Classical Lie algebras

Arun Ram
Department of Mathematics
University of Wisconsin
Madison, WI 53706
ram@math.wisc.edu

1 Classical Lie algebras

A *Lie algebra* is a vector space $\mathfrak{g}$ with a bilinear map $[,]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ such that

(a) $[x,y]=-[y,x]$, for $x,y\in\mathfrak{g}$, and

(b) (Jacobi identity) $[x,[y,z]]+[z,[x,y]]+[y,[z,x]]=0$, for all $x,y,z\in\mathfrak{g}$.

A bilinear form $\langle,\rangle\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ is ad-*invariant* if, for all $x,y,z\in\mathfrak{g}$,

$$\langle \mathrm{ad}_x(y),z\rangle=-\langle y,\mathrm{ad}_x(z)\rangle,\qquad \text{where}\qquad \mathrm{ad}_x(y)=[x,y], \tag{1.1}$$

for $x,y,\in\mathfrak{g}$. The *Killing form* is the inner product on $\mathfrak{g}$ given by

$$\langle x_1,x_2\rangle = Tr(\mathrm{ad}_x\mathrm{ad}_y)\rangle. \tag{1.2}$$

The Jacobi identity is equivalent to the fact that the Killing form is ad-invariant.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra with a nondegenerate ad-invariant bilinear form. The nondegeneracy of the form means that if $\{x_i\}$ be a basis of $\mathfrak{g}$ then the dual basis $\{x_i^*\}$ of $\mathfrak{g}$ with respect to $\langle,\rangle$ exists. The *Casimir element* of $\mathfrak{g}$ is

$$\kappa=\sum_i x_ix_i^*,\qquad \text{in } U\mathfrak{g}. \tag{1.3}$$

The element κ is central in $U\mathfrak{g}$ since, for $y\in\mathfrak{g}$,

$$\begin{aligned} y\kappa &= \sum_i yx_ix_i^* = \sum_i([y,x_i]+x_iy)x_i^* = \sum_{i,j}\langle [y,x_i],x_j^*\rangle x_jx_i^* + \sum_i x_iyx_i^* \\ &= \sum_{i,j} -x_j\langle x_i,[y,x_j^*]\rangle x_i^* + \sum_j x_jyx_j^* = \sum_j -x_j[y,x_j^*]+x_jyx_j^* = \sum_j x_jx_j^*y = \kappa y. \end{aligned}$$

Theorem 1.1. *Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$. The Casimir element κ acts on an $\mathfrak{g}$-module $L(\lambda)$ of highest weight λ by the constant*

$$\langle\lambda+\rho,\lambda+\rho\rangle-\langle\rho,\rho\rangle,\qquad \textit{where}\quad \rho=\frac{1}{2}\sum_{\alpha\in R^+}\alpha,$$

R^+ is the set of positive roots, and $\langle,\rangle$ is the form on $\mathfrak{h}^*$ obtained by restricting the Killing form to $\mathfrak{h}$ and identifying $\mathfrak{h}$ with $\mathfrak{h}^*$.

Proof. (a) Let us choose a basis of $\mathfrak{g}$ compatible with the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \Big(\bigoplus_{\alpha\in R} \mathfrak{g}_\alpha \Big) \qquad \text{with} \quad \dim \mathfrak{g}_\alpha = 1, \quad \text{for } \alpha \in R,$$

where $\mathfrak{h}$ is a Cartan subalgebra and R is the root system of $\mathfrak{g}$. Let $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$, $h_\alpha \in \mathfrak{h}$ be such that

$$[x_\alpha, y_\alpha] = h_\alpha, \qquad [h_\alpha, x_\alpha] = 2x_\alpha, \qquad [h_\alpha, y_\alpha] = -2y_\alpha.$$

Let $\langle , \rangle$ be the Killing form on $\mathfrak{g}$. Using the restriction of $\langle , \rangle$ to $\mathfrak{h}$ to identify $\mathfrak{h}$ and $\mathfrak{h}^*$,

$$\begin{matrix} \mathfrak{h} & \longrightarrow & \mathfrak{h}^* \\ h_\alpha & \longmapsto & \alpha^\vee \end{matrix} \qquad \text{where} \quad \alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha\rangle}.$$

Then

$$\langle x_\alpha, y_\alpha\rangle = \big\langle -\tfrac{1}{2}[x_\alpha, h_\alpha], y_\alpha \big\rangle = \tfrac{1}{2}\langle h_\alpha, [x_\alpha, y_\alpha]\rangle = \tfrac{1}{2}\langle h_\alpha, h_\alpha\rangle = \tfrac{1}{2}\langle \alpha^\vee, \alpha^\vee\rangle = \frac{2}{\langle \alpha, \alpha\rangle}.$$

and so

$$\left\langle x_\alpha, \frac{\langle\alpha,\alpha\rangle}{2} y_\alpha \right\rangle = 1 \qquad \text{and} \qquad \left[x_\alpha, \frac{\langle\alpha,\alpha\rangle}{2} y_\alpha \right] = \frac{\langle\alpha,\alpha\rangle}{2} h_\alpha.$$

Let $h_1, \dots, h_r$ be a basis of $\mathfrak{h}$ and let $h_1^*, \dots, h_r^*$ be the dual basis of $\mathfrak{h}$ with respect to $\langle , \rangle$. Then

$$\begin{matrix} \{h_1, \dots, h_r, x_\alpha, \frac{\langle\alpha,\alpha\rangle}{2} y_\alpha \mid \alpha \in R^+\} & \text{is a basis of } \mathfrak{g}\text{, and} \\ \{h_1^*, \dots, h_r^*, \frac{\langle\alpha,\alpha\rangle}{2} y_\alpha, x_\alpha \mid \alpha \in R^+\} & \text{is the dual basis of } \mathfrak{g}, \end{matrix}$$

with respect to $\langle , \rangle$.

