The basic construction

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1 The basic construction

In this section we shall assume that all algebras are finite dimensional algebras over an algebraically closed field $\mathbb{F}$. The fact that $\mathbb{F}$ is algebraically closed is only for convenience, to avoid the division rings that could arise in the decomposition of $\bar{A}$ just before (4.8) below.

Let $A \subseteq B$ be an inclusion of algebras. Then $B \otimes_{\mathbb{F}} B$ is an (A, A)-bimodule where A acts on the left by left multiplication and on the right by right multiplication. Fix an (A, A)-bimodule homomorphism

$$\varepsilon : B \otimes_{\mathbb{F}} B \longrightarrow A. \tag{1.1}$$

The *basic construction* is the algebra $B \otimes_A B$ with product given by

$$(b_1 \otimes b_2)(b_3 \otimes b_4) = b_1 \otimes \varepsilon(b_2 \otimes b_3)b_4, \qquad \text{for } b_1, b_2, b_3, b_4 \in B. \tag{1.2}$$

More generally, let A be an algebra and let L be a left A-module and R a right A-module. Let

$$\varepsilon : L \otimes_{\mathbb{F}} R \longrightarrow A, \tag{1.3}$$

be an (A, A)-bimodule homomorphism. The *basic construction* is the algebra $R \otimes_A L$ with product given by

$$(r_1 \otimes \ell_1)(r_2 \otimes \ell_2) = r_1 \otimes \varepsilon(\ell_1 \otimes r_2)\ell_2, \qquad \text{for } r_1, r_2 \in R \text{ and } \ell_1, \ell_2 \in L. \tag{1.4}$$

Theorem 4.18 below determines, explicitly, the structure of the algebra $R \otimes_A L$.

Let $N = \mathrm{Rad}(A)$ and let

$$\bar{A} = A/N, \qquad \bar{L} = L/NL, \qquad \text{and} \qquad \bar{R} = R/RN \tag{1.5}$$

Define an $(\bar{A}, \bar{A})$-bimodule homomorphism

$$\begin{array}{rccc} \bar{\varepsilon} : & \bar{L} \otimes_{\mathbb{F}} \bar{R} & \longrightarrow & \bar{A} \\ & \bar{\ell} \otimes \bar{r} & \mapsto & \overline{\varepsilon(\ell \otimes r)} \end{array} \tag{1.6}$$

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where $\bar{\ell} = \ell + NL$, $\bar{r} = r + RN$, and $\bar{a} = a + N$, for $\ell \in L, r \in R$ and $a \in A$. Then by basic tensor product relations [Bou1, Ch. II §3.3 Cor. to Prop. 2 and §3.6 Cor. to Prop. 6], the surjective algebra homomorphism

$$\begin{array}{rccc} \pi: & R \otimes_A L & \longrightarrow & \bar{R} \otimes_{\bar{A}} \bar{L} \\ & r \otimes \ell & \mapsto & \bar{r} \otimes \bar{\ell} \end{array} \qquad \text{has} \qquad \ker(\pi) = R \otimes_A NL. \tag{1.7}$$

The algebra $\bar{A}$ is a split semisimple algebra (an algebra isomorphic to a direct sum of matrix algebras). Fix an algebra isomorphism

$$\begin{array}{ccc} \bar{A} & \xrightarrow{\sim} & \bigoplus_{\mu \in \hat{A}} M_{d_\mu}(\mathbb{F}) \\ a^\mu_{PQ} & \leftarrow & E^\mu_{PQ} \end{array}$$

where $\hat{A}$ is an index set for the components and E^μ_{PQ} is the matrix with 1 in the (P, Q) entry of the μth block and 0 in all other entries. Also, fix isomorphisms

$$\bar{L} \cong \bigoplus_{\mu \in \hat{A}} \overrightarrow{A}^\mu \otimes L^\mu \qquad \text{and} \qquad \bar{R} \cong \bigoplus_{\mu \in \hat{A}} R^\mu \otimes \overleftarrow{A}^\mu \tag{1.8}$$

where $\overrightarrow{A}^\mu$, $\mu \in \hat{A}$, are the simple left $\bar{A}$-modules, $\overleftarrow{A}^\mu$, $\mu \in \hat{A}$, are the simple right $\bar{A}$-modules, and L^μ, R^μ, $\mu \in \hat{A}$ are vector spaces. The practical effect of this setup is that if $\hat{R}^\mu$ is an index set for a basis $\{r^\mu_Y | Y \in \hat{R}^\mu\}$ of R^μ, $\hat{L}^\mu$ is an index set for a basis $\{\ell^\mu_X | X \in \hat{L}^\mu\}$ of L^μ, and $\hat{A}^\mu$ is an index set for bases

$$\{\overrightarrow{a}^\mu_Q \mid Q \in \hat{A}^\mu\} \text{ of } \overrightarrow{A}^\mu \qquad \text{and} \qquad \{\overleftarrow{a}^\mu_P \mid P \in \hat{A}^\mu\} \text{ of } \overleftarrow{A}^\mu \tag{1.9}$$

such that

$$a^\lambda_{ST} \overrightarrow{a}^\mu_Q = \delta_{\lambda\mu}\delta_{TQ} \overrightarrow{a}^\mu_S \qquad \text{and} \qquad \overleftarrow{a}^\mu_P a^\lambda_{ST} = \delta_{\lambda\mu}\delta_{PS} \overleftarrow{a}^\mu_T \tag{1.10}$$

then

$$\begin{gathered} \bar{L} \text{ has basis } \{\overrightarrow{a}^\mu_P \otimes \ell^\mu_X \mid \mu \in \hat{A}, P \in \hat{A}^\mu, X \in \hat{L}^\mu\} \qquad \text{and} \\ \bar{R} \text{ has basis } \{r^\mu_Y \otimes \overleftarrow{a}^\mu_Q \mid \mu \in \hat{A}, Q \in \hat{A}^\mu, Y \in \hat{R}^\mu\}. \end{gathered} \tag{1.11}$$

With notations as in (4.9) and (4.11) the map $\bar{\varepsilon}: \bar{L} \otimes_{\mathbb{F}} \bar{R} \to \bar{A}$ is determined by the constants $\varepsilon^\mu_{XY} \in \mathbb{F}$ given by

$$\varepsilon(\overrightarrow{a}^\mu_Q \otimes \ell^\mu_X \otimes r^\mu_Y \otimes \overleftarrow{a}^\mu_P) = \varepsilon^\mu_{XY} a^\mu_{QP} \tag{1.12}$$

and ε^μ_{XY} does not depend on Q and P since

$$\begin{aligned} \varepsilon(\overrightarrow{a}^\lambda_S \otimes \ell^\lambda_X \otimes r^\mu_Y \otimes \overleftarrow{a}^\mu_T) &= \varepsilon(a^\lambda_{SQ} \overrightarrow{a}^\lambda_Q \otimes \ell^\lambda_X \otimes r^\mu_Y \otimes \overleftarrow{a}^\mu_P a^\mu_{PT}) && (1.13) \\ &= a^\lambda_{SQ} \varepsilon(\overrightarrow{a}^\lambda_Q \otimes \ell^\lambda_X \otimes r^\mu_Y \otimes \overleftarrow{a}^\mu_P) a^\mu_{PT} && (1.14) \\ &= \delta_{\lambda\mu} a^\mu_{SQ} \varepsilon^\mu_{XY} a^\mu_{QP} a^\mu_{PT} = \varepsilon^\mu_{XY} a^\mu_{ST}. && (1.15) \\ & && (1.16) \end{aligned}$$

For each $\mu \in \hat{A}$ construct a matrix

$$\mathcal{E}^\mu = (\varepsilon^\mu_{XY}) \tag{1.17}$$

and let $D^\mu = (D^\mu_{ST})$ and $C^\mu = (C^\mu_{ZW})$ be invertible matrices such that $D^\mu \mathcal{E}^\mu C^\mu$ is a diagonal matrix with diagonal entries denoted ε^μ_X,

$$D^\mu \mathcal{E}^\mu C^\mu = \mathrm{diag}(\varepsilon^\mu_X). \tag{1.18}$$

In practice D^μ and C^μ are found by row reducing $\mathcal{E}^\mu$ to its Smith normal form. The ε^μ_P are the *invariant factors* of $\mathcal{E}^\mu$.

