Almost semisimple algebras

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1 Convolution algebras

1.1 The decomposition theorem

Let M be a smooth G-variety and let N be a G-variety with finitely many G-orbits such that the orbit decomposition is an algebraic stratification of N,

$$N = \bigsqcup_{\varphi} Gx_\varphi, \qquad \text{and} \qquad \mu\colon M \longrightarrow N$$

is a G-equivariant projective morphism. Let $\mathcal{C}_M$ be the constant perverse sheaf on M. The decomposition theorem [CG, 8.4.12] says that

$$\mu_*\mathcal{C}_M = \bigoplus_{\substack{i\in\mathbb{Z} \\ \lambda=(\varphi,\chi)\in\hat{M}}} L(\lambda,i)\otimes IC^\lambda[i] \doteq \bigoplus_{\lambda\in\hat{M}} L(\lambda)\otimes IC^\lambda, \qquad \text{where} \qquad L(\lambda) = \bigoplus_{i\in\mathbb{Z}} L(\lambda,i),$$

μ_* is the derived functor of sheaf theoretic direct image, λ runs over the indexes of the intersection cohomology complexes IC^λ, $L(\lambda)$ are finite dimensional vector spaces, and $\doteq$ indicates an equality up to shifts in the derived category.

1.2 Convolution algebras

Let $\mu\colon M\to N$ be a proper map. The *convolution algebra* is

$$A = \mathrm{Ext}^*_{D^b(N)}(\mu_*\mathcal{C}_M, \mu_*\mathcal{C}_M) = \bigoplus_{k\in\mathbb{Z}} \mathrm{Ext}^k(\mu_*\mathcal{C}_M, \mu_*\mathcal{C}_M),$$

The decomposition theorem for $\mu_*\mathcal{C}_M$ induces a decomposition of A. Since the intersection cohomology complexes IC_ϕ are the simple objects in the category of perverse sheaves,

$$\mathrm{Ext}^0_{D^b(N)}(IC^\lambda, IC^\mu) = \delta_{\lambda\mu}\mathbb{C}, \qquad \text{and} \qquad \mathrm{Ext}^k_{D^b(N)}(IC^\lambda, IC^\mu) = 0, \qquad \text{for } k\in\mathbb{Z}_{<0},$$

and the decomposition of A simplifies to

$$A = \bigoplus_{\lambda\in\hat{M}} \mathrm{End}_{\mathbb{C}}(L(\lambda)) \bigoplus \left(\bigoplus_{k\in\mathbb{Z}_{>0}} \Big(\bigoplus_{\lambda,\mu\in\hat{M}} \mathrm{Hom}_{\mathbb{C}}(L(\lambda), L(\mu))\otimes \mathrm{Ext}^k_{D^b(N)}(IC^\lambda, IC^\mu) \Big) \right).$$

In this context there is a good theory of projective, standard and simple modules, and their decomposition matrices satisfy a BGG reciprocity. View elements of A as sums

$$\sum_{\lambda,\mu} \sum_{P\in\hat{L}(\lambda),Q\in\hat{L}(\mu)} c^{\lambda\mu}_{PQ} a^{\lambda\mu}_{PQ} \quad \text{where} \quad c^{\lambda\mu}_{PQ} \in \mathbb{C}, \quad \text{and} \quad a^{\lambda\mu}_{PQ} \in \bigoplus_{k>0} \mathrm{Ext}^k_{D^b(N)}(IC^\lambda, IC^\mu).$$

The algebra A is completely controlled by the dimensions of the $L(\lambda)$ and the multiplication in

$$A_{\mathrm{basic}} = \mathrm{Ext}^*(IC, IC) \qquad \text{where} \qquad IC = \bigoplus_{\lambda\in\hat{M}} IC^\lambda.$$

an algebra which has all one dimensional simple modules. The radical filtration of A is

$$\mathrm{Rad}^\ell(A) = \bigoplus_{\lambda,\mu\in\hat{M}} \mathrm{Hom}_{\mathbb{C}}(L(\lambda), L(\mu)) \otimes \Big(\bigoplus_{k\in\mathbb{Z}_{\geq\ell}} \mathrm{Ext}^k_{D^b(N)}(IC^\lambda, IC^\mu)\Big)$$

and the nonzero

$$L(\lambda) \text{ are the simple } A\text{-modules.}$$

1.3 Projective modules

Let e^λ be a minimal idempotent in $\bigoplus_\mu \mathrm{End}(L(\mu))$. Then

$$P(\lambda) = Ae^\lambda = L(\lambda) \bigoplus \left(\bigoplus_{\substack{k>0\\ \mu}} L(\mu) \otimes \mathrm{Ext}^k_{D^b(N)}(IC^\mu, IC^\lambda) \right)$$

is the projective cover of the simple A-module $L(\lambda)$. Define an A-module filtration

$$P(\lambda) \supseteq P(\lambda)^{(1)} \supseteq P(\lambda)^{(2)} \supseteq \cdots$$

by

$$P(\lambda)^{(m)} = \bigoplus_{\substack{k\geq m\\ \mu}} L(\mu) \otimes \mathrm{Ext}^k_{D^b(N)}(IC^\mu, IC^\lambda).$$

Then

$$L(\lambda) = P(\lambda)/P(\lambda)^{(1)} \qquad \text{and} \qquad \mathrm{gr}\big(P(\lambda)\big) \qquad \text{is a semisimple } A\text{-module.}$$

Thus the multiplicity of the simple A-module $L(\mu)$ in a composition series of $P(\lambda)$ is

$$[P(\lambda) : L(\mu)] = \dim\big(\mathrm{Ext}^*(IC_{\mathbb{O},\chi}, IC_{\mathbb{O}',\chi'})\big) = \sum_{k\geq 0} \dim\big(\mathrm{Ext}^k_{D^b(N)}(IC^\mu, IC^\lambda)\big).$$

1.4 Standard and costandard modules

Let $\lambda = (\varphi, \chi)$,

$$x \in \mathbb{O}^\varphi, \qquad \text{and let} \qquad i_x\colon \{x\} \hookrightarrow N \quad \text{be the injection.}$$

Then $i_x^! \mu_* \mathcal{C}_M$ is the *stalk* of $\mu_*\mathcal{C}_M$ at x and the Yoneda product makes

$$\Delta^\varphi = H^*(i_x^! \mathcal{C}_M) = \mathrm{Hom}_{D^b(\{x\})}(\mathbb{C}, i_x^! \mu_* \mathcal{C}_M[*]) = \mathrm{Hom}_{D^b(N)}((i_x)_! \mathbb{C}[-*], \mu_* \mathcal{C}_M), \quad \text{and}$$
$$\nabla^\varphi = H^*(i_x^* \mathcal{C}_M) = H^*(\{x\}, i_x^* \mu_* \mathcal{C}_M) = \mathrm{Hom}_{D^b(\{x\})}(\mathbb{D}, i_x^! \mu_* \mathcal{C}_M[*]) = \mathrm{Hom}_{D^b(N)}((i_x)_! \mathbb{C}[-*], \mu_* \mathcal{C}_M),$$

into right A-modules. The action of an element $a \in \mathrm{Ext}^k(\mu_*\mathcal{C}_M, \mu_*\mathcal{C}_M) = \mathrm{Hom}_{D^b(N)}(\mu_*\mathcal{C}_M, \mu_*\mathcal{C}_M[k])$ sends

$$H^*(\{x\}, i_x^! \mu_*\mathcal{C}_M) \longrightarrow H^{*+k}(\{x\}, i_x^! \mu_*\mathcal{C}_M).$$

A G-equivariant *local system* is a G-equivariant locally constant sheaf. The orbit $\mathbb{O}^\varphi$ can be identified with G/G_x where G_x is the stabilizer of x. $\pi_0(\mathbb{O}^\varphi, x) = G_x/G_x^\circ$ where G_x° is the connected component of the identity in G_x). There is a homomorphism $\pi_1(\mathbb{O}^\varphi, x) \to \pi_0(\mathbb{O}^\varphi, x) = G_x/G_x^\circ$ and the representations of $\pi_1(\mathbb{O}^\varphi, x)$ on the fibers $\mathcal{L}_x$ of G-equivariant local systems $\mathcal{L}$ are exactly the pullbacks of finite dimensional representations of $C = G_x/G_x^\circ$ to $\pi_1(\mathbb{O}^\varphi, x)$. In this way the irreducible G-equivariant local systems on $\mathbb{O}^\varphi$ can be indexed by (some of the) irreducible representations of G_x/G_x° [CG, Lemma 8.4.11]. There is an action of $C = G_x/G_x^\circ$ on Δ^φ which commutes with the action of A. Similar arguments apply to ∇^φ. As (A, C) bimodules,

$$\Delta^\varphi = \bigoplus_{\chi \in \hat{C}} \Delta(\varphi, \chi) \otimes \chi \qquad \text{and} \qquad \nabla^\varphi = \bigoplus_{\chi \in \hat{C}} \nabla(\varphi, \chi) \otimes \chi,$$

and the *standard* and *costandard* A-modules are

$$\Delta(\lambda) = \Delta(\varphi, \chi) \qquad \text{and} \qquad \nabla(\lambda) = \nabla(\varphi, \chi).$$

Using the decomposition theorem

$$\Delta(\lambda) = H^*(i_x^! \mathcal{C}_M)_\chi = \bigoplus_{\substack{k \in \mathbb{Z} \\ \mu}} L(\mu) \otimes H^k(i_x^! IC^\mu)_\chi,$$

where the subscript χ denotes the χ-isotypic component. Define a filtration

$$\Delta(\lambda) \supseteq \Delta(\lambda)^{(1)} \supseteq \Delta(\lambda)^{(2)} \supseteq \cdots \qquad \text{by} \qquad \Delta(\lambda)^{(m)} = \bigoplus_{j \geq m} \bigoplus_{\phi} L(\mu) \otimes H^j(i_x^! IC^\mu)_\chi.$$

Then $\Delta(\lambda)^{(m)}$ is an A-module and $\mathrm{gr}\big(\Delta(\lambda))\big)$ is a semisimple A-module. This (and a similar argument for $\nabla(\lambda)$) show that the multiplicity of the simple A-module $L(\mu)$ in composition series of $\Delta(\lambda)$ and $\nabla(\lambda)$ are

$$[\Delta(\lambda) : L(\mu)] = \sum_k \dim\big(H^k(i_x^! IC^\mu)_\chi\big) \qquad \text{and} \qquad [\nabla(\lambda) : L(\mu)] = \sum_k \dim\big(H^k(i_x^* IC^\mu)_\chi\big).$$

Define the *standard KL-polynomial* and the *costandard KL-polynomial* of A to be

$$P^\Delta_{\lambda\mu}(\mathtt{t}) = \sum_k \mathtt{t}^k \dim\big(H^k(i_x^! IC^\mu)_\chi\big) \qquad \text{and} \qquad P^\nabla_{\lambda\mu}(\mathtt{t}) = \sum_k \mathtt{t}^k \dim\big(H^k(i_x^* IC^\mu)_\chi\big),$$

respectively. Then ??? says that

$$[\Delta(\lambda) : L(\mu)] = P^\Delta_{\lambda\mu}(1) \qquad \text{and} \qquad [\nabla(\lambda) : L(\mu)] = P^*_{\lambda\mu}(1).$$

These identities are analogues of the original Kazhdan-Lusztig conjecture describing the multiplicities of simple $\mathfrak{g}$-modules in Verma modules.