Now compute the constant by which κ acts on $L(\lambda)$. If $L(\lambda)$ is an $\mathfrak{g}$-module generated by a highest weight vector v_λ^+ of weight λ so that

$$h_\alpha v_\lambda^+ = \langle \lambda, \alpha^\vee\rangle v_\lambda^+ \qquad \text{and} \qquad x_\alpha v_\lambda^+ = 0, \quad \text{for } \alpha \in R^+,$$

then

$$\begin{aligned}
\kappa v_\lambda^+ &= \left(\sum_i h_i h_i^* + \sum_{\alpha\in R^+} x_\alpha \tfrac{\langle\alpha,\alpha\rangle}{2} y_\alpha + \sum_{\alpha\in R^+} \tfrac{\langle\alpha,\alpha\rangle}{2} y_\alpha x_\alpha \right) v_\lambda^+ \\
&= \left(\sum_i h_i h_i^* + \sum_{\alpha\in R^+} \tfrac{\langle\alpha,\alpha\rangle}{2} \Big([x_\alpha, y_\alpha] + y_\alpha x_\alpha + y_\alpha x_\alpha \Big) \right) v_\lambda^+ \\
&= \left(\sum_i h_i h_i^* + \sum_{\alpha\in R^+} \tfrac{\langle\alpha,\alpha\rangle}{2} \big(h_\alpha + 2 y_\alpha x_\alpha \big) \right) v_\lambda^+ \\
&= \left(\langle\lambda,\lambda\rangle + \sum_{\alpha\in R^+} \tfrac{\langle\alpha,\alpha\rangle}{2} \big(\langle \lambda, \alpha^\vee\rangle + 0 \big) \right) v_\lambda^+ = \left\langle \lambda, \lambda + \sum_{\alpha\in R^+} \alpha \right\rangle v_\lambda^+ \\
&= \langle \lambda, \lambda + 2\rho\rangle v_\lambda^+ = \big(\langle \lambda+\rho, \lambda+\rho\rangle - \langle\rho,\rho\rangle \big) v_\lambda^+.
\end{aligned}$$

□

Let V be an n-dimensional vector space over $\mathbb{C}$. The Lie algebra

$$\mathfrak{gl}_n = \mathrm{End}(V) \quad \text{with bracket} \quad [x,y] = xy - yx, \quad \text{for } x, y, \in \mathfrak{gl}_n,$$

The Lie algebra

$$\mathfrak{sl}_n = \{x \in \mathfrak{gl}_n \mid \mathrm{tr}(x) = 0\}, \qquad \text{is a Lie subalgebra of } \mathfrak{gl}_n.$$

Suppose that $\langle , \rangle$ is a nondegenerate symmetric bilinear form on V. The Lie algebra

$$\mathfrak{so}_n = \{x \in \mathfrak{gl}_n \mid \ \langle xv_1, v_2\rangle + \langle v_1, xv_2\rangle = 0, \text{ for all } v_1, v_2 \in V\}.$$

Suppose that $\langle , \rangle$ is a nondegenerate skew symmetric bilinear form on V. The Lie algebra

$$\mathfrak{sp}_n = \{x \in \mathfrak{gl}_n \mid \ \langle xv_1, v_2\rangle + \langle v_1, xv_2\rangle = 0, \text{ for all } v_1, v_2 \in V\}.$$

The inner product

$$\langle , \rangle \colon \mathfrak{gl}_n \times \mathfrak{gl}_n \to \mathbb{C} \qquad \text{given by} \qquad \langle x, y\rangle = \mathrm{Tr}(xy), \tag{1.4}$$

is ad-invariant and nondegenerate on each of the Lie algebras $\mathfrak{gl}_n$, $\mathfrak{sl}_n$, $\mathfrak{so}_n$, and $\mathfrak{sp}_n$.

Identify $\mathfrak{gl}_n$ with $M_n(\mathbb{C})$ by choosing a basis $\{v_1, \ldots, v_n\}$ of V. Let E_{ij} be the matrix with 1 in the (i,j) entry and all other entries 0. Then

$$\mathfrak{gl}_n \qquad \text{has basis} \qquad \{E_{ij} \mid 1 \le i, j \le n\}$$

Let E_{ij} denote the matrix with 1 in the (i,j) entry and 0 in all other entries. Then

$$\mathfrak{gl}_n \qquad \text{has basis} \qquad \{E_{ij} \mid 1 \le i, j, \le n\}, \qquad \text{and} \qquad \{E_{ji} \mid 1 \le i, j, \le n\} \tag{1.5}$$

is the dual basis with respect to $\langle , \rangle$. Identify $\mathfrak{so}_n$ with a Lie subalgebra of $\mathfrak{gl}_n = M_n(\mathbb{C})$ by choosing an *orthonormal* basis $\{v_1, \ldots, v_n\}$ of V. Then

$$\mathfrak{so}_n = \{A \in M_n(\mathbb{C}) \mid A = -A^t\} \qquad \text{and has basis} \qquad \{E_{ij} - E_{ji} \mid 1 \le i < j \le n\}$$

and, with respect to $\langle , \rangle$,

$$\text{the dual basis is} \qquad \{-\tfrac{1}{2}(E_{ij} - E_{ji}) \mid 1 \le i, j \le n\}. \tag{1.6}$$

Let $\varepsilon_1, \ldots, \varepsilon_n$ be an orthonormal basis of the vector space $\mathbb{R}^n$. Then

$$\mathfrak{h}^* = \begin{cases} \mathbb{R}^n, & \text{if } \mathfrak{g} = \mathfrak{gl}_n, \\ \{\lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \mid \lambda_i \in \mathbb{R},\ \lambda_1 \cdots + \lambda_n = 0\}, & \text{if } \mathfrak{g} = \mathfrak{sl}_n, \\ \mathbb{R}^r, & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1}, \\ \mathbb{R}^r, & \text{if } \mathfrak{g} = \mathfrak{sp}_{2r}, \\ \mathbb{R}^r, & \text{if } \mathfrak{g} = \mathfrak{so}_{2r}, \end{cases}$$