For $\mu \in \hat{A}, X \in \hat{R}^\mu, Y \in \hat{L}^\mu$, define the following elements of $\bar{R} \otimes_{\bar{A}} \bar{L}$,

$$\bar{m}^\mu_{XY} = r^\mu_X \otimes \overrightarrow{a}^\mu_P \otimes \overleftarrow{a}^\mu_P \otimes \ell^\mu_Y, \qquad \text{and} \qquad \bar{n}^\mu_{XY} = \sum_{Q_1,Q_2} C^\mu_{Q_1 X} D^\mu_{YQ_2} \bar{m}^\mu_{Q_1 Q_2}. \tag{1.19}$$

Since

$$\begin{aligned}
(r^\lambda_S \otimes \overrightarrow{a}^\lambda_W \otimes \overleftarrow{a}^\mu_Z \otimes \ell^\mu_T) &= (r^\lambda_S \otimes \overrightarrow{a}^\lambda_P a^\lambda_{PW} \otimes \overleftarrow{a}^\mu_Z \otimes \ell^\mu_T) &(1.20)\\
&= (r^\lambda_S \otimes \overrightarrow{a}^\lambda_P \otimes a^\lambda_{PW} \overleftarrow{a}^\mu_Z \otimes \ell^\mu_T) &(1.21)\\
&= \delta_{\lambda\mu}\delta_{WZ}(r^\lambda_S \otimes \overrightarrow{a}^\lambda_P \otimes \overleftarrow{a}^\lambda_P \otimes \ell^\lambda_T) &(1.22)\\
& &(1.23)
\end{aligned}$$

the element $\bar{m}^\mu_{XY}$ does not depend on P and $\{\bar{m}^\mu_{XY} \mid \mu \in \hat{A}, X \in \hat{R}^\mu, Y \in \hat{L}^\mu\}$ is a basis of $\bar{R} \otimes_{\bar{A}} \bar{L}$.

The following theorem determines the structure of the algebras $R \otimes_A L$ and $\bar{R} \otimes_{\bar{A}} \bar{L}$. This theorem is used by W.P. Brown in the study of the Brauer algebra. Part (a) is implicit in [Bro1,§2.2] and part (b) is proved in [Bro2].

Theorem 1.1. *Let $\pi\colon R \otimes_A L \to \bar{R} \otimes_{\bar{A}} \bar{L}$ be as in (4.7) and let $\{k_i\}$ be a basis of $\ker(\pi) = R \otimes_A NL$. Let*

$$n^\mu_{YT} \in R \otimes_A L \quad \textit{be such that} \quad \pi(n^\mu_{YT}) = \bar{n}^\mu_{YT},$$

where the elements $\bar{n}^\mu_{YT} \in \bar{R} \otimes_{\bar{A}} \bar{L}$ are as defined in (4.16).

(a) The sets $\{\bar{m}^\mu_{XY} \mid \mu \in \hat{A}, X \in \hat{R}^\mu, Y \in \hat{L}^\mu\}$ and $\{\bar{n}^\mu_{XY} \mid \mu \in \hat{A}, X \in \hat{R}^\mu, Y \in \hat{L}^\mu\}$ (see (4.16)) are bases of $\bar{R} \otimes_{\bar{A}} \bar{L}$, which satisfy

$$\bar{m}^\lambda_{ST}\bar{m}^\mu_{QP} = \delta_{\lambda\mu}\varepsilon^\mu_{TQ}\bar{m}^\mu_{SP} \qquad \textit{and} \qquad \bar{n}^\lambda_{ST}\bar{n}^\mu_{QP} = \delta_{\lambda\mu}\delta_{TQ}\varepsilon^\mu_T\bar{n}^\mu_{SP},$$

where ε^μ_{TQ} and ε^μ_T are as defined in (4.12) and (4.15).

(b) The radical of the algebra $R \otimes_A L$ is

$$\mathrm{Rad}(R \otimes_A L) = \mathbb{F}\text{-}\textit{span}\{k_i, n^\mu_{YT} \mid \varepsilon^\mu_Y = 0 \textit{ or } \varepsilon^\mu_T = 0\}$$

and the images of the elements

$$e^\mu_{YT} = \frac{1}{\varepsilon^\mu_T} n^\mu_{YT}, \qquad \textit{for } \varepsilon^\mu_Y \neq 0 \textit{ and } \varepsilon^\mu_T \neq 0,$$

are a set of matrix units in $(R \otimes_A L)/\mathrm{Rad}(R \otimes_A L)$.

Proof. The first statement in (a) follows from the equations in (4.17). If $(C^{-1})^\mu$ and $(D^{-1})^\mu$ are the inverses of the matrices C^μ and D^μ then

$$\sum_{X,Y}(C^{-1})^\mu_{XS}(D^{-1})^\mu_{TY}\bar n_{XY} = \sum_{X,Y,Q_1,Q_2}(C^{-1})^\mu_{XS}C^\mu_{Q_1X}\bar m_{Q_1Q_2}D^\mu_{YQ_2}(D^{-1})^\mu_{TY} \tag{1.24}$$

$$= \sum_{Q_1,Q_2}\delta_{SQ_1}\delta_{Q_2T}\bar m^\mu_{Q_1Q_2} = \bar m^\mu_{ST}, \tag{1.25}$$

$$\tag{1.26}$$

and so the elements $\bar m^\mu_{ST}$ can be written as linear combinations of the $\bar n^\mu_{XY}$. This establishes the second statement in (a). By direct computation, using (4.10) and (4.12),

$$\begin{aligned}
\bar m^\lambda_{ST}\bar m^\mu_{QP} &= (r^\lambda_S\otimes \overrightarrow{a}^\lambda_W\otimes \overleftarrow{a}^\lambda_W\otimes \ell^\lambda_T)(r^\mu_Q\otimes \overrightarrow{a}^\mu_Z\otimes \overleftarrow{a}^\mu_Z\otimes \ell^\mu_P)\\
&= r^\lambda_S\otimes \overrightarrow{a}^\lambda_W\otimes \varepsilon(\overleftarrow{a}^\lambda_W\otimes \ell^\lambda_T\otimes r^\mu_Q\otimes \overrightarrow{a}^\mu_Z)\overleftarrow{a}^\mu_Z\otimes \ell^\mu_P\\
&= \delta_{\lambda\mu}(r^\lambda_S\otimes \overrightarrow{a}^\lambda_W\otimes \varepsilon^\lambda_{TQ}\bar a^\lambda_{WZ}\overleftarrow{a}^\lambda_Z\otimes \ell^\lambda_P)\\
&= \delta_{\lambda\mu}\varepsilon^\lambda_{TQ}(r^\lambda_S\otimes \overrightarrow{a}^\lambda_W\otimes \overleftarrow{a}^\lambda_W\otimes \ell^\lambda_P) = \delta_{\lambda\mu}\varepsilon^\lambda_{TQ}\bar m^\lambda_{SP},
\end{aligned}$$