1.5 The contravariant form

Note that there is a canonical homomorphism

$$\Delta(\lambda) \xrightarrow{c_\lambda} \nabla(\lambda)$$

coming from applying the functor H^* to the composition

$$(i_x)_!(i_x)^! \mu_* \mathcal{C}_M \longrightarrow \mu_* \mathcal{C}_M \longrightarrow (i_x)_*(i_x)^* \mu_* \mathcal{C}_M,$$

where the two maps arise from the canonical adjoint functor maps. Use the map c_λ to define a bilinear form on $\Delta(\lambda)$ by

$$\begin{array}{rccc} \langle , \rangle : & \Delta(\lambda) \otimes \Delta(\lambda) & \longrightarrow & \mathbb{C} \\ & m_1 \otimes m_2 & \longmapsto & m_1 \cap c_\lambda(m_2) \end{array}$$

Then

$$L(\lambda) = \Delta(\lambda)/\mathrm{Rad}(\langle , \rangle).$$

1.6 Contragradient modules

There is an involutive antiautomorphism ${}^t\colon A \to A$ on A (coming from switching the two factors in $Z = M \times_N M$). If M is an A-module the *contragredient* module is

$$M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C}) \qquad \text{with} \qquad (a\psi)(m) = \psi(a^t(m)), \quad \text{for } a \in A,\ \psi \in M^*, \text{ and } m \in M.$$

Then

$$\nabla(\lambda) \cong \Delta(\lambda)^*.$$

1.7 Reciprocity

If $\lambda = (\varphi, \rho)$ define

$$d_\lambda = \dim_{\mathbb{C}}(\mathbb{O}^\varphi), \qquad \text{and assume that} \qquad \mathrm{Ext}^{d_\psi + d_\varphi + k}_{D^b(N)}(IC^\varphi, IC^\psi) = 0, \quad \text{for all odd } k.$$

Then

$$
\begin{aligned}
[P(\lambda):L(\mu)] &= \sum_k \dim\mathrm{Ext}^k_{D^b(N)}(IC^\lambda, IC^\mu) \\
&= \sum_k \dim\mathrm{Ext}^{d_\lambda+d_\mu+k}_{D^b(N)}(IC^\lambda, IC^\mu) \\
&= \sum_k (-1)^k \dim\mathrm{Ext}^{d_\lambda+d_\mu+k}_{D^b(N)}(IC^\lambda, IC^\mu) \\
&= (-1)^{d_\phi+d_\psi} \sum_{\mathbb{O}} \chi(\mathbb{O}, i^!_{\mathbb{O}} IC^\vee_\phi \overset{!}{\otimes} i^!_{\mathbb{O}} IC_\psi) \\
&= (-1)^{d_\phi+d_\psi} \sum_{\mathbb{O}} \chi\left(\mathbb{O}, (-1)^{d_\phi} \sum_{\alpha,k} [\mathcal{H}^k i^!_{\mathbb{O}}(IC^\vee_\phi):\alpha]\alpha \overset{!}{\otimes} (-1)^{d_\psi} \sum_{\beta,\ell} [\mathcal{H}^\ell i^!_{\mathbb{O}}(IC_\psi):\beta]\beta\right) \\
&= \sum_{\mathbb{O},\alpha,\beta} \chi\left(\mathbb{O}, \sum_k [\mathcal{H}^k i^!_{\mathbb{O}}(IC_\phi):\alpha^*]\alpha \overset{!}{\otimes} \sum_\ell [\mathcal{H}^\ell i^!_{\mathbb{O}}(IC_\psi):\beta]\beta\right) \\
&= \sum_{\alpha,\beta}\sum_k \dim\mathcal{H}^k(i^!_\alpha IC_\phi)\left(\sum_{\mathbb{O}} \chi(\mathbb{O}, \alpha^* \overset{!}{\otimes} \beta)\right)\sum_\ell \dim\mathcal{H}^\ell(i^!_\beta IC_\psi) \\
&= \sum_{\alpha,\beta} [\mathcal{M}^!_\alpha : L_\phi]\left(\sum_{\mathbb{O}} \chi(\mathbb{O}, \alpha^*\otimes\beta)\right)[\mathcal{M}^!_\beta : L_\psi] \\
&= \sum_{\alpha,\beta} P_{\phi\alpha}(1) D_{\alpha\beta} P_{\psi\beta}(1) \\
&= (PDP^t)_{\phi\psi},
\end{aligned}
$$

where

(1) the third equality follows from the vanishing of Ext groups in odd degrees,

(2) χ denotes the Euler characteristic,

(3) P is the matrix $(P_{\phi\alpha}(1))$, and

(4) D is the matrix $(\sum_{\mathbb{O}} \chi(\mathbb{O}, \alpha^*\otimes\beta))$.

This identity is the "BGG reciprocity" for the algebra A.

1.8 The Steinberg variety

Let $x \in N$ and define

$$Z = M\times_N M = \{(m_1,m_2)\in M\times M \mid \mu(m_1)=\mu(m_2)\} \qquad \text{and} \qquad M_x = \mu^{-1}(x).$$

There are commutative diagrams

$$
\begin{array}{ccc}
Z = M\times_N M & \xrightarrow{\iota} & M\times M \\
\big\downarrow{\scriptstyle \mu_{12}} & & \big\downarrow{\scriptstyle \mu_1\times\mu_2} \\
N = N_\Delta & \xrightarrow{\Delta} & N\times N
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
M_x & \xrightarrow{\iota} & M \\
\big\downarrow{\scriptstyle \mu} & & \big\downarrow{\scriptstyle \mu} \\
\{x\} & \xrightarrow{i_x} & N
\end{array}
$$

which (via base change) provide isomorphisms

$$
\begin{aligned}
H_*(Z) &= \mathrm{Hom}_{D^b(Z_{12})}(\mathbb{C}_{Z_{12}}, (\mathbb{C}_{Z_{12}}[*])^\vee) \\
&= \mathrm{Hom}_{D^b(Z_{12})}(\mu_{12}^*\mathbb{C}_N, \iota^!\mathcal{C}_{M_1\times M_2}[m_1+m_2][-*]) \\
&= \mathrm{Hom}_{D^b(N)}(\mathbb{C}_N, (\mu_{12})_*\iota^!\mathcal{C}_{M_1\times M_2}[m_1+m_2-*]) \\
&= \mathrm{Hom}_{D^b(N)}(\mathbb{C}_N, \Delta^!(\mu_1\times\mu_2)_*(\mathcal{C}_{M_1}\boxtimes\mathcal{C}_{M_2})[m_1+m_2-*]) \\
&= \mathrm{Hom}_{D^b(N)}(\mathbb{C}_N, \Delta^!((\mu_1)_*\mathcal{C}_{M_1}\boxtimes(\mu_2)_*\mathcal{C}_{M_2})[m_1+m_2-*]) \\
&= \mathrm{Ext}^{m_1+m_2-*}_{D^b(N)}((\mu_1)_*\mathcal{C}_{M_1}, (\mu_2)_*\mathcal{C}_{M_2}),
\end{aligned}
$$

$$
\begin{aligned}
H_*(M_x) &= \mathrm{Hom}_{D^b(M_x)}(\mathbb{C}_{M_x}, (\mathbb{C}_{M_x}[*])^\vee) = \mathrm{Hom}_{D^b(M_x)}(\mu^*\mathbb{C}_{\{x\}}, ((\iota^*\mathbb{C}_M)[*])^\vee) \\
&= \mathrm{Hom}_{D^b(\{x\})}(\mathbb{C}_{\{x\}}, \mu_*(\iota^!\mathbb{C}_M[2m])[-*]) = \mathrm{Hom}_{D^b(\{x\})}(\mathbb{C}_{\{x\}}, i_x^!\mu_*\mathcal{C}_M[m-*]) \\
&= H^{m-*}(i_x^!\mu_*\mathcal{C}_M),
\end{aligned}
$$

and

$$
\begin{aligned}
H^*(M_x) &= \mathrm{Hom}_{D^b(M_x)}(\mathbb{C}_{M_x}, \mathbb{C}_{M_x}[*]) = \mathrm{Hom}_{D^b(M_x)}(\mu^*\mathbb{C}_{\{x\}}, \mathbb{C}_{M_x}[*]) \\
&= \mathrm{Hom}_{D^b(\{x\})}(\mathbb{C}_{\{x\}}, \mu_*\mathbb{C}_{M_x}[*]) = \mathrm{Hom}_{D^b(\{x\})}(\mathbb{C}_{\{x\}}, \mu_!\iota^*\mathbb{C}_M[*]) \\
&= \mathrm{Hom}_{D^b(\{x\})}(\mathbb{C}_{\{x\}}, i_x^*\mu_!\mathbb{C}_M[*]) = \mathrm{Hom}_{D^b(\{x\})}(\mathbb{C}_{\{x\}}, i_x^*\mu_*\mathcal{C}_M[*-m]) \\
&= H^{*-m}(i_x^*\mu_*\mathcal{C}_M).
\end{aligned}
$$

1.9 The category $D^b(N)$

The category $Comp^b(Sh(N))$ is the cateogry of all finite complexes

$$A = (0 \to A^{-m} \to A^{-m+1} \to \cdots \to A^{n-1} \to A^n \to 0), \qquad m, n \in \mathbb{Z}_{>>0},$$

of sheaves on N with morphisms being morphisms of complexes which commute with the differentials. The jth cohomology sheaf of A is

$$\mathcal{H}^j(A) = \frac{\ker(A^j \to A^{j+1})}{\mathrm{im}(A^{j-1} \to A^j)}.$$

A morphism in $Comp^b(Sh(N))$ is a *quasi-isomorphism* if it induces isomorphisms on cohomology. The category $D^b(Sh(N))$ is the category $Comp^b(Sh(N))$ with additional morphisms obtained by formally inverting all quasi-isomorphisms.

Assume that N is a G-variety with a finite number of orbits such that the G-orbit decomposition

$$N = \bigsqcup_\varphi \mathbb{O}^\varphi \qquad \text{is an algebraic stratification of } X.$$

A *constructible sheaf* is a sheaf that is locally constant on strata of N. A *constructible complex* is a complex such that all of its cohomology sheaves are constructible.

The *derived category of bounded constructible complexes of sheaves* on N is the full subcategory $D^b(N)$ of $D^b(Sh(N))$ consisting of constructible complexes. *Full* means that the morphisms in $D^b(N)$ are the same as those in $D^b(Sh(N))$.

The *shift functor* $[i]\colon D^b(N) \to D^b(N)$ is the functor that shifts all complexes by i.