The positive roots for $\mathfrak{gl}_n$ are the elements of

$$R^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n\}, \qquad \text{with} \qquad (\varepsilon_i - \varepsilon_j)^\vee = \varepsilon_i - \varepsilon_j. \tag{1.7}$$

The positive roots for $\mathfrak{so}_n$ are the elements of

$$R^+ = \left\{ \begin{matrix} \varepsilon_i \pm \varepsilon_j & \mid & 1 \le i < j \le n, \\ \varepsilon_i & \mid & 1 \le i \le n \end{matrix} \right\}, \qquad \text{with} \qquad \begin{aligned} (\varepsilon_i \pm \varepsilon_j)^\vee &= \varepsilon_i \pm \varepsilon_j, \quad \text{and} \\ (\varepsilon_i)^\vee &= 2\varepsilon_i. \end{aligned} \tag{1.8}$$

The fundamental weights are the generators of the $\mathbb{Z}_{\geq 0}$ module

$$P^+ = \sum_j \mathbb{Z}_{\geq 0}\omega_j, \qquad \text{of dominant integral weights,}$$

and the irreducible $\mathfrak{g}$-modules $L(\lambda)$ are indexed by the elements of P^+. The fundamental weights are given by

$$\begin{array}{rlll}
\omega_i & = \varepsilon_1 + \cdots + \varepsilon_i, & 1 \leq i \leq n, & \text{if } \mathfrak{g} = \mathfrak{gl}_n, \\
\omega_0 & = -(\varepsilon_1 + \cdots + \varepsilon_n), & & \\[2ex]
\omega_i & = \varepsilon_1 + \cdots + \varepsilon_i & 1 \leq i \leq n-1, & \text{if } \mathfrak{g} = \mathfrak{sl}_n, \\
& \quad -\frac{i}{n}(\varepsilon_1 + \cdots + \varepsilon_n), & & \\[2ex]
\omega_i & = \varepsilon_1 + \cdots + \varepsilon_i, & 1 \leq i \leq r-1, & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1}, \\
\omega_r & = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r), & & \\[2ex]
\omega_i & = \varepsilon_1 + \cdots + \varepsilon_i, & 1 \leq i \leq r, & \text{if } \mathfrak{g} = \mathfrak{sp}_{2r}, \\[2ex]
\omega_i & = \varepsilon_1 + \cdots + \varepsilon_i, & 1 \leq i \leq r-2, & \\
\omega_{r-1} & = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} - \varepsilon_r), & & \text{if } \mathfrak{g} = \mathfrak{so}_{2r}, \\
\omega_r & = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} + \varepsilon_r), & &
\end{array}$$

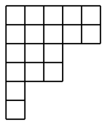
The dominant integral weights are

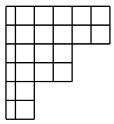
$$\begin{array}{lll}
\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n, & \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n, & \text{if } \mathfrak{g} = \mathfrak{gl}_n, \\
& \lambda_1, \ldots, \lambda_n \in \mathbb{Z}, & \\[2ex]
\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_{n-1}\varepsilon_{n-1} & \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq 0, & \text{if } \mathfrak{g} = \mathfrak{sl}_n, \\
\qquad -\frac{|\lambda|}{n}(\varepsilon_1 + \cdots + \varepsilon_n), & \lambda_1, \ldots, \lambda_{n-1} \in \mathbb{Z}, & \\[2ex]
\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r, & \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0, & \\
& \lambda_1, \ldots, \lambda_r \in \mathbb{Z}, \text{or} & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1}, \\
& \lambda_1, \ldots, \lambda_r \in \frac{1}{2} + \mathbb{Z}, & \\[2ex]
\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r, & \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0, & \text{if } \mathfrak{g} = \mathfrak{sp}_{2r}, \\
& \lambda_1, \ldots, \lambda_r \in \mathbb{Z}, & \\[2ex]
\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r, & \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r-1} \geq |\lambda_r| \geq 0, & \\
& \lambda_1, \ldots, \lambda_r \in \mathbb{Z}, \text{or} & \text{if } \mathfrak{g} = \mathfrak{so}_{2r}, \\
& \lambda_1, \ldots, \lambda_r \in \frac{1}{2} + \mathbb{Z}, &
\end{array}$$

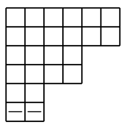
where $|\lambda| = \sum_i \lambda_i$. The element ρ is given by

$$2\rho = \sum_j \omega_j = \sum_i (y - 2i + 1)\varepsilon_i, \qquad \text{where} \quad y = \begin{cases} 2n-1, & \text{if } \mathfrak{g} = \mathfrak{gl}_n, \\ n, & \text{if } \mathfrak{g} = \mathfrak{sl}_n, \\ 2r, & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1}, \\ 2r+1, & \text{if } \mathfrak{g} = \mathfrak{sp}_{2r}, \\ 2r-1, & \text{if } \mathfrak{g} = \mathfrak{so}_{2r}, \end{cases} \tag{1.9}$$

Identify each dominant integral weight λ with the configuration of boxes which has λ_i boxes in row i ($1 \le i \le n$). If $\lambda_i \le 0$ put $|\lambda_i|$ boxes in row i and mark them with $-$ signs. For example