and

$$\begin{aligned}
\bar n^\lambda_{ST}\bar n^\mu_{UV} &= \sum_{Q_1,Q_2,Q_3,Q_4} C^\lambda_{Q_1S}D^\lambda_{TQ_2}\bar m^\lambda_{Q_1Q_2}C^\mu_{Q_3U}D^\mu_{VQ_4}\bar m^\mu_{Q_3Q_4}\\
&= \sum_{Q_1,Q_2,Q_3,Q_4}\delta_{\lambda\mu}C^\lambda_{Q_1S}D^\lambda_{TQ_2}\varepsilon^\mu_{Q_2Q_3}C^\mu_{Q_3U}D^\mu_{VQ_4}\bar m^\mu_{Q_1Q_4}\\
&= \delta_{\lambda\mu}\sum_{Q_1,Q_4}\delta_{TU}\varepsilon^\mu_T C^\mu_{Q_1S}D^\mu_{VQ_4}\bar m^\mu_{Q_1Q_4} = \delta_{\lambda\mu}\delta_{TU}\varepsilon^\mu_T\bar n^\mu_{SV}.
\end{aligned}$$

(b) Let $N = \mathrm{Rad}(A)$ as in (4.5). If $r_1\otimes n_1\ell_1, r_2\otimes n_2\ell_2 \in R\otimes_A NL$ with $n_1\in N^i$ for some $i\in\mathbb{Z}_{>0}$ then

$$(r_1\otimes n_1\ell_1)(r_2\otimes n_2\ell_2) = r_1\otimes \varepsilon(n_1\ell_1\otimes r_2)n_2\ell_2 = r_1\otimes n_1\varepsilon(\ell_1\otimes r_2)n_2\ell_2 \in R\otimes_A N^{i+1}L.$$

Since N is a nilpotent ideal of A it follows that $\ker(\pi) = R\otimes_A NL$ is a nilpotent ideal of $R\otimes_A L$. So $\ker(\pi)\subseteq \mathrm{Rad}(R\otimes_A L)$.

Let

$$I = \mathbb{F}\text{-span}\{k_i, n^\mu_{YT} \mid \varepsilon^\mu_Y = 0 \text{ or } \varepsilon^\mu_T = 0\}.$$

The multiplication rule for the $\bar n_{YT}$ implies that $\pi(I)$ is an ideal of $\bar R\otimes_{\bar A}\bar L$ and thus, by the correspondence between ideals of $\bar R\otimes_{\bar A}\bar L$ and ideals of $R\otimes_A L$ which contain $\ker(\pi)$, I is an ideal of $R\otimes_A L$.

If $\bar n^\mu_{Y_1T_1}, \bar n^\mu_{Y_2T_2}, \bar n^\mu_{Y_3T_3} \in \{\bar n^\mu_{YT} \mid \varepsilon^\mu_Y = 0 \text{ or } \varepsilon^\mu_T = 0\}$ then

$$\bar n^\mu_{Y_1T_1}\bar n^\mu_{Y_2T_2}\bar n^\mu_{Y_3T_3} = \delta_{T_1Y_2}\varepsilon^\mu_{Y_2}\bar n^\mu_{Y_1T_2}\bar n^\mu_{Y_3T_3} = \delta_{T_1Y_2}\delta_{T_2Y_3}\varepsilon^\mu_{Y_2}\varepsilon^\mu_{T_2}\bar n^\mu_{Y_1T_3} = 0,$$

since $\varepsilon^\mu_{Y_2} = 0$ or $\varepsilon^\mu_{T_2} = 0$. Thus any product $n^\mu_{Y_1T_1}n^\mu_{Y_2T_2}n^\mu_{Y_3T_3}$ of three basis elements of I is in $\ker(\pi)$. Since $\ker(\pi)$ is a nilpotent ideal of $R\otimes_A L$ it follows that I is an ideal of $R\otimes_A L$ consisting of nilpotent elements. So $I\subseteq \mathrm{Rad}(R\otimes_A L)$.

Since

$$e^\lambda_{YT}e^\mu_{UV} = \frac{1}{\varepsilon^\lambda_T}\frac{1}{\varepsilon^\mu_V}n^\lambda_{YT}n^\mu_{UV} = \delta_{\lambda\mu}\delta_{TU}\frac{1}{\varepsilon^\lambda_T\varepsilon^\lambda_V}\varepsilon^\lambda_T n^\lambda_{YV} = \delta_{\lambda\mu}\delta_{TU}e^\lambda_{YV} \qquad \mathrm{mod}\ I,$$

the images of the elements e^λ_{YT} in (4.7) form a set of matrix units in the algebra $(R\otimes_A L)/I$. Thus $(R\otimes_A L)/I$ is a split semisimple algebra and so $I\supseteq \mathrm{Rad}(R\otimes_A L)$. □

1.1 Basic constructions for $A \subseteq B$

Let $A \subseteq B$ be an inclusion of algebras. Let $\varepsilon_1 : B \to A$ be an (A, A) bimodule homomorphism and use the (A, A)-bimodule homomorphism

$$\begin{array}{rccc} \varepsilon: & B \otimes_{\mathbb{F}} B & \longrightarrow & A \\ & b_1 \otimes b_2 & \longmapsto & \varepsilon_1(b_1 b_2) \end{array} \tag{1.27}$$

and (4.2) to define the basic construction $B \otimes_A B$. Theorem 4.28 below provides the structure of $B \otimes_A B$ in the case that both A and B are split semisimple.

Let us record the following facts,

(4.20a) If $p \in A$ and $pAp = \mathbb{F}p$ then $(p \otimes 1)(B \otimes_A B)(p \otimes 1) = \mathbb{F} \cdot (p \otimes 1)$,

(4.20b) If p is an idempotent of A and $pAp = \mathbb{F}p$ then $\varepsilon_1(1) \in \mathbb{F}$,

(4.20c) If $p \in A$, $pAp = \mathbb{F}p$ and if $\varepsilon_1(1) \neq 0$, then $\frac{1}{\varepsilon(1)}(p \otimes 1)$ is a minimal idempotent in $B \otimes_A B$,

which are justified as follows. If $p \in A$ and $pAp = \mathbb{F}p$ and $b_1, b_2 \in B$ then $(p \otimes 1)(b_1 \otimes b_2)(p \otimes 1) = (p \otimes \varepsilon_1(b_1)b_2)(p \otimes 1) = p \otimes \varepsilon_1(b_1)\varepsilon_1(b_2 p) = p\varepsilon_1(b_1)\varepsilon_1(b_2)p \otimes 1 = \xi p \otimes 1$, for some constant $\xi \in \mathbb{F}$. This establishes (a). If p is an idempotent of A and $pAp = \mathbb{F}p$ then $p\varepsilon_1(1)p = \varepsilon_1(p^2) = \varepsilon_1(1 \cdot p) = \varepsilon_1(1)p$ and so (b) holds. If $p \in A$ and $pAp = \mathbb{F}p$ then $(p \otimes 1)^2 = \varepsilon_1(1)(p \otimes 1)$ and so, if $\varepsilon_1(1) \neq 0$, then $\frac{1}{\varepsilon(1)}(p \otimes 1)$ is a minimal idempotent in $B \otimes_A B$.