The *Verdier duality functor* ${}^\vee\colon D^b(N) \to D^b(N)$ is defined by requiring

$$\mathrm{Hom}_{D^b(N)}(A_1, A_2[i]) = \mathrm{Hom}_{D^b(N)}(\Delta^*(A_1 \boxtimes A_2^\vee)[-i], \mathbb{C}_N[2\mathrm{dim}_{\mathbb{C}}N]), \qquad \text{for all } i \in \mathbb{Z}, \text{ where}$$

$\Delta\colon N \to N \times N$ is the diagonal map.

The Verdier duality functor satisfies the properties

$$(A^\vee)^\vee = A, \qquad (A[i])^\vee = A^\vee[-i], \qquad \text{and} \qquad \mathrm{Hom}_{D^b(N)}(A_1, A_2) = \mathrm{Hom}_{D^b(N)}(A_2^\vee, A_1^\vee).$$

Define

$$\begin{array}{ll}
\mathrm{Ext}^k_{D^b(X)}(A_1, A_2) = \mathrm{Hom}_{D^b(X)}(A_1, A_2[k]), & \\
H^k(A) = H^k(X, A) = \mathrm{Hom}_{D^b(X)}(\mathbb{C}_X, A[k]), & \text{the } \textit{hypercohomology} \text{ of } A \in D^b(N), \\
H^k(N) = \mathrm{Hom}_{D^b(N)}(\mathbb{C}_N, \mathbb{C}_N[k]), & \text{the } \textit{cohomology} \text{ of } N, \\
H_k(N) = \mathrm{Hom}_{D^b(N)}(\mathbb{C}_N, (\mathbb{C}_N[k])^\vee), & \text{the } \textit{Borel-Moore homology} \text{ of } N, \\
\mathbb{D}_X = \mathbb{C}_X^\vee, & \text{the } \textit{dualizing complex},
\end{array}$$

respectively. The *Yoneda product*

$$\mathrm{Ext}^p_{D^b(N)}(A_1, A_2) \times \mathrm{Ext}^q_{D^b(N)}(A_2, A_3) \longrightarrow \mathrm{Ext}^{p+q}_{D^b(N)}(A_1, A_3)$$

is given by

$$\mathrm{Hom}_{D^b(N)}(A_1, A_2[p]) \times \mathrm{Hom}_{D^b(N)}(A_2[p], A_3[p+q]) \longrightarrow \mathrm{Hom}_{D^b(N)}(A_1, A_3[p+q]),$$

using the canonical identification $\mathrm{Hom}_{D^b(N)}(A_2, A_3[q]) \cong \mathrm{Hom}_{D^b(N)}(A_2[p], A_3[p+q])$.

If $f\colon X \to Y$ is a morphism define

$$f_* = \text{derived functor of sheaf theoretic direct image,}$$
$$f^* = \text{derived functor of sheaf theoretic inverse image,}$$

$$f^! A = (f^* A^\vee)^\vee, \text{ for } A \in D^b(Y), \qquad \text{and} \qquad f_! A = (f_* A^\vee)^\vee, \text{ for } A \in D^b(X).$$

Then

$$\begin{array}{ll}
\mathrm{Hom}_{D^b(X)}(f^* A_1, A_2) = \mathrm{Hom}_{D^b(Y)}(A_1, f_* A_2), & \text{and} \\
\mathrm{Hom}_{D^b(X)}(A_2, f^! A_1) = \mathrm{Hom}_{D^b(Y)}(f_! A_2, A_1). &
\end{array}$$

If $f\colon X \to Z$ and $g\colon Y \to Z$ define The *base change formula* is

$$\begin{array}{ccc}
X \times_Z Y & \xrightarrow{\pi_2} & Y \\
\big\downarrow \pi_1 & & \big\downarrow g \\
X & \xrightarrow{f} & Z
\end{array} \qquad g^! f_* A = (\pi_2)_* \pi_1^! A, \qquad \text{for } A \in D^b(X),$$

where $X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$.

The category of *perverse sheaves* on X is a full subcategory of $D^b(X)$ which is abelian. The simple objects in the category of perverse sheaves are the *intersection cohomology complexes*

$$IC_\phi \qquad \text{indexed by pairs} \qquad \phi = (\mathbb{O}, \chi),$$

where $\mathbb{O}$ is a G-orbit on X and χ is an irreducible local system on X. By ???, the local systems χ on $\mathbb{O}$ can be identified with (some of the) representations of the *component group* $Z_G(x)/Z_G(x)^\circ$ where x is a point in $\mathbb{O}$. If X is smooth the *constant perverse sheaf* $\mathcal{C}_X$ on X is given by

$$\mathcal{C}_X\big|_{X_i} = \mathbb{C}_{X_i}[\mathrm{dim}_{\mathbb{C}} X_i],$$

on the irreducible components of X. Since the intersection cohomology complexes IC_ϕ are the simple objects of the category of perverse sheaves,

$$\mathrm{Ext}^0_{D^b(N)}(IC_\phi, IC_\psi) = \mathbb{C} \cdot \delta_{\phi\psi} \qquad \text{and} \qquad \mathrm{Ext}^k_{D^b(N)}(IC_\phi, IC_\psi) = 0, \quad \text{if } k > 0.$$

2 Dlab-Ringel algebras

Let C and D be rings,

$$\begin{array}{l} L, \text{ a } (C,D) \text{ bimodule}, \\ R, \text{ a } (D,C) \text{ bimodule} \end{array} \qquad \text{and} \qquad \varepsilon\colon L\otimes_D R\to C,$$

a (C,C) bimodule homomorphism. Define an algebra

$$A = C\oplus D\oplus L\oplus R\oplus R\otimes_C L$$

and product determined by the multiplication in C and D, the module structure of R and L and the additional relations

$$cr=0,\quad d\ell=0,\quad rd=0,\quad \ell c=0,\quad \text{and}\quad (r_1\otimes\ell_1)(r_2\otimes\ell_2)=r_1\otimes\varepsilon(\ell_1\otimes r_2)\ell_2.$$

Let

e_C be the image of the identity of C in A, and

e_D be the image of the identity of D in A.

Then, if $e=e_C$ then

$$1=e_C+e_D,\qquad \begin{array}{ll} C=e_CAe_C, & L=e_CAe_D, \\ R=e_DAe_C, & D'=e_DAe_D, \end{array}$$

so that

$$A=\left\{\begin{pmatrix} c & \ell \\ r & d'\end{pmatrix}\ \Big|\ c\in C, \ell\in L, r\in R, d'\in D'\right\}$$

with matrix multiplication. Then

$e_DAe_D = D + R\otimes_C L$ is a subring of A, and

$R\otimes_C L$ is an ideal in e_DAe_D, and

$R\otimes_C L = e_DAe_CAe_D$.

2.1 Structure of $Z(\varepsilon)$

Let

$$\varepsilon\colon L\otimes_D R\longrightarrow C \qquad \text{be a } (C,C) \text{ bimodule homomorphism.}$$

Let *left radical* $L(\varepsilon)$ and the *right radical* $R(\varepsilon)$ of ε are defined by

$$\begin{array}{ll} L(\varepsilon) & =\{\ell\in L\mid \varepsilon(\ell\otimes r)\in \mathrm{Rad}(C), \text{ for all } r\in R\}, \\ R(\varepsilon) & =\{r\in R\mid \varepsilon(\ell\otimes r)\in \mathrm{Rad}(C), \text{ for all } \ell\in L\}, \end{array}$$

The map ε is *nondegenerate* if $\mathrm{Rad}(C)=0$, $L(\varepsilon)=0$, and $R(\varepsilon)=0$. Let

$$\begin{array}{l} \overline{C}=C/\mathrm{Rad}(C), \\ \overline{L}=L/L(\varepsilon), \\ \overline{R}=R/R(\varepsilon), \end{array} \qquad \text{and} \qquad \begin{array}{cccc} \phi\colon & R\otimes_C L & \longrightarrow & \bar R\otimes_{\bar C}\bar L \\ & \bar r\otimes\bar\ell & \longmapsto & \overline{r\otimes\ell} \end{array}$$

Then $\ker\varphi$ is generated by $R\otimes_C L(\varepsilon)$ and $R(\varepsilon)\otimes_C L$, and we have that $\ker\varphi\cdot R\subseteq R(\varepsilon)$ and $L\cdot\ker\varphi\subseteq L(\varepsilon)$. Then

$$I=\mathrm{Rad}(C)+L(\varepsilon)+R(\varepsilon)+\ker\varphi \qquad \text{is a nilpotent ideal of } A(\varepsilon),$$

and

$$\frac{A(\varepsilon)}{I}\cong A(\bar\varepsilon) \qquad \text{where the map} \qquad \begin{array}{rccc} \bar\varepsilon\colon & \bar L\otimes_D\bar R & \longrightarrow & \bar C \\ & \ell\otimes r & \longmapsto & \bar\ell\otimes\bar r \end{array}$$

is a nondegenerate $(\bar C,\bar C)$ bimodule homomorphism.

If $\varepsilon\colon L\otimes_D R\to C$ is nondegenerate and R is a projective C-module then there is a (D,C) bimodule isomorphism

$$\begin{array}{rccc} \tau\colon & R & \xrightarrow{\sim} & L^* \\ & r & \mapsto & \begin{array}{rccc} \lambda_r\colon & L & \to & C \\ & \ell & \mapsto & \varepsilon(\ell\otimes r) \end{array} \end{array} \qquad \text{so that} \qquad \varepsilon=\mathrm{ev}\circ(\mathrm{id}\otimes\tau)$$

and

$$A(\varepsilon)\cong A(\mathrm{ev}_L).$$

If C,D,L,R are finite dimensional vector spaces over $\mathbb{F}$ and $D=\mathbb{F}$ then

$$\varepsilon=\varepsilon_0\oplus\mathrm{ev}_P\colon (L_0\oplus P^*)\otimes_D(R_0\oplus P)\longrightarrow C,$$

with P projective and $\mathrm{im}\varepsilon_0\subseteq\mathrm{Rad}(C)$.