$\lambda =$  $= \begin{cases} 5\varepsilon_1 + 5\varepsilon_2 + 3\varepsilon_3 + 3\varepsilon_4 + \varepsilon_5 + \varepsilon_6 - \frac{18}{n+1}(\varepsilon_1 + \cdots + \varepsilon_{n+1}), & \text{for } \mathfrak{g} = \mathfrak{sl}_{n+1}, \\ 5\varepsilon_1 + 5\varepsilon_2 + 3\varepsilon_3 + 3\varepsilon_4 + \varepsilon_5 + \varepsilon_6, & \text{for } \mathfrak{g} = \mathfrak{gl}_n, \mathfrak{so}_{2r+1}, \mathfrak{sp}_{2r} \text{ of } \mathfrak{so}_{2r}, \end{cases}$

$\lambda =$  $= \frac{11}{2}\varepsilon_1 + \frac{11}{2}\varepsilon_2 + \frac{7}{2}\varepsilon_3 + \frac{7}{2}\varepsilon_4 + \frac{3}{2}\varepsilon_5 + \frac{3}{2}\varepsilon_6, \qquad \text{for } \mathfrak{g} = \mathfrak{so}_{2r+1} \text{ or } \mathfrak{g} = \mathfrak{so}_{2r},$

$\lambda =$  $= 6\varepsilon_1 + 6\varepsilon_2 + 4\varepsilon_3 + 4\varepsilon_4 + 2\varepsilon_5 - 2\varepsilon_6, \qquad \text{for } \mathfrak{g} = \mathfrak{so}_{12} \text{ (type } D_6\text{)},$

If b is a box in position (i, j) of λ the *content* of b is

$$c(b) = j - i = \text{the diagonal number of } b. \tag{1.10}$$

PICTURE

Theorem 1.2. *(b) Let E_{ij} be the matrix with 1 in the (i, j) entry and all other entries 0. The Casimir element of $U\mathfrak{g}$ is*

$$\kappa = \begin{cases} \displaystyle\sum_{1 \le i,j, \le n} E_{ij}E_{ji}, & \textit{for } \mathfrak{g} = \mathfrak{gl}_n, \\ ???, & \textit{for } \mathfrak{g} = \mathfrak{sl}_n, \\ \displaystyle -\frac{1}{4}\sum_{i,j=1}^{n} (E_{ij} - E_{ji})^2, & \textit{for } \mathfrak{g} = \mathfrak{so}_n, \\ ???, & \textit{for } \mathfrak{g} = \mathfrak{sp}_{2r}, \end{cases}$$

where we identify $\mathfrak{gl}_n$ with $M_n(\mathbb{C})$ by choosing a basis $\{v_1, \ldots, v_n\}$ of V which is orthonormal if the bilinear form $\langle,\rangle$ on V is symmetric and satisfies

$$\langle v_i, v_{r+j}\rangle = \delta_{ij}, \qquad \textit{if } n = 2r \textit{ and } \langle,\rangle \textit{ is skew symmetric.}$$

put in for $\le i, j \le r$??? The element κ acts on an $\mathfrak{g}$-module $L(\lambda)$ of highest weight λ by the constant

$$2\sum_{b \in \lambda} c(b) + \begin{cases} ???, & \textit{if } \mathfrak{g} = \mathfrak{gl}_n, \\ n|\lambda| - \frac{|\lambda^2|}{n}, & \textit{if } \mathfrak{g} = \mathfrak{sl}_n, \\ (n-1)|\lambda|, & \textit{if } \mathfrak{g} = \mathfrak{so}_n \textit{ and } \lambda_i \in \mathbb{Z}, \\ (n-1)|\lambda| + \frac{n}{4} + \frac{n^2}{2}, & \textit{if } \mathfrak{g} = \mathfrak{so}_n \textit{ and } \lambda_i \in \frac{1}{2} + \mathbb{Z}, \\ (2r+1)|\lambda|, & \textit{if } \mathfrak{g} = \mathfrak{sp}_{2r}, \end{cases}$$

Proof. (b) The formulas for κ follow from ??? and ???. Thus the Casimir element of $\mathfrak{so}_n$ is

$$\kappa = \sum_{1 \le i < j \le n} -\frac{1}{2}(E_{ij} - E_{ji})^2 = -\frac{1}{4}\sum_{i,j=1}^{n} (E_{ij} - E_{ji})^2.$$

PUT $\mathfrak{sp}_{2r}$ HERE.

Now compute the action of κ on a $\mathfrak{g}$ module $L(\lambda)$ generated by a highest weight vector of weight λ. In the case when $\mathfrak{g} = \mathfrak{gl}_n$,

$$\begin{aligned}
\kappa &= \sum_{1\le i,j\le n} E_{ij}E_{ji} = \sum_{i=1}^n E_{ii}E_{ii} + \sum_{1\le i<j\le n} E_{ij}E_{ji} + \sum_{1\le i<j\le n} [E_{ij}, E_{ji}] + E_{ji}E_{ij} \\
&= \sum_{i=1}^n E_{ii}E_{ii} + 2\sum_{1\le i<j\le n} E_{ij}E_{ji} + \sum_{1\le i<j\le n} (E_{ii} - E_{jj})
\end{aligned}$$

Since $E_{ii}v_\lambda^+ = \lambda_i v_\lambda^+$ and $E_{ij}v_\lambda^+ = 0$ for $i<j$, $\kappa v_\lambda^+ = c_\lambda v_\lambda^+$ where

$$\begin{aligned}
c_\lambda &= \sum_{i=1}^n \lambda_i^2 + 0 + \sum_{1\le i<j\le n} (\lambda_i - \lambda_j) = \sum_{i=1}^n \lambda_i^2 + (n-i)\lambda_i - (i-1)\lambda_i \\
&= \sum_{i=1}^n (\lambda_i + n - 2i + 1)\lambda_i = \sum_{i=1}^n (\lambda_i + 2n - 2i)\lambda_i - (n-1)\lambda_i = \langle \lambda, \lambda + 2\delta\rangle - (n-1)|\lambda| \\
&= \langle \lambda+\delta, \lambda+\delta\rangle - \langle \delta, \delta\rangle - (n-1)|\lambda|, \quad \text{where} \quad \delta = (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \cdots + \varepsilon_{n-1}.
\end{aligned}$$