Assume A and B are split semisimple. Let

$\hat{A}$ be an index set for the irreducible A-modules A^μ,

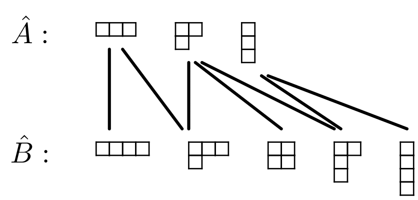
$\hat{B}$ be an index set for the irreducible B-modules B^λ, and let

$\hat{A}^\mu = \{ \, {\scriptstyle P \to \mu} \, \}$ be an index set for a basis of the simple A-module A^μ,

for each $\mu \in \hat{A}$ (the composite ${\scriptstyle P \to \mu}$ is viewed as a single symbol). We think of $\hat{A}^\mu$ as the set of "paths to μ" in the two level graph

$$\begin{array}{lll} \Gamma \quad \text{with} & \text{vertices on level A: } \hat{A}, \quad \text{vertices on level B: } \hat{B}, \quad \text{and} \\ & m_\mu^\lambda \text{ edges } \mu \to \lambda \text{ if } A^\mu \text{ appears with multiplicity } m_\mu^\lambda \text{ in } \mathrm{Res}_A^B(B^\lambda). \end{array} \tag{1.28}$$

For example, the graph Γ for the symmetric group algebras $A = \mathbb{C}S_3$ and $B = \mathbb{C}S_4$ is



If $\lambda \in \hat{B}$ then

$$\hat{B}^\lambda = \{ {\scriptstyle P \to \mu} \to \lambda \mid \mu \in \hat{A}, \, {\scriptstyle P \to \mu} \in \hat{A}^\mu \text{ and } \mu \to \lambda \text{ is an edge in } \Gamma \} \tag{1.29}$$

is an index set for a basis of the irreducible B-module B^λ. We think of $\hat{B}^\lambda$ as the set of paths to λ in the graph Γ. Let

$$\{ a_{\substack{PQ \\ \mu}} \mid \mu \in \hat{A}, {\scriptstyle P \to \mu}, {\scriptstyle Q \to \mu} \in \hat{A}^\mu \} \qquad \text{and} \qquad \{ b_{\substack{PQ \\ \mu\,\nu \\ \lambda}} \mid \lambda \in \hat{B}, {\scriptstyle P \to \mu} \to \lambda, {\scriptstyle Q \to \nu} \to \lambda \in \hat{B}^\lambda \}, \tag{1.30}$$

be sets of matrix units in the algebras A and B, respectively, so that

$$a_{\substack{PQ\\ \mu}} a_{\substack{ST\\ \nu}} = \delta_{\mu\nu}\delta_{QS}\, a_{\substack{PT\\ \mu}} \qquad \text{and} \qquad b_{\substack{PQ\\ \mu\gamma\\ \lambda}} b_{\substack{ST\\ \tau\nu\\ \sigma}} = \delta_{\lambda\sigma}\delta_{QS}\delta_{\gamma\tau}\, b_{\substack{PT\\ \mu\nu\\ \lambda}}, \tag{1.31}$$

and such that, for all $\mu \in \hat{A}$, $P, Q \in \hat{A}^{\mu}$,

$$a^{\mu}_{\substack{PQ\\ \mu}} = \sum_{\mu\to\lambda} b^{\lambda}_{\substack{PQ\\ \mu\,\mu\\ \lambda}} \tag{1.32}$$

where the sum is over all edges $\mu \to \lambda$ in the graph Γ.

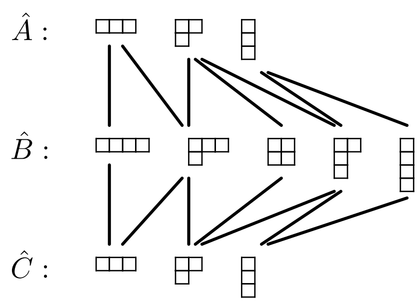
Though is not necessary for the following it is conceptually helpful to let $C = B \otimes_A B$, let $\hat{C} = \hat{A}$ and extend the graph Γ to a graph $\hat{\Gamma}$ with three levels, so that the edges between level B and level C are the reflections of the edges between level A and level B. In other words,

$$\begin{array}{lll} \hat{\Gamma} & \text{has} & \text{vertices on level } C\text{:} \quad \hat{C}, \quad \text{and} \\ & & \text{an edge } \lambda \to \mu,\ \lambda \in \hat{B},\ \mu \in \hat{C}, \text{ for each edge } \mu \to \lambda,\ \mu \in \hat{A},\ \lambda \in \hat{B}. \end{array} \tag{1.33}$$

For each $\nu \in \hat{C}$ define

$$\hat{C}^{\nu} = \left\{ {\scriptstyle P\to\mu} \to \lambda \to \nu \;\middle|\; \begin{array}{c} \mu \in \hat{A},\ \lambda \in \hat{B},\ \nu \in \hat{C},\ {\scriptstyle P\to\mu} \in \hat{A}^{\mu} \text{ and} \\ \mu \to \lambda \text{ and } \lambda \to \nu \text{ are edges in } \hat{\Gamma} \end{array} \right\}, \tag{1.34}$$

so that $\hat{C}^{\nu}$ is the set of "paths to ν" in the graph $\hat{\Gamma}$. Continuing with our previous example, $\hat{\Gamma}$ is



Theorem 1.2. *Assume A and B are split semisimple, and let the notations and assumption be as in (4.21-4.25).*

(a) The elements of $B \otimes_A B$ given by

$$b_{\substack{PT\\ \mu\gamma\\ \lambda}} \otimes b_{\substack{TQ\\ \gamma\nu\\ \sigma}}$$

do not depend on the choice of ${\scriptstyle T\to\gamma} \in \hat{A}^{\gamma}$ and form a basis of $B \otimes_A B$.

(b) For each edge $\mu \to \lambda$ in Γ define a constant $\varepsilon^{\lambda}_{\mu} \in \mathbb{F}$ by

$$\varepsilon_1\left(b_{\substack{PP\\ \mu\mu\\ \lambda}} \right) = \varepsilon^{\lambda}_{\mu}\, a_{\substack{PP\\ \mu}} \tag{1.35}$$

Then $\varepsilon^{\lambda}_{\mu}$ is independent of the choice of ${\scriptstyle P\to\mu} \in \hat{A}^{\mu}$ and

$$\Big(b_{\substack{PT\\ \mu\gamma\\ \lambda}} \otimes b_{\substack{TQ\\ \gamma\nu\\ \sigma}} \Big) \Big(b_{\substack{RX\\ \tau\pi\\ \rho}} \otimes b_{\substack{XS\\ \pi\xi\\ \eta}} \Big) = \delta_{\gamma\pi}\delta_{QR}\delta_{\nu\tau}\delta_{\sigma\rho}\varepsilon^{\sigma}_{\gamma} \Big(b_{\substack{PT\\ \pi\mu\\ \gamma}} \otimes b_{\substack{TS\\ \gamma\xi\\ \eta}} \Big).$$

$$\mathrm{Rad}(B\otimes_A B) \qquad \textit{has basis} \qquad \Big\{ b_{\substack{PT\\ \mu\,\gamma\\ \lambda}} \otimes b_{\substack{TQ\\ \gamma\,\nu\\ \sigma}} \mid \varepsilon_\mu^\lambda = 0 \ \textit{or}\ \varepsilon_\nu^\sigma = 0 \Big\},$$

and the images of the elements

$$e_{\substack{PQ\\ \mu\,\nu\\ \lambda\,\sigma\\ \gamma}} = \left(\frac{1}{\varepsilon_\gamma^\sigma}\right)\Big(b_{\substack{PT\\ \mu\,\gamma\\ \lambda}} \otimes b_{\substack{TQ\\ \gamma\,\nu\\ \sigma}}\Big), \qquad \textit{such that} \quad \varepsilon_\mu^\lambda \neq 0 \ \textit{and}\ \varepsilon_\nu^\sigma \neq 0,$$

form a set of matrix units in $(B\otimes_A B)/\mathrm{Rad}(B\otimes_A B)$.