If $\varepsilon=\varepsilon_0\oplus\mathrm{ev}_P$ with P finitely generated and projective then

$$\begin{array}{ccc} A(\varepsilon)\text{-mod} & \xrightarrow{\sim} & A(\varepsilon_0)\text{-mod} \\ M & \longmapsto & eM \end{array} \qquad \text{where} \qquad e=1-\sum_i p_i\otimes\alpha_i.$$

If $\mathrm{im}\,\varepsilon\subseteq\mathrm{Rad}(C)$ then

$$\mathrm{Rad}(A(\varepsilon_0))=I=\mathrm{Rad}(C)\oplus\mathrm{Rad}(D)\oplus L_0\oplus R_0\oplus R_0\otimes_C L_0$$

and

$$\frac{A(\varepsilon_0)}{\mathrm{Rad}(A(\varepsilon_0))}\cong\frac{C}{\mathrm{Rad}(C)}\oplus\frac{D}{\mathrm{Rad}(D)}.$$

2.2 The module category of $Z(\varepsilon)$

Let $\mathcal{C}$ and $\mathcal{D}$ be categories

$$F\colon\mathcal{C}\to\mathcal{D} \quad\text{and}\quad G\colon\mathcal{C}\to\mathcal{D} \quad\text{be functors,}\quad\text{and}\quad F\xrightarrow{\varepsilon}G,$$

a natural transformation. Define a category $\mathcal{A}$ with

Objects: $\left(M,V;\ \begin{tikzcd} FM \arrow[rr, "\varepsilon_M"] \arrow[dr, "m"'] & & GM \\ & V \arrow[ur, "n"'] & \end{tikzcd}\right)$, where $M\in\mathcal{C}$, $V\in\mathcal{D}$, and $m,n\in\mathrm{Mor}(\mathcal{D})$, Morphisms: (f,g) with $f \in \mathrm{Mor}(\mathcal{C})$, $g \in \mathrm{Mor}(\mathcal{D})$ such that

$$\begin{tikzcd}
FM \arrow[rr, "\varepsilon_M"] \arrow[dr, "m"'] \arrow[dd] & & GM \arrow[dd] \\
& V \arrow[ur, "n"'] \arrow[dd] & \\
FM' \arrow[rr, "\varepsilon_{M'}"] \arrow[dr, "m'"'] & & GM' \\
& V' \arrow[ur, "n'"'] &
\end{tikzcd}$$

commutes.

A fundamental case is when $\mathcal{D}$ is the category of vector spaces over $\mathbb{F}$.

The equivalence between the category $\mathcal{A}$ and the module category of $Z(\varepsilon)$ is given by letting $\mathcal{C} = C$-mod and $\mathcal{D} = D$-mod and

$$\begin{array}{rccc} F\colon & \mathcal{C} & \longrightarrow & \mathcal{D} \\ & M & \longmapsto & R \otimes_C M \end{array} \qquad \text{and} \qquad \begin{array}{rccc} G\colon & \mathcal{C} & \longrightarrow & \mathcal{D} \\ & M & \longmapsto & \mathrm{Hom}_C(L, M) \end{array}$$

where the D-action on $\mathrm{Hom}_C(L, M)$ is given by

$$(d\phi)(\ell) = \phi(\ell d), \qquad \text{for } d \in D,\ \ell \in L, \text{ and } \phi \in \mathrm{Hom}_C(L, M).$$

Then let $\varepsilon\colon F \to G$ be the natural transformation given by

$$\begin{array}{rccc} \varepsilon\colon & F & \longrightarrow & G \\ & R \otimes_C M & \xrightarrow{\varepsilon_M} & \mathrm{Hom}_C(L, M) \\ & r \otimes m & \longmapsto & \begin{array}{rccc} \tau\colon & L & \to & M \\ & \ell & \mapsto & \varepsilon(\ell \otimes r)m \end{array} \end{array}$$

Then

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\sim} & A\text{-mod} \\ (X, Y, \rho, \lambda) & \leftrightarrow & M \end{array} \qquad \text{where} \qquad X = eM, \quad Y = (1-e)M,$$

and the L-action and R-action on M define ρ and λ via

$$\ell y = (\lambda(y))(\ell) \qquad \text{and} \qquad rx = \rho(r \otimes x), \qquad \text{for } \ell \in L,\ r \in R,\ x \in X \text{ and } y \in Y.$$

Note that

$$\ell x = 0 \quad \text{and} \quad ry = 0, \qquad \text{for } \ell \in L,\ r \in R,\ x \in X,\ y \in Y,$$

and

$$\begin{tikzcd}
R \otimes_C X = FX \arrow[rr, "\varepsilon_X"] \arrow[dr, "\rho"'] & & GX = \mathrm{Hom}_C(L, X) \\
& Y \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

commutes.

2.3 Macpherson-Vilonen

Let X be a Thom-Mather stratified space with a fixed stratification such that all strata have even codimension. Let

$$P(X) \text{ be the category of perverse sheaves on } X.$$

Let S be a closed stratum such that S is contractible and let

$$\iota\colon X - S \hookrightarrow X,$$

be the inclusion. Let

$$j\colon L - K \hookrightarrow L, \qquad \text{where} \qquad \begin{array}{ll} L & = \text{the link of } S \\ \cup| & \\ K & = \text{perverse link of } S, \quad \text{a closed subset of } L. \end{array}$$

Let

$$\begin{array}{cccc} F\colon & P(X-S) & \longrightarrow & \{\text{vector spaces}\} \\ & P & \longmapsto & \mathbb{H}^{-d-1}(K;P) \end{array}$$

and

$$\begin{array}{cccc} G\colon & P(X-S) & \longrightarrow & \{\text{vector spaces}\} \\ & P & \longmapsto & \mathbb{H}^{-d}(L,K;P) = \mathbb{H}^{-d}(L, j_!P|_{L-K}), \end{array}$$

Let $\mathcal{A}$ be the corresponding category as in the previous section. Then the map

$$\begin{array}{ccc} P(X) & \xrightarrow{\sim} & \mathcal{A} \\ Q & \longmapsto & \left(Q|_{X-S}, \quad \begin{tikzcd} \mathbb{H}^{-d-1}(K,Q) \arrow[rr, "\varepsilon_X"] \arrow[dr, "\rho"'] & & \mathbb{H}^{-d}(L,K;Q) \\ & \mathbb{H}^{-d}(\mathbb{D},K;Q) \arrow[ur, "\lambda"'] & \end{tikzcd} \right) \end{array}$$

is an equivalence of categories, where $Q\big|_{X-S} = \iota^*Q$, and ε_Q is the coboundary homomorphism in the long exact sequence for the pair L, K. What is $\mathbb{D}$????

2.3.1 Examples

(1) The flag variety.

(2) The nilpotent cone.

3 Quasihereditary algebras

Let $\mathbb{F}$ be a field. A *separable algebra* over $\mathbb{F}$ is an algebra A such that

$$\frac{A}{\mathrm{Rad}(A)} \cong \bigoplus_{\lambda} \in \hat{A} M_{d_\lambda}(\mathbb{F}).$$

Two algebras A and B are *Morita equivalent* if Mod-A is equivalent to Mod-B (Check this in Gelfand-Manin).

A ring A is *semiprimary* if there is a nilpotent ideal $\mathrm{Rad}(A)$ such that $A/\mathrm{Rad}(A)$ is semisimple artinian. Note: If A is finite dimensional then A is semiprimary.

A *hereditary* ring is a ring A such that every submodule of a projective module is projective.

A *heredity ideal* is an ideal J such that

(a) J is projective as a right A-module,

(b) $J^2 = J$, and

(c) $J\mathrm{Rad}(A)J = 0$.

Note: $J^2 = J$ if and only if there is an idempotent $e \in A$ with $J = AeA$.
A *quasihereditary ring* is a semiprimary ring A with a chain of ideals

$$0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_m = A \qquad \text{such that} \qquad \frac{J_\ell}{J_{\ell-1}} \quad \text{is a heredity ideal of} \quad \frac{A}{J_{\ell-1}}$$

for each $1 \le \ell \le m-1$.

Theorem 3.1. *Let A be a quasihereditary algebra,*

$$0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_m = A.$$

Let e be a indempotent in A such that

$$J_{m-1} = AeA \qquad \text{and} \qquad eA(1-e) \subseteq \mathrm{Rad}(A).$$

Let

$$C = eAe \qquad \text{and} \qquad D = \frac{A}{AeA} = \frac{A}{J_{m-1}}$$

and

$$_CL_D = eA(1-e) \qquad \text{and} \qquad _DR_C = (1-e)Ae$$

and let

$$\begin{array}{rccc} \varepsilon\colon & L \otimes_D R & \longrightarrow & C \\ & \ell \otimes r & \longmapsto & \ell r \end{array}$$

Assume D is a separable k-algebra. Then

(a) $D + (1-e)AeA(1-e) = (1-e)A(1-e)$,

(b) $A = C(\varepsilon)$,

(c) C is qusihereditary with heredity chain

$$0 = I_0 \subseteq \cdots \subseteq I_{m-1} = C, \qquad \text{where} \quad I_\ell = eJ_\ell e.$$

3.1 Highest weight categories

Let A be a finite dimensional algebra and let $\hat{A}$ be an index set for

$L(\lambda)$, the simple A-modules.

Let $P(\lambda)$ be the projective cover of $L(\lambda)$, and
$I(\lambda)$ the injective hull of $L(\lambda)$.

Let $\le$ be a partial order on $\hat{A}$.

Let $\nabla(\lambda)$ be the largest subobject of $I(\lambda)$ with composition factors $L(\mu)$ with $\mu \le \lambda$,
$\Delta(\lambda)$ be the largest quotient of $P(\lambda)$ with composition factors $L(\mu)$ with $\mu \le \lambda$,

Then $\mathcal{A} = A$-mod is a *highest weight category* if $P(\lambda)$ has a filtration

$$0 = P(\lambda)^{(m)} \subseteq \cdots \subseteq P(\lambda)^{(1)} \subseteq P(\lambda),$$

with

$$\frac{P(\lambda)}{P(\lambda)^{(1)}} \cong \Delta(\lambda) \qquad \text{and} \qquad \frac{P(\lambda)^{(k)}}{P(\lambda)^{(k+1)}} \cong \Delta(\mu), \quad \text{with } \mu < \lambda,$$

for $1 \leq k \leq m-1$.

Theorem 3.2. *Highest weight categories satisfy BGG-reciprocity,*

$$[I(\lambda) : \nabla(\mu)] = [\Delta(\mu) : L(\lambda)].$$

Proof. Since

$$\mathrm{Ext}^1(\Delta(\lambda), \nabla(\mu)) = 0 \qquad \text{and} \qquad \mathrm{Hom}(\Delta(\lambda), \nabla(\mu)) = \begin{cases} \mathrm{End}(L(\mu)), & \text{if } \lambda = \mu, \\ 0, & \text{if } \lambda \neq \mu, \end{cases}$$

it follows that

$$\mathrm{Hom}(\Delta(\lambda), M) = (\text{number of of } \nabla(\lambda) \text{ in a } \nabla\text{-filtration of } M).$$

Thus

$$[I(\mu) : \nabla(\lambda)] = \frac{\dim\big(\mathrm{Hom}(\Delta(\lambda), I(\mu))\big)}{\dim\big(\mathrm{End}(\text{Ł}(\lambda))\big)} = [\Delta(\lambda) : L(\mu)].$$

How does this proof compare to the proof for convolution algebras in Chriss and Ginzburg? □

Examples of highest weight categories

(1) $G = G(\overline{\mathbb{F}})$, $\mathcal{A}$ the category of finite dimensional rational G-modules, and $\nabla(\lambda) = H^0(G/B, \mathcal{L}_\lambda)$,

(2) $\mathcal{A}$ the category $\mathcal{O}$, and $\nabla(\lambda) = M(\lambda)^\vee$.