Let λ and μ be partitions such that $\mu \subseteq \lambda$ and $\lambda/\mu = \square$. Suppose that the box where λ and μ differe is in the jth row so that $\lambda = \mu + \varepsilon_j$. Then, with c_λ as in the proof of (a),

$$\begin{aligned}
c_\lambda - c_\mu &= \big(\langle \lambda+\delta, \lambda+\delta\rangle - \langle \delta, \delta\rangle - (n-1)|\lambda|\big) - \big(\langle \mu+\delta, \mu+\delta\rangle - \langle \delta, \delta\rangle - (n-1)|\mu|\big) \\
&= \langle \mu+\varepsilon_j+\delta, \mu+\varepsilon_j+\delta\rangle - \langle \mu+\delta, \mu+\delta\rangle - (n-1)(|\lambda| - |\mu|) \\
&= 2\langle \mu+\delta, \varepsilon_j\rangle + \langle \varepsilon_j, \varepsilon_j\rangle - (n-1) \\
&= 2(\mu_j + n - j) + 1 - n + 1 = 2(\mu_j + 1 - j) + n = 2(c\lambda/\mu) + n.
\end{aligned}$$

Thus, by induction on the number of boxes in λ,

$$c_\lambda = 2\sum_{b\in\lambda} c(b) + n|\lambda|.$$

If λ and μ are partitions such that either $\lambda \subseteq \mu$ and $\mu/\lambda = \square$, or $\mu \subseteq \lambda$ and $\lambda/\mu = \square$, then

$$\begin{aligned}
&\big(\langle \lambda+\rho, \lambda+\rho\rangle - \langle \rho, \rho\rangle\big) - \big(\langle \mu+\rho, \mu+\rho\rangle - \langle \rho, \rho\rangle\big) = \langle \lambda, \lambda+2\rho\rangle - \langle \mu, \mu+2\rho\rangle \\
&\quad = \langle \mu \pm \varepsilon_j, \mu \pm \varepsilon_j + 2\rho\rangle - \langle \mu, \mu+2\rho\rangle = \langle \mu, \pm\varepsilon_j\rangle \pm \langle \varepsilon_j, \mu+2\rho\rangle + \langle \varepsilon_j, \varepsilon_j\rangle \\
&\quad = \pm\mu_j \pm \mu_j \pm (y - 2j + 1) + 1 = \begin{cases} 2(\mu_j - j + 1) + y, & \text{if } \lambda/\mu = \square, \\ -2(\mu_j - j) - y, & \text{if } \mu/\lambda = \square, \end{cases} \\
&\quad = \begin{cases} 2c(\lambda/\mu) + y, & \text{if } \lambda/\mu = \square, \\ -(2c(\mu/\lambda) + y), & \text{if } \mu/\lambda = \square, \end{cases}
\end{aligned}$$

Note that $c(\lambda/\lambda^-)$ may be a $\frac{1}{2}$-integer if μ_j is a $\frac{1}{2}$-integer. Also, if $\mathfrak{g} = \mathfrak{so}_{2r+1}$ or $\mathfrak{g} = \mathfrak{so}_2 r$ then

$$\langle \omega_r, \omega_r + 2\rho\rangle = \frac{r}{4} + \frac{1}{2}\sum_{i=1}^r (y - 2i + 1) = \frac{r}{4} + \frac{r}{2}\cdot y - \frac{r^2}{2} = \begin{cases} \frac{r^2}{2} + \frac{r}{4}, & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1}, \\ \frac{r^2}{2} - \frac{r}{4}, & \text{if } \mathfrak{g} = \mathfrak{so}_{2r}. \end{cases}$$

Using these formulas to compute $\langle \lambda, \lambda+2\rho\rangle$ for dominant integral weights λ gives

$$\langle \lambda, \lambda+2\rho\rangle = y|\lambda| + 2\sum_{b\in\lambda} c(b) + \begin{cases} 0\,, & \text{if } \mathfrak{g}=\mathfrak{gl}_n,\ \mathfrak{g}=\mathfrak{sp}_{2r} \text{ or } \mathfrak{g}=\mathfrak{so}_{2r+1} \text{ with } \lambda_i\in\mathbb{Z}, \\ -\dfrac{|\lambda|^2}{n}\,, & \text{if } \mathfrak{g}=\mathfrak{sl}_n, \\ ???\,, & \text{if } \mathfrak{g}=\mathfrak{so}_{2r}, \\ \dfrac{r}{4}+\dfrac{r^2}{2}\,, & \text{if } \mathfrak{g}=fso_{2r+1} \text{ with } \lambda_i\in\frac12+\mathbb{Z}. \end{cases}$$

□

We NEED the positive roots for the next statement, and define HOOK LENGTH.