(c) *Let* $tr_B : B \to \mathbb{F}$ *and* $tr_A : A \to \mathbb{F}$ *be traces on* B *and* A*, respectively, such that*

$$tr_A(\varepsilon_1(b)) = tr_B(b), \qquad \textit{for all } b \in B. \tag{1.36}$$

Let χ_A^μ*,* $\mu \in \hat{A}$*, and* χ_B^λ*,* $\lambda \in \hat{B}$*, be the irreducible characters of the algebras* A *and* B*, respectively. Define constants* tr_A^μ*,* $\mu\in\hat{A}$*, and* tr_B^λ*,* $\lambda \in \hat{B}$*, by the equations*

$$\mathrm{tr}_A = \sum_{\mu\in\hat{A}} \mathrm{tr}_A^\mu \chi_A^\mu \qquad \textit{and} \qquad \mathrm{tr}_B = \sum_{\lambda\in\hat{B}} \mathrm{tr}_B^\lambda \chi_B^\lambda, \tag{1.37}$$

respectively. Then the constants ε_μ^λ *defined in (4.29) satisfy*

$$\mathrm{tr}_B^\lambda = \varepsilon_\mu^\lambda \ \mathrm{tr}_A^\mu.$$

(d) *In the algebra* $B\otimes_A B$*,*

$$1\otimes 1 = \sum_{\substack{P\\ \downarrow\\ \mu\\ \lambda\swarrow\ \searrow\gamma}} b_{\substack{PP\\ \mu\,\mu\\ \lambda}} \otimes b_{\substack{PP\\ \mu\,\mu\\ \gamma}}$$

(g) *By left multiplication, the algebra* $B\otimes_A B$ *is a left* B*-module. If* $\mathrm{Rad}(B\otimes_A B)$ *is a* B*-submodule of* $B\otimes_A B$ *and* $\iota\colon B \to (B\otimes_A B)/\mathrm{Rad}(B\otimes_A B)$ *is a left* B*-module homomorphism then*

$$\iota\Big(b_{\substack{RS\\ \tau\,\beta\\ \pi}}\Big) = \sum_{\pi\to\gamma} e_{\substack{RS\\ \tau\,\beta\\ \pi\,\pi\\ \gamma}}.$$

Proof. By (4.11) and (4.25),

$$\begin{array}{ccc} B & \xrightarrow{\sim} & \bigoplus_{\mu\in\hat{A}} \overrightarrow{A}^\mu \otimes L^\mu \\ b_{\substack{PQ\\ \mu\,\nu\\ \lambda}} & \longmapsto & \overrightarrow{a}_{\substack{P\\ \mu}} \otimes \ell^\mu_{\substack{Q\\ \mu\,\nu\\ \lambda}} \end{array} \quad \text{and} \quad \begin{array}{ccc} B & \xrightarrow{\sim} & \bigoplus_{\nu\in\hat{A}} R^\nu \otimes \overleftarrow{A}^\nu \\ b_{\substack{PQ\\ \mu\,\nu\\ \lambda}} & \longmapsto & r^\nu_{\substack{P\\ \mu\,\nu\\ \lambda}} \otimes \overleftarrow{a}_{\substack{Q\\ \nu}} \end{array} \tag{1.38}$$

as left A-modules and as right A-modules, respectively. Identify the left and right hand sides of these isomorphisms. Then, by (4.17), the elements of $C = B\otimes_A B$ given by

$$\bar{m}_{\substack{PQ\\ \mu\,\nu\\ \lambda\,\sigma\\ \gamma}} = r^\gamma_{\substack{P\\ \mu\,\gamma\\ \lambda}} \otimes \overleftarrow{a}_{\substack{T\\ \gamma}} \otimes \overrightarrow{a}_{\substack{T\\ \gamma}} \otimes \ell^\gamma_{\substack{Q\\ \gamma\,\nu\\ \sigma}} = b_{\substack{PT\\ \mu\,\gamma\\ \lambda}} \otimes b_{\substack{TQ\\ \gamma\,\nu\\ \sigma}} \tag{1.39}$$

do not depend on $T\to\gamma \in \hat{A}^\gamma$ and form a basis of $B\otimes_A B$.

(b) By (4.12), the map $\varepsilon\colon B \otimes_{\mathbb{F}} B \to A$ is determined by the values

$$\varepsilon^{\mu}_{\substack{TQ\\ \gamma\,\tau\\ \lambda\,\sigma\\ \mu}} \in \mathbb{F} \qquad \text{given by} \qquad \varepsilon^{\mu}_{\substack{TQ\\ \gamma\,\tau\\ \lambda\,\sigma\\ \mu}} a_{\substack{PP\\ \mu}} = \varepsilon\big(\overrightarrow{a}_{\substack{P\\ \mu}} \otimes \ell^{\mu}_{\substack{T\\ \mu\,\gamma\\ \lambda}} \otimes r^{\mu}_{\substack{Q\\ \tau\,\mu\\ \sigma}} \otimes \overleftarrow{a}_{\substack{P\\ \mu}}\big). \tag{1.40}$$

Since

$$\begin{aligned}\varepsilon^{\mu}_{\substack{TQ\\ \gamma\,\tau\\ \lambda\,\sigma\\ \mu}} a_{\substack{PP\\ \mu}} &= \varepsilon\big(b_{\substack{PT\\ \mu\,\gamma\\ \lambda}} \otimes b_{\substack{QP\\ \tau\,\mu\\ \sigma}}\big) = \varepsilon_1\big(b_{\substack{PT\\ \mu\,\gamma\\ \lambda}} \otimes b_{\substack{QP\\ \tau\,\mu\\ \sigma}}\big)\\ &= \delta_{\substack{TQ\\ \gamma\,\tau\\ \lambda\,\sigma}}\varepsilon_1\big(b_{\substack{PP\\ \mu\,\mu\\ \lambda}}\big) = \delta_{\substack{TQ\\ \gamma\,\tau\\ \lambda\,\sigma}}\varepsilon_1\big(b_{\substack{PP\\ \mu\,\mu\\ \lambda}} b_{\substack{PP\\ \mu\,\mu\\ \lambda}}\big) = \delta_{\substack{TQ\\ \gamma\,\tau\\ \lambda\,\sigma}}\varepsilon^{\mu}_{\substack{PP\\ \mu\mu\mu\\ \lambda\lambda}} a_{\substack{PP\\ \mu}}.\end{aligned}$$