Vogan, Irreducible characters of semisimple Lie groups II; The Kazhdan-Lusztig conjectures

$$P_{yw} = \sum_i q^i \dim(Ext^{\ell(w)-\ell(y)-2i}(M_y, L_w)), \qquad \text{for } y \leq w.$$

Theorem 3.3. *Let A be a finite dimensional algebra and let $\mathcal{A} = A$-mod. The $\mathcal{A}$ is a highest weight category if and only if A is a quasihereditary algebra.*

Proof. $\Rightarrow$: Assume $\mathcal{A}$ is a highest weight category. Let λ be a maximal weight and let

$$P(\lambda) = Ae_\lambda \qquad \text{and} \qquad JAe_\lambda A.$$

Then J is projective as a left A-module,

$$\mathrm{Hom}_A(J, A/J) = 0, \qquad J \cdot \mathrm{Rad}(J) = 0.$$

So J is a heredity ideal. Finally, (A/J)-mod is a highest weight category with $\widehat{(A/J)} = \hat{A} - \{\lambda\}$.

$\Leftarrow$: Assume A is a quasihereditary algebra,

$$0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_m = A.$$

Define $\lambda < \mu$ if

$$L(\lambda) \ \text{ appears in } \ \frac{J_i/J_{i-1}}{\mathrm{Rad}(J_i/J_{i-1})} \qquad \text{and} \qquad L(\mu) \ \text{ appears in } \ \frac{J_j/J_{j-1}}{\mathrm{Rad}(J_j/J_{j-1})},$$

with $i < j$. Suppose i is (the unique integer) such that $L(\lambda)$ appears in $(J_i/J_{i-1})/((\mathrm{Rad}(J_i/J_{i-1}))$ and let

$$\Delta(\lambda) \ \text{ be the projective cover of } L(\lambda), \quad \text{as an } A/J_{i-1} \text{ module.}$$

Then $L(\lambda)$ is the simple head of $A(\lambda)$ and, since $J_{i-1} \cdot \mathrm{Rad}(A/J_{i-1}) \cdot J_{i-1} = 0$, all other composition factors of $A(\lambda)$ are lower.

If $L(\lambda)$ is a simple A-module then there is an idempotent $e_\lambda \in A$ such that $P(\lambda) = Ae_\lambda$ (e_λ is a minimal idempotent). Then

$$0 = J_0 e_\lambda \subseteq J_1 e_\lambda \subseteq \cdots \subseteq J_m e_\lambda = Ae_\lambda = P(\lambda)$$

is a good filtration of $P(\lambda)$. $\square$

3.2 Duals and Projectives

Let L be a C-module and let

$$Z = \mathrm{End}_C(L)$$

so that L is a (C, Z) bimodule. The *dual module* to L is the (Z, C) bimodule

$$L^* = \mathrm{Hom}_C(L, C).$$

The *evaluation map* is the (C, C) bimodule homomorphism

$$\begin{array}{rccc} \mathrm{ev}\colon & L \otimes_Z L^* & \longrightarrow & C \\ & \ell \otimes \lambda & \longmapsto & \lambda(\ell) \end{array}$$

and the *centralizer map* is the (Z, Z) bimodule homomorphism

$$\begin{array}{rccc} \xi\colon & L^* \otimes_C L & \longrightarrow & Z \\ & \lambda \otimes \ell & \longmapsto & z_{\lambda,\ell}\colon \begin{array}{ccc} L & \to & L \\ m & \mapsto & \lambda(m)\ell \end{array} \end{array}$$

Recall that [Bou, Alg. II §4.2 Cor.]

(a) L is a projective C-module if and only if $1 \in \operatorname{im} \xi$,

(b) If L is a projective C-module then ξ is injective,

(c) If L is a finitely generated projective C-module then ξ is bijective,

(d) If L is a finitely generated free module then

$$\xi^{-1}(z) = \sum_i b_i^* \otimes z(b_i),$$

where $\{b_1, \ldots, b_d\}$ is a basis of L and $\{b_1^*, \ldots, b_d^*\}$ is the dual basis in M^*.

Statement (a) says that L is projective if and only if there exist $b_i \in L$ and $b_i^* \in L^*$ such that

$$\text{if } \ell \in L \quad \text{then} \quad \ell = \sum_i b_i^*(\ell) b_i, \qquad \text{so that} \quad \xi\Big(\sum_i b_i^* \otimes b_i\Big) = 1.$$

4 Cellular algebras

A *cellular algebra* is an algebra A with

$$\begin{array}{ccc} \text{a basis} & \{a^\lambda_{ST} \mid \lambda \in \hat{A},\ S,T \in \hat{A}^\lambda\} & \\ \text{an involutive antihomomrphism} & {}^*\colon A \to A, & \text{and} \\ \text{a partial order} & \leq \text{ on } \hat{A} & \end{array}$$

such that

(a) $(a^\lambda_{ST})^* = a^\lambda_{TS}$,

(b) If $A(<\lambda) = \text{span-}\{a^\mu_{ST} \mid \mu < \lambda\}$

then

$$aa^\lambda_{ST} = \sum_{Q\in\hat{A}^\lambda} A^\lambda(a)_{QT} a^\lambda_{QT} \qquad \text{mod} \ \ A(<\lambda), \qquad \text{for all } a \in A.$$

Applying the involution * to (b) and using (a) gives that

$$a^\lambda_{TS} a^* = \sum_{Q\in\hat{A}^\lambda} A^\lambda(a)_{QS} a^\lambda_{TQ} \qquad \text{mod} \ \ A(<\lambda), \qquad \text{for all } a \in A.$$

The concept of a cellular algebra is not really the "right" one. The "right" one comes from the structure of a convolution algebra whenever the decomposition theorem holds [CG, 8.6.9].

5 Peter Webb's generalized reciprocity

Let $\mathfrak{o}$ be a complete discrete valuation ring, $k = \mathfrak{o}/\mathfrak{p}$ its residue field and let ${}_{\mathfrak{o}}A$ be an algebra over $\mathfrak{o}$,

$$\begin{array}{ccccc} k & \longleftarrow & \mathfrak{o} & \longrightarrow & \mathbb{K} \\ {}_kA & \longleftarrow & {}_{\mathfrak{o}}A & \longrightarrow & {}_{\mathbb{K}}A \end{array}$$

Theorem 5.1. *The diagram*

$$\begin{array}{ccc} K_0({}_{\mathbb{K}}A) & \xrightarrow{c_A} & G_0({}_{\mathbb{K}}A) \\ \uparrow{\scriptstyle e=D^t} & & \downarrow{\scriptstyle D} \\ K_0({}_kA) & \xrightarrow{c_\lambda} & G_0({}_kA) \end{array}$$

commutes, where e is defined by lifting idempotents. Furthermore $e = D^t$.

Proof. If P is projective, U any finitely generated module, put

$$\langle P, U\rangle = \dim\mathrm{Hom}(P,U).$$

This is well defined on $K_0({}_{\mathbb{K}}A) \times G_0({}_{\mathbb{K}}A)$ and $K_0({}_k\mathbb{A}) \times G_0({}_kA)$. Then

$$e(P) = \mathbb{K} \otimes_{\mathfrak{o}} \hat{P}, \qquad \text{where} \qquad k \otimes_{\mathfrak{o}} \hat{P} = P.$$

Lemma 5.2. *Let U_0 be a $\mathfrak{o}$-form of U and let P be projective. Then $\mathrm{Hom}_{{}_{\mathfrak{o}}A}(\hat{P}, U_0)$ is an $\mathfrak{o}$-lattice in $\mathrm{Hom}_{{}_{\mathbb{K}}A}(K \otimes_{\mathfrak{o}} \hat{P}, U)$ and the morphism $\mathrm{Hom}_{{}_{\mathfrak{o}}A}(\hat{P}, U_0) \to \mathrm{Hom}_{{}_kA}(P, U_0/\mathfrak{p}U_0)$ is reduction mod $\mathfrak{p}$.*

Corollary 5.3.

$$\dim\mathrm{Hom}_{\mathbb{K}A}(K \otimes_{\mathfrak{o}} \hat{P}, U) = \mathrm{rank}_{\mathfrak{o}}\mathrm{Hom}_{\mathfrak{o}A}(\hat{P}, U_0) = \dim\mathrm{Hom}_{kA}(P, U_0/\mathfrak{p}U_0).$$

This shows that e and D are the transpose of each other with respect to the forms. The diagram commutes from the definition of e. $\square$

Corollary 5.4. *The Cartan matrix*

$$C_{kA} = DC_{\mathbb{K}A}D^t$$

where $C_{\mathbb{K}A}$ is the Cartan matrix of A.

If $\mathbb{K}A$ is semisimple then $C_{\mathbb{K}A} = \mathrm{id}$.

6 The category $\mathcal{O}$

Let U be a $\mathbb{Z}$ graded algebra with

(a) U_0 reductive,

(b) U_i finite dimensional,

(c) U semisimple under the adjoint action.

The *category* $\mathcal{O}$ is the category of $\mathbb{Z}$ graded U modules which are

(a) U_0 semisimple, and

(b) $U_{\geq 0}$ locally finite.

Define

$$\mathcal{O}_{\leq n} = \{M \in \mathcal{O} \mid M_i = 0 \text{ if } i > n\}.$$

6.1 Standard and costandard modules

Let $\hat{U}_0$ be an index set for the finite dimensional $\mathbb{Z}$-graded U_0 modules. The *Verma module* or *standard module* and the *coVerma module* or *costandard module* are given by

$$\Delta(\lambda) = U \otimes_{U_{\geq 0}} U_0^{\lambda} \qquad \text{and} \qquad \nabla(\lambda) = \mathrm{Hom}_{U_{\leq 0}}(U, U_0^{\lambda}), \qquad \text{for } \lambda \in \hat{U}_0.$$

Let $M \in \mathcal{O}$. A Δ-*flag* for M is an increasng filtration

$$0 = M^{(0)} \subseteq M^{(1)} \subseteq M^{(2)} \subseteq \cdots \qquad \text{such that} \qquad M = \bigcup_i M^{(i)},$$

and, for each $i \geq 1$, $M^{(i)}/M^{(i-1)} \cong \Delta(\lambda^{(i)})$ for some $\lambda^{(i)} \in \hat{U}_0$.

Proposition 6.1. *(a) $\Delta(\lambda)$ has simple head $L(\lambda)$.*

(b) $\nabla(\lambda)$ has simple socle $L(\lambda)$.

(c) $\{L(\lambda) \mid \lambda \in \hat{U}_0\}$ are the simple objects in $\mathcal{O}$.

Proposition 6.2. *(a) $\Delta(\lambda)$ is the projective cover of $L(\lambda)$ in $\mathcal{O}_{\leq|\lambda|}$.*

(b) $\nabla(\lambda)$ *is the injective hull of* $L(\lambda)$ *in* $\mathcal{O}_{\leq|\lambda|}$.

(c) $\mathrm{Hom}_{\mathcal{O}}(\Delta(\mu), \nabla(\lambda)) = \begin{cases} 0, & \text{if } \lambda \neq \mu, \\ \mathbb{C}, & \text{if } \lambda = \mu. \end{cases}$

(d) $\mathrm{Ext}^1_{\mathcal{O}}(\Delta(\mu), \nabla(\lambda)) = 0$.