Proposition 1.3. *(a)*

$$\dim(L_{\mathfrak{gl}_n}(\lambda)) = \prod_{\alpha\in R^+}\frac{\langle\lambda+\delta,\alpha^\vee\rangle}{\langle\delta,\alpha^\vee\rangle} = \prod_{n\geq j>i\geq 1}\frac{(\lambda_i+n-i)-(\lambda_j+n-j)}{(n-i)-(n-j)} = \prod_{1\leq i<j\leq n}\frac{n+c(b)}{h(b)}.$$

(b) Let

$$\rho = \tfrac12\sum_{\alpha\in R^+}\alpha = \tfrac12\sum_{i=1}^{n}(2n-(2i-1))\varepsilon_i.$$

Then

$$\begin{aligned}\dim(L_{\mathfrak{so}_n}(\lambda)) &= \prod_{\alpha\in R^+}\frac{\langle\lambda+\rho,\alpha^\vee\rangle}{\langle\rho,\alpha^\vee\rangle} \\ &= \prod_{1\leq i<j\leq n}\left(\frac{(\lambda_i+n-i+\frac12)-(\lambda_j+n-j+\frac12)}{(n-i+\frac12)-(n-j+\frac12)}\right)\left(\frac{(\lambda_i+n-i+\frac12)+(\lambda_j+n-j+\frac12)}{(n-i+\frac12)+(n-j+\frac12)}\right) \\ &\qquad\cdot\prod_{i=1}^{n}\frac{2(\lambda_i+n-i+\frac12)}{2(n-i+\frac12)} \\ &= \prod_{b\in\lambda}\frac{2n+r(b)}{h(b)},\end{aligned}$$

where

$$r(b) = \begin{cases}\lambda_i-i+\lambda_j-j+1, & \text{if } i\geq j, \\ -(\lambda_i'-i+\lambda_j'-j+1), & \text{if } i<j.\end{cases}$$

Proof. (a) Think about the derivation of the Weyl dimension formula in the context of Schur functions.

$$s_\lambda(e^{t\delta}) = s_\lambda(e^{t(n-1)}, e^{t(n-2)}, \ldots, e^t, 1) = s_\lambda(q^{(n-1)}, q^{(n-2)}, \ldots, q, 1)$$

$$= \frac{\det((q^{n-i})^{\lambda_j+n-j})}{\det((q^{n-i})^{n-j})} = \frac{\det((q^{\lambda_j+n-j})^{n-i})}{\det((q^{n-i})^{n-j})} = \frac{a_\delta(q^{\lambda_1+n-1}, \ldots, q^{\lambda_n+n-n})}{a_\delta(q^{n-1}, \ldots, q^{n-n})}$$

$$= \prod_{i<j} \frac{q^{\lambda_j+n-j} - q^{\lambda_i+n-i}}{q^{n-j} - q^{n-i}} = q^{\sum(i-1)\lambda_i} \left(\prod_{i<j} \frac{1 - q^{(\lambda_j+n-j)-(\lambda_i+n-i)}}{q^{(n-j)-(n-i)}} \right)$$

$$= q^{n(\lambda)} \frac{\displaystyle\prod_{i\geq 1} \prod_{k=1}^{\lambda_i+n-i} (1-q^k)}{\left(\displaystyle\prod_{b\in\lambda} 1 - q^{h(b)}\right) \displaystyle\prod_{i<j} (1 - q^{i-j})} = q^{n(\lambda)} \prod_{b\in\lambda} \frac{1 - q^{n+c(b)}}{1 - q^{h(b)}}$$

For example, if $\lambda = 5\varepsilon_1 + 4\varepsilon_2 + 4\varepsilon_3 + 3\varepsilon_4 + 2\varepsilon_5$,

2	3	5	7	9	8	6	4	1	$\lambda_1 + n - 1 = 9$
	1	3	5	7	6	4	2		$\lambda_2 + n - 2 = 7$
		2	4	6	5	3	1		$\lambda_3 + n - 3 = 6$
			2	4	3	1			$\lambda_4 + n - 4 = 4$
				2	1				$\lambda_5 + n - 5 = 2$

Then

1	2	3	4	5	6	7	8	9
	1	2	3	4	5	6	7	
		1	2	3	4	5	6	
			1	2	3	4		
				1	2			

So

$$\dim(U^\lambda) = \frac{5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 4\cdot 5\cdot 6\cdot 7\cdot 3\cdot 4\cdot 5\cdot 6\cdot 2\cdot 3\cdot 4\cdot 1\cdot 2}{9\cdot 8\cdot 6\cdot 4\cdot 1\cdot 7\cdot 6\cdot 4\cdot 2\cdot 5\cdot 4\cdot 3\cdot 1\cdot 4\cdot 3\cdot 1\cdot 2\cdot 1} = \frac{5\cdot 7\cdot 5\cdot 6}{6} = 5\cdot 7\cdot 5 = 175.$$

The first equality is Weyl's dimension formula and the second equality results from the formulas in ???. By using the argument of (???) on the partition $\lambda_{\geq i} = (\lambda_i, \ldots, \lambda_n)$ gives

$$\begin{aligned}\{1, 2, \ldots, \lambda_i + n - i\} &= \{(\lambda_i - i) + (\lambda'_j - j) + 1 \mid \lambda_i \geq j \geq 1\} \sqcup \{(\lambda_i - i) - (\lambda_j - j) \mid n \geq j > 1\} \\ &= \{h(b) \mid b \text{ is in row } i \text{ of } \lambda\} \sqcup \{(\lambda_i + n - i) - (\lambda_j + n - j) \mid n \geq j > 1\}.\end{aligned}$$

On the other hand

$$\begin{aligned}\{1, 2, \ldots, \lambda_i + n - i\} &= \{1, 2\ldots, n - i\} \sqcup \{n + j - i \mid \lambda_i \geq j \geq 1\} \\ &= \{(n - i) - (n - j) \mid n \geq j > i\} \sqcup \{n + c(b) \mid b \text{ is in row } i \text{ of } \lambda\}.\end{aligned}$$

Thus

$$\left(\prod_{b\in\lambda} h(b)\right)\prod_{n\geq j>i\geq 1}((\lambda_i+n-i)-(\lambda_j+n-j)) = \left(\prod_{b\in\lambda} n+c(b)\right)\prod_{n\geq j>i\geq 1}((n-i)-(n-j))$$

and the third equality follows.