the matrix $\mathcal{E}^{\mu}$ given by (4.14) is diagonal with entries $\varepsilon^{\lambda}_{\mu}$ given by (4.15) and, by (4.17), $\varepsilon^{\lambda}_{\mu}$ is independent of $P{\to}\mu \in \hat{A}^{\mu}$. By Theorem 4.18(a),

$$\bar{m}_{\substack{PQ\\ \mu\,\nu\\ \lambda\,\sigma\\ \gamma}} \bar{m}_{\substack{RS\\ \tau\,\xi\\ \rho\,\eta\\ \pi}} = \delta_{\gamma\pi}\varepsilon_{\substack{QR\\ \nu\,\tau\\ \sigma\,\rho\\ \gamma}} \bar{m}_{\substack{PS\\ \mu\,\xi\\ \lambda\,\eta\\ \gamma}} = \delta_{\gamma\pi}\delta_{\substack{QR\\ \nu\,\tau\\ \sigma\,\rho}}\varepsilon^{\sigma}_{\gamma} \bar{m}_{\substack{PS\\ \mu\,\xi\\ \lambda\,\eta\\ \gamma}}$$

in the algebra C. The rest of the statements in part (b) follow from Theorem 4.18(b).
(c) Evaluating the equations in (4.31) and using (4.29) gives

$$\mathrm{tr}^{\lambda}_{B} = \mathrm{tr}_B(b_{\substack{PP\\ \mu\,\mu\\ \lambda}}) = \mathrm{tr}_A(\varepsilon_1(b_{\substack{PP\\ \mu\,\mu\\ \lambda}})) = \varepsilon^{\lambda}_{\mu}\mathrm{tr}_A(a_{\substack{PP\\ \mu}}) = \varepsilon^{\lambda}_{\mu}\mathrm{tr}^{\mu}_{A}, \tag{1.41}$$

(d) Since

$$1 = \sum_{P\to\mu\to\lambda} b_{\substack{PP\\ \mu\,\mu\\ \lambda}} \qquad \text{in the algebra } B,$$

it follows from part (b) and (4.16) that

$$1\otimes 1 = \Big(\sum_{P\to\mu\to\lambda} b_{\substack{PP\\ \mu\,\mu\\ \lambda}}\Big) \otimes \Big(\sum_{Q\to\nu\to\gamma} b_{\substack{QQ\\ \nu\,\nu\\ \gamma}}\Big) = \sum_{\substack{P\to\mu\to\lambda\\ Q\to\nu\to\gamma}} \delta_{PQ}\delta_{\mu\nu}\Big(b_{\substack{PP\\ \mu\,\mu\\ \lambda}} \otimes b_{\substack{QQ\\ \nu\,\nu\\ \gamma}}\Big) = \sum_{\substack{P\\ \downarrow\\ \mu\\ \swarrow\ \ \searrow\\ \lambda\qquad\gamma}} \bar{m}_{\substack{PP\\ \mu\,\mu\\ \lambda\,\gamma\\ \mu}}$$

giving part (d).
(e) By left multiplication, the algebra $B\otimes_A B$ is a left B-module. If $\varepsilon^{\lambda}_{\gamma} \neq 0$ and $\varepsilon^{\sigma}_{\gamma} \neq 0$ then

$$b_{\substack{RS\\ \tau\,\beta\\ \pi}} e_{\substack{PQ\\ \mu\,\nu\\ \lambda\,\sigma\\ \gamma}} = \left(\frac{1}{\varepsilon^{\sigma}_{\gamma}}\right) b_{\substack{RS\\ \tau\,\beta\\ \pi}}\Big(b_{\substack{PT\\ \mu\,\gamma\\ \lambda}} \otimes b_{\substack{TQ\\ \gamma\,\nu\\ \sigma}}\Big) = \left(\frac{1}{\varepsilon^{\sigma}_{\gamma}}\right) \delta_{\substack{SP\\ \beta\,\mu\\ \pi\,\lambda}}\Big(b_{\substack{RT\\ \tau\,\gamma\\ \lambda}} \otimes b_{\substack{TQ\\ \gamma\,\nu\\ \sigma}}\Big) = \delta_{\substack{SP\\ \beta\,\mu\\ \pi\,\lambda}} e_{\substack{RQ\\ \tau\,\nu\\ \pi\,\sigma\\ \gamma}}.$$

Thus, if $\iota\colon B \to (B\otimes_A B)/\mathrm{Rad}(B\otimes_A B)$ is a left B-module homomorphism then

$$\iota\Big(b_{\substack{RS\\ \tau\,\beta\\ \pi}}\Big) = \iota\Big(b_{\substack{RS\\ \tau\,\beta\\ \pi}}\Big)\cdot 1 = b_{\substack{RS\\ \tau\,\beta\\ \pi}} \sum_{P\to\mu\to\lambda\to\gamma} e_{\substack{PP\\ \mu\,\mu\\ \lambda\,\lambda\\ \gamma}} = \sum_{P\to\mu\to\lambda\to\gamma} \delta_{\substack{SP\\ \beta\,\mu\\ \pi\,\lambda}} e_{\substack{RP\\ \tau\,\mu\\ \pi\,\lambda\\ \gamma}} = \sum_{\pi\to\gamma} e_{\substack{RS\\ \tau\,\beta\\ \pi\,\pi\\ \gamma}}.$$

$\square$

2 Quasihereditary algebras

Let $\mathbb{F}$ be a field. A *separable algebra* over $\mathbb{F}$ is an algebra A such that

$$\frac{A}{\mathrm{Rad}(A)} \cong \bigoplus_{\lambda} \in \hat{A} M_{d_\lambda}(\mathbb{F}).$$

Two algebras A and B are *Morita equivalent* if Mod-A is equivalent to Mod-B (Check this in Gelfand-Manin).

A ring A is *semiprimary* if there is a nilpotent ideal $\mathrm{Rad}(A)$ such that $A/\mathrm{Rad}(A)$ is semisimple artinian. Note: If A is finite dimensional then A is semiprimary.

A *hereditary* ring is a ring A such that every submodule of a projective module is projective.

A *heredity ideal* is an ideal J such that

(a) J is projective as a right A-module,

(b) $J^2 = J$, and

(c) $J\mathrm{Rad}(A)J = 0$.

Note: $J^2 = J$ if and only if there is an idempotent $e \in A$ with $J = AeA$.

A *quasihereditary ring* is a semiprimary ring A with a chain of ideals

$$0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_m = A \qquad \text{such that} \quad \frac{J_\ell}{J_{\ell-1}} \quad \text{is a heredity ideal of} \quad \frac{A}{J_{\ell-1}}$$

for each $1 \le \ell \le m-1$.

2.1 The Dlab-Ringel theorem

Let C and D be rings,

$$\begin{array}{l} L, \text{ a } (C,D) \text{ bimodule,} \\ R, \text{ a } (D,C) \text{ bimodule} \end{array} \qquad \text{and} \qquad \varepsilon\colon L \otimes_D R \to C,$$

a (C,C) bimodule homomorphism. Define an algebra

$$A = C \oplus D \oplus L \oplus R \oplus R \otimes_C L$$

and product determined by the multiplication in C and D, the module structure of of R and L and the additional relations

$$cr = 0, \quad , d\ell = 0, \quad rd = 0, \quad \ell c = 0, \quad \text{and} \quad (r_1 \otimes \ell_1)(r_2 \otimes \ell_2) = r_1 \otimes \varepsilon(\ell_1 \otimes r_2)\ell_2.$$

Let

e_C be the image of the identity of C in A, and

e_D be the image of the identity of D in A.