6.2 Projectives

If $K = \bigoplus K_i$ is a $\mathbb{Z}$ graded $U_{\geq 0}$ module define

$$\tau_{\leq n} = \frac{K}{\bigoplus_{i>n} K_i} = \bigoplus_{i \leq n} K_i.$$

If $\lambda \in \hat{U}_0$ define

$$Q = U \otimes_{U_{\geq 0}} \tau_{\leq n}(U_{\geq 0} \otimes_{U_0} U_0^{\lambda}),$$

and let $P_{\leq n}(\lambda)$ be an indecomposable summand of Q which has $L(\lambda)$ as a quotient and define $K_{m,n}$, for $m \geq n$ by the exact sequence

$$0 \longrightarrow K_{m,n} \longrightarrow P_{\leq m}(\lambda) \longrightarrow P_{\leq n}(\lambda) \longrightarrow 0.$$

Proposition 6.3. *(a)* Q *is projective and* $Q \to L(\lambda) \to 0$.

(b) $P_{\leq n}(\lambda)$ *is a projective cover of* $L(\lambda)$ *in* $\mathcal{O}_{\leq n}$.

(c) $P_{\leq n}(\lambda)$ *has a* Δ *flag.*

(d) $K_{m,n}$ *has a* Δ *flag.*

(e) $L(\lambda)$ *has a projective cover in* $P(\lambda)$ *in* $\mathcal{O}$ *if and only if the projective system* $P_{\leq m}(\lambda) \to P_{\leq n}(\lambda)$ *stabilizes, in which case*

$$P(\lambda) \cong P_{\leq n}(\lambda), \qquad \text{for } n >> 0.$$

6.3 Injective module

6.4 Tilting modules

Let $\lambda \in \hat{U}_0$. A *tilting module* is a module that has both a Δ flag and a *nabla* flag.

There is a unique indecomposable tilting module $T(\lambda)$ of highest weight λ.

6.5 Blocks

Define $\geq$ on $\hat{U}_0$ by

$$\mu \geq \lambda \qquad \text{if} \qquad [\Delta(\mu) : L(\lambda)] \neq 0 \quad \text{or} \quad [\nabla(\mu) : L(\lambda)] \neq 0.$$

Let $[\lambda]$ denote the equivalence class of λ with respect to the equivalence relation generated by $\geq$. Define

$$\mathcal{O}^{[\lambda]} = \{M \in \mathcal{O} \mid \text{if } [M : L(\mu)] \neq 0 \text{ then } \mu \in [\lambda]\},$$

and for $M \in \mathcal{O}$ define

$$M^{[\lambda]} = U\left(\sum \mathrm{im}\big(P_{\leq n}(\lambda) \xrightarrow{\varphi} M\big)\right),$$

the submodule of M generated by the images of morphisms $\varphi \colon P_{\leq n}(\lambda) \to M$.

Theorem 6.4.

$$\mathcal{O} = \bigoplus \mathcal{O}^{[\lambda]} \qquad \text{and} \qquad M = \bigoplus M^{[\lambda]}, \quad \text{for } M \in \mathcal{O}.$$

6.6 Multiplicities

Let $\mathcal{A}$ be an abelian category and let L be simple. Let $m \in \mathcal{A}$ The *multiplicity* of L in M is

$$[M : L] = \sup_F \mathrm{Card}\{i \mid F_i M/F_{i+1}M \cong L\},$$

where the supremum is over all (finite) filtrations of M.

$$\text{If } \; 0 \to M' \to M \to M'' \to 0 \; \text{ is exact} \quad \text{then} \quad [M : L] = [M' : L] + [M'' : L].$$

If $M \in \mathcal{O}_{\le n}$ and $N \in \mathcal{O}$ with a Δ-flag then

$$[M : L(\lambda)] = \mathrm{dimHom}_{\mathcal{O}}(P_{\le n}(\lambda), M) \qquad \text{and} \qquad [N : \Delta(\mu)] = \mathrm{dimHom}(N, \nabla(\mu)).$$

Thus

$$[P_{\le n}(\lambda) : \Delta(\mu)] = [\nabla(\mu) : L(\lambda)], \qquad \text{for } \lambda, \mu \in \hat{U}_0 \text{ and } n \ge \max\{|\lambda|, |\mu|\}.$$

7 The category $\mathcal{O}_{\mathrm{int}}$

Start with $U = U_{<0}U_0U_{>0}$.

$$\mathcal{O}_{\mathrm{int}} = \{M \in U-\mathrm{mod} \mid M \in U_0^{\mathrm{ss}}, M \in U_{>0}^{\mathrm{nilp}}, M \in U_{<0}^{\mathrm{nilp}}\}.$$

8 Finite dimensional algebras

Let A be a finite dimensional algebra.

The projective indecomposables are Ae for a minimal idempotent e of A.

The simples $L(\lambda)$ are the simple heads of the projective indecomposables $P(\lambda)$.

The blocks are Az for a minimal central idempotent z of A.

The *Cartan matrix* is

$$[P(\lambda) : L(\mu)].$$

9 Temperley-Lieb algebras

9.1 Computation of the $\varepsilon_\sigma^\gamma$

The quantum dimensions of the finite dimensional simple $U_q\mathfrak{sl}_2$ modules are

$$\dim_q(L(k-2j)) = \prod_{b \in (k-j)} \frac{[2+c(b)]}{[h(b)]} = \prod_{i=0}^{k-j-1} \frac{[2+i]}{[k-j-i]} = [k-j+1] = [\dim(L(k-2j))].$$

As a $(U_q\mathfrak{sl}_2, TL_k(n))$ bimodule

$$V^{\otimes k} \cong \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} L(k-2j) \otimes TL_k^{(k-j,j)}.$$

Thus

$$\mathrm{tr}_q(b) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \dim_q(L(k-2j))\chi_{TL_k}^{(k-j,j)}(b), \qquad \text{for } b \in TL_k(n),$$

and

$$\mathrm{tr}_q\big(a_{\substack{ZX\\ \sigma}}\big) = \delta_{ZX}\dim_q(L(\sigma)) \qquad \text{and} \qquad \mathrm{tr}_q\Big(b_{\substack{ZX\\ \sigma\,\mu\\ \gamma}}\Big) = \delta_{\substack{ZX\\ \sigma\,\mu}}\dim_q(L(\gamma)).$$

If $a \in A$ then

$$\begin{aligned}
\mathrm{tr}_q(ae_k) &= \mathrm{tr}_q(a)\mathrm{tr}_q(e_k) = n\mathrm{tr}_q(a), \qquad \text{and so} \\
\mathrm{tr}_q(\varepsilon_1(b)) &= \frac{1}{n}\mathrm{tr}_q(\varepsilon_1(b)e_k) = \frac{1}{n}\mathrm{tr}_q(e_k b e_k) = \frac{1}{n}\mathrm{tr}_q(be_k^2) \\
&= \mathrm{tr}_q(be_k) = \mathrm{tr}_q(b(T_k - q)) = (z-q)\mathrm{tr}_q(b) = \left(\frac{q^2}{n} - q\right)\mathrm{tr}_q(b) = \frac{1}{n}\mathrm{tr}_q(b).
\end{aligned}$$

So

$$\tfrac{1}{n}\dim_q(L(\gamma)) = \tfrac{1}{n}\mathrm{tr}_q\Big(b_{\substack{ZX\\ \sigma\,\mu\\ \gamma}}\Big) = \mathrm{tr}_q\Big(\varepsilon_1\Big(b_{\substack{ZX\\ \sigma\,\mu\\ \gamma}}\Big)\Big) = \mathrm{tr}_q\big(\varepsilon_\sigma^\gamma a_{\substack{ZX\\ \sigma}}\big) = \varepsilon_\sigma^\gamma \dim_q(L(\sigma))$$

Thus

$$\varepsilon_\sigma^\gamma = \frac{[\dim(L(\gamma))]}{n \cdot [\dim(L(\sigma))]}. \tag{9.1}$$

9.2 Generators and relations

The *Temperley-Lieb algebra*, $\mathbb{C}T_k(n)$, is the algebra over $\mathbb{C}$ given by generators $E_1, E_2, ..., E_{k-1}$ and relations

$$\begin{aligned}
&E_iE_j = E_jE_i, && \text{if } |i-j| > 1, \\
&E_iE_{i\pm1}E_i = E_i, && \text{and} \\
&E_i^2 = nE_i.
\end{aligned}$$

If

$$[2] = q + q^{-1} = n \qquad \text{then} \qquad q = \frac{1}{2}(n + \sqrt{n^2-4}), \quad q^{-1} = \frac{1}{2}(n - \sqrt{(n^2-4}),$$

since $q^2 - nq + 1 = 0$. Then

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}} = \frac{1}{2^{k-1}} \sum_{m=1}^{(k+1)/2} \binom{k}{2m-1} n^{k-2m+1}(n^2-4)^{m-1}.$$

The problem with this expression is that it is not clear that $[k]$ is a polynomial in n with integer coefficients (which alternate in sign?).

The *Iwahori-Hecke algebra* $H_k(q)$ is the algebra over $\mathbb{C}$ with generators $T_1, T_2, ..., T_k - 1$ and relations

$$\begin{aligned}
&T_iT_j = T_jT_i, && \text{if } |i-j| > 1, \\
&T_iT_{i\pm1}T_i = T_{i+1}T_iT_{i+1}, && \text{if } 2 \le i \le k-1, \\
&T_i^2 = (q - q^{-1})T_i + 1.
\end{aligned}$$

There is a surjective algebra homomorphism

$$\varphi\colon H_k(q) \longrightarrow T_k(n) \qquad \text{given by} \qquad \varphi(T_i) = E_i - q^{-1} \quad \text{and} \quad \varphi(q + q^{-1}) = n.$$

with

$$\ker\varphi = \langle T_iT_{i+1}T_i + T_iT_{i+1} + T_{i+1}T_i + T_i + T_{i+1} + 1\rangle$$

Composing with the surjective homomorphism

$$\begin{array}{ccc} \tilde{H}_k(q) & \longrightarrow & H_k(q) \\ X^{\varepsilon_i} & \longmapsto & T_{i-1}\cdots T_2T_1^1T_2\cdots T_{i-1} \\ T_i & \longmapsto & T_i \end{array}$$

9.3 Murphy elements

Let us write

$$T_i = E_i - q^{-1}, \qquad \text{so that} \quad X^{\varepsilon_1} = 1, \quad \text{and} \quad X^{\varepsilon_i} = T_{i-1}X^{\varepsilon_{i-1}}T_{i-1}$$

in the Temperley-Lieb algebra. Then define $m_1, \ldots, m_k$ by

$$m_1 = 0 \qquad \text{and} \qquad (q - q^{-1})m_j = q^{i-2}X^{\varepsilon_i}) - q^{i-4}X^{\varepsilon_{i-1}} \quad \text{for } 2 \le i \le k.$$

