The center of the affine Hecke algebra

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1 The center of $\tilde{H}$

Theorem 1.1. *The center of the affine Hecke algebra is the ring*

$$Z(\tilde{H}) = \mathbb{K}[P]^W = \{f \in \mathbb{K}[P] \mid wf = f \text{ for all } w \in W\}$$

of symmetric functions in $\mathbb{K}[P]$.

Proof. If $z \in \mathbb{K}[P]^W$ then by the fourth relation in (???), $T_i z = (s_i z)T_i + (q-q^{-1})(1-x^{-\alpha_i})^{-1}(z - s_i z) = zT_i + 0$, for $1 \le i \le n$, and by the third relation in (???), $zx^\lambda = x^\lambda z$, for all $\lambda \in P$. Thus z commutes with all the generators of $\tilde{H}$ and so $z \in Z(\tilde{H})$.

Assume

$$z = \sum_{\lambda \in P, w \in W} c_{\lambda,w} x^\lambda T_w \in Z(\tilde{H}).$$

Let $m \in W$ be maximal in Bruhat order subject to $c_{\gamma,m} \neq 0$ for some $\gamma \in P$. If $m \neq 1$ there exists a dominant $\mu \in P$ such that $c_{\gamma+\mu-m\mu,m} = 0$ (otherwise $c_{\gamma+\mu-m\mu,m} \neq 0$ for every dominant $\mu \in P$, which is impossible since z is a finite linear combination of $x^\lambda T_w$). Since $z \in Z(\tilde{H})$ we have

$$z = x^{-\mu} z x^\mu = \sum_{\lambda \in P, w \in W} c_{\lambda,w} x^{\lambda-\mu} T_w x^\mu.$$

Repeated use of the fourth relation in (???) yields

$$T_w x^\mu = \sum_{\nu \in P, v \in W} d_{\nu,v} x^\nu T_v$$

where $d_{\nu,v}$ are constants such that $d_{w\mu,w} = 1$, $d_{\nu,w} = 0$ for $\nu \neq w\mu$, and $d_{\nu,v} = 0$ unless $v \le w$. So

$$z = \sum_{\lambda \in P, w \in W} c_{\lambda,w} x^\lambda T_w = \sum_{\lambda \in P, w \in W} \sum_{\nu \in P, v \in W} c_{\lambda,w} d_{\nu,v} x^{\lambda-\mu+\nu} T_v$$

and comparing the coefficients of $x^\gamma T_m$ gives $c_{\gamma,m} = c_{\gamma+\mu-m\mu,m} d_{m\mu,m}$. Since $c_{\gamma+\mu-m\mu,m} = 0$ it follows that $c_{\gamma,m} = 0$, which is a contradiction. Hence $z = \sum_{\lambda \in P} c_\lambda x^\lambda \in \mathbb{K}[P]$.

The fourth relation in (???) gives

$$zT_i = T_i z = (s_i z)T_i + (q - q^{-1})z'$$

where $z' \in \mathbb{K}[P]$. Comparing coefficients of x^λ on both sides yields $z' = 0$. Hence $zT_i = (s_i z)T_i$, and therefore $z = s_i z$ for $1 \le i \le n$. So $z \in \mathbb{K}[P]^W$. □