Then, if $e = e_C$ then

$$\begin{aligned} 1-e=e_D, \qquad & C = eAe, & L = eA(1-e), \\ & D' = (1-e)A(1-e), & R = (1-e)Ae, \end{aligned}$$

so that

$$A = \left\{ \begin{pmatrix} c & \ell \\ r & d' \end{pmatrix} \;\middle|\; c \in C, \ell \in L, r \in R, d' \in D' \right\}$$

with matrix multiplication. Then

$D' = D + R \otimes_C L$ is a subring of A, and

$R \otimes_C L$ is an ideal in A, and

$R \otimes_C L = (1-e)AeA(1-e)$.

Theorem 2.1. *Let A be a quasihereditary algebra,*

$$0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_m = A.$$

Let e be a indempotent in A such that

$$J_{m-1} = AeA \qquad \text{and} \qquad eA(1-e) \subseteq \mathrm{Rad}(A).$$

Let

$$C = eAe \qquad \text{and} \qquad D = \frac{A}{AeA} = \frac{A}{J_{m-1}}$$

and

$${}_C L_D = eA(1-e) \qquad \text{and} \qquad {}_D R_C = (1-e)Ae$$

and let

$$\begin{array}{rccc} \varepsilon\colon & L \otimes_D R & \longrightarrow & C \\ & \ell \otimes r & \longmapsto & \ell r \end{array}$$

Assume D is a separable k-algebra. Then

(a) $D + (1-e)AeA(1-e) = (1-e)A(1-e)$,

(b) $A = C(\varepsilon)$,

(c) C is qusihereditary with heredity chain

$$0 = I_0 \subseteq \cdots \subseteq I_{m-1} = C, \qquad \text{where} \quad I_\ell = eJ_\ell e.$$

2.2 Nondegeneracy

Let

$$\varepsilon\colon L \otimes_D R \qquad \text{be a } (C,C) \text{ bimodule homomorphism.}$$

Let *left radical* $L(\varepsilon)$ and the *right radical* $R(\varepsilon)$ are defined by

$$\begin{aligned} L(\varepsilon) &= \{\ell \in L \mid \varepsilon(\ell \otimes r) \in \mathrm{Rad}(C), \text{ for all } r \in R\}, \\ R(\varepsilon) &= \{r \in R \mid \varepsilon(\ell \otimes r) \in \mathrm{Rad}(C), \text{ for all } \ell \in L\}, \end{aligned}$$

The map ε is *nondegenerate* if

$$\mathrm{Rad}(C) = 0, \qquad L(\varepsilon) = 0, \qquad \text{and} \qquad R(\varepsilon) = 0.$$

Let

$$\overline{C} = \frac{C}{\mathrm{Rad}(C)}, \qquad \overline{L} = \frac{L}{L(\varepsilon)}, \quad \overline{R} = \frac{R}{R(\varepsilon)}, \qquad \text{and define} \qquad \begin{array}{rccc} \bar\varepsilon\colon & \bar L \otimes_D \bar R & \longrightarrow & \bar C \\ & \bar\ell \otimes \bar r & \longmapsto & \overline{\varepsilon(\ell \otimes r)} \end{array}$$

Then $\bar\varepsilon$ is nondegenerate. If

$$\begin{array}{rccc} \phi\colon & R \otimes_C L & \longrightarrow & \bar R \otimes_{\bar C} \bar L \\ & \bar r \otimes \bar\ell & \longmapsto & \overline{r \otimes \ell} \end{array}$$

then $\ker\phi$ is generated by $R \otimes_C L(\varepsilon)$ and $R(\varepsilon) \otimes_C L$, $\ker\phi \cdot R \subseteq R(\varepsilon)$, $L \cdot \ker\phi \subseteq L(\varepsilon)$, and

$$A(\bar\varepsilon) \cong \frac{A(\varepsilon)}{I}, \qquad \text{where} \qquad I = \mathrm{Rad}(C) + L(\varepsilon) + R(\varepsilon) + \ker\phi.$$

If $\varepsilon\colon L \otimes_D R \to C$ is nondegenerate then the map

$$\begin{array}{rccc} \tau\colon & R & \xrightarrow{\sim} & L^* \\ & r & \mapsto & \begin{array}{rccc} \lambda_r\colon & L & \to & C \\ & \ell & \mapsto & \varepsilon(\ell \otimes r) \end{array} \end{array}$$

is an isomorphism and

$$\varepsilon = \mathrm{ev} \circ (\mathrm{id} \otimes \tau).$$

Thus

$$A(\varepsilon) \cong A(\mathrm{ev}).$$

2.3 Duals and Projectives

Let L be a C-module and let

$$Z = \mathrm{End}_C(L)$$

so that L is a (C, Z) bimodule. The *dual module* to L is the (Z, C) bimodule

$$L^* = \mathrm{Hom}_C(L, C).$$

The *evaluation map* is the (C, C) bimodule homomorphism

$$\begin{array}{rccc} \mathrm{ev}\colon & L \otimes_Z L^* & \longrightarrow & C \\ & \ell \otimes \lambda & \longmapsto & \lambda(\ell) \end{array}$$

and the *centralizer map* is the (Z, Z) bimodule homomorphism

$$\begin{array}{rccc} \xi\colon & L^* \otimes_C L & \longrightarrow & Z \\ & \lambda \otimes \ell & \longmapsto & \begin{array}{rccc} z_{\lambda,\ell}\colon & L & \to & L \\ & m & \mapsto & \lambda(m)\ell \end{array} \end{array}$$

Recall that [Bou, Alg. II §4.2 Cor.]

(a) L is a projective C-module if and only if $1 \in \mathrm{im}\,\xi$,

(b) If L is a projective C-module then ξ is injective,

(c) If L is a finitely generated projective C-module then ξ is bijective,

(d) If L is a finitely generated free module then

$$\xi^{-1}(z) = \sum_i b_i^* \otimes z(b_i),$$

where $\{b_1, \dots, b_d\}$ is a basis of L and $\{b_1^*, \dots, b_d^*\}$ is the dual basis in M^*.

Statement (a) says that L is projective if and only if there exist $b_i \in L$ and $b_i^* \in L^*$ such that

$$\text{if } \ell \in L \quad \text{then} \quad \ell = \sum_i b_i^*(\ell) b_i, \qquad \text{so that} \quad \xi\Big(\sum_i b_i^* \otimes b_i\Big) = 1.$$

2.4 The Macpherson-Vilonen construction

Let $\mathcal{C}$ and $\mathcal{D}$ be categories

$$F\colon \mathcal{C} \to \mathcal{D} \quad \text{and} \quad G\colon \mathcal{C} \to \mathcal{D} \quad \text{be functors,} \qquad \text{and} \qquad F \xrightarrow{\varepsilon} G,$$

a natural transformation. Define a category $\mathcal{A}$ with

Objects: $(M, V; PICTURE)$, where $M \in \mathcal{C}$, $V \in \mathcal{D}$, and $m, n \in \mathrm{Mor}(\mathcal{D})$,

Morphisms: (f, g) with $f \in \mathrm{Mor}(\mathcal{C})$, $g \in \mathrm{Mor}(\mathcal{D})$ such that

$$PICTURE \qquad \text{commutes.}$$

A fundamental case is when $\mathcal{D}$ is the category of vector spaces over $\mathbb{F}$.