Soling for X^{ε_i} in terms of the m_i gives

$$X^{\varepsilon_i} = (q - q^{-1})(q^{-(i-2)}m_i + q^{-(i-2+1)}m_{i-1} + \cdots + q^{-(2i-4)}m_2) + q^{-2(i-1)},$$

from which one obtains

$$q^{(k-2)}(X^{\varepsilon_1} + X^{\varepsilon_2} + \cdots + X^{\varepsilon_k}) - q[k] = (q - q^{-1})(m_k + [2]m_{k-1} + \cdots + [k-1]m_2).$$

Using the definition of X^{ε_i} and substituting for $X^{\varepsilon_{i-1}}$ in terms of the m_i gives

$$\begin{aligned}
(q - q^{-1})m_i &= q^{i-2}X^{\varepsilon_i} - q^{i-4}X^{\varepsilon_{i-1}} \\
&= q^{i-2}(E_{i-1} - q^{-1})X^{\varepsilon_{i-1}}(E_{i-1} - q^{-1}) - q^{i-4}X^{\varepsilon_{i-1}} \\
&= q^{i-2}E_{i-1}X^{\varepsilon_{i-1}}E_{i-1} - q^{i-3}(E_{i-1}X^{\varepsilon_{i-1}} + X^{\varepsilon_{i-1}}E_{i-1}) \\
&= q^{i-2}E_{i-1}\big((q - q^{-1})(q^{-(i-3)}m_i + q^{-(i-3+1)}m_{i-1} + \cdots + q^{-(2i-6)}m_2) + q^{-2(i-2)}\big)E_{i-1} \\
&\quad - q^{i-3}E_{i-1}\big((q - q^{-1})(q^{-(i-3)}m_i + q^{-(i-3+1)}m_{i-1} + \cdots + q^{-(2i-6)}m_2) + q^{-2(i-2)}\big) \\
&\quad - q^{i-3}\big((q - q^{-1})(q^{-(i-3)}m_i + q^{-(i-3+1)}m_{i-1} + \cdots + q^{-(2i-6)}m_2) + q^{-2(i-2)}\big)E_{i-1} \\
&= q^{i-2}(q - q^{-1})q^{-(i-3)}E_{i-1}m_{i-1}E_{i-1} - q^{i-3}(q - q^{-1})q^{-(i-3)}(E_{i-1}m_{i-1} + m_{i-1}E_{i-1}) \\
&\quad + q^{i-2}(q + q^{-1})E_{i-1}\big((q - q^{-1})(q^{-(i-3+1)}m_{i-1} + \cdots + q^{-(2i-6)}m_2) + q^{-2(i-2)}\big) \\
&\quad - 2q^{i-3}E_{i-1}\big((q - q^{-1})(q^{-(i-3+1)}m_{i-1} + \cdots + q^{-(2i-6)}m_2) + q^{-2(i-2)}\big) \\
&= q^{i-2}(q - q^{-1})q^{-(i-3)}E_{i-1}m_{i-1}E_{i-1} - q^{i-3}(q - q^{-1})q^{-(i-3)}(E_{i-1}m_{i-1} + m_{i-1}E_{i-1}) \\
&\quad + q^{i-2}(q - q^{-1})E_{i-1}\big((q - q^{-1})(q^{-(i-3+1)}m_{i-1} + \cdots + q^{-(2i-6)}m_2) + q^{-2(i-2)}\big)
\end{aligned}$$

since E_{i-1} commutes with $m_2, m_3, \ldots, m_{i-1}$. Thus

$$\begin{aligned}
m_i = q^{-(i-2)}E_{i-1} + qE_{i-1}m_{i-1}E_{i-1} - (E_{i-1}m_{i-1} + m_{i-1}E_{i-1}) \\
+ (q - q^{-1})(m_{i-2} + q^{-1}m_{i-3} + q^{-2}m_{i-4} + \cdots + q^{-(i-4)}m_2)E_{i-1}.
\end{aligned}$$

It seems to me that this formula provides the easiest way to compute m_i in terms of the Es. I would not be too worried about the coefficients of E_1E_4 and E_2E_4 in m_4 looking strange. One expects diagrams that are equal to their own flip to act a bit differently in m_k. Note also that

$$[3] - 1 = \frac{[4]}{[2]} \quad \text{and} \quad [3] + 1 = [2]^2,$$

so these are pretty nice q-versions of 2. Let's have a look at m_6 and see if we can get an induction going. It might help to categorize the terms according to what their flip is to see where the next level is coming from.

For n such that $\mathbb{C}T_k(n)$ is semisimple, the simple $T_k(n)$ are indexed by partitions in the set

$$\hat{T}_k = \{\lambda \vdash k \mid \lambda \text{ has at most two columns}\}.$$

The irreducible $\mathbb{C}T_k(n)$ modules have seminormal basis

$$\{v_T \mid T \text{ is a standard tableau of shape } \lambda\}$$

and

$$X^{\varepsilon_i} v_T = q^{2c(T(i))} v_T.$$

Since $c(T(i)) = c(T(i-1)) - 1$ if the boxes $T(i)$ and $T(i-1)$ are in the same column and $c(T(i)) + c(T(i-1)) = 3 - i$ if the boxes $T(i)$ and $T(i-1)$ are in different columns it follows that

$$m_i v_T = \frac{q^{i-2} q^{2c(T(i))} - q^{i-4)} q^{2c(T(i-1))}}{q - q^{-1}} = c_T(i) v_T,$$

where

$$c_T(i) = \begin{cases} 0, & \text{if } T(i) \text{ and } T(i-1) \text{ are in the same column,} \\ [i - 2 + 2c(T(i))], & \text{if } T(i) \text{ and } T(i-1) \text{ are in different columns.} \end{cases}$$

Now we want to define pseudomatrix units in $\mathbb{C}T_k(n)$ according to the left and right eigenspaces of the m_i. Let

$$p_{ST} \in L_S \cap R_T,$$

normalized so that the coefficients are in $\mathbb{Z}[n]$ with greatest common divisor 1. Then

$$\begin{aligned} p_{ST} p_{UV} &= \gamma_T \delta_{UV} p_{SV}, \\ p_{ST} &= \sum_{S^+, T^+} c_{S^+T^+} p_{S^+T^+}, \\ p_{ST} e_k p_{UV} &= \beta_{T^-} \delta_{T^-U^-} p_{S^+V^+}, \\ e_{k+1} p_{ST} e_{k+1} &= \varepsilon_{S^+T^+} \delta_{S(k)T(k)} p_{ST} e_{k+1} \end{aligned}$$

9.4 Examples

Let's start with generic n. Here

$$e_{ST} = \frac{[a]}{[b]} e_{S^-U^-} E_{k-1} e_{U^-T^-}.$$

Then

$$E_k = \sum \frac{[b]}{[a]} e_{ST} \qquad \text{and} \qquad m_k = \sum \mu_k(S) e_{SS}.$$

whee the first sum is over all pairs (S, T) such that $S = T$ or S and T only differ at the $k-1$st level.

In $\mathbb{C}T_2(n)$ let

$$\begin{pmatrix} p_{12,12} & \\ & p_{\substack{1,1\\2\ 2}} \end{pmatrix} = \begin{pmatrix} [2]e_{12,12} & \\ & [2]e_{\substack{1,1\\2\ 2}} \end{pmatrix}$$

In $\mathbb{C}T_3(n)$ let

$$\begin{pmatrix} p_{\begin{smallmatrix}1&2\\3&\end{smallmatrix},\begin{smallmatrix}1&2\\3&\end{smallmatrix}} & p_{\begin{smallmatrix}1&2\\3&\end{smallmatrix},\begin{smallmatrix}1&3\\2&\end{smallmatrix}} & \\ p_{\begin{smallmatrix}1&3\\2&\end{smallmatrix},\begin{smallmatrix}1&2\\3&\end{smallmatrix}} & p_{\begin{smallmatrix}1&3\\2&\end{smallmatrix},\begin{smallmatrix}1&3\\2&\end{smallmatrix}} & \\ & & p_{\begin{smallmatrix}1\\2\\3\end{smallmatrix},\begin{smallmatrix}1\\2\\3\end{smallmatrix}} \end{pmatrix} = \begin{pmatrix} [2]e_{\begin{smallmatrix}1&2\\3&\end{smallmatrix},\begin{smallmatrix}1&2\\3&\end{smallmatrix}} & [3][2]e_{\begin{smallmatrix}1&2\\3&\end{smallmatrix},\begin{smallmatrix}1&3\\2&\end{smallmatrix}} & \\ [2]e_{\begin{smallmatrix}1&3\\2&\end{smallmatrix},\begin{smallmatrix}1&2\\3&\end{smallmatrix}} & [3][2]e_{\begin{smallmatrix}1&3\\2&\end{smallmatrix},\begin{smallmatrix}1&3\\2&\end{smallmatrix}} & \\ & & [3]e_{\begin{smallmatrix}1\\2\\3\end{smallmatrix},\begin{smallmatrix}1\\2\\3\end{smallmatrix}} \end{pmatrix}$$

In $\mathbb{C}T_4(n)$ let

$$\begin{pmatrix} p_{\begin{smallmatrix}1&2\\3&4\end{smallmatrix},\begin{smallmatrix}1&2\\3&4\end{smallmatrix}} & p_{\begin{smallmatrix}1&2\\3&4\end{smallmatrix},\begin{smallmatrix}1&3\\2&4\end{smallmatrix}} & & & & \\ p_{\begin{smallmatrix}1&3\\2&4\end{smallmatrix},\begin{smallmatrix}1&2\\3&4\end{smallmatrix}} & p_{\begin{smallmatrix}1&3\\2&4\end{smallmatrix},\begin{smallmatrix}1&3\\2&4\end{smallmatrix}} & & & & \\ & & p_{\begin{smallmatrix}1&2\\3&\\3&\end{smallmatrix},\begin{smallmatrix}1&2\\3&\\3&\end{smallmatrix}} & p_{\begin{smallmatrix}1&2\\3&\\3&\end{smallmatrix},\begin{smallmatrix}1&3\\2&\\2&\end{smallmatrix}} & p_{\begin{smallmatrix}1&2\\3&\\3&\end{smallmatrix},\begin{smallmatrix}1&4\\2&\\2&\end{smallmatrix}} & \\ & & p_{\begin{smallmatrix}1&3\\2&\\2&\end{smallmatrix},\begin{smallmatrix}1&2\\3&\\3&\end{smallmatrix}} & p_{\begin{smallmatrix}1&3\\2&\\2&\end{smallmatrix},\begin{smallmatrix}1&3\\2&\\2&\end{smallmatrix}} & p_{\begin{smallmatrix}1&3\\2&\\2&\end{smallmatrix},\begin{smallmatrix}1&4\\2&\\2&\end{smallmatrix}} & \\ & & p_{\begin{smallmatrix}1&4\\2&\\2&\end{smallmatrix},\begin{smallmatrix}1&2\\3&\\3&\end{smallmatrix}} & p_{\begin{smallmatrix}1&4\\2&\\2&\end{smallmatrix},\begin{smallmatrix}1&3\\2&\\2&\end{smallmatrix}} & p_{\begin{smallmatrix}1&4\\2&\\2&\end{smallmatrix},\begin{smallmatrix}1&4\\2&\\2&\end{smallmatrix}} & \\ & & & & & p_{\begin{smallmatrix}1\\2\\3\\4\end{smallmatrix},\begin{smallmatrix}1\\2\\3\\4\end{smallmatrix}} \end{pmatrix}$$