(b) The first equality is Weyl's dimension formula and the second equality results from the formulas in ???.

$$\begin{aligned}
\{h(b)\mid b \text{ is in row } i \text{ of } \lambda\} &\sqcup \{(\lambda_i+n-i+\tfrac12)-(\lambda_j+n-j+\tfrac12)\mid n\geq j>i\}\\
&\sqcup \{(\lambda_i+n-i+\tfrac12)+(\lambda_j+n-j+\tfrac12)\mid n\geq j\geq i\}\\
&\sqcup \{2n+(\lambda_i-i)-(\lambda'_j-j)\mid (i,j)\in\lambda\}\\
= \{(\lambda_i-i)-(\lambda'_j-j)+1\mid \lambda_i\geq j\geq 1\}&\\
&\sqcup \{(\lambda_i+n-i)-(\lambda_j+n-j)\mid n\geq j>i\}\\
&\sqcup \{2(\lambda_i+n-i+\tfrac12)-((\lambda_i-i)-(\lambda_j-j))\mid n\geq j\geq i\}\\
&\sqcup \{2(\lambda_i+n-i+\tfrac12)-((\lambda_i-i)-(\lambda'_j-j)+1)\mid \lambda_i\geq j\geq 1\}\\
= \{1,2,\ldots,\lambda_i+n-i\} &\sqcup \{\lambda_i+n-i+1,\ldots,2(\lambda_i+n-i+\tfrac12)\}\\
= \{1,2,\ldots,2(\lambda_i+n-i+\tfrac12)\}&
\end{aligned}$$

Let $r(\lambda)$ be the *Frobenius rank* of the partition λ, i.e. the largest r such that λ contains the "diagonal" box in position (r,r). Then, for $1\leq i\leq r(\lambda)$,

$$\begin{aligned}
\{1,2,\ldots,2n-1-2(\lambda'_i-i)\} &\sqcup \{2n+1+(\lambda_i-i)+(\lambda_j-j))\mid j\geq i,(j,i)\in\lambda\}\\
&\sqcup \{2n+(\lambda_i-i)-(\lambda'_j-j))\mid j>i,(i,j)\in\lambda\}\\
&\sqcup \{2n-(\lambda'_i-i)+(\lambda_j-j))\mid j\geq i,(j,i)\in\lambda\}\\
&\sqcup \{2n-1-(\lambda'_i-i)-(\lambda'_j-j))\mid j>i,(i,j)\in\lambda\}\\
= \{1,2,\ldots,2n-1-2(\lambda'_i-i)\} &\sqcup \{2n+1+2(\lambda_i-i)-((\lambda_i-i)-(\lambda_j-j))\mid \lambda'_i\geq j\geq i\}\\
&\sqcup \{2n+1+2(\lambda_i-i)-((\lambda_i-i)-(\lambda'_j-j)+1)\mid \lambda_i\geq j>i\}\\
&\sqcup \{2n-(\lambda'_i-i)+(\lambda_i-i)-((\lambda_i-i)-(\lambda_j-j))\mid \lambda'_i\geq j\geq i\}\\
&\sqcup \{2n-(\lambda'_i-i)+(\lambda_i-i)-((\lambda_i-i)-(\lambda'_j-j)+1)\mid \lambda_i\geq j>i\}\\
= \{1,2,\ldots,2n-1-2(\lambda'_i-i)\} &\sqcup \{2n+1+(\lambda_i-i)-(\lambda'_i-i),\ldots,2n+1+2(\lambda_i-i)\}\\
&\sqcup \{2n-2(\lambda'_i-i),\ldots,2n+(\lambda_i-i)-(\lambda'_i-i)\}\\
= \{1,2,\ldots,2n+1+2(\lambda_i-i)\}.&
\end{aligned}$$

Thus

$$\Big(\prod_{b\in\lambda} h(b)\Big) \prod_{n\ge j>i\ge 1} ((\lambda_i+n-i+\tfrac{1}{2})-(\lambda_j+n-j+\tfrac{1}{2}))$$

$$\cdot \prod_{n\ge j\ge i\ge 1} ((\lambda_i+n-i+\tfrac{1}{2})+(\lambda_j+n-j+\tfrac{1}{2})) \prod_{(i,j)\in\lambda} (2n+(\lambda_i-i)-(\lambda'_j-j))$$

$$= \prod_{i=1}^{n} (2(\lambda_i+n-i+\tfrac{1}{2}))! = \prod_{i=1}^{r(\lambda)} (2(\lambda_i+n-i+\tfrac{1}{2}))! \prod_{i=r(\lambda)+1}^{n} (2(\lambda_i+n-i+\tfrac{1}{2}))!$$

$$= \prod_{i=r(\lambda)+1}^{n} (2(\lambda_i+n-i+\tfrac{1}{2}))!$$

$$\cdot \prod_{i=1}^{r(\lambda)} \Big((2n-1-2(\lambda'_i-i))!$$

$$\cdot \prod_{\substack{j\ge i\\(j,i)\in\lambda}} (2n+1+(\lambda_i-i)+(\lambda_j-j)) \prod_{\substack{j>i\\(i,j)\in\lambda}} (2n+(\lambda_i-i)-(\lambda'_j-j))$$

$$\cdot \prod_{\substack{j\ge i\\(j,i)\in\lambda}} (2n-(\lambda'_i-i)+(\lambda_j-j)) \prod_{\substack{j>i\\(i,j)\in\lambda}} (2n-1-(\lambda'_i-i)+(\lambda'_j-j))\Big)$$