The connection between the Dlab-Ringel construction and the Macpherson-Vilonen construction is given by letting $\mathcal{C} = C$-mod and $\mathcal{D} = D$-mod and

$$\begin{array}{cccc} F\colon & \mathcal{C} & \longrightarrow & \mathcal{D} \\ & M & \longmapsto & R \otimes_C M \end{array} \qquad \text{and} \qquad \begin{array}{cccc} G\colon & \mathcal{C} & \longrightarrow & \mathcal{D} \\ & M & \longmapsto & \mathrm{Hom}_C(L, M) \end{array}$$

where the D-action on $\mathrm{Hom}_C(L, M)$ is given by

$$(d\phi)(\ell) = \phi(\ell d), \qquad \text{for } d \in D,\ \ell \in L, \text{ and } \phi \in \mathrm{Hom}_C(L, M).$$

Then let $\varepsilon\colon F \to G$ be the natural transformation given by

$$\begin{array}{cccc} \varepsilon\colon & F & \longrightarrow & G \\ & R \otimes_C M & \xrightarrow{\varepsilon_M} & \mathrm{Hom}_C(L, M) \\ & r \otimes m & \longmapsto & \tau\colon \begin{array}{ccc} L & \to & M \\ \ell & \mapsto & \varepsilon(\ell \otimes r) m \end{array} \end{array}$$

Then

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\sim} & A\text{-mod} \\ (X, Y, \rho, \lambda) & \leftrightarrow & M \end{array} \qquad \text{where} \qquad X = eM, \quad Y = (1-e)M,$$

and the L-action and R-action on M define ρ and λ via

$$\ell y = (\lambda(y))(\ell) \qquad \text{and} \qquad rx = \rho(r \otimes x), \qquad \text{for } \ell \in L,\ r \in R,\ x \in X \text{ and } y \in Y.$$

Note that

$$\ell x = 0 \quad \text{and} \quad ry = 0, \qquad \text{for } \ell \in L,\ r \in R,\ x \in X,\ y \in Y,$$

and

$$PICTURE$$

commutes.

2.5 Highest weight categories

Let A be a finite dimensional algebra and let $\hat{A}$ be an index set for

$$L(\lambda), \quad \text{the simple } A\text{-modules.}$$

$$\begin{array}{lll} \text{Let} & P(\lambda) & \text{be the projective cover of } L(\lambda) \text{, and} \\ & I(\lambda) & \text{the injective hull of } L(\lambda). \end{array}$$

$$\begin{array}{lll} \text{Let} & \leq & \text{be a partial order on } \hat{A}. \end{array}$$

$$\begin{array}{lll} \text{Let} & \nabla(\lambda) & \text{be the largest subobject of } I(\lambda) \text{ with composition factors } L(\mu) \text{ with } \mu \leq \lambda, \\ & \Delta(\lambda) & \text{be the largest quotient of } P(\lambda) \text{ with composition factors } L(\mu) \text{ with } \mu \leq \lambda, \end{array}$$

Then $\mathcal{A} = A$-mod is a *highest weight category* if $P(\lambda)$ has a filtration

$$0 = P(\lambda)^{(m)} \subseteq \cdots \subseteq P(\lambda)^{(1)} \subseteq P(\lambda),$$

with

$$\frac{P(\lambda)}{P(\lambda)^{(1)}} \cong \Delta(\lambda) \qquad \text{and} \qquad \frac{P(\lambda)^{(k)}}{P(\lambda)^{(k+1)}} \cong \Delta(\mu), \quad \text{with } \mu < \lambda,$$

for $1 \leq k \leq m-1$.

Theorem 2.2. *Highest weight categories satisfy BGG-reciprocity,*

$$[I(\lambda) : \nabla(\mu)] = [\Delta(\mu) : L(\lambda)].$$

Proof. Since

$$\mathrm{Ext}^1(\Delta(\lambda), \nabla(\mu)) = 0 \qquad \text{and} \qquad \mathrm{Hom}(\Delta(\lambda), \nabla(\mu)) = \begin{cases} \mathrm{End}(L(\mu)), & \text{if } \lambda = \mu, \\ 0, & \text{if } \lambda \neq \mu, \end{cases}$$

it follows that

$$\mathrm{Hom}(\Delta(\lambda), M) = (\text{number of of } \nabla(\lambda) \text{ in a } \nabla\text{-filtration of } M).$$

Thus

$$[I(\mu) : \nabla(\lambda)] = \frac{\dim\big(\mathrm{Hom}(\Delta(\lambda), I(\mu))\big)}{\dim\big(\mathrm{End}(\text{Ł}(\lambda))\big)} = [\Delta(\lambda) : L(\mu)].$$

How does this proof compare to the proof for convolution algebras in Chriss and Ginzburg? $\square$

Examples of highest weight categories

(1) $G = G(\overline{\mathbb{F}})$, $\mathcal{A}$ the category of finite dimensional rational G-modules, and $\nabla(\lambda) = H^0(G/B, \mathcal{L}_\lambda)$,

(2) $\mathcal{A}$ the category $\mathcal{O}$, and $\nabla(\lambda) = M(\lambda)^\vee$.

Theorem 2.3. *Let A be a finite dimensional algebra and let $\mathcal{A} = A$-mod. The $\mathcal{A}$ is a highest weight category if and only if A is a quasihereditary algebra.*

Proof. $\Rightarrow$: Assume $\mathcal{A}$ is a highest weight category. Let λ be a maximal weight and let

$$P(\lambda) = Ae_\lambda \qquad \text{and} \qquad JAe_\lambda A.$$

Then J is projective as a left A-module,

$$\mathrm{Hom}_A(J, A/J) = 0, \qquad J \cdot \mathrm{Rad}(J) = 0.$$

So J is a heredity ideal. Finally, (A/J)-mod is a highest weight category with $\widehat{(A/J)} = \hat{A} - \{\lambda\}$.
$\Leftarrow$: Assume A is a quasihereditary algebra,

$$0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_m = A.$$

Define $\lambda < \mu$ if

$$L(\lambda) \ \text{ appears in } \ \frac{J_i/J_{i-1}}{\mathrm{Rad}(J_i/J_{i-1})} \qquad \text{and} \qquad L(\mu) \ \text{ appears in } \ \frac{J_j/J_{j-1}}{\mathrm{Rad}(J_j/J_{j-1})},$$

with $i < j$. Suppose i is (the unique integer) such that $L(\lambda)$ appears in $(J_i/J_{i-1})/((\mathrm{Rad}(J_i/J_{i-1}))$ and let

$$Delta(\lambda) \ \text{ be the projective cover of } L(\lambda), \quad \text{as an } A/J_{i-1} \text{ module.}$$

Then $L(\lambda)$ is the simple head of $A(\lambda)$ and, since $J_{i-1} \cdot \mathrm{Rad}(A/J_{i-1}) \cdot J_{i-1} = 0$, all other composition factors of $A(\lambda)$ are lower.

If $L(\lambda)$ is a simple A-module then there is an idempotent $e_\lambda \in A$ such that $P(\lambda) = Ae_\lambda$ (e_λ is a minimal idempotent). Then

$$0 = J_0 e_\lambda \subseteq J_1 e_\lambda \subseteq \cdots \subseteq J_m e_\lambda = Ae_\lambda = P(\lambda)$$

is a good filtration of $P(\lambda)$. □

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