$$= \begin{pmatrix} [2]^2e_{\begin{smallmatrix}1&2\\3&4\end{smallmatrix},\begin{smallmatrix}1&2\\3&4\end{smallmatrix}} & [2]^2e_{\begin{smallmatrix}1&2\\3&4\end{smallmatrix},\begin{smallmatrix}1&3\\2&4\end{smallmatrix}} & & & & \\ [2]^2e_{\begin{smallmatrix}1&3\\2&4\end{smallmatrix},\begin{smallmatrix}1&2\\3&4\end{smallmatrix}} & [2]^2e_{\begin{smallmatrix}1&3\\2&4\end{smallmatrix},\begin{smallmatrix}1&3\\2&4\end{smallmatrix}} & & & & \\ & & [3][2]^2e_{\begin{smallmatrix}1&2\\3&\\3&\end{smallmatrix},\begin{smallmatrix}1&2\\3&\\3&\end{smallmatrix}} & [3][2]^2e_{\begin{smallmatrix}1&2\\3&\\3&\end{smallmatrix},\begin{smallmatrix}1&3\\2&\\2&\end{smallmatrix}} & [3][2]^2e_{\begin{smallmatrix}1&2\\3&\\3&\end{smallmatrix},\begin{smallmatrix}1&4\\2&\\2&\end{smallmatrix}} & \\ & & [3][2]^2e_{\begin{smallmatrix}1&3\\2&\\2&\end{smallmatrix},\begin{smallmatrix}1&2\\3&\\3&\end{smallmatrix}} & [3][2]^2e_{\begin{smallmatrix}1&3\\2&\\2&\end{smallmatrix},\begin{smallmatrix}1&3\\2&\\2&\end{smallmatrix}} & [3][2]^2e_{\begin{smallmatrix}1&3\\2&\\2&\end{smallmatrix},\begin{smallmatrix}1&4\\2&\\2&\end{smallmatrix}} & \\ & & [3][2]^2e_{\begin{smallmatrix}1&4\\2&\\2&\end{smallmatrix},\begin{smallmatrix}1&2\\3&\\3&\end{smallmatrix}} & [3][2]^2e_{\begin{smallmatrix}1&4\\2&\\2&\end{smallmatrix},\begin{smallmatrix}1&3\\2&\\2&\end{smallmatrix}} & [3][2]^2e_{\begin{smallmatrix}1&4\\2&\\2&\end{smallmatrix},\begin{smallmatrix}1&4\\2&\\2&\end{smallmatrix}} & \\ & & & & & [4][3][2]e_{\begin{smallmatrix}1\\2\\3\\4\end{smallmatrix},\begin{smallmatrix}1\\2\\3\\4\end{smallmatrix}} \end{pmatrix}$$

The special value $n = \pm\sqrt{2}$*, i.e. when* $[4] = 0$.

Then

$$p_{\begin{smallmatrix}1&4\\2&\\2&\end{smallmatrix},\begin{smallmatrix}1&43\\2&\end{smallmatrix}} = p_{\begin{smallmatrix}1\\2\\3\\4\end{smallmatrix},\begin{smallmatrix}1\\2\\3\\4\end{smallmatrix}} \qquad \text{and we let} \qquad p^{(2)}_{\begin{smallmatrix}1&4\\2&\\2&\end{smallmatrix},\begin{smallmatrix}1&4\\2&\\2&\end{smallmatrix}} = 1 - e_{\begin{smallmatrix}1&2\\3&\end{smallmatrix},\begin{smallmatrix}1&2\\3&\end{smallmatrix}}.$$

In this basis

$$\mathrm{Rad}(\mathbb{C}T_4) = \mathrm{span}\begin{pmatrix} 0 & 0 & & & & \\ 0 & 0 & & & & \\ & & 0 & 0 & 1 & \\ & & 0 & 0 & 1 & \\ & & 1 & 1 & 1 & \\ & & & & & 0 \end{pmatrix} \qquad \text{and} \qquad \mathrm{Rad}^2(\mathbb{C}T_4) = \mathrm{span}\begin{pmatrix} 0 & 0 & & & & \\ 0 & 0 & & & & \\ & & 0 & 0 & 0 & \\ & & 0 & 0 & 0 & \\ & & 0 & 0 & 1 & \\ & & & & & 0 \end{pmatrix}$$

$$\mathbb{C}T_1 = \{(a)\} = \left\{ \begin{pmatrix} a & \\ & a \end{pmatrix} \right\} = \left\{ \begin{pmatrix} a & 0 & \\ 0 & a & \\ & & a \end{pmatrix} \right\}$$

and

$$\mathbb{C}T_2 = \left\{ \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} a_1 & 0 & \\ 0 & a_2 & \\ & & a_2 \end{pmatrix} \right\}$$

and

$$\mathbb{C}T_3 = \left\{ \begin{pmatrix} a_{11} & a_{12} & \\ a_{21} & a_{22} & \\ & & a_3 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} a_{11} & a_{12} & & & & \\ a_{21} & a_{22} & & & & \\ & & a_{11} & a_{12} & 0 & \\ & & a_{21} & a_{22} & 0 & \\ & & 0 & 0 & a_3 & \\ & & & & & a_3 \end{pmatrix} \right\}$$

The special value $n = \pm 1$, i.e. when $[3] = 0$

Then

$$p_{\substack{1\,3\\2},\substack{1\,3\\2}} = p_{\substack{1\\2\\3},\substack{1\\2\\3}} \qquad \text{and we let} \qquad p^{(2)}_{\substack{1\,3\\2},\substack{1\,3\\2}} = 1 - e_{\substack{1\,2\\3},\substack{1\,2\\3}}.$$

In this basis

$$\mathrm{Rad}(\mathbb{C}T_3) = \mathrm{span} \begin{pmatrix} 0 & 1 & \\ 1 & 1 & \\ & & 0 \end{pmatrix} \qquad \text{and} \qquad \mathrm{Rad}^2(\mathbb{C}T_3) = \mathrm{span} \begin{pmatrix} 0 & 0 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}$$

Then

$$E_1 = \begin{pmatrix} 1 & 0 & \\ 0 & 0 & \\ & & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 1 & 1 & \\ 1 & 1 & \\ & & 0 \end{pmatrix}, \qquad 1 = \begin{pmatrix} 1 & 0 & \\ 0 & 0 & \\ & & 1 \end{pmatrix},$$

$$m_2 = \begin{pmatrix} 1 & 0 & \\ 0 & 0 & \\ & & 0 \end{pmatrix}, \qquad m_3 = \begin{pmatrix} -1 & 0 & \\ 0 & 1 & \\ & & 0 \end{pmatrix}.$$

$$\mathbb{C}T_1 = \left\{ \begin{pmatrix} a & \\ & a \end{pmatrix} \right\} = \left\{ \begin{pmatrix} a & 0 & \\ 0 & 0 & \\ & & a \end{pmatrix} \right\}$$

and

$$\mathbb{C}T_2 = \left\{ \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} a_1 & 0 & \\ 0 & 0 & \\ & & a_2 \end{pmatrix} \right\}$$

The special value $n = 0$, i.e. when $[2] = 0$.

Then

$$p_{12,12} = -p_{\substack{1\\2},\substack{1\\2}} \qquad \text{and we let} \qquad p^{(2)}_{12,12} = 1.$$

In the basis

$$\begin{pmatrix} p_{12,12} & \\ & p^{(2)}_{12,12} \end{pmatrix}$$

$$e_1 = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \qquad m_2 = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \qquad \text{and} \qquad \mathrm{Rad}(\mathbb{C}T_2) = \mathrm{span} \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$$

With respect to this basis there is a new matrix

$$\mathcal{E} = \begin{pmatrix} e_2 p_{12,12}^2 e_2 & e_2 p_{12,12} p_{12,12}^{(2)} e_2 \\ e_2 p_{12,12}^{(2)} p_{12,12} e_2 & e_2 (p_{12,12}^{(2)})^2 e_2 \end{pmatrix} = \begin{pmatrix} n & 1 \\ 1 & n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is *not diagonal*. In $\mathbb{C}T_3$ the basis elements

$$\begin{pmatrix} p^{(2)}_{\substack{1\,2\\3},\substack{1\,2\\3}} & p_{\substack{1\,2\\3},\substack{1\,3\\2}} & \\ p^{(2)}_{\substack{1\,3\\2},\substack{1\,2\\3}} & p^{(2)}_{\substack{1\,3\\2},\substack{1\,3\\2}} & \\ & & p_{\substack{1\\2\\3},\substack{1\\2\\3}} \end{pmatrix} = \begin{pmatrix} p_{12,12} e_2 p^{(2)}_{12,12} & p^{(2)}_{12,12} e_2 p^{(2)}_{12,12} & \\ p_{12,12} e_2 p_{12,12} & p_{12,12} e_2 p^{(2)}_{12,12} & \\ & & 1 - p^{(2)}_{\substack{1\,2\\3},\substack{1\,2\\3}} - p^{(2)}_{\substack{1\,3\\2},\substack{1\,3\\2}} \end{pmatrix}$$

form a set of matrix units. In this basis

$$E_1 = \begin{pmatrix} 0 & 1 & \\ 0 & 0 & \\ & & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}, \qquad 1 = \begin{pmatrix} 1 & 0 & \\ 0 & 1 & \\ & & 1 \end{pmatrix},$$

$$m_2 = \begin{pmatrix} 0 & 1 & \\ 0 & 0 & \\ & & 0 \end{pmatrix}, \qquad m_3 = \begin{pmatrix} -1 & 0 & \\ 0 & -1 & \\ & & 0 \end{pmatrix},$$

$$\mathbb{C}T_1 = \{(a)\} = \left\{ \begin{pmatrix} 0 & \\ & a \end{pmatrix} \right\}. = \left\{ \begin{pmatrix} a & 0 & \\ 0 & a & \\ & & a \end{pmatrix} \right\}$$

and

$$\mathbb{C}T_2 = \left\{ \begin{pmatrix} a_2 & \\ & a_1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} a_1 & a_2 & \\ 0 & a_1 & \\ & & a_1 \end{pmatrix} \right\}.$$

References

[GW1] F. Goodman and H. Wenzl, *The Temperley-Lieb algebra at roots of unity*, Pacific J. Math. **161** (1993), no. 2, 307–334.

[GL1] J. Graham and G. Lehrer, *Diagram algebras, Hecke algebras and decomposition numbers at roots of unity*, Ann. Sci. École Norm. Sup. (4) **36** (2003), no. 4, 479–524.

[GL2] J. Graham and G. Lehrer, *The two-step nilpotent representations of the extended affine Hecke algebra of type A*, Compositio Math. **133** (2002), no. 2, 173–197.