}$ is a G-orbit on X and χ is an irreducible local system on X. By ???, the local systems χ on $\mathbb{O}$ can be identified with (some of the) representations of the *component group* $Z_G(x)/Z_G(x)^\circ$ where x is a point in $\mathbb{O}$. If X is smooth the *constant perverse sheaf* $\mathcal{C}_X$ on X is given by

$$\mathcal{C}_X\big|_{X_i} = \mathbb{C}_{X_i}[\dim_{\mathbb{C}} X_i],$$

on the irreducible components of X. Since the intersection cohomology complexes IC_ϕ are the simple objects of the category of perverse sheaves,

$$\mathrm{Ext}^0_{D^b(N)}(IC_\phi, IC_\psi) = \mathbb{C} \cdot \delta_{\phi\psi} \qquad \text{and} \qquad \mathrm{Ext}^k_{D^b(N)}(IC_\phi, IC_\psi) = 0, \quad \text{if } k > 0.$$

References

[Dr1] .G. Drinfel'd, *A new realization of Yangians and quantized affine algebras*, Soviet Math. Dokl. **36** No, 2 (1998), 212–216.