Dividing each side by

$$\prod_{(i,j)\in\lambda} (2n+(\lambda_i-i)-(\lambda'_j-j))$$

gives

$$\Big(\prod_{b\in\lambda} h(b)\Big) \prod_{n\ge j>i\ge 1} ((\lambda_i+n-i+\tfrac{1}{2})-(\lambda_j+n-j+\tfrac{1}{2})) \prod_{n\ge j\ge i\ge 1} ((\lambda_i+n-i+\tfrac{1}{2})+(\lambda_j+n-j+\tfrac{1}{2}))$$

$$= \prod_{i=1}^{r(\lambda)} \Big((2n-1-2(\lambda'_i-i))! \prod_{i=r(\lambda)+1}^{n} (2(\lambda_i+n-i+\tfrac{1}{2}))!$$

$$\cdot \prod_{\substack{j\ge i\\(j,i)\in\lambda}} (2n+1+(\lambda_i-i)+(\lambda_j-j)) \prod_{\substack{j>i\\(i,j)\in\lambda}} (2n-1-(\lambda'_i-i)+(\lambda'_j-j))$$

Then, from

$$\begin{aligned}\{2(n-(\lambda'_i-i)-\tfrac{1}{2}))\mid 1\le i\le r(\lambda)\} &\sqcup \{2(\lambda_i+n-i+\tfrac{1}{2})\mid r(\lambda)+1\le i\le n\}\\ &= \{2(n+\tfrac{1}{2}-(\lambda'_j-j)+1\mid r(\lambda)\ge j\ge 1\} \sqcup \{2(n+\tfrac{1}{2}-(\lambda_j-j)\mid n\ge j\ge r(\lambda)\}\\ &= \{2(n+\tfrac{1}{2}-1,\cdots,2(n+\tfrac{1}{2}-n\} = \{2n-1,2n-3,\ldots,3,1\} = \{2i-1\mid 1\le i\le n\},\end{aligned}$$

it follows that

$$\prod_{i=1}^{r(\lambda)} \Big((2n-1-2(\lambda'_i-i))! \prod_{i=r(\lambda)+1}^{n} (2(\lambda_i+n-i+\tfrac{1}{2}))!$$

$$= \prod_{i=1}^{n} (2i-1)! = \prod_{1\le i<j\le n} ((n-i+\tfrac{1}{2})-(n-j+\tfrac{1}{2})) \prod_{1\le i\le j\le n} ((n-i+\tfrac{1}{2})+(n-j+\tfrac{1}{2}))$$

which establishes the third equality in (b). □

The group $GL_n(\mathbb{C})$ and the Lie algebra $\mathfrak{gl}_n$ act on the n-dimensional complex vector space V with basis $v_1, \ldots, v_n$ by

$$gv_i = \sum_{j=1}^{n} g_{ji} v_j, \qquad \text{and} \qquad xv_i = \sum_{j=1}^{n} x_{ji} v_j, \tag{1.11}$$

for $g = (g_{ij}) \in GL_n(\mathbb{C})$ and $x = (x_{ij}) \in \mathfrak{gl}_n$. Let $\langle , \rangle : V \times V \to \mathbb{C}$ be the inner product defined by making the basis $v_1, \ldots, v_n$ orthonormal. The complex Lie group

$$O_n(\mathbb{C}) = \{g \in GL_n(\mathbb{C}) \mid \langle gv, gw \rangle = \langle v, w \rangle \text{ for all } v, w \in V\} = \{g \in GL_n(\mathbb{C}) \mid gg^t = 1\}$$

has Lie algebra

$$\mathfrak{so}_n = \{x \in M_n(\mathbb{C}) \mid \langle xv, w \rangle + \langle v, xw \rangle = 0 \text{ for all } v, w \in V\}, = \{x \in M_n(\mathbb{C}) \mid x + x^t = 0\}$$

a Lie subalgebra of $\mathfrak{gl}_n$. The complex Lie group

$$SO_n(\mathbb{C}) = \{g \in O_n(\mathbb{C}) \mid \det g = 1\},$$

also has Lie algebra $\mathfrak{so}_n$. Since an element $g \in O_n(\mathbb{C})$ has $\det g = \pm 1$ the group $O_n(\mathbb{C})$ is a union of two cosets

$$O_n(\mathbb{C}) = SO_n(\mathbb{C}) \sqcup rSO_n(\mathbb{C}), \qquad \text{where} \quad r = -E_{11} + \sum_{\ell=2}^{n} E_{\ell\ell}, \tag{1.12}$$

A matrix A is in $\mathfrak{so}_n$ if and only if

$$A_{ij} = \Big\langle \sum_{j=1}^{n} A_{ji} v_j, v_j \Big\rangle = \langle Av_i, v_j \rangle = -\langle v_i, Av_j \rangle = -\Big\langle v_i \sum_{i=1}^{n} A_{ij} v_i \Big\rangle = -A_{ij},$$

for all $1 \le i, j \le n$.

For all dominant integral weights λ

$$L(\lambda) \otimes L(\omega_1) = \begin{cases} \displaystyle\bigoplus_{\lambda^+} L(\lambda^+), & \text{if } \mathfrak{g} = \mathfrak{gl}_n \text{ or } \mathfrak{g} = \mathfrak{sl}_n, \\ L(\lambda) \bigoplus \left(\displaystyle\bigoplus_{\lambda^\pm} L(\lambda^\pm) \right), & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1} \text{ and } \lambda_r > 0, \\ \left(\displaystyle\bigoplus_{\lambda^\pm} L(\lambda^\pm) \right), & \text{if } \mathfrak{g} = \mathfrak{sp}_{2r},\ \mathfrak{g} = \mathfrak{so}_{2r}, \text{ or} \\ & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1} \text{ and } \lambda_n = 0, \end{cases}$$

where the sum over λ^+ is a sum over all partitions (of length $\le r$) obtained by adding a box to λ, and the sum over $\lambda^\pm$ denotes a sum over all dominant weights obtained by adding or removing a box from λ. If $\mathfrak{g} = \mathfrak{so}_{2r}$ then addition and removal of a box should include the possibility of addition and removal of a box marked with a $-$ sign, and removal of a box from row r when $\lambda_r = \frac{1}{2}$ changes λ_r to $-\frac{1}{2}$.

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